Chapter 2
The Basic Theory of Polycyclic Groups

Group Classes

This is effectively just a language, developed by P. Hall in the 1950's and 60's to make certain types of group theoretic arguments more concise while highlighting the essential components of the proof.

A group class is a class $\mathbf{X}$ of groups such that $H \cong G \in \mathbf{X}$ implies $H \in \mathbf{X}$ (the main condition; effectively we are dealing with isomorphism classes of groups rather than the groups themselves) and such that $\langle 1 \rangle \in \mathbf{X}$ (a convenient convention). For certain commonly used classes we have special notations. These include the following.

$\mathbf{F}$	the class of all finite groups	$\mathbf{A}$	the class of all abelian groups
$\mathbf{G}$	the class of all finitely generated groups	$\mathbf{S}$	the class of all soluble groups
$\mathbf{G}_1$	the class of all cyclic (= 1-generator) groups	$\mathbf{N}$	the class of all nilpotent groups
$\mathbf{I}$	the class of all trivial groups $\langle 1 \rangle$	$\mathbf{U}$	the class of all groups
$\mathbf{C}_n$	the class of cyclic groups of order n or 1		

An *operator* X on group classes is a function from group classes to group classes such that $X\mathbf{X} \subseteq X\mathbf{Y}$ whenever $\mathbf{X} \subseteq \mathbf{Y}$, such that $\mathbf{X} \subseteq X\mathbf{X}$ and such that $X\mathbf{I} = \mathbf{I}$ (necessarily $X\mathbf{U} = \mathbf{U}$). Operators multiply in the obvious way; viz. $YX\mathbf{X} = Y(X\mathbf{X})$. This multiplication of operators is associative but not commutative. If $X\mathbf{X} = \mathbf{X}$ we say $\mathbf{X}$ is X-closed. We say X is a *closure operator* if $X = X^2$; that is if $X(X\mathbf{X}) = X\mathbf{X}$, in other words if $X\mathbf{X}$ is X-closed for all $\mathbf{X}$.

We have the following commonly used closure operators. S is the subgroup operator; $S\mathbf{X}$ is the class of all subgroups of $\mathbf{X}$-groups. Now subgroups of soluble groups are soluble. Thus $S\mathbf{S} = \mathbf{S}$ and so $\mathbf{S}$ is S-closed (or say $\mathbf{S}$ is subgroup closed). Q is

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the quotient operator (some authors write H for Q); $Q\mathbf{X}$ is the class of all (homomorphic) images of $\mathbf{X}$-groups. Images of soluble groups are soluble. Hence $Q\mathbf{S} = \mathbf{S}$ and $\mathbf{S}$ is Q-closed (or image closed). In the same way we have that $\mathbf{A}, \mathbf{N}, \mathbf{F}, \mathbf{G}_1, \mathbf{U}$ and $\mathbf{I}$ are S-closed and Q-closed; $\mathbf{G}$ is Q-closed but not S-closed (try a free group).

Exercise Find a soluble example; we will see such examples later.

There is a product of group classes. If $\mathbf{X}$ and $\mathbf{Y}$ are group classes, then $\mathbf{XY}$ is the class of all groups G with a normal subgroup N such that N is an $\mathbf{X}$-group and G/N is a $\mathbf{Y}$-group. We say G in $\mathbf{XY}$ is an extension of an $\mathbf{X}$-group by a $\mathbf{Y}$-group or just say G is $\mathbf{X}$ by $\mathbf{Y}$. For example $\mathbf{SS} = \mathbf{S}$ while $\mathbf{S} \supset \mathbf{NN} \supset \mathbf{N}$. Clearly $\mathbf{IX} = \mathbf{XI} = \mathbf{X}$ and $\mathbf{UX} = \mathbf{XU} = \mathbf{U}$. Warning: in general $(\mathbf{XY})\mathbf{Z} \neq \mathbf{X}(\mathbf{YZ})$. For example, the alternating group Alt(4) of order 12 lies in $(\mathbf{G}_1\mathbf{G}_1)\mathbf{G}_1$ but not in $\mathbf{G}_1(\mathbf{G}_1\mathbf{G}_1)$.

Exercise For any group classes $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$, prove that $(\mathbf{XY})\mathbf{Z}$ always contains $\mathbf{X}(\mathbf{YZ})$.

An important closure operator, especially for us here, is the poly operator P (some authors write E for P). If G is a group with a series of subgroups

$$\langle 1 \rangle = G_0 \lhd G_1 \lhd \cdots \lhd G_r = G, \qquad (*)$$

where r is an integer, and if each $G_{i+1}/G_i \in \mathbf{X}$, we write $G \in P\mathbf{X}$ and say G is poly $\mathbf{X}$. Then P is a closure operator; further $\mathbf{X}$ is P-closed if and only if whenever $N \lhd G \in \mathbf{U}$ with N and G/N in $\mathbf{X}$, then $G \in \mathbf{X}$. For example, $P\mathbf{A} = \mathbf{S}$, $P\mathbf{N} = \mathbf{S}$ and $P\mathbf{S} = \mathbf{S}$. The primary object of study of this book is $\mathbf{P} = P\mathbf{G}_1$, the class of polycyclic groups.

2.1 For any group class $\mathbf{X}$ we have the following.

(a) $SP\mathbf{X} \subseteq PS\mathbf{X}$ (we write $SP \leq PS$).
(b) $QP\mathbf{X} \subseteq PQ\mathbf{X}$ (we write $QP \leq PQ$).

Proof (a) If H is a subgroup of G and we are given the series $(*)$ above, then

$$\langle 1 \rangle = G_0 \cap H \lhd G_1 \cap H \lhd \cdots \lhd G_r \cap H = H$$

is a series for H and

$$\begin{aligned}(G_i \cap H)/(G_{i-1} \cap H) &= (G_i \cap H)/(G_{i-1} \cap (G_i \cap H)) \\ &\cong (G_i \cap H)G_{i-1}/G_{i-1} \leq G_i/G_{i-1}.\end{aligned}$$

If follows that if $G \in P\mathbf{X}$ then $H \in PS\mathbf{X}$ and hence that $SP \leq PS$.

(b) If N is a normal subgroup of G and again we are given $(*)$, then

$$\langle 1 \rangle = G_0N/N \lhd G_1N/N \lhd \cdots \lhd G_rN/N = G/N$$

is a series for G/N and

$$(G_i N/N)/(G_{i-1}N/N) \cong G_i N/G_{i-1}N \cong G_i/(G_i \cap G_{i-1}N)$$
$$= G_{i/}/G_{i-1}(G_i \cap N),$$

which is an image of G_i/G_{i-1}. Thus if $G \in P\mathbf{X}$ then $G/N \in PQ\mathbf{X}$ and so $QP \leq PQ$. □

2.2 Corollary Subgroups and images of polycyclic groups are polycyclic.

For $S\mathbf{P} = SP\mathbf{G}_1 \subseteq PS\mathbf{G}_1 = P\mathbf{G}_1 = \mathbf{P}$. Similarly $Q\mathbf{P} = \mathbf{P}$.

If $\mathbf{X}$ is any group class, a group G is *residually* $\mathbf{X}$ (and we write $G \in R\mathbf{X}$) if for each g in $G\backslash\langle 1\rangle$ there is a normal subgroup N of G such that $g \notin N$ and $G/N \in \mathbf{X}$; that is, if the intersection of the normal subgroups N of G with G/N an $\mathbf{X}$-group is trivial. Equivalently, if for each $g \in G\backslash\langle 1\rangle$ there is a homomorphism ϕ of G onto an $\mathbf{X}$ group with $g\phi \neq 1$. It is easy to check that R is another closure operator.

Exercise If $\mathbf{X}$ is an S-closed class prove that the group $G \in R\mathbf{X}$ if and only if G is isomorphic to a subgroup of a cartesian product of $\mathbf{X}$-groups.

There is one further standard operator that we shall have occasional recourse to, namely the *local* operator. A group $G \in L\mathbf{X}$, that is lies in the class of *locally* $\mathbf{X}$-groups, if for every finite *subset* F of G there is an $\mathbf{X}$-subgroup H of G containing F. Again L is a closure operator. Note that this is not quite the same as demanding that the finitely generated subgroups of G be $\mathbf{X}$-groups. If $\mathbf{X}$ is S-closed then $G \in L\mathbf{X}$ if and only if every finitely generated subgroup of G is an X-group (simply replace H by the subgroup generated by F). Thus a group is locally soluble (resp. locally nilpotent; resp. locally finite) if each of its finitely generated subgroups is soluble (resp. nilpotent; resp. finite). Clearly $L\mathbf{A} = \mathbf{A}$.

Exercise Determine the classes $L\mathbf{G}_1$ (consider the sections of the additive group of the rationals) and $L\mathbf{G}$.

Polycyclic Groups

We are now ready to start on our main area of study. Just to recall what we have above, a group G is polycyclic if it has a series of finite length with cyclic factors and subgroups and images of polycyclic groups are polycyclic. Further a polycyclic-by-finite group is a group with a polycyclic normal subgroup of finite index. We will see below that for many purposes the class **PF** of polycyclic-by-finite groups is a more natural object of study than **P** itself. It follows from the polycyclic case that subgroups and images of polycyclic-by-finite groups are also polycyclic-by-finite.

2.3 Let G be a group. The following are equivalent.

(a) Every subgroup of G is finitely generated.

(b) Every ascending chain $G_1 \le G_2 \le \cdots \le G_i \le \cdots$, where $i = 1, 2, \ldots$, of subgroups of G contains only finitely many distinct members.
(c) Every non-empty set of subgroups of G has a maximal member.

Proof This is a special case of a well-known bit of universal algebra depending ultimately on the axiom of choice.

(a) implies (b). The union H of the G_i is finitely generated by (a), say by $g_1, g_2, \ldots, g_n$. Then there exists j such that G_j contains all the g_k. Thus $G_i = G_j$ for $i \ge j$.

(b) implies (c). Let $\mathcal{S}$ be a non-empty set of subgroups of G with no maximal member. There exists G_1 in $\mathcal{S}$ and G_1 is not a maximal member of $\mathcal{S}$. Hence there exists G_2 in $\mathcal{S}$ with $G_1 < G_2$. Again G_2 is not maximal so there exists G_3 in $\mathcal{S}$ with $G_2 < G_3$. Keep going infinitely often (this is where the axiom of choice kicks in). We obtain a contradiction of (b).

(c) implies (a). Consider $H \le G$. Let $\mathcal{S}$ denote the set of all finitely generated subgroups of H. Then $\langle 1 \rangle$ lies in $\mathcal{S}$, so $\mathcal{S}$ is not empty. By (c) there is a maximal member M of $\mathcal{S}$. If $h \in H$, then $\langle M, h \rangle$ lies in $\mathcal{S}$ and contains M. Consequently $M = \langle M, h \rangle$ and $h \in M$. Therefore $H = M$, which lies in $\mathcal{S}$, so H is finitely generated. □

If the group G satisfies the conditions of 2.3 we say that G satisfies the *maximal condition on subgroups*, a phrase which we shorten to max, and write $G \in \mathbf{Max}$; that is **Max** denotes the class of groups satisfying max. Clearly $\mathbf{G}_1 \cup \mathbf{F} \subseteq \mathbf{Max}$.

2.4

(a) **G** is Q- and P-closed.
(b) **Max** is S-, Q- and P-closed.

The class **G** is not S-closed: let G denote the following subgroup of 2 by 2 rational matrices.

$$\left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Then G contains the subgroup

$$\left\langle \begin{pmatrix} 1 & 0 \\ r2^s & 1 \end{pmatrix} : \text{all integers } r, s \right\rangle,$$

which is isomorphic to the non-finitely generated, additive group of the ring $\mathbf{Z}[1/2]$.

Proof of 2.4 (a) This is obvious.

(b) The S-closure follows from 2.3(a) and the Q-closure from 2.3(a) and part (a). Using 2.1 and 2.3 we have

$$SP\mathbf{Max} \subseteq PS\mathbf{Max} \subseteq P\mathbf{G} = \mathbf{G} \quad \text{and} \quad P\mathbf{Max} \subseteq \mathbf{Max}. \qquad \square$$

2.5 Corollary (Hirsch 1938a, 1938b) $P(\mathbf{G}_1 \cup \mathbf{F}) \subseteq \mathbf{Max}$ and $\mathbf{G} \cap \mathbf{A} \subseteq P\mathbf{G}_1 \subseteq \mathbf{Max}$.

Exercise Prove that $\mathbf{P} = \mathbf{S} \cap \mathbf{Max} \subseteq \mathbf{S} \cap \mathbf{G}$ and $\mathbf{G} \cap \mathbf{A} = \mathbf{Max} \cap \mathbf{A}$.

For a long time the $P(\mathbf{G}_1 \cup \mathbf{F})$ groups were the only known groups with max. Then Ol'shanskii (1979) produced examples of infinite groups all of whose proper subgroups are cyclic of prime order. Such groups clearly satisfy max (and also min for that matter). As time went by more examples of this general type were discovered. For example groups were constructed with the proper subgroups all of the same prime order. None of these constructions are easy. See Ol'shanskii (1991) for an account of this.

2.6 Theorem (Hirsch 1938a, 1938b, 1946) Let G be a group. The following are equivalent.

(a) G has a series $\langle 1 \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_r = G$ with each factor cyclic or finite.
(b) G has a series $\langle 1 \rangle = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_s \triangleleft G$ with each H_i/H_{i-1} infinite cyclic and G/H_s finite.
(c) G has a series $\langle 1 \rangle = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_t \triangleleft G$ with each $K_i \triangleleft G$, with each K_i/K_{i-1} free abelian of finite rank and with G/K_t finite.

2.7 Corollary The following hold.

(a) $P(\mathbf{G}_1 \cup \mathbf{F}) = \mathbf{PF}$. (Thus poly (cyclic or finite) groups are polycyclic by finite.)
(b) Polycyclic groups are torsion-free by finite.
(c) If G is a polycyclic-by-finite group, the finite subgroups of G have bounded order.
(d) $\mathbf{P} \subseteq (P\mathbf{Z})\mathbf{F}$; that is, polycyclic groups are (poly infinite-cyclic) by finite.

Proof of 2.6 Trivially (c) implies (b) and (b) implies (a). Suppose (a) holds. We prove (b), which is the main part of the proof. We induct on r. By induction applied to G_{r-1} we may assume that G_i/G_{i-1} is infinite cyclic for $i \le r-2$ and G_{r-1}/G_{r-2} is finite, of order n say. Set $N = G_{r-1}^n = \langle g^n : g \in G_{r-1} \rangle$. Then N is normal in G and is contained in G_{r-2} with G_{r-1}/N finite. Also

$$\langle 1 \rangle \triangleleft N \cap G_1 \triangleleft N \cap G_2 \triangleleft \cdots \triangleleft N \cap G_{r-3} \triangleleft N$$

is a series for N with infinite cyclic factors. If G/G_{r-1} is finite, then G/N is finite and (b) holds.

Suppose G/G_{r-1} is infinite (and hence cyclic). Set $P = G_{r-1}/N$ and $Q = G/N$. Put $C = C_Q(P)$. Then P is finite and Q/C embeds into $\operatorname{Aut} P$, so Q/C is finite. Also $C/(C \cap P)$ is infinite cyclic, being isomorphic to a non-trivial subgroup of $Q/P \cong G/G_{r-1}$, and $C \cap P$ is central in C. Thus C is abelian and if $m = |C \cap P|$ then C^m is infinite cyclic and normal in Q and $(Q : C^m) = (Q : C)(C : C^m)$ is finite. Set $L/N = C^m$. Then L/N is infinite cyclic, G/L is finite and

$$\langle 1 \rangle \triangleleft N \cap G_1 \triangleleft N \cap G_2 \triangleleft \cdots \triangleleft N \cap G_{r-3} \triangleleft N \triangleleft L \triangleleft G$$

is a series of the type required by (b).

It remains to prove that (b) implies (c). Now in (b) the group H_s is clearly soluble; let K_1 be the last but one term of its derived series. Then K_1 is polycyclic, abelian, torsion-free and normal in G. In particular K_1 is free abelian of finite rank. Repeat with G/K_1 to define K_2/K_1. Continue in this way. By 2.5 this process stops, say at K_t, after a finite number of steps. Clearly then G/K_t is finite. □

The Hirsch Number

This numerical invariant is an important tool for inductive proofs involving polycyclic groups. Let $\langle 1\rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_r = G$ be a series of the group G with factors cyclic or finite. Now if A is a subgroup of the infinite cyclic group B, then either $A = \langle 1\rangle$ and B/A is infinite cyclic, or A is infinite cyclic and B/A is finite. Thus any refinement of the above series has the same number of infinite cyclic factors as the original series. Now any two series have isomorphic refinements (a theorem of Schreier from 1928, e.g. see Wehrfritz (1999), Theorem 1.3). Thus any two poly (cyclic or finite) series of a group G have the same number of infinite cyclic factors (Hirsch 1938a, 1938b). This invariant of the polycyclic-by-finite group G is called the Hirsch number of G; we denote it by $h(G)$. Frequently arguments use induction on this non-negative integer $h(G)$.

Exercise If H is a normal subgroup and K is a subgroup of the polycyclic-by-finite group G, prove that

$$h(G) = h(H) + h(G/H) \quad \text{and} \quad h(K) \leq h(G).$$

Polycyclic groups are greatly influenced by their finite images. Our first example of this is the following further theorem of Hirsch.

2.8 (Hirsch 1946) If every finite image of the polycyclic-by-finite group G is nilpotent, then G is nilpotent.

Proof Clearly we may assume that G is infinite, so by 2.6 there is a non-trivial torsion-free abelian normal subgroup A of G. By induction on $h(G)$ each G/A^p is nilpotent for each prime p. If $r = h(A)$ (equivalently $r = \operatorname{rank} A$), then $(A : A^p) = p^r$ and $[A, {}_r G] \leq \bigcap_p A^p = \langle 1\rangle$. Also G/A is nilpotent. Therefore G is nilpotent. □

Developments of 2.8

This theorem of Hirsch stimulated a long chain of results. Baer (1957) proved that a polycyclic-by-finite group with all its finite images supersoluble is itself supersoluble. Unlike 2.8 this is not an elementary result; it involves some substantial number theory. For a proof see Segal (1983) p. 54, Theorem 1 or Wehrfritz (1973a) Theorem 11.11. Independently Robinson (1970) and Wehrfritz (1970) proved that a finitely generated soluble group with each of its finite images nilpotent is nilpotent (see Robinson (1980) 15.5.3 for a proof). Platonov (1966) and Wehrfritz (1968), again independently and with different approaches, proved that a finitely generated group of matrices over a field with each of its finite images nilpotent, is nilpotent, see Wehrfritz (1973a) 4.16 and 10.5 for the two different approaches. The example given between 2.4 and 2.5 above is a finitely generated soluble (even metabelian) matrix group with all its finite images supersoluble that is not supersoluble. The more complex situation here is completely analysed in Segal (1975a). More recently Endimioni (1998) proved that if every finite image of a polycyclic group G has a series of length n with nilpotent factors, then so does G. For $n = 1$ this becomes Hirsch's theorem 2.8.

2.9 Theorem (Mal'cev 1958) Let H be any subgroup of the polycyclic-by-finite group G. Then H is the intersection of all the subgroups of G of finite index containing H.

Proof Note first that if B is a finitely generated abelian group, then $\bigcap_{r\geq 1} B^r = \langle 1\rangle$, for if B is the direct product $\bigtimes_i \langle b_i\rangle$, then $B^r = \bigtimes_i \langle b_i^r\rangle$ and $\bigcap_r B^r = \bigtimes_i \bigcap_r \langle b_i^r\rangle = \langle 1\rangle$.

Again we induct on $h(G)$. If G is finite there is nothing to prove, so assume otherwise. Then we have a non-trivial torsion-free abelian normal subgroup A of G. Set $H^\wedge$ equal to the intersection of all the subgroups of G of finite index containing H. By induction applied to each G/A^r we have

$$H \leq H^\wedge \leq \bigcap_{r\geq 1} HA^r \leq HA.$$

Thus

$$H^\wedge \leq H\left(A \cap \bigcap_{r\geq 1} HA^r\right) = H\left(\bigcap_{r\geq 1} (H\cap A)A^r\right) = H(H\cap A) = H,$$

since $A/(H\cap A)$ is a finitely generated abelian group. Therefore $H = H^\wedge$. □

2.10 Corollary (Hirsch 1952) If G is a polycyclic-by-finite group, then G is residually finite.

Proof If $L \leq G$, then $L_G = \bigcap_{g\in G} L^g$ is the largest normal subgroup of G contained in L. If L has finite index in G, then $G = \bigcup_{x\in X} Lx$ for some finite subset X of G, $L_G = \bigcap_{x\in X} L^x$ and L_G has finite index in G. Therefore

$$\begin{aligned} H^\wedge &= \bigcap (L : H \leq L \leq G \text{ with } (G:L) \text{ finite}) \\ &= \bigcap (HN : N \triangleleft G \text{ with } (G:N) \text{ finite}). \end{aligned}$$

In particular if G is polycyclic-by-finite and if $H = \langle 1 \rangle$, this and 2.9 say that G is residually finite. □

There are too many results related to 2.10 to be able to mention more than just a couple of them. By a very deep theorem (Roseblade 1973, 1976 with Hall 1959 and Jategaonkar 1974) a finitely generated abelian-by-polycyclic-by-finite group is always residually finite. Much of the latter part of this book revolves around this result, see 9.13 below. Also (Mal'cev 1940) any finitely generated matrix group (over a field) is residually finite (see Wehrfritz 1973a 4.2 for a proof).

If p is a prime and π a set of primes, we denote the class of finite p-groups by $\mathbf{F}_{\{p\}}$ and the class of finite π-groups by $\mathbf{F}_\pi$

2.11 Theorem (Shmel'kin 1968 and Wehrfritz 1970 independently)

$$\mathbf{PF} \subseteq \bigcap_{\text{primes } p} ((R\mathbf{F}_{\{p\}})\mathbf{F}).$$

That is, if G is a polycyclic-by-finite group and p is any prime, then there exists N_p normal of finite index in G such that N_p is residually a finite p-group.

We will give a very easy proof of this theorem in Chap. 4, see 4.10.

Exercise Give a direct proof of 2.11 from the above—see Shmel'kin (1968).

2.12 Corollary (Learner 1964) If $G$is polycyclic-by-finite there exists a finite set π of primes such that G is residually a finite π-group (that is, $G \in R\mathbf{F}_\pi$).

Proof With N_2 as in the statement of 2.11, let π denote the set of prime divisors of $2(G : N_2)$. □

2.13 Theorem (Hirsch 1938a, 1938b) $\mathbf{G} \cap \mathbf{N} \subseteq \mathbf{P}$. That is, finitely generated nilpotent groups are polycyclic.

Proof Let $\gamma^c G$ be the last non-trivial term of the lower central series of G. By induction we may assume that $G/\gamma^c G$ is polycyclic. In particular $\gamma^{c-1}G/\gamma^c G$ is finitely generated. Suppose $G = \langle x_1, x_2, \ldots, x_m \rangle$ and $\gamma^{c-1}G = \langle y_1, y_2, \ldots, y_n \rangle \gamma^c G$. Then

$$\gamma^c G = [\langle x_1, x_2, \ldots, x_m \rangle, \langle y_1, y_2, \ldots, y_n \rangle] = \langle [x_i, y_j] : 1 \le i \le m \;\&\; 1 \le j \le n \rangle$$

by 1.2(b), since $\gamma^c G$ is central in G. Thus $\gamma^c G$ is finitely generated and it follows that G is polycyclic.

Alternatively if G is any group with G/G' finitely generated then part (b) of the proof of 1.23 and a trivial induction shows that $\gamma^n G/\gamma^{n+1} G$ is finitely generated for each $n \ge 1$ and then 2.13 follows easily. (A third proof will be given in Chap. 3.) □

2.14

(a) If G is a nilpotent group, then G has normalizer condition, meaning that $H < G$ implies that $H < N_G(H)$.
(b) If G is a group with normalizer condition, then the set of all elements of G of finite order is a subgroup of G and is a direct product of p-groups, one for each prime p.
(c) (Hirsch 1938a, 1938b) Finitely generated nilpotent groups are finite by torsion-free.

Proof (a) The group G has a central series $\langle 1\rangle = G_0 \le G_1 \le \cdots \le G_r = G$ of finite length. Since $H \ne G$ there exists s with $G_s \le H$ and G_{s+1} not contained in H. Also $[H, G_{s+1}] \le [G, G_{s+1}] \le G_s \le H$, so $G_{s+1} \le N_G(H)$. Therefore $H \ne N_G(H)$.

(b) Let p be a prime and let G_p be any maximal p-subgroup of G (such exists by 1.13). Then G_p is characteristic in $N_G(G_p)$, so G_p is normal in $N_G(N_G(G_p))$. The latter is therefore equal to $N_G(G_p)$ and hence is G by hypothesis. Thus G_p is normal in G and hence is the unique maximal p-subgroup of G. It follows that the set of elements of G if finite order is equal to $\langle G_p : p \text{ prime}\rangle$ and that the latter is just the direct product of the G_p.

(c) This follows from 2.13, 2.7(b) and parts (a) and (b). □

Note that polycyclic groups need not be finite by torsion-free. For example the infinite dihedral group

$$D_\infty = \langle a, x : a^x = a^{-1}, x^2 = 1\rangle = \langle x\rangle[\langle a\rangle$$

is clearly polycyclic, being infinite-cyclic by cyclic-of-order-2. Also

$$(xa)^2 = xaxa = x^{-1}axa = a^{-1}a = 1.$$

Thus xa and x both have finite order (namely 2) and clearly $D_\infty = \langle xa, x\rangle$, so D_∞ is not finite by torsion-free.

Exercise Prove that D_∞ is isomorphic to the following matrix group over the integers

$$\left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

2.15 Let G be a nilpotent group. If $Z = \zeta_1(G)$ is torsion-free, then so are G, G/Z and each $\zeta_i(G)/\zeta_{i-1}(G)$. If $Z^n = \langle 1\rangle$, then $\zeta_i(G)^n \le \zeta_{i-1}(G)$ for all $i \ge 1$ and G has finite exponent (meaning that $G^m = \langle 1\rangle$ for some positive integer m).

Proof Suppose Z is torsion-free and let $z \in \zeta_2(G)$ with $z^r \in Z$. If $g \in G$, then $1 = [z^r, g] = [z, g]^r$ by 1.2(b), since $[z, g] \in Z$ is central. Then $[z, g] = 1$, for all g, so $z \in Z$ and $\zeta_2(G)/\zeta_1(G)$ is torsion-free. By an elementary induction each $\zeta_i(G)/\zeta_{i-1}(G)$ is torsion-free. Consequently so too are G and G/Z.

Exercise Poly torsion-free is torsion-free.

Now assume that $Z^n = \langle 1 \rangle$. If $z \in \zeta_2(G)$, then $[z^n, g] = [z, g]^n = 1$. It follows that $\zeta_2(G)^n \le \zeta_1(G)$. The remaining claims follow easily by induction. □

If G is any group we denote by $\tau(G)$ the unique maximal periodic normal subgroup of G. (A *periodic* group is one all of whose elements have finite order.)

Exercise Any group has a unique maximal periodic normal subgroup.

Warning: in general $\tau(G)$ is not equal to the set of elements in G of finite order and $G/\tau(G)$ need not be torsion-free. For example $\tau(D_\infty) = \langle 1 \rangle$.

2.16 Theorem (Gruenberg 1957) Let G be a finitely generated nilpotent group.

(a) If $\tau(G) = \langle 1 \rangle$ (equivalently by 2.14, if G is torsion-free), then $G \in \bigcap_p R\mathbf{F}_{\{p\}}$, the intersection being over all primes p; that is, G is residually a finite p-group for every prime p.
(b) If $\tau(G) \neq \langle 1 \rangle$ is a π-group, then $G \in R\mathbf{F}_\pi$.

Proof (a) Set $Z = \zeta_1(G)$. By 2.15 and induction on the nilpotency class $G/Z \in R\mathbf{F}_{\{p\}}$. Let $z \in Z \backslash \langle 1 \rangle$. Now Z is free abelian of finite rank by 2.13, so $z \notin Z^q$ for $q = p^i$ and some i. Pick (by 2.5) N with $Z^q \le N \lhd G$ and maximal in G subject to $z \notin N$. Now G/N is polycyclic (2.13), so G/N is residually finite by 2.10. Hence there is $M \ge N$, normal of finite index in G with $zN \notin M/N$, that is with $z \notin M$. By the choice of N we have $M = N$, so G/N is finite. By 2.14 the group G/N is a direct product of its Sylow subgroups. The choice of N shows that G/N is an r-group for some prime r. But $|zN| \neq 1$ divides $q = p^i$. Consequently $r = p$ and G/N is a p-group. Therefore $G \in R\mathbf{F}_{\{p\}}$.

(b) Now $\tau(G) = \times_{p\in\pi} G_p$, where G_p is a p-group. Set $G_{p'} = \langle G_q : q \neq p \rangle$. If $z \in \tau(G) \backslash G_{p'}$, pick $N \lhd G$ maximal subject to $G_{p'} \le N$ and $z \notin N$. As in the previous case G/N is a finite p-group. Also $G/\tau(G) \in R\mathbf{F}_{\{p\}}$ by part (a). Hence $G/G_{p'} \in R\mathbf{F}_{\{p\}}$. Clearly $\bigcap_p G_{p'} = \langle 1 \rangle$. Thus $G \cong G/\bigcap_p G_{p'} \in R(\bigcup_{p\in\pi} \mathbf{F}_{\{p\}}) \subseteq R\mathbf{F}_\pi$. □

There are other residual properties known to hold in finitely generated nilpotent groups. For example there is the following result of Higman (1955). If G is such a group and if π is any infinite set of primes, then $\bigcap_{p\in\pi} G^p$ is finite; equivalently, if G is also torsion-free, then $\bigcap_{p\in\pi} G^p = \langle 1 \rangle$. For generalizations of this (and a proof) see Wehrfritz (1972). It is easy to see that this result of Higman's does not extend to polycyclic groups (just try the infinite dihedral group). The same applies to Gruenberg's Theorem, as the following theorem shows. (It also shows the necessity of the finite pieces in 2.11.)

2.17 Theorem (Seksenbaev 1965) Let π be an infinite set of primes. If G is a polycyclic-by-finite group with $G \in \bigcap_{p\in\pi} R\mathbf{F}_{\{p\}}$, then G is torsion-free and nilpotent.

To prove 2.17 we need the following special case of something known as Learner's Lemma. We will apply it with $\mathbf{X} = \mathbf{F}_{\{p\}}$.

2.18 If A is a maximal abelian normal subgroup of a group G and if $G \in R\mathbf{X}$, then $G/A \in RQ\mathbf{X}$.

Proof Set $B = \bigcap(AN : N \lhd G \text{ with } G/N \in \mathbf{X})$. Now AN/N is abelian, so $(AN)' \leq N$ and $B' \leq \bigcap_{G/N\in\mathbf{X}} N = \langle 1\rangle$, since G is residually $\mathbf{X}$. Thus B is an abelian normal subgroup of G containing A. Consequently $B = A$. Clearly $G/AN \in Q\mathbf{X}$. Therefore $G/A \in RQ\mathbf{X}$.

Proof of 2.17 If $x \in H \in R\mathbf{F}_{\{p\}}$ with $|x|$ finite, then there is a normal subgroup N_i of H with H/N_i a p-group and $x^i \notin N_i$, this for each i with $1 \leq i < |x|$. Set $N = \bigcap_i N_i$. Then H/N is a p-group and $\langle x\rangle \cap N = \langle 1\rangle$. Thus $\langle x\rangle$ is a p-group. This shows that any element of finite order in a residually (finite p-group) is a p-element. Therefore in 2.17 the group G is torsion-free.

Let A be a maximal abelian normal subgroup of G (A exists by 2.5 for example). By 2.18 we have $G/A \in \bigcap_{p\in\pi} R\mathbf{F}_{\{p\}}$. By induction on the Hirsch number G/A is nilpotent. Pick $B \leq A$ with B normal in G, G/B nilpotent and B of least Hirsch number (= rank here) amongst subgroups of G with these properties. Clearly we may assume that $B \neq \langle 1\rangle$.

Let $p \in \pi$. There exists a normal subgroup N of G of finite index with G/N a finite p-group and B not contained in N, that is, with $B > B\cap N$. Now G/N is nilpotent, say of class c, so $[B, {}_cG] \leq B\cap N$. Thus $[B, G](B\cap N) < B$. Also $B/(B\cap N)$ is a p-group, so the cyclic group of order p is an image of $B/[B, G]$. This is for all p in the infinite set π. Hence $B/[B, G]$ is infinite and $h(B) > h([B, G])$. Also G/B is nilpotent, so $G/[B, G]$ is nilpotent. This contradicts the choice of B, so $B = \langle 1\rangle$ and G is nilpotent. □

H is a *subnormal* subgroup of the group G (we write $H \lhd\lhd G$) if there exists finitely many subgroups H_i of G with

$$H = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_r = G.$$

For example, cf. 2.14(a), every subgroup of a nilpotent group G is subnormal in G, while not every subgroup of the finite metabelian group Sym(3) is subnormal.

2.19 Theorem (Kegel 1966) Let H be a subgroup of the polycyclic-by-finite group G. Then H is subnormal in G if and only if HN/N is subnormal in G/N for every normal subgroup N of finite index in G. The latter condition is equivalent to $HN \lhd\lhd G$ for all such N.

So again the finite images of G determine the property for the group itself. Notice that in any group G, if

$$N \le H = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_r = G$$

with N normal in G, then

$$H/N = H_0/N \lhd H_1/N \lhd H_2/N \lhd \cdots \lhd H_r/N = G/N$$

and conversely, so the final claim is immediate. If G is polycyclic-by-finite and $H \le G$, then $H = \bigcap_N HN$ by 2.9, see proof of 2.10, where N ranges over the normal subgroups of G of finite index. Thus it is immediate that H is normal in G if and only if all the HN are normal in G. A problem in 2.19 is that the subnormal chains for the HN in G may have unbounded length as N varies. If

$$G = \langle a, x, y : a^x = a^{-1}, a^y = a^2, xy = yx, x^2 = 1\rangle,$$

then G is finitely generated and soluble (even metabelian). If $H = \langle a, x\rangle \le G$, then H is not subnormal in G (since $[y^{-n}, a, {}_{n-1}x] \notin H$ for every positive integer n) but every HN for N a normal subgroup of G of finite index is normal in G ($a \in H$ and $A/\langle a\rangle$ for $A = \langle a^G\rangle$ is a Prüfer 2-group, so $A \le HN \lhd G$). For further details see Lennox (1976).

Exercise Prove that this G is isomorphic to the rational matrix group

$$\left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Proof Clearly we may assume that G is infinite, so G has a non-trivial free-abelian normal subgroup A. By induction on the Hirsch number we may also suppose that AH/A is subnormal in G/A; consequently AH is now subnormal in G. It suffices to prove that H is subnormal in AH. Now A is normal in G, so $A \cap H$ is normal in H. Also A is abelian, so $A \cap H$ is normal in A. Therefore $A \cap H$ is normal in AH. Let $x \mapsto x^*$ denote the natural projection of AH onto $AH/(A \cap H)$. Then $(AH)^* = A^*]H^*$, the split extension of A^* by H^*.

Also by induction $A^n H$ is subnormal in G for $n = 1, 2, \ldots$, so $(A^n)^* H^*$ is subnormal in A^*H^*, say in $k(n)$ steps. Then $[A^*H^*, {}_{k(n)}(A^n)^*H^*] \le (A^n)^*H^*$, after $k(n)$ applications of 1.4. Hence

$$[A^*, {}_{k(n)}H^*] \le (A^n)^*H^* \cap A^* = (A^n)^*(H^* \cap A^*) = (A^n)^*.$$

Set $t = |\tau(A^*)|$ and $r = h(A^*)$. If p is a prime not dividing t, then $|A^*/(A^p)^*| = p^r$ and so $[A^*, {}_r H^*] \le \bigcap_p (A^p)^*$, where p runs over all primes p not dividing t. This intersection is $\tau(A^*)$. Hence

$$[A^*, {}_{\max\{r,k(t)\}}H^*] \le \tau(A^*) \cap (A^t)^* = \langle 1\rangle.$$

Set $m = \max\{r, k(t)\}$. Then

$$H^* \triangleleft H^*[A^*, {}_{m-1}H^*] \triangleleft H^*[A^*, {}_{m-2}H^*] \triangleleft \cdots \triangleleft H^*A^*.$$

Therefore, since $A \cap H \leq H$, we have that H is subnormal in AH, which is subnormal in G. The proof is complete. □

In fact 2.19 extends 2.8. To see this one needs the following.

2.20 If G is a polycyclic-by-finite group the following are equivalent.

(a) G is nilpotent.
(b) Every subgroup of G is subnormal in G.
(c) G has normalizer condition.

Proof Since $G \in$ **Max** it is easy to see that (b) and (c) are equivalent. Also (a) implies (c) by 2.14(a). Suppose (c). If G^* is any finite image of G, then G^* is a direct product of its p-subgroups and hence is nilpotent by 2.14(b). By 2.8 the group G is nilpotent; that is, (a) holds. □

We summarize some further examples of the influence of the finite images of a polycyclic-by-finite group on the group itself.

2.21 Theorem (Lennox and Wilson 1977) Let H and K be subgroups of the polycyclic-by-finite group G. The following are equivalent.

(a) $HK = KH$.
(b) $H\phi K\phi = K\phi H\phi$ for all homomorphisms ϕ of G to finite groups.
(c) $HKN = KHN$ for all normal subgroups N of G of finite index.

Clearly (a) implies (b) and it is very easy to see that (b) and (c) are equivalent. Thus (b) implies (a) is the meat of the theorem. See Lennox and Wilson (1977, 1979) for proofs.

2.22 Theorem (Remeslennikov 1969; Formanek 1970, 1976) Let G be a polycyclic-by-finite group. If x and y are elements of G, then x and y are conjugate in G if and only if $x\phi$ and $y\phi$ are conjugate in $G\phi$ for every homomorphism ϕ of G to a finite group.

Apart from the original references, see Segal (1983), p. 59 for a proof. A group in which the conjugacy of elements is determined by the finite images of the group, as in 2.22, is said to be *conjugacy separable*. Behind 2.22 lies some substantial number theory, so its proof is not elementary. For the nilpotent case see 5.14 below and for the more recent result that the free product of two polycyclic-by-finite groups amalgamating a cyclic subgroup is conjugacy separable, see Ribes et al. (1998).

The reader may be beginning to wonder whether a polycyclic group is actually determined by its finite images. This is not the case. There exist non-isomorphic polycyclic groups (even nilpotent ones) with the same set of finite images. For example Baumslag (1974) shows that

$$\langle a, b | a^{25} = 1, a^b = a^6 \rangle \quad \text{and} \quad \langle a, b | a^{25} = 1, a^b = a^{11} \rangle$$

are not isomorphic but have the same set of finite images (these groups are in $\mathbf{C}_{25}\mathbf{C}_{\infty}$ and $\mathbf{C}_{\infty}(\mathbf{C}_{25}\mathbf{C}_5)$). However we do have the following very deep theorem of Grunewald et al. (1980). A full account of its proof is given in Segal (1983).

2.23 Theorem (Grunewald et al. 1980) The collection of all polycyclic-by-finite groups with a given set of finite images is the union of a finite set of isomorphism classes.

In other words, you cannot get an infinite set of pairwise non-isomorphic polycyclic-by-finite groups all having the same set of finite images.

The Profinite Topology

This gives another way of thinking about some of the theorems above. Let G be any group. Topologize G by taking the set of normal subgroups of G of finite index to be a basis of open neighborhoods of the identity. Thus a subset X of G is open by definition if X is a union of cosets of various normal subgroups of G of finite index. Check that this does impose a topology on G. (Main step: if M and N are normal subgroups of G of finite index and if $x, y \in G$, either $xM \cap yN = \emptyset$ or there exists z in $xM \cap yN$; in the latter case $xM \cap yN = zM \cap zN = z(M \cap N)$ and $M \cap N$ is a normal subgroup of G of finite index). A subset Y of G is closed if and only if $Y = \bigcap_N YN$, where N runs over the normal subgroups of G of finite index. In general set $\bigcap_N YN = Y^\wedge$. Clearly $Y^\wedge \supseteq Y$. Now if $G \backslash Y$ is open and $g \in G \backslash Y$, then $gN \subseteq G \backslash Y$ for some $N \triangleleft G$ of finite index and so $g \notin Y^\wedge$. Hence if Y is closed then $Y^\wedge = Y$. If $Y^\wedge = Y$ and $g \in G \backslash Y$, then $g \notin YN$ for some N. Thus $gN \subseteq G \backslash Y$. Consequently $G \backslash Y$ is open and Y is closed.

This topology is called the profinite topology of G. It makes G into a topological group; that is, $x \mapsto x^{-1}$ of G to G and $(x, y) \mapsto xy$ of $G \times G$ (with product topology) to G are both continuous. If H is an open subgroup of G, then $(G : H)$ is finite by definition of open set. If H is a subgroup of G of finite index, then $H_G = \bigcap_{g \in G} Hg$ is normal of finite index in G. Thus H_G is open and H as a union of cosets of H_G is also open. The profinite topology is the weakest topology making G into a topological group with every subgroup of finite index open.

2.24 Let H be a subgroup of the polycyclic-by-finite group G.

(a) H is closed in G in the profinite topology of G.
(b) If $K \leq H$ has finite index in H, then there exists $L \leq G$ of finite index in G such that $K = H \cap L$.
(c) The profinite topology of G induces on H the profinite topology of H.

Proof (a) This is just 2.9 rephrased.

(b) From (a) we have $K = \bigcap_F F$, where F ranges over all the subgroups of G of finite index containing K. Since $(H : K)$ is finite there exists finitely many of these F, say $F_1, F_2, \ldots, F_r$, with $K = H \cap F_1 \cap F_2 \cap \cdots \cap F_r$. Set $L = F_1 \cap F_2 \cap \cdots \cap F_r$.

(c) If N is normal in G of finite index, then $H \cap N$ is normal in H of finite index and hence $H \cap N$ is open in the profinite topology of H. Suppose now that N is a normal subgroup of H of finite index. By (b) there exists $L \leq G$ of finite index with $N = H \cap L$. Thus N is open in H in the topology induced on H by the profinite topology of G. The claim follows. □

Let G be a polycyclic-by-finite group. Then 2.22 is equivalent to each conjugacy class of elements of G being closed in the profinite topology. In Lennox and Wilson (1979) is proved that if H and K are subgroups of G then the product set HK is closed in the profinite topology of G. Then 2.21 can easily be derived from this result.

Exercise If G is any group, prove that its profinite topology is Hausdorff if and only if G is residually finite.

Exercise If G and H are any groups taken with their profinite topologies, prove that any (group) homomorphism of G into H is continuous.

We complete this chapter by just mentioning a few further results about polycyclic groups that can be proved using the machinery we have available so far. Wehrfritz (1994) shows that the polycyclic group

$$\langle a, b, x | ab = ba, ax = xa^3b^4, bx = xa^2b^3 \rangle$$

contains no subgroup H of finite index with H/H' torsion-free (compare 2.7(d)). In the same paper it is shown that a finitely generated nilpotent-by-finite group has a nilpotent subgroup H of finite index such that H and H/H' are torsion-free.

Suppose ϕ is an endomorphism of the polycyclic-by-finite group G. Under any one of the following conditions ϕ is an automorphism of G.

(a) G is polycyclic and ϕ restricts to an automorphism of the centre $\zeta_1\eta_1(G)$ of the Fitting subgroup of G.
(b) ϕ is monic and restricts to an automorphism of $\zeta_1\eta_1(G)$.
(c) ϕ restricts to an automorphism of $\tau(G).\zeta_1\eta_1(G)$.
(d) $\tau(G)$ is soluble and ϕ restricts to an automorphism of $\zeta_1\eta_1(G)$.
(e) Farkas (1982) ϕ restricts to an automorphism of D, where $D/\tau(G) = \zeta_1\eta_1(G/\tau(G))$.

This subgroup D of the polycyclic-by-finite group G is called the Zalesskii subgroup of G. Clearly (a) here is a special case of (d). The above can be found in

Wehrfritz (2009a), a paper that extends ideas from Farkas (1982) and Wehrfritz (1983).

Nikolov and Segal (2007) prove the following. Let G be a polycyclic-by-finite group and M a normal subgroup of G. Then M is a direct factor of G if and only if MG^n/G^n is a direct factor of G/G^n for every positive integer n. They deduce that if G is isomorphic to the direct product of M and G/M, then M is a direct factor of G.

Exercise Let G be a polycyclic-by-finite group. Prove that G is nilpotent if and only if for each prime p there is a normal subgroup N_p of G with G/N_p a finite p-group such that $\bigcap_p N_p = \langle 1 \rangle$. (Hint: Slightly modify 2.18 and the proof of 2.17 for one direction and use Higman (1955) for the other.)

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