2
Topological Properties

This chapter, in contrast to the last, shows that sets are *not* homeomorphic. To show that sets S and T are *not* homeomorphic we construct a suitable topological property P such that S has property P but T does not. From the definition of a topological property, if S were homeomorphic to T, then T would also have property P. So S and T are not homeomorphic. Thus every topological property is a tool for proving sets to be non-homeomorphic.

We give a collection of elementary topological properties that will be useful in the next chapter as well.

The topological properties we consider here are based on the intuitively clear idea of a *path*, which mathematicians tend to think of, not as a static object, but as a moving point. We denote by $[a, b]$ the closed interval of all x such that $a \leq x \leq b$.

Definition 2.1

A *path* α *in* S is a continuous mapping from the closed interval $[a, b]$ to S for some a, b where $a < b$. If $\alpha(a) = p$ and $\alpha(b) = q$, we say that α *joins* p *to* q.

Definition 2.2

A subset S of $\mathbb{R}^n$ is *path-connected* if every pair p, q of points in S can be joined by a path in S.

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Example 2.1

The plane is path-connected. For let p, q be any points in the plane. The straight path given by

$$\alpha(u) = p + u(q - p), \quad 0 \leq u \leq 1,$$

joins p to q. Hence the plane is path-connected. There is nothing particularly two-dimensional about the argument, so we have also shown that $\mathbb{R}^n$ is path-connected for all $n \geq 1$.

Even with the sphere it would be getting complicated to write down explicit formulae for paths. Life is made easy for us by the following result.

Theorem 2.1

The continuous image of a path-connected set is path-connected.

Proof

Suppose that S is path-connected and that T is the image of S under the continuous mapping f. Take any points p, q in T. We must construct a path in T joining p to q. Now there are points x, y in S such that $f(x) = p$ and $f(y) = q$. Because S is path-connected, there is a path $\alpha : [a, b] \to S$ joining x and y. Define $\beta(t) = f(\alpha(t))$ for t in $[a, b]$. Then β is a continuous mapping from $[a, b]$ to T, and so is a path in T. Also $\beta(a) = p$ and $\beta(b) = q$, so β joins p to q. This completes the proof.

Example 2.2

The circle and the sphere are path-connected. The circle is the image of the real line under the continuous mapping $u \to (\cos u, \sin u)$, and the sphere is the continuous image of the plane under the continuous mapping $(u, v) \to (\sin u \cos v, \sin u \sin v, \cos u)$.

Example 2.3

The torus is path-connected, being the image of the plane under the continuous mapping

$$(u, v) \to (\cos u, \sin u, \cos v, \sin v).$$

Example 2.4

The punctured plane—the plane with the origin removed—is path-connected because it is the image of the whole plane under the continuous mapping $(x, y) \to e^x(\cos y, \sin y)$.

Theorem 2.2

Path-connectedness is a topological property.

Proof

Suppose that S is path-connected and that f is a homeomorphism from S to T. Then T is the image of S under the continuous mapping f so the path-connectedness of T follows from Theorem 2.1. This completes the proof.

Example 2.5

Let S be the real line with the origin removed. Any path in the real line from 1 to -1 must pass through the origin. So S is *not* path-connected. We now have our first proof that sets are not homeomorphic because S cannot be homeomorphic to the real line or any of the sets we showed to be path-connected.

Example 2.6

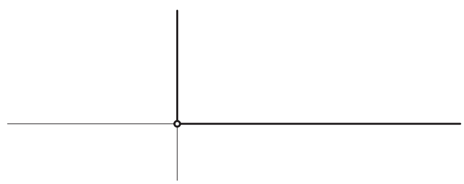

Figure 2.1 Example 2.6

Let T be the union of the half-axes $\{(0, y) : y > 0\}$ and $\{(x, 0) : x > 0\}$. Then T is *not* path-connected: any plane path joining points on these two axes must meet the line $y = x$. Alternatively, we could show that T is homeomorphic to the set S of Example 2.5.

We use path-connectedness to make another simple topological tool. The next step is to introduce language to express the idea that a non-path-connected set is formed of separate pieces, or *components*. Let S be a set. We say that points p, q in S are *together* if there is a path in S joining p to q. We wish to define the component containing the point p to consist of all points that are together with p, so we require the following result.

Theorem 2.3

Togetherness is an equivalence relation.

Proof

Let S be a set. Certainly, if p is in S, then p and p are together because mapping every point of $[0, 1]$ to p is a path in S joining p to p: thus togetherness is reflexive. Now suppose that p, q are together and that $\alpha : [a, b] \to S$ joins p to q. Go backwards along α: put $\omega(t) = \alpha(a+b-t)$. Then ω is a path in S joining q to p, so togetherness is symmetric. Finally, to show transitivity, let α join p to q as before and let β join q to r, where $\beta : [c, c+h] \to S$. Join p to r by going along α and then β. Define $\gamma : [a, b+h] \to S$ by

$$\gamma(t) = \alpha(t), \quad a \leq t \leq b$$

$$\gamma(t) = \beta(c+t-b), \quad b \leq t \leq b+h.$$

Then γ is a path in S joining p to r. This completes the proof.

Example 2.7

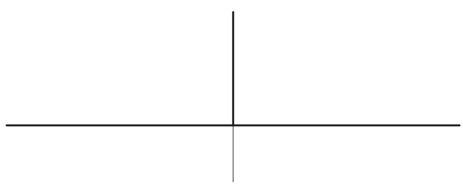

Figure 2.2 Example 2.7

Let T be the union of $\{(0, y) : y > 0\}$ and the whole real line. Then each point of T can be joined to the origin by a straight path in T. So each point is together with the origin, and, by transitivity, any two points in T are together. Thus T *is* path-connected.

Definition 2.3

A *component* is an equivalence class of togetherness.

Thus each point p of a set belongs to just one component, which consists of all those points that are together with p. To prove that sets with different numbers of components are not homeomorphic we need the following theorem.

Theorem 2.4

Homeomorphic sets have the same number of components.

Proof

Let f be a homeomorphism from S to T and suppose that α joins p to q in S. Then, as in Theorem 1, the path $t \to f(\alpha(t))$ joins $f(p)$ to $f(q)$ in T. Hence points that are together in S are sent to points that are together in T. Also points that are not together in S are sent to points that are not together in T, as otherwise the inverse of f would send points together in T to points not together in S. So the image of a component of S is a component of T, and the images of different components of S are different components of T. This completes the proof.

The previous result is not quite enough to deal with the following problems.

Example 2.8

Let S be the real line with the origin removed, and let T consist of those real numbers x such that $x < 0$ or $x = 1$. Then both S and T have two components. But each component of S is homeomorphic to $]0, 1[$, whereas T has one component homeomorphic to $]0, 1[$ and the other consisting of the single point 1. Now $]0, 1[$ and $\{1\}$ are not homeomorphic, for the superficial reason that they do not have the same number of points. Hence there is no way of pairing the components of S with those of T so that paired components are homeomorphic. The following theorem then tells us that S and T are not homeomorphic.

Theorem 2.5

The components of homeomorphic sets are homeomorphic in pairs.

Proof

Let f be a homeomorphism from S to T. Theorem 2.4 was proved by showing that a component of S can be paired with its image under f. But a set and its image under a homeomorphism are homeomorphic, so paired components are homeomorphic. This completes the proof.

Theorem 2.5 reduces the problem of deciding whether two sets are homeomorphic to the case where the sets are both path-connected.

Valuable information about a path-connected set is found by counting the number of pieces remaining when the set is "cut" by the removal of one point.

Definition 2.4

Let S be a path-connected set. We call a point p of S an *n-point* of S if removing p from S cuts S into n pieces, that is, $S \backslash \{p\}$ has n components. An n-point is also called a cut-point of type n, and a 1-point is called a not-cut-point.

Example 2.9

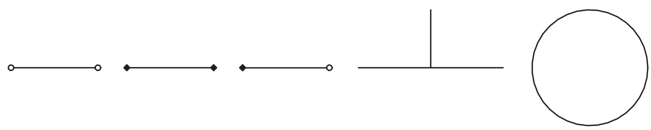

Figure 2.3 Example 2.9

Each point of the open interval $]0, 1[$ is a 2-point. The end points of $[0, 1]$ are 1-points, all other points being 2-points. The half-open interval $[0, 1[$ has one 1-point, all other points being 2-points. The set T of Example 2.7 has one 3-point, all other points being 2-points. The circle consists of not-cut-points.

Example 2.10

The set indicated in Figure 2.4 is path-connected, has infinitely many not-cut-points, but just one n-point for each $n \geq 2$.

For the calculation of Example 2.9 to provide proof that no two of the five sets are homeomorphic, we appeal to the next theorem.

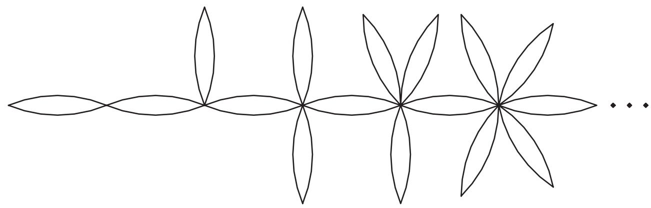

Figure 2.4 Example 2.10

Theorem 2.6

Homeomorphic sets have the same number of cut-points of each type.

Proof

Let f be a homeomorphism from S to T. We show that f sends each n-point of S to an n-point of T, so that, for each n, f gives a correspondence between the n-points of S and the n-points of T. Let p be an n-point of S. Then $S\backslash\{p\}$ has n components. But $S\backslash\{p\}$ is homeomorphic to its image $T\backslash\{f(p)\}$ under f. Consequently $S\backslash\{p\}$ and $T\backslash\{f(p)\}$ have the same number of components. So $T\backslash\{f(p)\}$ has n components, and $f(p)$ is an n-point. This completes the proof.

Example 2.11

Let S and T be the sets shown in Figure 2.5, the end points of the "arms" being missing. Both S and T have infinitely many 2-points and infinitely many not-cut-points. But the 2-points of S and T form the arms, including the points joining them to the circle. Hence the set of 2-points of S is path-connected, whereas the set of 2-points of T is not.

The next theorem gives the justification for saying that S and T are therefore not homeomorphic.

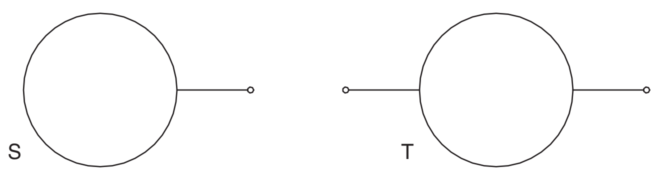

Figure 2.5 Example 2.11

Theorem 2.7

Homeomorphic sets have homeomorphic sets of each type of cut-point.

Proof

For any set X denote the set of n-points of X by X_n. Suppose that S and T are homeomorphic. We show that, for each n, S_n and T_n are homeomorphic. Let f be a homeomorphism from S to T. From the proof of Theorem 2.6 we know that f sends points in S_n to points in T_n and points not in S_n to points not in T_n. Hence the image of S_n under the homeomorphism f is T_n, and it follows that S_n and T_n are homeomorphic. This completes the proof.

Example 2.12

> The circle and plane both consist of infinitely many not-cut-points. Removing any pair of points from the plane leaves a path-connected set, whereas removing any pair of points from the circle does not. To prove that the circle and the plane are not homeomorphic, we need to adapt our theory of cut-points to *cut-pairs*.

Definition 2.5

Let S be a path-connected set, and let p, q be distinct points of S. We call $\{p, q\}$ an *n-pair* of S if $S \backslash \{p, q\}$ has n components. An n-pair is also called a cut-pair of type n, and a 1-pair is called a not-cut-pair.

Theorem 2.8

Homeomorphic sets have the same number of cut-pairs of each type.

Proof

Let f be a homeomorphism from S to T. We show that f sends each n-pair of S to an n-pair of T. For each n, therefore, f gives a correspondence between the n-pairs of S and the n-pairs of T. Let $\{p, q\}$ be an n-pair of S. Then $S \backslash \{p, q\}$ has n components. But $S \backslash \{p, q\}$ is homeomorphic to its image $T \backslash \{f(p), f(q)\}$ under f. Consequently $S \backslash \{p, q\}$ and $T \backslash \{f(p), f(q)\}$ have the same number of components. So $T \backslash \{f(p), f(q)\}$ has n components, and $\{f(p), f(q)\}$ is an n-pair. This completes the proof.

EXERCISES

2.1. The plane set S is path-connected and is the union of three line segments, each segment being not only homeomorphic to $]0,1[$ but also straight. Find twelve examples of such a set S, no two of your examples being homeomorphic. Show that no two of your examples are homeomorphic.

2.2. The plane set S is path-connected and is the union of three line segments, each segment being not only homeomorphic to $[0,1]$ but also straight. Find eighteen examples of such a set S, no two of your examples being homeomorphic. Show that no two of your examples are homeomorphic.

2.3. The plane set S is path-connected and is the union of the axes and a circle (a round circle, not just a set homeomorphic to a circle). Find eight such sets S, no two being homeomorphic. Show that no two of the sets are homeomorphic.

2.4. The plane set S is path-connected and is the union of the vertical lines $\{(0,y) : 0 \leq y \leq 1\}$ and $\{(1,y) : 0 \leq y \leq 1\}$ and two horizontal closed line segments of length 1. Find eleven examples of such a set S, no two being homeomorphic. Show that no two of your sets are homeomorphic.

2.5. Let L_1 be the set

$$\{(x,0) : 0 \leq x < 1\} \cup \{(0,y) : 0 \leq y < 1\}$$

and let L_2 be congruent to L_1. The plane set T is $L_1 \cup L_2$. Sketch eight examples of such a set T, no two being homeomorphic. Show that no two of your examples are homeomorphic.

2.6. The sets S and T of Figure 2.5 have points where three lines emanate: in fact S has one whereas T has two. Give a precise definition of an *n-node,* a point where n lines emanate, and show that a homeomorphism sends an n-node to an n-node. The plane set X is path-connected and is the union of three circles (round circles, not just sets homeomorphic to a circle). Sketch eleven examples of such a set X, no two being homeomorphic. Show that no two of your examples are homeomorphic.

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