Chapter 2
Review of Lyapunov Functions

Abstract We turn next to some of the basic notions of Lyapunov functions. Roughly speaking, a Lyapunov function for a given nonlinear system is a positive definite function whose decay along the trajectories of the system can be used to establish a stability property of the system. In general, one also requires Lyapunov functions to be proper, but one can prove stability using non-proper Lyapunov-like functions as well. Even when a system is known to be stable, one often still needs explicit strict Lyapunov functions, e.g., to design stabilizing feedbacks, or to find closed form expressions for the comparison functions in the ISS condition.

Non-strict Lyapunov functions cannot in general be used for these purposes. As we will see, strict Lyapunov functions are also important when estimating domains of attraction and $\mathcal{L}_2$ gains. We also address the issue of whether a given time-invariant system admits a Lyapunov function that has a globally bounded gradient. This is important, because the existence of such a Lyapunov function guarantees robustness with respect to additive uncertainty in the dynamics. We illustrate these ideas in several examples.

2.1 Strict Lyapunov Function

2.1.1 Definition

A strict Lyapunov function is a CLF for a system with no controls. Strict Lyapunov functions are also called strong Lyapunov functions. We therefore begin by defining CLFs. In the rest of this section, we consider only *continuous* time nonlinear systems

$$\dot{x} = f(t, x, u) \tag{2.1}$$

under the assumptions of the previous chapter, where the state x and input vector u are valued in an open set $\mathcal{X} \subseteq \mathbb{R}^n$ and a set $U \subseteq \mathbb{R}^p$, respectively. We discuss analogs for discrete time systems in Sect. 2.3.

We assume that the system (2.1) has equilibrium state 0. Let $V : [0, \infty) \times \mathcal{X} \to [0, \infty)$. We say that V is *proper* provided the set $\{x \in \mathcal{X} : \sup_t V(t, x) \leq L\}$ is compact for each constant $L > 0$; we call it *positive definite* provided $\inf_t V(t, x) = 0$ if and only if $x = 0$. When V is C^1, we use the notation

$$\dot{V}(t, x, u) \doteq \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x, u).$$

A C^1, proper, positive definite function $V : [0, \infty) \times \mathcal{X} \to [0, \infty)$ is then called a CLF for the system (2.1) provided for each $x \in \mathcal{X} \setminus \{0\}$, there exists a value $u \in U$ such that

$$\dot{V}(t, x, u) < 0$$

for all $t \geq 0$. When the system has no controls, we indicate this decay condition by $\dot{V}(t, x) < 0$. Also, when $\mathcal{X} = \mathbb{R}^n$, we use the term *radially bounded* to mean properness, which in this case gives the condition that

$$\lim_{|x| \to +\infty} \inf_t V(t, x) = +\infty.$$

For the special case of time-invariant control affine systems

$$\dot{x} \;=\; \varphi_1(x) + \varphi_2(x) u \tag{2.2}$$

with $\mathcal{X} = \mathbb{R}^n$, a positive definite time-invariant function $V(x)$ is a CLF for (2.2) provided the following hold:

1. V is radially unbounded; and
2. $L_{\varphi_1} V(x) \geq 0 \Rightarrow [x = 0 \text{ or } L_{\varphi_2} V(x) \neq 0]$.

We say that a CLF for (2.2) has the *small control property* provided: For each $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 \neq |x| < \delta$, then there is a $u \in U$ such that $|u| < \varepsilon$ and $\nabla V(x)\varphi_1(x) + \nabla V(x)\varphi_2(x)u < 0$. A special case of Artstein's Theorem [10] says the following: *Let $V(x)$ be a positive definite radially unbounded function. There exists a continuous feedback $K(x)$ so that V is a strict Lyapunov function for (2.2) in closed-loop with $u = K(x)$ if and only V is a CLF for (2.2) that satisfies the small control property.*

Specializing to systems with no controls and $\mathcal{X} = \mathbb{R}^n$, the strict Lyapunov function decay condition $\dot{V}(t, x) < 0$ for all $x \neq 0$ and all $t \geq 0$ means that

$$\frac{d}{dt} V(t, x(t, t_0, x_0)) \;<\; 0 \tag{2.3}$$

for all $t \geq t_0 \geq 0$ as long as the trajectory $x(t, t_0, x_0)$ is not at 0. The decay condition (2.3) is equivalent to the existence of a positive definite function α such that

$$\dot{V}(t,x) \leq -\alpha(|x|) \quad \forall x \in \mathbb{R}^n \text{ and } \forall t \geq 0;$$

this is shown in [157] for time-invariant systems but the generalization to time-varying systems is straightforward. Using suitable transformations $\Gamma(V)$ of the Lyapunov function gives different possible functions α. In fact, a slight variant of an argument from [141, Sect. 4] shows that a suitable transformation $V_1 = \Gamma(V)$ that is C^1 on $\mathbb{R}^n \setminus \{0\}$, proper, and positive definite satisfies $\dot{V}_1(t,x) \leq -V_1(t,x)$ for all x and t. We then call V_1 an *exponential decay Lyapunov function*, although the norm of the trajectories will not in general decay exponentially.

It is sometimes useful to relax the properness requirement on Lyapunov functions. A positive definite function that satisfies all of the requirements for being a strict Lyapunov function except properness is called a *strict Lyapunov-like function*. Strict Lyapunov-like functions were constructed in [106], under Matrosov Conditions; see Chap. 3. Throughout the chapter, we use the convention that all (in)equalities should be understood to hold globally unless otherwise indicated, and we leave out the arguments of our functions when they are clear from the context.

2.1.2 Lemmas

The existence of a strict Lyapunov function for our system

$$\dot{x} = f(t,x), \quad x \in \mathcal{X} \tag{2.4}$$

is sufficient for the system to be UGAS. Strict Lyapunov-like functions can be used to prove asymptotic stability as well. The following result from [70, Sect. 4.5] illustrates these points:

Lemma 2.1. *Let* 0 *be an equilibrium for (2.4), and* $V : [0,\infty) \times \mathcal{X} \to [0,\infty)$ *be a* C^1 *function that admits continuous positive definite functions* W_i *so that the following conditions hold:*

1. $W_1(x) \leq V(t,x) \leq W_2(x)$*; and*
2. $\dot{V}(t,x) \leq -W_3(x)$ *for all* $t \geq 0$ *and* $x \in \mathcal{X}$*.*

Then 0 *is a uniformly asymptotically stable equilibrium for (2.4). If the preceding conditions hold with* $\mathcal{X} = \mathbb{R}^n$ *and* W_1 *is radially unbounded, then* 0 *is a UGAS equilibrium for (2.4). In the special case where there exist positive constants* c_i *and* p *so that the preceding assumptions hold with* $W_i(x) = c_i|x|^p$ *and* $\mathcal{X} = \mathbb{R}^n$*, then the equilibrium is GES.*

The preceding theorem reduces the stability analysis to a search for an appropriate Lyapunov function. On the other hand, even if a system (2.4) is known to be UGAS, it is often important to be able to go in the converse direction, by constructing a strict Lyapunov function for the system. As a simple time-invariant example, assume that we know that a control affine system (2.2) is rendered GAS to the origin by a given feedback $u_s(x)$. In general, there is no reason to expect the closed-loop system

$$\dot{x} = \phi_1(x) + \phi_2(x)[K(x) + d] \tag{2.5}$$

with the disturbance d to be ISS when we pick $K(x) = u_s(x)$.[1] On the other hand, if we know a strict Lyapunov function V for the closed-loop system

$$\dot{x} = f(x) \doteq \phi_1(x) + \phi_2(x)u_s(x) \tag{2.6}$$

for which $-L_f V$ is radially unbounded, then (2.5) is ISS if we choose

$$K(x) = u_s(x) - \left(L_{\phi_2} V(x)\right)^\top . \tag{2.7}$$

Standard converse Lyapunov function theory guarantees the *existence* of a strict Lyapunov function for the GAS system (2.6).

However, to have an implementable stabilizer (2.7), we need an explicit expression for the Lie derivative $L_{\phi_2} V(x)$, hence an explicit strict Lyapunov function V. The strict Lyapunov functions provided by converse Lyapunov theory are usually not explicit, even if the system is UGES. The following result from [70, Sect. 4.7] illustrates this point:

Lemma 2.2. *Assume that there exist constants $D > 1$ and $\lambda > 0$ such that all trajectories of (2.4) satisfy the UGES condition*

$$|x(t, t_0, x_0)| \ \le\ D|x_0|e^{-\lambda(t-t_0)} \quad \forall x_0 \in \mathcal{X} \ \ \textit{and} \ \ \forall t \ge t_0 \ge 0 \tag{2.8}$$

and that there exists a constant $K > \lambda$ such that

$$\left|\frac{\partial f}{\partial x}(t, x)\right| \ \le\ K \quad \forall x \in \mathbb{R}^n \ \ \textit{and} \ \ \forall t \in [0, \infty). \tag{2.9}$$

Then the function

$$V(t, \xi) = 2\int_t^{t+\delta} |x(\tau, t, \xi)|^2 \mathrm{d}\tau, \ \ \textit{where} \ \ \delta = \frac{\ln(2D^2)}{2\lambda} \tag{2.10}$$

admits constants $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned} &c_1|\xi|^2 \ \le\ V(t, \xi) \ \le\ c_2|\xi|^2\,, \ \ |V_\xi(t, \xi)| \ \le\ c_3|\xi|\,, \ \ \textit{and} \\ &V_t(t, \xi) + V_\xi(t, \xi) f(t, \xi) \ \le\ -|\xi|^2 \end{aligned} \tag{2.11}$$

hold for all $t \in [0, \infty)$ and $\xi \in \mathbb{R}^n$, and therefore is a strict Lyapunov function for the system.

[1] For example, $\dot{x} = -\arctan(x) + u$ is GAS when we choose $u \equiv 0$, but $\dot{x} = -\arctan(x) + d$ is not ISS, because the bounded disturbance $d \equiv 2$ produces unbounded trajectories.

Formula (2.10) is non-explicit, because the flow map in the integrand cannot ordinarily be obtained in closed form, except in basic cases where (2.4) is linear and time-invariant. In Chap. 10, we explicitly construct strict Lyapunov functions for a class of *nonlinear* time-varying systems that satisfy the conclusions of Lemma 2.2. Lemma 2.2 can be extended to time-varying systems that are not necessarily exponentially stable. For example, we have the following from [70, Chap. 4, p.167]:

Theorem 2.1. *Assume that (2.4) is UGAS to the origin, and that* $f : [0,\infty) \times \mathcal{X} \to \mathbb{R}^n$ *is* C^1*. Let* $r > 0$ *be any constant such that* $r\mathcal{B}_n \subseteq \mathcal{X}$*, and assume that* $\frac{\partial f}{\partial x}$ *is bounded on* $[0,\infty) \times r\mathcal{B}_n$*. Let* $\beta \in \mathcal{KL}$ *and the constant* $r_0 > 0$ *be such that* $r_0 \le r$*,* $\beta(r_0, 0) < r$ *and*

$$|x(t, t_0, x_0)| \le \beta(|x_0|, t - t_0) \quad \forall t \ge t_0 \ge 0 \;\; \text{and} \;\; x_0 \in \mathcal{X}. \tag{2.12}$$

Then the following conclusions hold: (a) There exist a C^1 *function* $V : [0,\infty) \times (r_0\mathcal{B}_n) \to \mathbb{R}$ *and continuous positive definite increasing functions* $\alpha_i : [0, r_0] \to [0,\infty)$ *such that the following hold on* $[0,\infty) \times r_0\mathcal{B}_n$*:*

$$\begin{aligned} \alpha_1(|x|) \;&\le\; V(t,x) \;\le\; \alpha_2(|x|); \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \;&\le\; -\alpha_3(|x|); \;\; \text{and} \\ \left|\frac{\partial V}{\partial x}(t,x)\right| \;&\le\; \alpha_4(|x|). \end{aligned} \tag{2.13}$$

(b) If $\mathcal{X} = \mathbb{R}^n$ *and* $\frac{\partial f}{\partial x}$ *is bounded, then we can find a* C^1 *function* V *and functions* $\alpha_1, \dots \alpha_4 \in \mathcal{K}_\infty$ *such that (2.13) hold for all* $t \ge 0$ *and* $x \in \mathbb{R}^n$*. If, in addition, the system (2.4) is time-invariant, then* V *can be taken to be time-invariant; while if (2.4) is periodic in* t*, then* V *can be taken to be periodic in* t *as well.*

The strict Lyapunov function in the proof of Theorem 2.1 is also expressed in terms of the flow map and so is non-explicit; see Appendix B.1 for the main ideas from the proof. The challenge is to obtain explicit formulas for global strict Lyapunov functions that do not involve the flow map.

2.1.3 ISS Lyapunov Function

Consider the system with inputs

$$\dot{x} = f(t,x,u), \;\; x \in \mathcal{X}, \; u \in U \tag{2.14}$$

satisfying our standing assumptions from the previous chapter. For simplicity, we assume in the rest of this subsection that the state space $\mathcal{X}$ for (2.14) is all of $\mathbb{R}^n$. When $\mathcal{X} = \mathbb{R}^n$, we call a function $V : [0,\infty) \times \mathcal{X} \to [0,\infty)$ a *storage function* provided that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that $\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|)$ everywhere; in this case, we also say that V is *uniformly proper and positive definite*, or of class UPPD. For systems with inputs, we typically make the following more stringent assumption on V: A C^1 function $V : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ is said to be of class UBPPD (written $V \in$ UBPPD) provided (i) it is a storage function and (ii) its gradient is uniformly bounded in t, meaning there exists a function $\alpha_3 \in \mathcal{K}_\infty$ such that for all $t \geq 0$ and $x \in \mathbb{R}^n$, we have

$$|\nabla V(t,x)| \leq \alpha_3(|x|). \tag{2.15}$$

Notice that (2.15) is redundant when $V \in C^1$ is periodic in t. The corresponding notions of iISS and ISS Lyapunov functions are as follows:

Definition 2.1. Assume that $V : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ is a C^1 storage function. We say that V is an iISS Lyapunov function for (2.14) provided there exist a positive definite function α_3 and a function $\gamma \in \mathcal{K}_\infty$ such that

$$\dot{V}(t,x,u) \;\leq\; -\alpha_3(|x|) + \gamma(|u|) \tag{2.16}$$

for all $x \in \mathbb{R}^n$, $t \geq 0$, and $u \in U$. If, in addition, $\alpha \in \mathcal{K}_\infty$, then we call V an ISS Lyapunov function.

The following was established by Sontag and Wang in [169] for time-invariant ISS systems but the time-varying systems version can be shown by similar arguments [39]. The iISS statement was shown in [8].

Lemma 2.3. *Let (2.14) be periodic in t. The system (2.14) is iISS (resp., ISS) if and only if it admits a C^1 iISS (resp., ISS) Lyapunov function.*

As in the case where there are no controls, the converse parts of this lemma do not in general lead to explicit Lyapunov functions. On the other hand, if we know an explicit ISS Lyapunov function for (2.14), then we can use standard arguments to derive explicit formulas for the functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ in the ISS estimate. Let us sketch the derivation.

Let V be an ISS Lyapunov function for (2.14). Choose $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that $\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|)$ everywhere, and let the functions $\alpha_3, \gamma \in \mathcal{K}_\infty$ satisfy the requirements of Definition 2.1. Setting $\alpha(s) = \min\{s, \alpha_3 \circ \alpha_2^{-1}(s)\}$ gives

$$\dot{V} \leq -\alpha(V) + \gamma(|u|_\infty)$$

along all trajectories of (2.14). Hence, along any trajectory of (2.14),

$$V \geq \alpha^{-1}\big(2\gamma(|u|_\infty)\big) \quad\Rightarrow\quad \dot{V} \leq -\tfrac{1}{2}\alpha(V).$$

Let $u \in \mathcal{M}(U)$, $x_0 \in \mathcal{X}$, and $t_0 \geq 0$ be given, and let $x(t)$ denote the corresponding trajectory of (2.14) satisfying $x(t_0) = x_0$.

Arguing as in [82, Lemma 4.4, p.135] with the function $y(r) = V(t_0 + r, x(t_0 + r))$ shows that if $\dot{V} \leq -0.5\alpha(V)$ on any interval $[t_0, \bar{t}]$, then

$$V(t, x(t)) \ \le \ \beta_\alpha(V(t_0, x(t_0)), t - t_0)$$

on that interval, where $\beta_\alpha \in \mathcal{KL}$ is

$$\beta_\alpha(s,t) = \begin{cases} 0, & \text{if } s = 0 \\ s + \Phi^{-1}(\Phi(s) + t), & \text{if } s > 0 \end{cases} \tag{2.17}$$

and

$$\Phi(s) = \begin{cases} +\infty, & \text{if } s = 0 \\ -2\int_1^s \frac{\mathrm{d}r}{\alpha(r)}, & \text{if } s > 0 \end{cases} . \tag{2.18}$$

The fact that $\Phi(s) \to \infty$ as $s \to 0^+$ is used to show that $\beta(s,t) \to 0$ as $s \to 0^+$ for each $t \ge 0$ [82]. A standard invariance argument that is analogous to the one used in [157] shows that if

$$V(t, x(t)) \le \alpha^{-1}\big(2\gamma(|u|_\infty)\big)$$

for a given trajectory $x(t))$ of (2.14) at a given time $t = \tilde{t}$, then this inequality remains true for all $t \ge \tilde{t}$. Hence, we can take

$$\beta(s,t) = \alpha_1^{-1} \circ \beta_\alpha\big(\alpha_2(s), t\big) \ \text{ and } \ \gamma(r) = \alpha_1^{-1} \circ \alpha^{-1}\big(2\gamma(r)\big)$$

to satisfy our requirements.

This makes it possible to explicitly quantify the effects of the disturbance, while at the same time obtaining the decay rate on the norm of the state, which is valuable in applications. See Sect. 2.4.2 for a specific example where β and γ are computed. Analogous arguments can be carried out for the iISS case; see [8]. This motivates our search for explicit iISS Lyapunov functions as well.

2.2 Non-strict Lyapunov Function

Our main building blocks for strict Lyapunov functions will be non-strict Lyapunov functions (which are also called weak Lyapunov functions). Non-strict Lyapunov functions V are defined in exactly the same way as strict Lyapunov functions except instead of the decay condition $\dot{V} < 0$ outside the equilibrium state, we have $\dot{V} \le 0$. A positive definite function V that satisfies all requirements for being a (non-)strict Lyapunov function except for properness is called a *(non-)strict Lyapunov-like function.* In this subsection, we discuss three contexts in which non-strict Lyapunov functions naturally arise. In Chapters 3-5, we provide systematic mechanisms for building strict Lyapunov functions in each of these contexts.

2.2.1 Matrosov Theorems

Matrosov's Theorem [97] provides a Lyapunov approach to proving stability without having to construct a strict Lyapunov function. In its original formulation, it concludes uniform asymptotic stability of time-varying systems by using a non-strict Lyapunov function and an auxiliary function whose time derivative along trajectories is non-zero at all points $x \in \mathbb{R}^n \setminus \{0\}$ where the derivative of the non-strict Lyapunov function is zero. There are various generalizations of the original Matrosov result, involving an arbitrary number of auxiliary functions [86]. These generalizations are referred to as Matrosov Theorems, and they prove uniform asymptotic stability as well.

While the original motivation for Matrosov's Theorem was to eliminate the need for a strict Lyapunov function, it is still important to be able to construct explicit strict Lyapunov functions for systems satisfying Matrosov's Conditions [111]. However, the proofs in [86, 97] do not construct strict Lyapunov functions. Instead, they conclude uniform asymptotic stability by directly analyzing the trajectories of the system. One standard formulation is the following result from [145], where we maintain our standing assumptions on (2.4) from the previous chapter:

Theorem 2.2. *Assume that there are constants* $\bar{R} > \underline{R} > 0$ *and* $L > 0$*, functions* $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}$*, and continuous functions*

$$\begin{aligned} V_1 &: [0,\infty) \times \mathrm{int}(\bar{R}\mathcal{B}_n) \to \mathbb{R}, \\ V_2 &: [0,\infty) \times \mathrm{int}(\bar{R}\mathcal{B}_n) \to \mathbb{R}, \text{ and} \\ W &: \bar{R}\mathcal{B}_n \to \mathbb{R} \end{aligned}$$

for which $\dot{V}_1(t,x)$ *and* $\dot{V}_2(t,x)$ *are continuous and the following hold:*

1. $V_1(t,0) = \dot{V}_1(t,0) = 0$ *for all* $t \geq 0$*;*
2. $\max\{|V_2(t,x)|, |f(t,x)|\} \leq L$ *for all* $(t,x) \in [0,\infty) \times \underline{R}\mathcal{B}_n$*;*
3. $\underline{\alpha}(|x|) \leq V_1(t,x) \leq \bar{\alpha}(|x|)$ *for all* $(t,x) \in [0,\infty) \times \underline{R}\mathcal{B}_n$*;*
4. $\dot{V}_1(t,x) \leq W(x) \leq 0$ *for all* $(t,x) \in [0,\infty) \times \underline{R}\mathcal{B}_n$*; and*
5. $\dot{V}_2(t,x)$ *is non-zero definite on* $\{x \in \underline{R}\mathcal{B}_n : W(x) = 0\}$*.*

Then $\lim_{t\to+\infty} x(t,t_0,x_0) = 0$ *for each solution* $x(\cdot,t_0,x_0)$ *of (2.4) that remains in* $\underline{R}\mathcal{B}_n$ *for all* $t \geq t_0$*.*

For the proof, see [145]. By non-zero definiteness of a function $G : [0,\infty) \times \bar{R}\mathcal{B}_n \to \mathbb{R}$ on a closed set $M \subseteq \bar{R}\mathcal{B}_n$, we mean that for each pair of constants (ν, ε) for which $0 < \nu < \varepsilon \leq \bar{R}$, there are values $\gamma, \delta > 0$ such that:

$$\Big[\big\{\nu \leq |x| \leq \varepsilon\big\} \text{ and } \big\{|x|_M < \gamma\big\} \text{ and } \big\{t \in [0,\infty)\big\}\Big] \;\Rightarrow\; \big|G(t,x)\big| > \delta,$$

where $|x|_M \doteq \inf\{|x - q| : q \in M\}$ is the distance of x from M. When G is independent of t, non-zero definiteness simply says that G is bounded away from zero on the part of any annulus around 0 that is close enough to M.

2.2.2 LaSalle Invariance Principle

Recall that a set $M \subseteq \mathbb{R}^n$ is called *positively invariant* for a forward complete time-invariant dynamics

$$\dot{x} = f(x) \tag{2.19}$$

evolving on an open set $\mathcal{X}$ provided for each $x_0 \in M$, the corresponding solution $t \mapsto x(t, x_0)$ remains in M for all times $t \geq 0$. The set M is called *invariant* for (2.19) if the system is complete and each such trajectory is in M for all $t \in \mathbb{R}$. LaSalle's Invariance Theorem is the following result, which is shown, e.g., in [70, Sect. 4.2]:

Lemma 2.4. *Let the compact set $\Omega \subseteq \mathcal{X}$ be positively invariant for (2.19), and $V : \mathcal{X} \to \mathbb{R}$ be a C^1 function for which $L_f V(x) \leq 0$ on Ω. Let M be the largest invariant subset of*

$$E \doteq \{x \in \Omega : L_f V(x) = 0\}$$

for (2.19). Then every solution of (2.19) converges to M as $t \to +\infty$. If the preceding assumptions hold except with $\Omega = \mathbb{R}^n$ and $f(0) = 0$, and if no solution of (2.19) can stay in E for all times $t \geq 0$ except for the trivial solution $x(t) \equiv 0$, then the origin is GAS.

The preceding result can be extended to time-varying systems. For example, we have the following from [148, Sect. 5.4]:

Lemma 2.5. *Consider the system (2.4) with state space $\mathcal{X} = \mathbb{R}^n$. Assume that $f(t, x)$ and $V(t, x)$ have the same period $T > 0$ in t, where V is a C^1 storage function. If $\dot{V}(t, x) \leq 0$ for all $t \geq 0$ and all $x \in \mathbb{R}^n$, and if the largest invariant set for (2.4) in*

$$S \doteq \left\{x \in \mathbb{R}^n : \dot{V}(t, x) = 0 \ \forall t \geq 0\right\}$$

is $\{0\}$, then 0 is a UGAS equilibrium for (2.4).

See also [148, Sect. 5.5] for generalized LaSalle Theorems for time-varying systems that are not necessarily periodic in time. LaSalle Invariance provides another method for proving stability without having to find a strict Lyapunov function, but it is of limited use when the system is subject to disturbances. This is because small perturbations of the dynamics can cause $\dot{V}(t, x)$ to become positive at some pairs (t, x).

2.2.3 Jurdjevic-Quinn Theorem

Consider a general control affine system

$$\dot{x} \;=\; f(x) + g(x)u \;\;, \;\; \text{where} \;\; g(x) = (g_1(x), \ldots, g_p(x)) \tag{2.20}$$

evolving on $\mathbb{R}^n$ in which the vector fields $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ are smooth and $f(0) = 0$. We use the standard Lie bracket notation

$$\begin{aligned} &\mathrm{ad}_f^0(g) = g, \quad \mathrm{ad}_f(g) = [f, g] = g_* f - f_* g, \\ &\text{and} \;\; \mathrm{ad}_f^k(g) = \mathrm{ad}_f \left(\mathrm{ad}_f^{k-1}(g) \right) \end{aligned}$$

for all $k > 1$ and all smooth vector fields $f, g : \mathbb{R}^n \to \mathbb{R}^n$, where the star subscript indicates a Jacobian. In [68], Jurdjevic and Quinn proved the following single input result:

Theorem 2.3. *(Jurdjevic-Quinn Theorem) Consider the system (2.20) with state space $\mathcal{X} = \mathbb{R}^n$ and $p = 1$. Assume the following:*

1. *$f(x) = Ax$ for some skew symmetric matrix A; and* [2]
2. *for all $x \in \mathbb{R}^n \setminus \{0\}$, we have* $\mathrm{span}\{(ad_f^k(g))(x) : k = 0, 1, 2, \ldots\} = \mathbb{R}^n$.

Then the feedback $u(x) = -x^\top g(x)$ renders (2.20) GAS to zero.

Proof. By a simple calculation,

$$\frac{d}{dt}|x(t, x_0)|^2 \;=\; -2u^2\big(x(t, x_0)\big) \leq 0$$

for all $t \geq 0$ and $x_0 \in \mathbb{R}^n$ along all trajectories $t \mapsto x(t, x_0)$ of the closed-loop system, because $x^\top Ax = 0$ for all x. We show that no solution can stay in $E \doteq \{x : u(x) = 0\}$ except for the trivial solution. The GAS property will then follow from the LaSalle Invariance Principle.

Fix $x_0 \in E$ and consider the function $\Gamma(t) \doteq \langle e^{tA}x_0, g(e^{tA}x_0)\rangle$. Since $x(t, x_0) = e^{tA}x_0 \in E$ for all $t \geq 0$, we get $\Gamma(t) = 0$ for all $t \geq 0$. A simple inductive argument gives

$$0 = \frac{d^k}{dt^k}\Gamma(t) = \left\langle x_0, e^{-tA}\mathrm{ad}_f^k(g)(e^{tA}x_0)\right\rangle$$

for all $t \geq 0$. Evaluating these higher time derivatives at $t = 0$ shows that $\langle x_0, (ad_f^k(g))(x_0)\rangle = 0$ for all $k \geq 0$. Hence, $x_0 = 0$, by Assumption 2. □

[2] A slight variant of the argument that we are about to give applies if the skew symmetry assumption on A is replaced by the assumption that there is an invertible matrix M such that $MAM^{-1} = MAM^\top = J$ is skew symmetric, by showing that the dynamics for $y = Mx$ is GAS to the origin with the feedback $u(y) = -\langle y, Mg(M^\top y)\rangle$. In fact, the dynamics are $\dot{y} = F(y) + G(y)u$, where $F(y) = Jy$ and $G(y) = Mg(M^\top y)$, so the proof follows from the relations $(\mathrm{ad}_F^k(G))(y) = M(\mathrm{ad}_f^k(g))(M^\top y)$.

The Jurdjevic-Quinn Theorem has been generalized in several works. In general, conditions that provide a smooth asymptotically stabilizing control law using a first integral of the drift vector field, under some controllability conditions, are now called Jurdjevic-Quinn Conditions. A general set of Jurdjevic-Quinn Conditions for cases where the vector field f in (2.20) is not required to be linear is as follows.

Definition 2.2. We say that (2.20) satisfies the *(Weak) Jurdjevic-Quinn Conditions* provided there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying:

1. V is positive definite and radially unbounded;
2. for all $x \in \mathbb{R}^n$, $L_f V(x) \leq 0$; and
3. there exists an integer l such that the set

$$W(V) = \left\{ \begin{array}{l} x \in \mathbb{R}^n : \forall k \in \{1, \ldots, p\} \text{ and } i \in \{0, \ldots, l\}, \\ L_f V(x) = L_{ad_f^i(g_k)} V(x) = 0 \end{array} \right\}$$

equals $\{0\}$.

If (2.20) satisfies the Weak Jurdjevic-Quinn Conditions, then it is globally asymptotically stabilized by any feedback

$$u = -\xi(x) L_g V(x)^\top,$$

where ξ is any everywhere positive function of class C^1 [41]. The proof of this result also follows from the LaSalle Invariance Principle. However, it is far from clear how to construct CLFs for systems satisfying the Weak Jurdjevic Quinn Conditions. We address this CLF construction problem in Chap. 4.

2.3 Discrete Time Lyapunov Function

The preceding definitions have analogs for discrete time systems

$$x_{k+1} = f(k, x_k, u_k) \tag{2.21}$$

with equilibrium state 0 by replacing the continuous time $t \geq 0$ with the discrete time $k \in \{0, 1, 2, \ldots\}$ and replacing the time derivative $\dot{V}(t, x, u)$ of the Lyapunov function along trajectories with the first difference

$$\Delta V(k, x, u) = V\big(k+1, f(k, x, u)\big) - V(k, x) \tag{2.22}$$

in the conditions defining Lyapunov functions.

The definition of discrete time Lyapunov functions does not require V to be C^1 because there are no derivatives of V in (2.22). Also, discrete time strict Lyapunov functions have the property that along each trajectory sequence

$\{x_k\}$ of (2.21), the sequence $V(k, x_k)$ is decreasing in k, as long as $x_k \neq 0$. If instead V is only a *non-strict* Lyapunov function, then $V(k, x_k)$ is non-increasing. We focus on continuous time systems in much of the sequel, but most of the results to follow have discrete time analogs.

2.4 Illustrations

2.4.1 Strict Lyapunov Function

Let us again consider the system

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 u_1 \end{cases} \tag{2.23}$$

from Sect. 1.4. In closed-loop with the stabilizing feedbacks

$$\begin{array}{l} u_1 = -x_1 + \sin(t)(\cos(t)x_1 + x_2) \\ u_2 = -\sin(t) - \cos(t), \end{array} \tag{2.24}$$

the system becomes

$$\begin{cases} \dot{x}_1 = -x_1 + \sin(t)\big[\cos(t)x_1 + x_2\big] \\ \dot{x}_2 = \big[-\sin(t) - \cos(t)\big]\big[-x_1 + \sin(t)\big(\cos(t)x_1 + x_2\big)\big]. \end{cases} \tag{2.25}$$

We now show that (2.25) admits the global strict Lyapunov function

$$V_s(t, x) \ = \ \frac{1}{2}x_1^2 + \left(4 + \frac{\pi}{2} - 2\sin(t)\cos(t)\right)[\cos(t)x_1 + x_2]^2. \tag{2.26}$$

In later chapters, we provide general methods for constructing global strict Lyapunov functions.

Since

$$\begin{aligned} [\cos(t)x_1 + x_2]^2 &\geq \cos^2(t)x_1^2 + x_2^2 - \left(\frac{3}{4}x_2^2 + \frac{4}{3}\cos^2(t)x_1^2\right) \\ &\geq -\frac{1}{3}x_1^2 + \frac{1}{4}x_2^2, \end{aligned}$$

one easily checks that the inequalities

$$\frac{1}{6}[x_1^2 + x_2^2] \ \leq \ V_s(t, x) \ \leq \ 17[x_1^2 + x_2^2] \tag{2.27}$$

are satisfied everywhere. Also, the time derivative of V_s along trajectories of (2.25) is

$$\begin{aligned}\dot{V}_s(t,x) = &-x_1^2 + \sin(t)x_1[\cos(t)x_1 + x_2] \\ &+2\left(\sin^2(t) - \cos^2(t)\right)[\cos(t)x_1 + x_2]^2 \\ &+2\left(4 + \tfrac{\pi}{2} - 2\sin(t)\cos(t)\right)[\cos(t)x_1 + x_2] \\ &\quad\times[-\sin(t)x_1 + \cos(t)\dot{x}_1 + \dot{x}_2].\end{aligned} \tag{2.28}$$

Setting $\zeta = \cos(t)x_1 + x_2$, we obtain

$$\begin{aligned}\dot{V}_s(t,x) &= -x_1^2 + \sin(t)x_1\zeta + 2\left(\sin^2(t) - \cos^2(t)\right)\zeta^2 \\ &\quad +2\left(4 + \tfrac{\pi}{2} - 2\sin(t)\cos(t)\right)\zeta \\ &\quad \times[-\sin(t)x_1 + \cos(t)(-x_1 + \sin(t)\zeta) \\ &\quad\quad +\{-\sin(t) - \cos(t)\}(-x_1 + \sin(t)\zeta)] \\ &= -x_1^2 + \sin(t)x_1\zeta + 2\left(\sin^2(t) - \cos^2(t)\right)\zeta^2 \\ &\quad -2\left(4 + \tfrac{\pi}{2} - 2\sin(t)\cos(t)\right)\sin^2(t)\zeta^2 \\ &= -x_1^2 + \sin(t)x_1\zeta \\ &\quad + \left[-2 - (4+\pi)\sin^2(t) + 4\sin^3(t)\cos(t)\right]\zeta^2 \\ &\leq -\tfrac{1}{2}x_1^2 - \zeta^2 \\ &= -\tfrac{1}{2}x_1^2 - (\cos(t)x_1 + x_2)^2 \leq -\tfrac{1}{6}[x_1^2 + x_2^2],\end{aligned} \tag{2.29}$$

where the first inequality used the relation

$$\sin(t)x_1\zeta \leq \frac{1}{2}x_1^2 + \frac{1}{2}\zeta^2$$

and the second inequality used

$$-(\cos(t)x_1 + x_2)^2 \leq -\cos^2(t)x_1^2 - x_2^2 + \frac{4}{3}\cos^2(t)x_1^2 + \frac{3}{4}x_2^2.$$

Since $\dot{V}_s(t,x)$ has a negative definite upper bound, it follows from (2.27) that V_s is a strict Lyapunov function for the system (2.25).

2.4.2 ISS Lyapunov Function

We next consider the case where there is additive noise in the u_1 input in (2.23). We show that the resulting closed-loop system

$$\begin{cases} \dot{x}_1 = -x_1 + \sin(t)[\cos(t)x_1 + x_2] + \delta_1(t) \\ \dot{x}_2 = [-\sin(t) - \cos(t)][-x_1 + \sin(t)(\cos(t)x_1 + x_2) + \delta_1(t)] \end{cases} \tag{2.30}$$

with the controllers (2.24) is ISS with respect to the disturbance δ_1. Our strategy is to show that (2.26) is an ISS Lyapunov function for (2.30), which will lead to explicit functions β and γ in the ISS estimate.

To this end, first note that (2.26) satisfies

$$\max_{i=1,2} \left| \frac{\partial V_s}{\partial x_i}(t,x) \right| \leq 17|x|_1 \ \ \forall x = (x_1\ x_2) \in \mathbb{R}^2 \text{ and } t \geq 0. \tag{2.31}$$

Hence, the last inequality of (2.29) implies that the time derivative of V_s along the trajectories of (2.30) satisfies

$$\begin{aligned} \dot{V}_s(t,x) &\leq -\tfrac{1}{6}|x|^2 + 51|x|_1|\delta_1(t)| \\ &\leq -\tfrac{1}{6}|x|^2 + \{|x|\}\{102|\delta_1(t)|\} \\ &\leq -\tfrac{1}{12}|x|^2 + 3 \times 102^2 \delta_1^2(t), \end{aligned} \tag{2.32}$$

by the triangle inequality

$$pq \leq \tfrac{1}{2\varepsilon}p^2 + \tfrac{\varepsilon}{2}q^2$$

applied to the terms in braces with $p = |x|$, $q = 102|\delta_1(t)|$, and $\varepsilon = 6$. This and the proper positive definiteness condition (2.27) imply that V_s is an ISS Lyapunov function for (2.30). As we saw in Sect. 2.1.3, explicit ISS Lyapunov functions lead to explicit expressions for the functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ in the ISS estimate. We derive these expressions next for the dynamics (2.30).

Combining (2.32) with the inequalities (2.27) gives

$$\dot{V}_s(t,x) \leq -\tfrac{1}{204}V_s(t,x) + 3 \times 102^2 \delta_1^2(t). \tag{2.33}$$

By integrating this inequality, we deduce that for any initial condition $x(t_0) = x_0$, the corresponding trajectories satisfy

$$V_s(x(t),t) \leq e^{-\frac{t-t_0}{204}} V_s(x(t_0),t_0) + 612 \times 102^2 |\delta_1|^2_{[t_0,t]}. \tag{2.34}$$

Combining (2.34) with (2.27), and then using the relation

$$\sqrt{p+q} \ \leq \ \sqrt{p} + \sqrt{q}$$

for $p, q \geq 0$ gives the desired ISS estimate

$$|x(t,t_0,x_0,u)| \leq \beta(|x_0|, t-t_0) + \gamma(|\delta_1|_\infty)$$

with the choices $\beta(r,s) = 11re^{-s/408}$ and $\gamma(r) = 102\sqrt{3672}r$.

2.4.3 iISS Lyapunov Function

If we allow additive disturbances in both the u_1 and u_2 channels in the dynamics (2.23) and use the feedbacks (2.24) as before, then the corresponding closed-loop system

$$\begin{cases} \dot{x}_1 = -x_1 + \sin(t)\zeta + \delta_1(t) \\ \dot{x}_2 = [-\sin(t) - \cos(t) + \delta_2(t)][-x_1 + \sin(t)\zeta + \delta_1(t)] \end{cases} \tag{2.35}$$

where

$$\zeta = \cos(t)x_1 + x_2$$

is not ISS with respect to the disturbance $\delta = (\delta_1, \delta_2)$. This follows by applying Lemma 1.2 with the disturbance

$$\delta = (0, \sin(t) + \cos(t) + 1),$$

$Lx = (x_2 - x_1, x_2)$, and $k = 1$. On the other hand, (2.35) is iISS with respect to δ. We show this next using the strict Lyapunov function (2.26). Our arguments are a time-varying analog of those of [8, pp.1091-2].

Using (2.31) and the first inequality in (2.32) gives

$$\begin{aligned} \dot{V}_s(t,x) &\leq -\tfrac{1}{6}|x|^2 + 51|x|_1|\delta| + 17|x|_1(2|x|_1 + |\delta|)|\delta| \\ &\leq -\alpha_3(|x|) + \lambda(|x|)\Delta(|\delta|), \end{aligned} \tag{2.36}$$

and

$$\alpha_1(|x|) \ \leq \ V_s(t,x) \ \leq \ \alpha_2(|x|)$$

everywhere, where

$$\begin{aligned} &\alpha_1(r) = \alpha_3(r) = \tfrac{1}{6}r^2, \ \ \alpha_2(r) = 17r^2, \\ &\lambda(r) = 136(r + r^2), \ \text{ and } \ \Delta(r) = r + r^2. \end{aligned}$$

Taking

$$\begin{aligned} &W(t,x) = \Pi(V_s), \ \text{ where } \ \Pi(r) = \textstyle\int_0^r \dfrac{\mathrm{d}s}{1 + \chi(s)} \ \text{ and} \\ &\chi(s) = \lambda \circ \alpha_1^{-1}(s) = 136\sqrt{6s} + 816s, \end{aligned}$$

it follows that

$$\dot{W}(t,x) \ \leq -\rho(|x|) + \Delta(|\delta|), \ \text{ where } \ \rho(r) = \frac{\alpha_3(r)}{1 + \lambda(\alpha_1^{-1}(\alpha_2(r)))}.$$

Moreover, W is also proper and positive definite, and ρ is positive definite. Therefore, W is an iISS Lyapunov function for (2.35). It follows from Lemma 2.3 that (2.35) is iISS.

2.4.4 LaSalle Invariance Principle

Consider the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2^3 \end{cases} \tag{2.37}$$

with $x \in \mathbb{R}^2$. The function

$$V(x) = \frac{1}{2}[x_1^2 + x_2^2] \tag{2.38}$$

satisfies

$$\dot{V}(x) = -W(x), \tag{2.39}$$

where $W(x) = x_2^4$. However, this is not enough to conclude that the system is GAS to 0 because W is non-negative definite but not positive definite. Instead, we use W in conjunction with the LaSalle Invariance Principle to show the GAS property for (2.37).

Let $x(t)$ be any solution of (2.37) with any initial state x_0. By (2.39),

$$x(t) \in \Omega_{x_0} = \{x \in \mathbb{R}^2 : |x| \leq |x_0|\} \;\; \forall t \geq 0.$$

We deduce from Lemma 2.4 that $x(t)$ converges to the largest invariant set $\mathcal{S}$ contained in

$$E_{x_0} = \{x \in \Omega_{x_0} : x_2 = 0\}.$$

We show that $\mathcal{S} = \{0\}$. Let $z(t) = (z_1(t)\ z_2(t))$ be any solution of the system (2.37) with initial condition $z_0 \in S$. Since $\mathcal{S}$ is positively invariant, we have $z_2(t) = 0$ for all $t \geq 0$. Therefore $\dot{z}_2(t) = 0$ for all $t \geq 0$, so

$$-z_1(t) - z_2^3(t) = 0$$

for all $t \geq 0$. It follows that $z_1(t) = 0$ for all $t \geq 0$. We deduce that $\mathcal{S} = \{0\}$ which implies that all the solutions of (2.37) converge to the origin, by LaSalle Invariance.

2.4.5 Matrosov Theorems

The GAS of the origin of (2.37) can be established through the version of the Matrosov Theorem we gave in Theorem 2.2, as follows. We show that for all positive constants $\underline{R}$, all solutions of the system with initial states in $\underline{R}\mathcal{B}_2$ remain in $\underline{R}\mathcal{B}_2$ and converge to 0 as $t \to +\infty$. We apply the theorem with

$$\begin{aligned} &V_1(x) \;=\; \tfrac{1}{2}[x_1^2 + x_2^2]\,, \quad V_2(x) \;=\; x_1 x_2\,, \quad \text{and} \\ &W(x) = -x_2^4. \end{aligned} \tag{2.40}$$

The conditions are verified as follows:

1. $V_1(0) = \dot{V}_1(0) = 0$ along the trajectories of (2.37);
2. for all $x \in \underline{R}\mathcal{B}_2$, we have $\max\{|V_2(x)|, |f(x)|\} \leq |x|^2 + \sqrt{x_2^2 + (x_1 + x_2^3)^2} \leq \underline{R}^2 + (\underline{R}^2 + (\underline{R} + \underline{R}^3)^2)^{1/2}$;
3. $\underline{\alpha}(|x|) \leq V_1(x) \leq \bar{\alpha}(|x|)$ for all x when we choose $\bar{\alpha}(r) = \underline{\alpha}(r) = \frac{1}{2}r^2$;
4. $\dot{V}_1(x) = W(x) = -x_2^4 \leq 0$ for all $x \in \underline{R}\mathcal{B}_2$; and
5. $\dot{V}_2(x) = \dot{x}_1 x_2 + x_1 \dot{x}_2 = x_2^2 + x_1(-x_1 - x_2^3)$ so for any constants $\nu, \varepsilon > 0$, we can find a constant $\gamma > 0$ so that $|\dot{V}_2(x)|$ is bounded away from zero on $\{x \in \mathbb{R}^2 : \nu \leq |x| \leq \varepsilon, |x_2| < \gamma\}$. Hence, $\dot{V}_2$ is non-zero definite on $\{x \in \underline{R}\mathcal{B}_2 : x_2^4 = 0\}$.

We conclude from Matrosov's Theorem and the fact that $\dot{V}_1 \leq 0$ that every solution of (2.37) with initial condition in $\underline{R}\mathcal{B}_2$ remains in $\underline{R}\mathcal{B}_2$ and converges to 0 as $t \to +\infty$. We deduce that the origin of (2.37) is GAS.

2.4.6 Non-strict Lyapunov-Like Function

Consider an experimental anaerobic digester used to treat waste water [16, 89, 172]. This process degrades a polluting organic substrate s with the anaerobic bacteria x and produces a methane flow rate y_1. The methane and substrate can generally be measured, so the system with output y is

$$\begin{cases} \dot{s} = u(s_{in} - s) - kr(s, x) \\ \dot{x} = r(s, x) - \alpha u x \\ y = (\lambda r(s, x), s) \end{cases} \tag{2.41}$$

where the biomass growth rate r is any non-negative C^1 function that admits everywhere positive functions $\underline{\Delta}$ and $\bar{\Delta}$ such that

$$s\bar{\Delta}(s, x) \ \geq \ r(s, x) \ \geq \ xs\underline{\Delta}(s, x) \tag{2.42}$$

for all $s \geq 0$ and $x \geq 0$; u is the non-negative input (i.e., the dilution rate); α is a known positive real number representing the fraction of the biomass in the liquid phase; and λ, k, and s_{in} are positive constants representing methane production and substrate consumption yields and the influent substrate concentration, respectively. Hence, the methane flow rate is $y_1 = \lambda r(s, x)$. This includes the single species undisturbed chemostat model from Sect. 1.5.3, in which case $r(s, x)$ is the product of the species concentration and the Monod growth rate function, namely,

$$r(s, x) = \frac{Asx}{B + s}$$

for appropriate positive constants A and B. However, (2.42) is far more general because it allows other growth laws such as those of Haldane and Cantois; see [14] for details.

Assume now that $s_* \in (0, s_{in})$ is a given constant. We wish to regulate s to s_*. We assume that there are known constants $\gamma_M > \gamma_m > 0$ such that

$$\gamma_* \doteq \frac{k}{\lambda(s_{in} - s_*)} \in (\gamma_m, \gamma_M) \quad \text{and} \quad \frac{k}{\lambda s_{in}} < \gamma_m \tag{2.43}$$

and we use the notation

$$v_* = s_{in} - s_* \text{ and } x_* = \frac{v_*}{k\alpha}$$

in the sequel.

The work [89] leads to a non-strict Lyapunov-like function and an adaptive controller for an error dynamics associated with (2.41). We next review these earlier results. We treat adaptive control in detail in Chap. 9. Later, we will see how the constructions from [89] lead to a strict Lyapunov-like function for the error dynamics of $(\tilde{s}, \tilde{x}) = (s - s_*, x - x_*)$.

We introduce the dynamics

$$\dot{\gamma} = y_1(\gamma - \gamma_m)(\gamma_M - \gamma)\nu$$

evolving on (γ_m, γ_M), where ν is a function to be selected that is independent of x. With $u = \gamma y_1$, the system (2.41) with its dynamic extension becomes

$$\begin{cases} \dot{s} = y_1\left[\gamma(s_{in} - s) - \frac{k}{\lambda}\right] \\ \dot{x} = y_1\alpha\left[\frac{1}{\alpha\lambda} - \gamma x\right] \\ \dot{\gamma} = y_1(\gamma - \gamma_m)(\gamma_M - \gamma)\nu \end{cases} \tag{2.44}$$

by the definition of y_1, with the same output y as before. The dynamics (2.44) evolves on the invariant domain $E = (0, \infty) \times (0, \infty) \times (\gamma_m, \gamma_M)$. The following is easily checked:

Lemma 2.6. *For each initial value $(s(t_0), x(t_0), \gamma(t_0)) \in E$, there is a compact set $K_0 \subseteq (0, \infty)^2$ so that the corresponding solution of (2.44) is such that $(s(t), x(t)) \in K_0$ for all $t \geq t_0$.*

It follows from Lemma 2.6 and (2.42) that we can re-parameterize (2.44) in terms of the Erdmann Transformation

$$\tau = \int_{t_0}^{t} y_1(l)\, \mathrm{d}l \ .$$

Doing so and setting

$$\tilde{x} = x - x_*, \quad \tilde{s} = s - s_*, \quad \text{and} \quad \tilde{\gamma} = \gamma - \gamma_*$$

yields the error dynamics

$$\begin{cases} \dot{\tilde{s}} = -\gamma\tilde{s} + \tilde{\gamma}v_* \\ \dot{\tilde{x}} = \alpha\left[-\gamma\tilde{x} - \tilde{\gamma}x_*\right] \\ \dot{\tilde{\gamma}} = (\gamma - \gamma_m)(\gamma_M - \gamma)\nu \end{cases} \tag{2.45}$$

for $t \mapsto (\tilde{s}, \tilde{x}, \tilde{\gamma})(\tau^{-1}(t))$. The state space of (2.45) is

$$D = (-s_*, \infty) \times (-x_*, \infty) \times (\gamma_m - \gamma_*, \gamma_M - \gamma_*).$$

The system (2.45) has an uncoupled triangular structure; i.e., its $(\tilde{s}, \tilde{\gamma})$-subsystem does not depend on $\tilde{x}$ (because ν is independent of x), and the $\tilde{x}$-subsystem is globally input-to-state stable with respect to $\tilde{\gamma}$ with the ISS Lyapunov function $\tilde{x}^2$. Therefore, (2.45) is GAS to 0 if the system

$$\begin{cases} \dot{\tilde{s}} = -\gamma\tilde{s} + \tilde{\gamma}v_* \\ \dot{\tilde{\gamma}} = (\gamma - \gamma_m)(\gamma_M - \gamma)\nu \end{cases} \tag{2.46}$$

with state space

$$\mathcal{X} = (-s_*, \infty) \times (\gamma_m - \gamma_*, \gamma_M - \gamma_*)$$

is GAS to 0. Hence, we may limit our analysis to (2.46) in the following analysis.

For a given tuning parameter $K > 0$, the non-strict Lyapunov-like function for (2.46) provided by [89] is

$$V_1(\tilde{s}, \tilde{\gamma}) = \frac{1}{2\gamma_m}\tilde{s}^2 + \frac{v_*}{K\gamma_m}\int_0^{\tilde{\gamma}} \frac{l}{(l + \gamma_* - \gamma_m)(\gamma_M - \gamma_* - l)}dl, \tag{2.47}$$

which is positive definite on $\mathcal{X}$. In fact,

$$\dot{V}_1 = \frac{1}{\gamma_m}\left[-\gamma\tilde{s}^2 + \tilde{s}\tilde{\gamma}v_*\right] + \frac{v_*}{K\gamma_m}\tilde{\gamma}\nu$$

along the trajectories of (2.46), so choosing

$$\nu(\tilde{s}) = -K\tilde{s} \tag{2.48}$$

gives

$$\dot{V}_1 = -\frac{\gamma}{\gamma_m}\tilde{s}^2 \;\leq\; -\tilde{s}^2.$$

This follows because $\gamma(t) \in (\gamma_m, \gamma_M)$ for all t. Using the LaSalle Invariance Principle, it follows [89] that (2.46) is globally asymptotically stable to 0 when we make the choice (2.48).

2.5 Basin of Attraction Revisited

Strict Lyapunov functions can be used to estimate the basins of attractions for locally asymptotically stable time-varying nonlinear systems

$$\dot{x} = f(t, x). \tag{2.49}$$

Let us review how this can be done. Assume that we know a C^1 storage function $V : [0, \infty) \times \mathbb{R}^n \to [0, \infty)$, a continuous positive definite function $W : \mathbb{R}^n \to [0, \infty)$, and an open set $D \subseteq \mathbb{R}^n$ containing $x = 0$ such that

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x) \leq -W(x) \quad \forall t \geq 0 \ \text{ and } \ x \in D\,. \tag{2.50}$$

Choose functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \quad \forall t \geq 0 \ \text{ and } \ x \in \mathbb{R}^n\,. \tag{2.51}$$

For convenience, we continue to write

$$\dot{V} := \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x)\,.$$

Since D is an open set that contains the origin, there exists a constant $r > 0$ such that $r\mathcal{B}_n \subseteq D$. The non-decreasing function $\gamma : [0, r] \to [0, \infty)$ defined by

$$\gamma(s) = \inf_{\{\xi \in r\mathcal{B}_n : |\xi| \geq s\}} W(\xi)$$

then satisfies $\gamma(0) = 0$ and $\gamma(s) > 0$ for all $s \in (0, r)$, because W is positive definite. Moreover, for all $t \geq 0$ and $x \in r\mathcal{B}_n$, we have

$$\dot{V} \leq -\gamma(|x|)\,. \tag{2.52}$$

Also, (2.51) in combination with the facts that $\alpha_2 \in \mathcal{K}_\infty$ and γ is non-decreasing give

$$0 \leq \gamma\big(\alpha_2^{-1}\big(V(t, x)\big)\big) \leq \gamma(|x|) \quad \forall t \geq 0 \ \text{ and } \ x \in r\mathcal{B}_n \tag{2.53}$$

Combining (2.52) and (2.53) yields

$$\dot{V} \leq -\overline{\gamma}\big(V(t, x)\big) \quad \forall t \geq 0 \ \text{ and } \ x \in r\mathcal{B}_n \tag{2.54}$$

where $\overline{\gamma}(s) = \gamma(\alpha_2^{-1}(s))$.

Let us now consider the set $E_r = \{x \in r\mathcal{B}_n : V(t, x) < \alpha_1(r) \ \forall t \geq 0\}$. One readily checks that E_r is a positively invariant set for (2.49). To see why, let $x_0 \in E_r$ and $t_0 \geq 0$, and let $x(t)$ denote the solution of (2.49) such that $x(t_0) = x_0$. Suppose that there exists a time $t_1 > t_0$ such that $V(t_1, x(t_1)) = \alpha_1(r)$ and $V(t, x(t)) < \alpha_1(r)$ for all $t \in [t_0, t_1)$. Then (2.51) implies that

$\alpha_1(|x(t)|) < \alpha_1(r)$ for all $t \in [t_0, t_1)$, i.e., $x \in r\mathcal{B}_n$ for all $t \in [t_0, t_1)$. This allows us to conclude from (2.54) that

$$V(t_1, x(t_1)) \;\leq\; V(t_0, x(t_0)) \;<\; \alpha_1(r),$$

which is a contradiction. We deduce that for all $t \geq t_0$, $V(t, x(t)) < \alpha_1(r)$ and $|x(t)| < r$, which gives the positive invariance of E_r. Moreover, (2.54) implies that all solutions of the system starting in E_r converge to the origin. Also, $0.5\alpha_2^{-1}(\alpha_1(r))\mathcal{B}_n \subseteq E_r$ and $\alpha_2^{-1}(\alpha_1(r)) > 0$. Therefore E_r is a non-empty subset of the basin of attraction of (2.49). Hence, knowing a strict Lyapunov function for the system leads to an approximation of the basin of attraction.

2.6 $\mathcal{L}_2$ Gains

2.6.1 Basic Theorem

Strict Lyapunov functions can also help estimate the effect of a disturbance on a specific output. For instance, consider a locally Lipschitz forward complete nonlinear control affine system with output

$$\dot{x} = f(x) + g(x)u, \;\; y = h(x) \in \mathbb{R}^q \tag{2.55}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$. We wish to determine a constant $\gamma > 0$ and a function $\Gamma \in \mathcal{K}_\infty$ such that for any continuous input $u(t)$ and any initial state x_0, the corresponding solution $x(t)$ of (2.55) satisfies

$$\sqrt{\int_0^T |y(s)|^2 \, \mathrm{d}s} \;\leq\; \gamma\sqrt{\int_0^T |u(s)|^2 \, \mathrm{d}s} + \Gamma(|x_0|) \tag{2.56}$$

for all constants $T > 0$. Here is a useful result in that direction from [70]:

Theorem 2.4. *Consider the system with output (2.55). Assume that f, g, h are Lipschitz continuous and $f(0) = 0$ and $h(0) = 0$. Assume that there is a C^1 function $V : \mathbb{R}^n \to [0, \infty)$ such that*

$$L_f V(x) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x}(x)g(x)\left(\frac{\partial V}{\partial x}(x)g(x)\right)^\top + \frac{1}{2}h(x)^\top h(x) \leq 0 \tag{2.57}$$

for all $x \in \mathbb{R}^n$. Then for all $x_0 \in \mathbb{R}^n$, the inequality

$$\sqrt{\int_0^T |y(s)|^2 \, \mathrm{d}s} \;\leq\; \gamma\sqrt{\int_0^T |u(s)|^2 \, \mathrm{d}s} + \sqrt{2V(x_0)} \tag{2.58}$$

is satisfied for all $T \geq 0$.

Proof. The time derivative of V along the trajectories of (2.55) satisfies

$$\begin{aligned}\dot{V} &= L_f V(x) + \frac{\partial V}{\partial x}(x)g(x)u \\ &= L_f V(x) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x}(x)g(x)\left(\frac{\partial V}{\partial x}(x)g(x)\right)^\top + \frac{1}{2}h(x)^\top h(x) \\ &\quad -\frac{\gamma^2}{2}u^\top u + \frac{\partial V}{\partial x}(x)g(x)u - \frac{1}{2\gamma^2}\frac{\partial V}{\partial x}(x)g(x)\left(\frac{\partial V}{\partial x}(x)g(x)\right)^\top \\ &\quad +\frac{\gamma^2}{2}u^\top u - \frac{1}{2}h(x)^\top h(x)\end{aligned} \tag{2.59}$$

From (2.57) and the fact that

$$\begin{aligned}&-\frac{\gamma^2}{2}u^\top u + \frac{\partial V}{\partial x}(x)g(x)u - \frac{1}{2\gamma^2}\frac{\partial V}{\partial x}(x)g(x)\left(\frac{\partial V}{\partial x}(x)g(x)\right)^\top \\ &= -\frac{1}{2}\left|\gamma u^\top - \frac{1}{\gamma}\frac{\partial V}{\partial x}(x)g(x)\right|^2 \leq 0,\end{aligned}$$

it follows that

$$\dot{V} \;\leq\; \frac{\gamma^2}{2}|u|^2 - \frac{1}{2}|y|^2. \tag{2.60}$$

By integrating (2.60) over $[0, T]$, we obtain

$$V(x(t)) - V(x_0) \leq \tfrac{\gamma^2}{2}\int_0^T |u(s)|^2\,\mathrm{d}s - \frac{1}{2}\int_0^T |y(s)|^2\,\mathrm{d}s \tag{2.61}$$

Using the non-negativity of $V(x(t))$, we get

$$\tfrac{1}{2}\int_0^T |y(s)|^2\,\mathrm{d}s \leq V(x_0) + \tfrac{\gamma^2}{2}\int_0^T |u(s)|^2\,\mathrm{d}s \tag{2.62}$$

and therefore (2.58) follows from the relation $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for nonnegative values a and b. □

2.6.2 Illustration

The $\mathcal{L}_2$ gain of the system (2.55) is defined to be the infimum of the set of all constants $\gamma > 0$ such that the inequality (2.58) holds for all x_0, u, and T. Theorem 2.4 provides constants γ that are larger than the $\mathcal{L}_2$ gain of the system (2.55). The constants depend on the function V selected. Moreover, there are cases where the $\mathcal{L}_2$ gain is a finite number, but where an inadequate

Lyapunov function V does not allow us to determine an approximate upper bound for this gain.

We illustrate this using the simple one-dimensional nonlinear system

$$\begin{cases} \dot{x} = -x - x^3 + u \\ y = x \ . \end{cases} \tag{2.63}$$

If we choose $V_a(x) = \frac{1}{2}x^2$, then the inequality (2.57) becomes

$$-x^4 + \frac{1-\gamma^2}{2\gamma^2}x^2 \ \leq \ 0 \tag{2.64}$$

and 1 is the smallest value of γ such that (2.64) holds for all x. If we instead choose $V_b(x) = x^2$, then the inequality (2.57) becomes

$$\frac{4-3\gamma^2}{2\gamma^2}x^2 - 2x^4 \ \leq \ 0 \tag{2.65}$$

and $2/\sqrt{3}$ is the smallest γ such that (2.64) holds for all x. Finally, if we take $V_c(x) = \frac{1}{4}x^4$, then the inequality (2.57) reads

$$-x^4 - x^6 + \frac{1}{2\gamma^2}x^6 + \frac{1}{2}x^2 \ \leq \ 0 \ , \tag{2.66}$$

which cannot be satisfied for all values of x. Hence, we cannot use V_c to get an upper bound for the $\mathcal{L}_2$ gain of the system (2.55).

This provides yet another reason for wanting explicit Lyapunov functions. Indeed, no estimate for the $\mathcal{L}_2$ gain of a system (2.55) can be deduced from the mere existence of a strict Lyapunov function, as provided by the converse Lyapunov theorem.

2.7 Lyapunov Functions with Bounded Gradients

The converse Lyapunov theorem ensures that UGAS systems admit global strict Lyapunov functions. However, it does not in general give information on the type of Lyapunov function that is associated with a given UGAS system, e.g., whether or not the gradient of the Lyapunov function is uniformly bounded in norm. This is a shortcoming of converse theory.

In fact, if $V(x)$ is a strict Lyapunov function for a system $\dot{x} = f(x)$, and if $\nabla V(x)$ is known to be globally bounded, then V is also an iISS Lyapunov function for $\dot{x} = f(x) + d$ with disturbance d, and then the iISS estimate bounds the trajectories if d is exponentially decaying to zero. This is yet another motivation for constructing strict Lyapunov functions. Many UGAS systems admit strict Lyapunov functions with globally bounded gradients; see Sect. 4.6 for related results.

2.7.1 Effect of Exponentially Decaying Disturbances

It is possible to construct a UGAS system that has unbounded trajectories in the presence of an additive exponentially decaying disturbance, and therefore cannot admit a strict Lyapunov function with a globally bounded gradient. For example, consider the following system from [177]:

$$\begin{cases} \dot{x}_1 = g(x_1 x_2) x_1 \\ \dot{x}_2 = -2x_2 + d\,, \end{cases} \tag{2.67}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $d \in \mathbb{R}$ is a disturbance, and the function g is such that:

1. g is Lipschitz continuous;
2. $|g(s)| \leq 1$ for all s;
3. $g(s) = -1$ for all $s \in \left(-\infty, \frac{1}{2}\right] \cup \left[\frac{3}{2}, \infty\right)$; and
4. $g(1) = 1$.

The system (2.67) has the additional property of being globally *exponentially* stable when $d = 0$. In fact, we can prove:

Proposition 2.1. *When $d \equiv 0$, the solutions of (2.67) satisfy*

$$|x(t)| \;\leq\; e^4 e^{-t} |x(0)| \tag{2.68}$$

for all $t \geq 0$ and all initial states $x(0) \in \mathbb{R}^2$.

Proof. Let

$$P(r) = \int_0^r H(m)\, \mathrm{d}m \tag{2.69}$$

where

$$H(m) = \begin{cases} \dfrac{1 + g(m)}{m[2 - g(m)]}, & \text{if } m \neq 0 \\ 0, & \text{if } m = 0\,. \end{cases}$$

Then

$$0 \leq P(r) \leq 4 \tag{2.70}$$

for all $r \in \mathbb{R}$. We introduce the variable

$$\xi = x_1 x_2\ . \tag{2.71}$$

Then

$$\begin{cases} \dot{x}_1 = g(\xi)x_1 \\ \dot{x}_2 = -2x_2 \\ \dot{\xi} = [-2 + g(\xi)]\xi \ . \end{cases} \tag{2.72}$$

Let

$$Z_1 = e^{P(\xi)} x_1 \ . \tag{2.73}$$

Then

$$\begin{aligned} \dot{Z}_1 &= \Big(g(\xi) + H(\xi)\big[-2 + g(\xi) \big]\xi \Big) Z_1 \\ &= -Z_1 \ . \end{aligned} \tag{2.74}$$

Hence,

$$|Z_1(t)| \ \leq \ e^{-t}|Z_1(0)| \quad \text{and} \quad |x_2(t)| \ \leq \ e^{-2t}|x_2(0)| \tag{2.75}$$

for all $x(0) \in \mathbb{R}^2$ and all $t \geq 0$. We deduce that

$$|x_1(t)| \leq e^{-P(\xi(t))} e^{-t} e^{P(x_1(0)x_2(0))} |x_1(0)| \ . \tag{2.76}$$

Finally, from (2.70), we conclude that (2.68) holds, as claimed. □

Remark 2.1. In [177], it is proved that

$$|x(t)| \ \leq \ 9e^{-t}|x(0)|$$

along all trajectories of (2.67). We establish the inequality (2.68) because its proof relies on a slightly different technique from the one given in [177] that will be useful later to establish complementary results.

The next result establishes that the behavior of the solutions of the system (2.67) may change drastically in the presence of a decaying disturbance.

Proposition 2.2. *When*

$$x_1(0) \neq 0, \ \ x_2(0) = x_1(0)^{-1}, \ \text{and} \ \ d(t) = x_2(0)e^{-t}, \tag{2.77}$$

the solutions of (2.67) satisfy

$$x_1(t) = e^t x_1(0) \tag{2.78}$$

for all $t \geq 0$.

Proof. The functions

$$x_1(t) = e^t x_1(0) \ \ \text{and} \ \ x_2(t) = x_2(0)e^{-t}$$

are such that

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = -2x_2 + x_2(0)e^{-t} \ . \end{cases} \tag{2.79}$$

Since $g(1) = 1$ and $d(t) = x_2(0)e^{-t}$, we have

$$\begin{cases} \dot{x}_1 = g(1)x_1 \\ \dot{x}_2 = -2x_2 + d(t) \ . \end{cases} \tag{2.80}$$

In addition, for all t, we get $x_1(t)x_2(t) = e^t x_1(0)x_2(0)e^{-t} = x_1(0)x_2(0) = 1$. Therefore,

$$\begin{cases} \dot{x}_1 = g(x_1 x_2)x_1 \\ \dot{x}_2 = -2x_2 + d(t) \ . \end{cases} \tag{2.81}$$

This proves the proposition. □

An immediate consequence of Propositions 2.1 and 2.2 is:

Proposition 2.3. *If $V(x)$ is a strict Lyapunov function for the system (2.67) with the disturbance $d \equiv 0$, then there does not exist a constant $C > 0$ such that*

$$\left| \frac{\partial V}{\partial x}(x) \right| \leq C \tag{2.82}$$

for all $x \in \mathbb{R}^2$.

Proof. Suppose the contrary. Since V satisfies (2.82) for some constant C, the time derivative of V along the trajectories of (2.67) satisfies

$$\dot{V} \leq C|d(t)| \ . \tag{2.83}$$

In particular, the choices (2.77) give

$$\dot{V} \leq C|x_2(0)e^{-t}| \ . \tag{2.84}$$

By integrating this inequality we deduce that

$$V(x(t)) \leq V(x(0)) + C|x_2(0)| \tag{2.85}$$

for all $t \geq 0$. According to Proposition 2.2, $x_1(t) = e^t x_1(0)$ for all $t \geq 0$, so

$$\lim_{t \to +\infty} V(x(t)) = +\infty.$$

This and (2.85) yield a contradiction. □

2.7.2 Dependence on Coordinates

We next show that the property of having no strict Lyapunov function with a bounded gradient is coordinate dependent.

Proposition 2.4. *Let P be as defined in (2.69). Then the variables*

$$Z_1 = e^{P(x_1x_2)}x_1 \quad \textit{and} \quad Z_2 = x_2 \tag{2.86}$$

define a global change of coordinates that transforms (2.67) with $d \equiv 0$ into

$$\begin{cases} \dot{Z}_1 = -Z_1 \\ \dot{Z}_2 = -2Z_2 \ . \end{cases} \tag{2.87}$$

Hence, (2.67) is globally linearizable.

Proof. Routine calculations yield

$$\begin{aligned} \tfrac{\partial Z_1}{\partial x_1}(x) &= e^{P(x_1x_2)}\left[1 + x_1x_2P'(x_1x_2)\right] \\ &= e^{P(x_1x_2)}\left[1 + \frac{1+g(x_1x_2)}{2-g(x_1x_2)}\right] \;=\; e^{P(x_1x_2)}\frac{3}{2-g(x_1x_2)} > 0 \ . \end{aligned} \tag{2.88}$$

We deduce that (2.86) defines a global change of coordinates that transforms the system (2.67) with $d \equiv 0$ into the linear system (2.87). □

Remark 2.2. Notice that $\mathcal{V}(Z) = \ln(1+Z_1^2+Z_2^2)$ is a strict Lyapunov function for (2.87) that has a bounded gradient.

2.7.3 Strictification

It is tempting to surmise that non-strict Lyapunov functions with globally bounded gradients can be transformed into strict Lyapunov functions with globally bounded gradients. The following result shows how such a strictification transformation can sometimes be carried out:

Proposition 2.5. *Let*

$$\dot{x} = f(t, x) \tag{2.89}$$

be a UGAS system that is periodic in t and admits a non-strict Lyapunov function V such that

$$\left|\frac{\partial V}{\partial t}(t,x)\right| \le 1 \quad \textit{and} \quad \left|\frac{\partial V}{\partial x}(t,x)\right| \le 1 \tag{2.90}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Assume that $\frac{\partial f}{\partial x}$ is bounded. Then the system (2.89) admits a strict Lyapunov function U such that

$$\left|\frac{\partial U}{\partial t}(t,x)\right| \le 1 \quad \textit{and} \quad \left|\frac{\partial U}{\partial x}(t,x)\right| \le 1 \tag{2.91}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Proof. By Theorem 2.1, there exists a C^1 function $\nu(t,x)$, a positive definite function $w(x)$, and functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that

$$\gamma_1(|x|) \;\leq\; \nu(t,x) \;\leq\; \gamma_2(|x|) \;\text{ and} \tag{2.92}$$

$$\max\left\{\left|\frac{\partial \nu}{\partial t}(t,x)\right|, \left|\frac{\partial \nu}{\partial x}(t,x)\right|\right\} \;\leq\; \gamma_3(|x|) \tag{2.93}$$

hold for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and such that the time derivative of $\nu(t,x)$ along the trajectories of (2.89) satisfies

$$\dot{\nu}(t,x) \leq -w(x)\ . \tag{2.94}$$

Let $\Gamma : [0,\infty) \to [1,\infty)$ be a continuous increasing function such that

$$1 + \gamma_3(\gamma_1^{-1}(s)) \leq \Gamma(s) \tag{2.95}$$

for all $s \geq 0$. By (2.92), we have

$$|x| \;\leq\; \gamma_1^{-1}(\nu(t,x)),$$

so (2.95) with the choice $s = \nu(t,x)$ gives

$$1 + \gamma_3(|x|) \;\leq\; 1 + \gamma_3(\gamma_1^{-1}(\nu(t,x))) \;\leq\; \Gamma(\nu(t,x))$$

everywhere. This inequality and (2.93) imply that

$$\left|\frac{\partial \nu}{\partial t}(t,x)\right| \leq \Gamma(\nu(t,x)) \quad\text{and}\quad \left|\frac{\partial \nu}{\partial x}(t,x)\right| \leq \Gamma(\nu(t,x)) \tag{2.96}$$

everywhere.

Next, consider the function

$$U(t,x) = \frac{1}{2}\left[V(t,x) + \int_0^{\nu(t,x)} \frac{1}{\Gamma(m)}\mathrm{d}m\right]\ . \tag{2.97}$$

Then

$$\left|\frac{\partial U}{\partial t}(t,x)\right| \;\leq\; \frac{1}{2}\left[\left|\frac{\partial V}{\partial t}(t,x)\right| + \frac{1}{\Gamma(\nu(t,x))}\left|\frac{\partial \nu}{\partial t}(t,x)\right|\right] \;\leq\; 1 \tag{2.98}$$

everywhere, by (2.90) and (2.96). Similarly, one can show that

$$\left|\frac{\partial U}{\partial x}(t,x)\right| \leq 1\ . \tag{2.99}$$

Finally, observe that

$$\frac{1}{2}V(t,x) \le U(t,x) \le \frac{1}{2}[V(t,x) + \nu(t,x)] \tag{2.100}$$

implies that

$$\frac{1}{2}\alpha_1(|x|) \le U(t,x) \le \frac{1}{2}[\alpha_2(|x|) + \gamma_2(|x|)] \ , \tag{2.101}$$

where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ are from the proper and positive definite conditions on V. Therefore, U is a global strict Lyapunov for (2.89) with a bounded gradient. This proves the proposition. □

2.8 Comments

Throughout our work, we assume that our (control) Lyapunov functions are at least C^1. One can generalize the definitions to allow Lyapunov functions that are continuous but not necessarily differentiable; see, e.g., the books [17, 184] for some early results. Nonsmooth analysis provides a unifying method for the analysis of nondifferentiable functions [22]. One pioneering result [155] by Sontag showed that a system $\dot{x} = f(x,u)$ evolving on $\mathbb{R}^n$ is asymptotically controllable to the origin if and only if it admits a continuous positive definite proper function V that satisfies: *For each $x \in \mathbb{R}^n$, there exists a relaxed control w such that $\dot{V}_w(x) < 0$. Moreover, there exist positive constants k and η such that w can be chosen to satisfy $||w|| < k$ whenever $|x| < \eta$.* Here

$$\dot{V}_w(x) = \liminf_{t \to 0^+} \frac{1}{t}\{V(\phi(t,x,w)) - V(x)\}$$

is the lower Dini Derivative, ϕ is the flow map of the system, and relaxed controls are measurable mappings of $[0, \infty)$ into the set of all probability measures on the control set U. Also, $||w||$ is the infimum of the set of all r's so that w is supported in $r\mathcal{B}_m$. One motivation for using nonsmooth analysis in control theory is that a system $\dot{x} = f(x,u)$ admits a C^1 control Lyapunov function $V(x)$ if and only if it is C^1 stabilizable by a time-invariant feedback $u_s(x)$, so C^1 control Lyapunov functions cannot exist unless Brockett's Condition is satisfied. This motivation becomes less important when we allow time-varying feedback stabilizers.

The system (2.67) is important because it shows how a globally exponentially stable system can be destabilized by an exponentially decaying disturbance that is arbitrarily small in the $\mathcal{L}_1$ norm. See [166] for an earlier example of a time-invariant GAS system $\dot{x} = f(x)$ that admits an integrable disturbance $\mathbf{d}$ for which $\dot{x} = f(x) + \mathbf{d}$ has unbounded solutions.

One can also develop Lyapunov function theory for *retarded functional differential equations*, which have the form

$$\dot{x}(t) = f(t, x_t) \tag{2.102}$$

where $x(t) \in \mathbb{R}^n$, $f : \mathbb{R} \times \mathcal{C}_n([-r, 0]) \to \mathbb{R}^n$ for a given choice of the constant $r > 0$, the function x_t is defined by $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-r, 0]$, and the initial value is $\phi \in \mathcal{C}_n([-r, 0])$. In this context, $\mathcal{C}_n(\mathcal{I})$ is the set of all continuous functions $\phi : \mathcal{I} \to \mathbb{R}^n$ on any interval $\mathcal{I} \subseteq \mathbb{R}$. We assume that f has enough regularity to guarantee the existence and uniqueness of a maximal solution for each initial condition; see [109] for sufficient conditions for the existence of maximal solutions. Equations of this type are also called *(time) delayed differential equations.* Their study is an important subject that is best considered in books devoted only to systems with delays [51, 114, 124]. Here we only summarize some Lyapunov results for delay systems.

Lyapunov related functions are key for the stability analysis and control design for systems with delay. Two important theorems for delayed systems are the Razumikhin Theorem and the Lyapunov-Krasovski Theorem. Both rely on delayed Lyapunov functions or functionals, which are often constructed by first building Lyapunov functions for the corresponding undelayed systems (obtained by setting the delays equal to zero). For a given constant $r > 0$, the Razumikhin Theorem is the following [52]:

Theorem 2.5. *(Razumikhin Theorem) Let* $f : \mathbb{R} \times \mathcal{C}_n([-r, 0]) \to \mathbb{R}^n$ *map* $\mathbb{R}\times$ *(bounded subsets of* $\mathcal{C}_n([-r, 0])$*) into bounded subsets of* $\mathbb{R}^n$*. Let* $u, v, w : [0, \infty) \to [0, \infty)$ *be continuous non-decreasing functions for which* u *and* v *are positive definite and* v *is increasing. Assume the following:*

1. *There exists a continuously differentiable function* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *whose time derivatives along the solutions* $x(t)$ *of (2.102) satisfy*

$$\dot{V}(t, x(t)) \leq -w(|x(t)|) \tag{2.103}$$

whenever

$$V(t + \theta, x(t + \theta)) \leq V(t, x(t)) \tag{2.104}$$

for all $\theta \in [-r, 0]$*. Also,* $u(|x|) \leq V(t, x) \leq v(|x|)$ *everywhere.*

Then the system (2.102) is uniformly stable. If, in addition,

2. $w(s) > 0$ *for all* $s > 0$ *and there exists a continuous non-decreasing function* p *such that* $p(s) > s$ *for all* $s > 0$*, and such that*

$$\dot{V}(t, x(t)) \leq -w(|x(t)|) \tag{2.105}$$

whenever

$$V(t + \theta, x(t + \theta)) \leq p(V(t, x(t))) \tag{2.106}$$

for all $\theta \in [-r, 0]$*.*

then the system (2.102) is uniformly asymptotically stable. If 1. and 2. hold and

$$\lim_{s \to +\infty} u(s) = +\infty,$$

then the system (2.102) is UGAS.

The stability concepts for (2.102) are defined as in the undelayed case, except that the initial state $x_0 \in \mathbb{R}^n$ is replaced by an initial *function* $\phi \in \mathcal{C}_n([-r,0])$. The Lyapunov-Krasovski Theorem is the following, where we use the notation

$$\dot{V}(t,\phi) = \frac{d}{dt}V(t,x_t)|_{t=\tau,x_t=\phi}$$

for any C^1 function $V : \mathbb{R} \times \mathcal{C}_n([-r,0]) \to \mathbb{R}$ [52]:

Theorem 2.6. *(Lyapunov-Krasovski Theorem) Let* $f : \mathbb{R} \times \mathcal{C}_n([-r,0]) \to \mathbb{R}^n$ *map* $\mathbb{R}\times$*(bounded subsets of* $\mathcal{C}_n([-r,0])$*) into bounded subsets of* $\mathbb{R}^n$*. Let* $u, v, w : [0,\infty) \to [0,\infty)$ *be continuous non-decreasing functions for which* u *and* v *are positive definite and* v *is increasing. Assume the following:*

1. *There exists a continuously differentiable function* $V : \mathbb{R}\times\mathcal{C}_n([-r,0]) \to \mathbb{R}$ *such that*
$$u(|\phi(0)|) \le V(t,\phi) \le v(|\phi|_{[-r,0]}) \tag{2.107}$$
and
$$\dot{V}(t,\phi) \le -w(|\phi(0)|) \tag{2.108}$$
for all $\phi \in \mathcal{C}_n([-r,0])$ *and* $t \in \mathbb{R}$*.*

Then the trivial solution of (2.102) is uniformly stable. If, in addition,

2. $w(s) > 0$ *for all* $s > 0$*,*

then (2.102) is uniformly asymptotically stable. Finally, if 1. and 2. hold and if we also have

$$\lim_{s\to+\infty} u(s) = +\infty,$$

then the system (2.102) is UGAS.

Over the last two decades, Lyapunov-Krasovski functionals have been used extensively for the analysis of linear systems. For linear systems, Lyapunov-Krasovski functionals give stability criteria in terms of linear matrix inequalities, which can be analyzed through numerical methods; see for instance [125].

The ISS paradigm can be extended to delayed systems, using either the Razumikhin Theorem (as was done, e.g., in [62, 175]) or Lyapunov-Krasovski functionals (as in [109, 130]). For example, if we consider

$$\dot{x}(t) = f\big(x(t),t\big) + g(x(t),t)\big[u_s\big(\xi_\tau(t),t\big) + d(t)\big], \;\; x(t) \in \mathbb{R}^n \tag{2.109}$$

with

$$\xi_\tau(t) = \big(x_1(t-\tau_1), x_2(t-\tau_2), \cdots, x_n(t-\tau_n)\big)$$

and constant delays $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ satisfying $0 \le \tau_i \le \bar{\tau}$ for all i for any bound $\bar{\tau} \ge 0$, and with unknown but bounded disturbances d and a known feedback u_s, an appropriate notion of Lyapunov-Krasovski functionals is as follows:

Definition 2.3. A continuous functional $U : [0, \infty) \times \mathcal{C}_n(\mathbb{R}) \to [0, \infty)$ is called an *ISS Lyapunov-Krasovski functional (ISS-LKF)* for (2.109) provided that for all $\tau \in (0, \bar{\tau}]^n$ and all trajectories $x(t) \doteq x(t; t_0, x_0, d, \tau)$ of (2.109) (corresponding to all possible initial conditions $x(t_0) = x_0$ and measurable essentially bounded disturbances d), the function $t \mapsto U(t, x_t)$ is locally absolutely continuous and there exist functions $\alpha_i \in \mathcal{K}_\infty$ for $i = 1, 2, 3, 4$ and $\kappa \in \mathbb{N}$ such that for all $\phi \in \mathcal{C}_n([-\kappa\bar{\tau}, 0])$, all trajectories $x(t)$ of (2.109), and all $t \geq t_0 + \kappa\bar{\tau}$, we have (a) $\alpha_1(|\phi(0)|) \leq U(t, \phi) \leq \alpha_2(|\phi|_{[-\kappa\bar{\tau},0]})$ and (b) the time derivative $D_t U(t, x_t)$ of $U(t, x_t)$ satisfies $D_t U(t, x_t) \leq -\alpha_3(U(t, x_t)) + \alpha_4(|d|_{[t_0,t]})$ almost everywhere.

In this context, $\kappa \in \mathbb{N}$ represents the length of the time lag. The key ingredient in the ISS-LKF definition is that instead of being defined for points in the state space, $U(t, \phi)$ is evaluated at continuous $\mathbb{R}^n$-valued functions $\phi \in \mathcal{C}_n(\mathbb{R})$ defined on the real line and times $t \geq 0$, hence the term function*al* instead of function in the definition.

The explicit construction of ISS-LKFs is a challenging problem. One such construction in [109] shows that (2.109) admits an ISS-LKF of the form

$$U(t, x_t) \;=\; V\big(t, x(t)\big) + \frac{1}{4\bar{\tau}} \int_{t-2\bar{\tau}}^{t} \left(\int_{r}^{t} \sigma^2(\sqrt{n}|x(l)|) \mathrm{d}l \right) \mathrm{d}r \qquad (2.110)$$

where the proper positive definite function V, the undelayed dynamics

$$\dot{x} = F(t, x, u_s) \doteq f(t, x) + g(t, x) u_s(t, x),$$

and the function σ satisfy appropriate conditions, including the decay condition

$$V_t(l, x) + V_x(l, x) F(l, x, u_s) \leq -\sigma^2\left(\sqrt{n}|x|\right).$$

The function σ is of class $\mathcal{K}_\infty$, making V a strict Lyapunov function for the undelayed dynamics. In particular, u_s is assumed to stabilize the undelayed dynamics. The Lyapunov-Krasovski functional is valid when the delay bound $\bar{\tau}$ is small enough, but one can take any desired positive constant bound $\bar{\tau}$ when the drift term f in (2.109) is identically zero; see [109] for the explicit computation of the admissible delay bound $\bar{\tau}$ for (2.109). Using (2.110), we can get an explicit ISS decay estimate for (2.109) which is analogous to the usual ISS estimate, except with an initial function replacing the initial state. Therefore, we can quantify the combined effects of feedback delays τ_i and uncertainty d on the stability performance of the feedback u_s. Lyapunov functions are also useful for partial differential equations [28, 29, 76] and stochastic systems [72]. This book will focus on deterministic ODE systems without delays, although it would be of interest to extend many of the results to time-varying delayed systems or stochastic PDEs.

Stabilization problems for biological dynamics such as (2.41) have been studied by numerous authors. Many of the results are based on the (local) linearization approach [57] or linearizing state controllers that assume perfect knowledge of the model. For an alternative approach to uncertain biosystems based on interval observers, see [49]. Our biosystems analysis assumes that the reactor is well mixed, meaning all organisms have equal access to the nutrient; dropping this assumption leads to PDE models. Our approach takes the non-negativity of the state components into account and leads to an explicit strict Lyapunov function; see Sect. 3.1 for details.

http://www.springer.com/978-1-84882-534-5

Constructions of Strict Lyapunov Functions
Malisoff, M.; Mazenc, F.
2009, XVI, 386 p., Hardcover
ISBN: 978-1-84882-534-5