2

Preliminaries

This chapter presents several probabilistic representation methods of the random nature of input parameters for structural models. The concept of the random field and its discretization are discussed with graphical interpretations. In later sections, we discuss linear regression and polynomial regression procedures which can be applied to stochastic approximation. A procedure for checking the adequacy of a regression model is also given with a representative example of the regression problem.

2.1 Basic Probabilistic Description

There are many ways to specify probabilistic characteristics of systems under uncertainty. Random variables are measurable values in the probability space associated with events of experiments. Accordingly, random vectors are sequences of measurements in the context of random experiments. Random variables are analyzed by examining underlying features of their probability distributions. A PDF indicates a relative probability of observing each random variable x and can be expressed as a formula, graph, or table. Since the computation of the PDF is not always easy, describing the data through numerical descriptive measures, such as the mean and variance, is also popular. In this section, elementary statistical formulas and several definitions of probability theory, random field, and regression analysis are briefly described in order to facilitate an introduction to the later sections.

2.1.1 Characteristics of Probability Distribution

Random Variable

A random variable X takes on various values x within the range $-\infty < x < \infty$. A random variable is denoted by an uppercase letter, and its particular value is represented by a lowercase letter. Random variables are of two types: discrete and continuous. If the random variable is allowed to take only discrete

values, $x_1, x_2, x_3 \ldots, x_n$, it is called a *discrete random variable*. On the other hand, if the random variable is permitted to take any real value within a specified range, it is called a *continuous random variable*.

Probability Density and Cumulative Distribution Function

If a large number of observations or data records exist, then a *frequency diagram* or *histogram* can be drawn. A histogram is constructed by dividing the range of data into intervals of approximately similar size and then constructing a rectangle over each interval with an area proportional to the number of observations that fell within the interval.

The histogram is a useful tool for visualizing characteristics of the data such as the spread in the data and locations. If the rectangular areas are normalized so that the total sum of their areas is unity, then the histogram would represent the probability distribution of the sample population, and the ordinate would represent the probability density. The probability that a randomly chosen sample will fall within a certain range can be calculated by summing up the total area within that range. In this sense, it is analogous to calculating mass as density times volume where

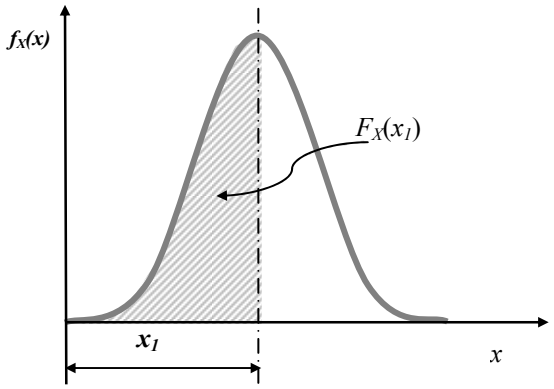
$$\text{Probability} = \text{Probability density} \times \text{Interval size}.$$

There are an infinite number of values a continuous variable can take within an interval, although there is a limit on measurement resolution. One can see that if the histogram were constructed with a very large number of observations and the intervals were to become infinitesimally small as the number of observations grew, the probability distribution would become a continuous curve. The mathematical function that describes the distribution of a random variable over the sample space of the continuous random variable, *X*, is called the probability density function and is designated as $f_X(x)$. The PDF is only defined for continuous random variables. The *Probability Mass Function* (PMF) describes the distribution of discrete random variables and is denoted as $p_X(x)$. Another way to describe the probability distribution for both discrete and continuous random variables is the *Cumulative Distribution Function* (CDF), $F_X(x)$. The CDF is defined for all values of random variables *X* from $-\infty$ to $+\infty$ and is equal to the probability that *X* is less than or equal to a realized value *x*.

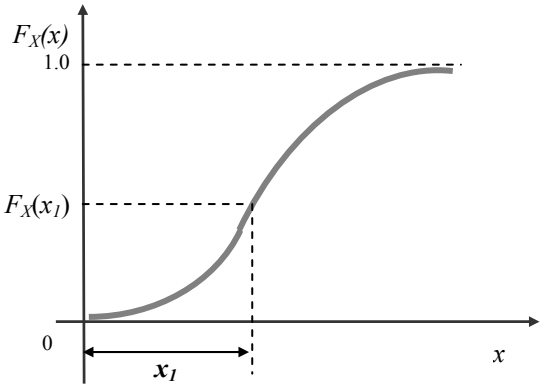
For a continuous random variable, $F_X(x)$ is calculated by integrating the PDF for all values of *X* less than or equal to *x*:

$$F_X(x) = \int_{-\infty}^{x} f_X(s)ds \tag{2.1}$$

Furthermore, if $F_X(x)$ is continuous, then the probability of *X* having a value between *a* and *b* can be calculated as



(a) Probability Density Function



(b) Cumulative Distribution Function

Figure 2.1. PDF and Associated CDF

$$F_X(b) - F_X(a) = \int_a^b f_X(x)dx \qquad \text{(for all real numbers } a \text{ and } b\text{)} \tag{2.2}$$

If the random variable X is continuous and if the first derivative of the distribution function exists, then the probability density function $f_X(x)$ is given by the first derivative of the CDF, $F_x(x)$:

$$f_X(x) = \frac{dF_X(x)}{dx} \tag{2.3}$$

If Y is a one-to-one function of the random variable X, $Y=h(X)$; then the *derived density function* of Y is given by [1]

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X(h^{-1}) \left| \frac{dh^{-1}}{dy} \right| \tag{2.4a}$$

If Y is a many-to-one function of X, then $Y=h(X)$ may be inverted for each case,

$$f_Y(y) = \sum_{i=1}^{k} \frac{dF_Y(y)}{dy} = \sum_{i=1}^{k} f_X(h_i^{-1}) \left| \frac{dh_i^{-1}}{dy} \right| \tag{2.4b}$$

where h_i^{-1} is the inverse function of h_i, that is, $h_i^{-1}(y) = x_i$. For example, if $y = x^2 = h(X)$, then $x = \pm\sqrt{y}$ or $x_i = h_i^{-1}(y)$ where $h_1^{-1}(y) = \sqrt{y}$ and $h_2^{-1}(y) = -\sqrt{y}$.

If X is a discrete random variable, then the CDF of X is discontinuous at points x_i and is obtained as

$$F_X(x) = \sum_{x_i \le x} p_X(x_i) \tag{2.5}$$

The CDF is a non-decreasing function of x (its slope is always greater than or equal to zero) with lower and upper limits of 0 and 1, respectively. The CDF is also referred to at times as a distribution function, and the corresponding distribution functions are shown in Figure 2.1. Because the CDF is defined by integrating the PDF, $F_X(x_1)$ is obtained by integrating the PDF $f_X(x)$ between the limits $-\infty$ and x_1, as shown in Figure 2.1.

Joint Density and Distribution Functions

Joint probability expresses the probability that two or more random events will happen simultaneously. In general, if there are n random variables, the outcome is an n-dimensional random vector. For instance, the probability of the two-dimensional case is calculated as

$$P[a<X<b,\ c<Y<d] = \int_c^d \int_a^b f_{XY}(x,y)\,dx\,dy \tag{2.6}$$

where $f_{XY}(x,y)$ is the joint PDF of the random variables X and Y ($f_{XY}(x,y) \ge 0$, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$).

The probability density of X for all possible values of y is the *marginal density* of x. The marginal density of x is determined by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy \tag{2.7}$$

The PDF of X for a specified y represents the *conditional probability* of X given by:

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)}, \; f_Y > 0 \tag{2.8}$$

If X and Y are independent, then

$$f_{X|Y}(x \mid y) = f_X(x) \;\text{ and }\; f_{Y|X}(y \mid x) = f_Y(y) \tag{2.9}$$

The conditional PDF becomes the marginal PDF, and the joint PDF becomes the product of the marginals:

$$f_{XY}(x, y) = f_X(x) f_Y(y) \tag{2.10}$$

In general, the joint PDF is equal to the product of the marginals when all the variables are mutually independent:

$$f_X(X) = f_{X_1}(x_1) f_{X_1}(x_2) \ldots f_{X_{n-1}}(x_{n-1}) f_{X_n}(x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) \tag{2.11}$$

Central Measures

The population *mean*, also referred to as the *expected value* or *average*, is used to describe the central tendency of a random variable. This is a weighted average of all the values that a random variable may take. If $f_X(x)$ is the probability density function of X, the mean is given by

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x \; f_X(x)dx \tag{2.12}$$

Thus, μ_X is the distance from the origin to the centroid of the PDF. It is called the *first moment* since it is the first moment of area of the PDF. The mean is analogous to the centroidal distance of a cross-section.

According to the definition of a random variable, any function of a random variable is itself a random variable. Therefore, if $g(x)$ is an arbitrary function of x, the expected value of $g(x)$ is defined as

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx \tag{2.13}$$

The *expectation operator*, $E[\cdot]$, possesses the following useful properties: If X and Y are independent,

$$E[XY] = E[X]E[Y] \tag{2.14}$$

And if c is a constant

$$E[c] = c \tag{2.15}$$

$$E[cX] = cE[X] \tag{2.16}$$

But, in general

$$E[g(X)] \neq g(E[X]) \tag{2.17}$$

Given $Z = X_1 + X_2 + ... + X_n$, the expected value of Z is a linear combination of individual values:

$$E(Z) = E(X_1) + E(X_2)..... + E(X_n) \tag{2.18}$$

Other useful central measures are the *median* and *mode* of the data: the median is the value of X at which the cumulative distribution function has a value of 0.5, and the mode is the value of X corresponding to the peak value of the probability density function.

Dispersion Measures

The expected value or mean value is a measure of the central tendency, which indicates the location of the distribution on the coordinate axis representing the random variable. The *variance*, $V(X)$, a second central moment of X, is a measure of spread in the data about the mean:

$$\begin{aligned} V(X) &= E[(X - \mu_X)^2] \\ &= E(X^2) - 2E(X)\mu_X + \mu_X^2 = E(X^2) - \mu_X^2 \end{aligned} \tag{2.19}$$

Geometrically, it represents the moment of inertia of the probability density function about the mean value. The variance of a random variable is analogous to the moment of inertia of a weight about its centroid. A measure of the variability of the random variable is usually given by a quantity known as the *standard deviation*. The standard deviation is a square root of the variance:

$$\sigma_X = \sqrt{V(X)} \tag{2.20}$$

The standard deviation is often preferred over the variance as a measure of dispersion because the units are consistent with the variable X and its mean value μ_X.

Nondimensionalizing the standard deviation will result in the *Coefficient of Variation* (COV), δ_X, which indicates the relative amount of uncertainty or randomness:

$$\delta_X = \frac{\sigma_X}{\mu_X} \tag{2.21}$$

Therefore, if we know any two of the mean (expected value), standard deviation, or coefficient of variation, the third term can be determined.

Measures of Correlation

If two random variables (X and Y) are correlated, the likelihood of X can be affected by the value taken by Y. In this case, the *covariance*, σ_{XY}, can be used as a measure to describe a linear association between two random variables:

$$\begin{aligned}\sigma_{XY} &= Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)f_{XY}(x,y)dxdy\end{aligned} \tag{2.22}$$

The *correlation coefficient* is a nondimentional measure of the correlation

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \tag{2.23}$$

If x and y are statistically independent, the variables are uncorrelated and the covariance is 0 (Figure 2.2a). Therefore, the correlation coefficients of ± 1 indicate a perfect correlation (Figure 2.2b).

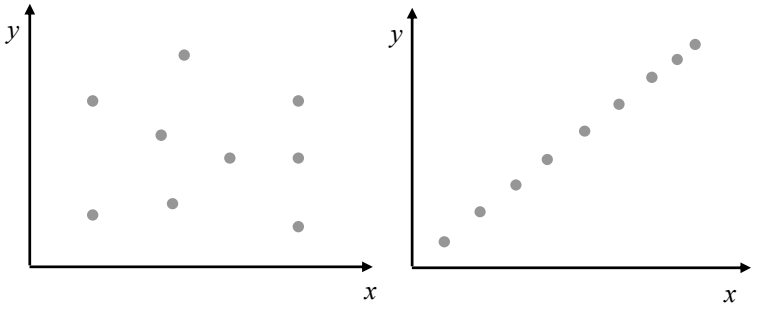
If $Y = a_1X_1 + a_2X_2$, where a_1 and a_2 are constants, the variance of Y can be obtained as

$$Var[Y] = E[\{a_1X_1 + a_2X_2 - (a_1\mu_{X_1} + a_2\mu_{X_2})^2] \tag{2.24}$$

$$= E[\{a_1(X_1-\mu_{X_1}) + a_2(X_2-\mu_{X_2})\}^2]$$

$$= E[a_1^2(X_1-\mu_{X_1})^2 + 2a_1a_2(X_1-\mu_{X_1})(X_2-\mu_{X_2}) + a_2^2(X_2-\mu_{X_2})^2]$$

$$= a_1^2Var[X_1] + a_2^2Var[X_2] + 2a_1a_2Cov(X_1,X_2)$$



(a) Covariance near Zero (b) Positive Covariance

Figure 2.2. Examples of Paired Data Sets

Table 2.1. Properties of Central and Dispersion Measures

Central	$E[a_0] = a_0,\ E[a_1X_1] = a_1E[X_1]$ $E[X_1X_2] = E[X_1]E[X_2]$ $E[a_0 + a_1X_1 + a_2X_2] = a_0 + a_1E[X_1] + a_2E[X_2]$
Dispersion	$Var[a_0] = 0,\ Var[a_1X_1] = a_1^2[X_1]$ $Var[a_0 + a_1X_1 + a_2X_2]$ $= a_1^2Var[X_1] + a_2^2Var[X_2] + 2a_1a_2Cov(X_1, X_2)$ $= a_1^2Var[X_1] + a_2^2Var[X_2] + 2a_1a_2\rho_{X_1X_2}\sigma_{X_1}\sigma_{X_2}$ $Cov[a_1X_1, X_2] = a_1Cov[X_1, X_2]$ $Cov[X_1, X_2 + X_3] = Cov[X_1, X_2] + Cov[X_1, X_3]$ $Cov[a_1 + X_1, a_2 + X_2] = Cov[X_1, X_2]$

In general, if $Y = \sum_{i=1}^{n} a_i X_i$, then the corresponding variance is

$$Var[Y] = \sum_{i=1}^{n} a_i^2 Var[X_i] + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j Cov(X_i, X_j),\ i \neq j \tag{2.25}$$

$$= \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \rho_{ij} \sigma_{X_i} \sigma_{X_j}$$

Furthermore, if another linear function of X is given as $Z = \sum_{i=1}^{n} b_i X_i$, the covariance between Y and Z can be obtained as [1]

$$Cov[Y,Z] = \sum_{i=1}^{n} a_i b_i Var[X_i] + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j Cov(X_i, X_j), i \neq j \tag{2.26}$$
$$= \sum_{i=1}^{n} a_i b_i \sigma_{X_i}^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \rho_{ij} \sigma_{X_i} \sigma_{X_j}$$

Useful properties for the central and dispersion measures of the random variables X_1, X_2 and X_3 are summarized in Table 2.1 (a_0, a_1, and a_2 are constants).

Other Measures

The expected value of the cube of the deviation of the random variable from its mean value (also known as the third moment of the distribution about the mean) is taken as a measure of the skewness, or lack of symmetry, of the distribution. Therefore, the *skewness*, the third central moment of X, describes the degree of asymmetry of a distribution around its mean:

$$skewness = E[(X-\mu_x)^3] = \int_{-\infty}^{\infty} (X-\mu_x)^3 f_X(x)dx \tag{2.27}$$

The value of $E[(X-\mu_X)^3]$ can be positive or negative.

A nondimensional measure of skewness known as the *skewness coefficient* is denoted as

$$\theta_X = \frac{E[(X-\mu_X)^3]}{\sigma_X^3} \tag{2.28}$$

Any symmetric data have zero θ_x; if θ_x is positive, the dispersion is more above the mean than below the mean (Figure 2.3a); and, if it is negative, the dispersion is more below than above the mean (Figure 2.3b). Therefore, the skewness coefficient is known as a measure of the symmetry of density functions.

The *kurtosis*, the fourth central moment of X, is a measure of the flatness of a distribution:

$$kurtosis = \frac{E[(X-\mu_X)^4]}{\sigma_X^4} \tag{2.29}$$

An alternative definition of the kurtosis is given by

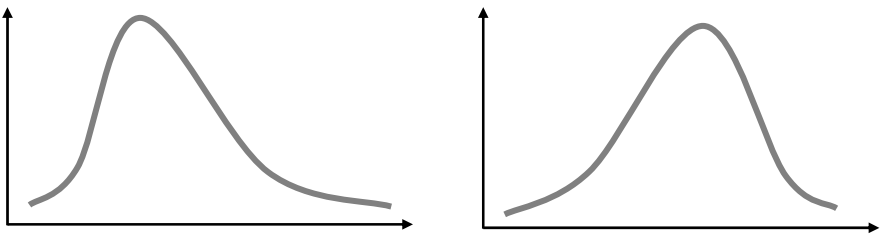
$$kurtosis = \frac{1}{n}\sum_{i=1}^{n}\frac{(x_i - \mu_X)^4}{\sigma_X} - 3 \tag{2.30}$$

In this definition, the kurtosis of the normal distribution is zero, a positive value of the kurtosis describes a distribution that has a sharp peak, and a negative value of the kurtosis indicates a flat distribution compared to the normal distribution.

Recall that the first and second moments of X are defined in Equation 2.12 and Equation 2.19, respectively. The n^{th}-order central moments are traditionally defined in terms of differences from the mean:

$$m_X^n = E[(X - \mu_x)^n] = \int_{-\infty}^{\infty}(X - \mu_x)^n f_X(x)dx \tag{2.31}$$

where, $\mu_X = E(X) = \int_{-\infty}^{\infty} x\ f_X(x)dx$.



(a) Positively Skewed (b) Negatively Skewed

Figure 2.3. Skewed Density Functions

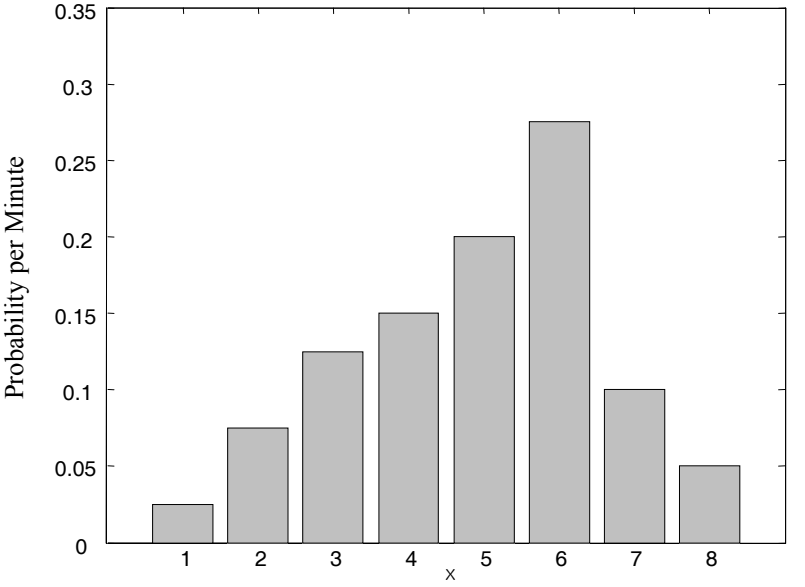
Example 2.1

The probability that a given number of cars per minute will arrive at a tollbooth is given in the table below. (a) Sketch the probability distribution as a function of X and find the mean, median, and mode. (b) Determine $E(X^2)$ and $E(X^3)$, the standard deviation, and the skewness coefficient.

No. of cars arriving per minute (X)	1	2	3	4	5	6	7	8
Probability per minute	0.025	0.075	0.125	0.150	0.200	0.275	0.100	0.050

Solution:

(a)



Mean:

$$\mu = \sum_{i=1}^{8} x_i P_i = 1(0.025) + 2(0.075) + 3(0.125) + 4(0.150)$$
$$+5(0.2) + 6(0.275) + 7(0.10) + 8(0.05) = 4.9$$

Mode: The peak in the probability density function is at $x = 6$, therefore this is the mode.

Median: Examination of the data shows that the cumulative probability of 0.5 lies between 4 and 5 cars per minute. A quadratic interpolation of CDF using 4, 5, and 6 provides a value of 4.75.

(b) $$E(X^2) = \sum_{i=1}^{8} x_i^2 P_i = 1(0.025)+4(0.075)+9(0.125)+16(0.150)$$
$$+25(0.2)+36(0.275)+49(0.10)+64(0.05) = 26.85$$

$$E(X^3) = \sum_{i=1}^{8} x_i^3 P_i = 1(0.025)+8(0.075)+27(0.125)+64(0.150)+125(0.2)$$
$$+216(0.275)+ 343(0.10)+512(0.05) = 157.9$$

$$\sigma_X^2 = \sum_{i=1}^{8} (x_i - \mu)^2 P_i = E(X^2) - \mu_X^2 = 26.85 - 4.9^2 = 2.84$$

$$\sigma_X = \sqrt{2.84} = 1.69$$

$$\theta_X = \frac{E[(x-\mu)^3]}{\sigma_X^3} = \frac{\sum_{i=1}^{8}(x_i-\mu)^3 P_i}{\sigma_X^3} = \frac{-1.497}{1.69^3} = -0.313$$

2.1.2 Common Probability Distributions

In evaluating structural reliability, several types of standardized probability distributions are used to model the design parameters or random variables. Selection of the distribution function is an essential part of obtaining probabilistic characteristics of structural systems. The selection of a particular type of distribution depends on

- The nature of the problem
- The underlying assumptions associated with the distribution
- The shape of the curve between $f_X(x)$ or $F_X(x)$ and x obtained after estimating data
- The convenience and simplicity afforded by the distribution in subsequent computations

The selection or determination of the distribution functions of random variables is known as *statistical tolerancing*. In general, the first few moments (mean, variance, skewness, etc.) of the distribution need to be estimated and matched through the use of several techniques, including the Taylor series approximation, the Taguchi method, and the Monte Carlo method. Detailed discussions of these methods can be found in [3] and [6]. In this section, the properties of some of the more commonly used distributions are presented.

Gaussian Distribution

The *Gaussian* (or *normal*) *distribution* is used in many engineering and science fields due to its simplicity and convenience, especially a theoritical basis of the central limit theorem. The *central limit theorem* states that the sum of many arbitrary distribution random variables asymptotically follows a normal distribution when the sample size becomes large.

This distribution is often used for small coefficients of variation cases, such as Young's modulus, Poisson's ratio, and other material properties. The Gaussian distribution is given by

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right], \quad -\infty < x < \infty \tag{2.32}$$

where the parameters of the distribution μ_X and σ_X denote the mean and standard deviation of the variable *X*, respectively, and *X* is identified as $N(\mu_X, \sigma_X)$. The location (μ_X) and scale (σ_X) parameters generate a family of distributions.

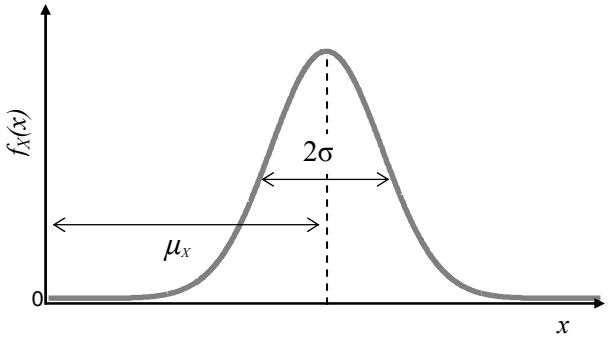


Figure 2.4. Normal Density Function

The density function and corresponding parameters are shown in Figure 2.4. The PDF of the Gaussian distribution is also known as a *bell curve* because of its shape in the graph. The Gaussian distribution is symmetric with respect to the mean and has inflection points at $x = \mu \pm \sigma$. The areas under the curve within one, two, and three standard deviations are about 68%, 95.5%, and 99.7% of the total area, respectively.

The Gaussian distribution has the following useful properties:

1) Any linear functions of normally distributed random variables are also normally distributed. For instance, let *Z* be the sum of *n* normally distributed random variables:

$$Z = a_0 + a_1 X_1 + a_2 X_2 + ... + a_n X_n \tag{2.33}$$

where X_i are independent random variables, and a_i's are constants.

Then, *Z* will also be normal with the following properties:

$$\mu_Z = a_0 + \sum_{i=1}^{n} a_i \mu_i \quad \sigma_Z = \sqrt{\sum_{i=1}^{n} (a_i \sigma_i)^2} \tag{2.34}$$

2) A nonlinear function of normally distributed random variables may or may not be normal. For Example, the function $y = \sqrt{X_1^2 + X_2^2}$ of two independent standard normally distributed random variables X_1 and X_2 with $N(0, \sigma^2)$ is a Rayleigh distribution. Its density and distribution functions are computed as

$$f_Y(y) = \frac{y}{\sigma^2} e^{-\frac{y^2}{2\sigma^2}}, \quad y \geq 0 \tag{2.35}$$

$$F_Y(y) = 1 - e^{-\frac{y^2}{2\sigma^2}}, \; y \geq 0 \tag{2.36}$$

The Gaussian distribution can be normalized by defining $\xi = (x-\mu)/\sigma$ and yields the *standard normal distribution N(0,1)*. The density function of a *standard normally distributed variable* ξ is given by

$$f_\Xi(\xi) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\xi^2}{2}\right), \quad -\infty < \xi < \infty \tag{2.37}$$

The notation $\Phi(\cdot)$ is commonly used for the cumulative distribution function of the standard normally distributed variable ξ and is given by

$$\Phi(\xi) = F_\Xi(\xi) = \int_{-\infty}^{\xi} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\xi^2}{2}\right) d\xi \tag{2.38}$$

If $\Phi(\xi_p) = p$ is given, the standard normal variate ξ_p corresponding to the cumulative probability (p) is denoted as

$$\xi_p = \Phi^{-1}(p) \tag{2.39}$$

The values of the standard normal cumulative distribution function, $\Phi(\cdot)$, are tabulated (Appendix C). Usually, the probabilities are given in tables only for positive values of ξ and for negative values

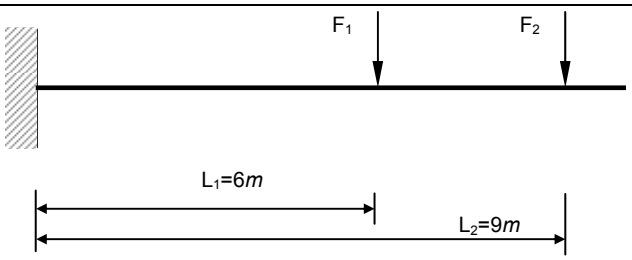
$$\Phi(-\xi) = 1 - \Phi(\xi) \tag{2.40}$$

due to the symmetry of the density function about zero. Similarly, we can find that

$$\xi_p = \Phi^{-1}(p) = -\Phi^{-1}(1-p) \tag{2.41}$$

Example 2.2

If a cantilever beam supports two random loads with means and standard deviations of $\mu_1 = 20\,\text{kN}, \sigma_1 = 4$ kN and $\mu_2 = 10$ kN, $\sigma_2 = 2$ kN as shown in the accompanying drawing, the bending moment (M) and the shear force (V) at the fixed end due to the two loads are $M = L_1F_1 + L_2F_2$ and $V = F_1 + F_2$, respectively.



(a) If two loads are independent, what are the mean and the standard deviation of the shear and the bending moment at the fixed end?
(b) If two random loads are normally distributed, what is the probability that the bending moment will exceed 235 kNm?
(c) If two loads are independent, what is the correlation coefficient between V and M?

Solution:

(a) From the properties of the expected value operator (Equation 2.16 and Equation 2.18), the mean and the standard deviation of V and M can be obtained as

$$V = F_1 + F_2 \rightarrow E[V] = \mu_V = E[F_1] + E[F_2] = 20 + 10 = 30 \text{ kN}$$
$$Var[V] = Var[F_1] + Var[F_2] + 2Cov(F_1, F_2)$$
$$= 4^2 + 2^2 + 0 = 20 \text{ kN}^2$$
$$\therefore \sigma_V = \sqrt{20} = 4.47 \text{ kN}$$

$$M = L_1 F_1 + L_2 F_2 \rightarrow$$
$$E[M] = \mu_M = L_1 E[F_1] + L_2 E[F_2] = 6 \times 20 + 9 \times 10 = 210 \text{ kNm}$$
$$Var[M] = L_1^2 Var[F_1] + L_2^2 Var[F_2] + 2L_1 L_2 Cov(F_1, F_2)$$
$$= 6^2 \times 4^2 + 9^2 \times 2^2 + 0 = 900 \text{ kNm}^2$$
$$\therefore \sigma_M = \sqrt{900} = 30 \text{ kNm}$$

(b) From the calculated results of (a), $\mu_M = 210$ kNm and

$$\sigma_M = \sqrt{900} = 30 \text{ kNm}$$

Therefore, the probability that the bending moment exceeds 235 kNm is

$$P(M > 235) = P\left(\xi > \frac{235 - 210}{30}\right)$$
$$= P(\xi > 0.8333) = 1 - \Phi(0.83) = 0.2023$$

(c) Regardless of the independence of F_1 and F_2, V and M can be correlated.

From Equation 2.22,

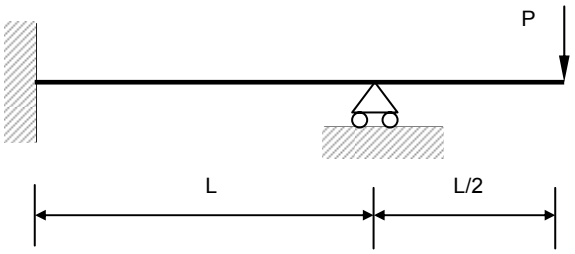
$$\begin{aligned}\sigma_{VM} &= Cov(V,M)\\ &= E[(V-\mu_V)(M-\mu_M)] = E[VM]-\mu_V\mu_M\\ &= E[(F_1+F_2)(L_1F_1+L_2F_2)]-(\mu_1+\mu_2)(L_1\mu_1+L_2\mu_2)\\ &= L_1E[F_1^2]+L_2E[F_2^2]+(L_1+L_2)E[F_1F_2]\\ &\qquad -L_1\mu_1^2-L_2\mu_2^2-(L_1+L_2)\mu_1\mu_2\\ &= L_1\sigma_1^2+L_2\sigma_2^2 \quad (\because \text{Equation 2.19})\end{aligned}$$

Thus, the correlation coefficient is obtained as

$$\rho_{VM} = \frac{\sigma_{VM}}{\sigma_V\sigma_M} = \frac{L_1\sigma_1^2+L_2\sigma_2^2}{\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{L_2^2\sigma_1^2+L_2^2\sigma_2^2}} = 0.98387$$

Example 2.3

Consider a cantilever beam structure subjected to a force P.



The displacement at the tip is given by

$$u = \frac{5PL^3}{48EI}$$

where E is Young's modulus and I is the area moment of the cross section. If E has a Gaussian distribution with $\mu_E = 10\,\text{kN}, \sigma_E = 2$ kN, derive the PDF of the displacement.

Solution:

According to Equation 2.4, $h_i^{-1}(u)$ and $\left|\frac{dh_i^{-1}}{du}\right|$ must be obtained. From the given formulation of the displacement:

$$E = \frac{5PL^3}{48I}\frac{1}{u} = \frac{c}{u} = h^{-1}(u)$$

where $c = 5PL^3 / 48I$.

Thus,

$$\frac{dE}{du} = -\frac{c}{u^2} \rightarrow \left|\frac{dh^{-1}}{du}\right| = \left|\frac{dE}{du}\right| = \frac{c}{u^2}$$

Finally, the derived density of the displacement is calculated as

$$f_U(u) = \frac{1}{\sigma_E\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\frac{c}{u}-\mu_E}{\sigma_E}\right)^2\right]\frac{c}{u^2}$$

$$= \frac{c}{u^2\sigma_E\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{c-u\mu_E}{u\sigma_E}\right)^2\right]$$

Lognormal Distribution

The *lognormal distribution* plays an important role in probabilistic design because negative values of engineering phenomena are sometimes physically impossible. Typical uses of the lognormal distribution are found in descriptions of fatigue failure, failure rates, and other phenomena involving a large range of data. Examples are cycles to failure, material strength, loading variables, etc.

A situation may arise in reliability analysis where a random variable *X* is the product of several random variables $x_i : x = x_1 x_2 x_3 .... x_n$. Taking the natural logarithm of both sides,

$$ln\ x = ln\ x_1 + ln\ x_2 + ... + ln\ x_n$$

if no one term on the right side dominates, then by Equation 2.33, *ln x* should be normally distributed. In the equation *Y* = *ln X*, the random variable *X* is said to follow lognormal distribution (Figure 2.5), and *Y* follows a normal distribution.

Thus the PDF of y is given by

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right], \quad -\infty < y < \infty \tag{2.42}$$

Since $Y = ln\ X$, the above equation can be rewritten in terms of X as

$$f_X(x) = \frac{1}{\sqrt{2\pi}x\sigma_Y} \exp\left[-\frac{1}{2}\left(\frac{\ln x-\mu_Y}{\sigma_Y}\right)^2\right], \quad 0 \le x < \infty \tag{2.43}$$

where

$$\sigma_Y^2 = \ln\left[\left(\frac{\sigma_X}{\mu_X}\right)^2 + 1\right] \tag{2.44}$$

and

$$\mu_Y = \ln \mu_X - \frac{1}{2}\sigma_Y^2 \tag{2.45}$$

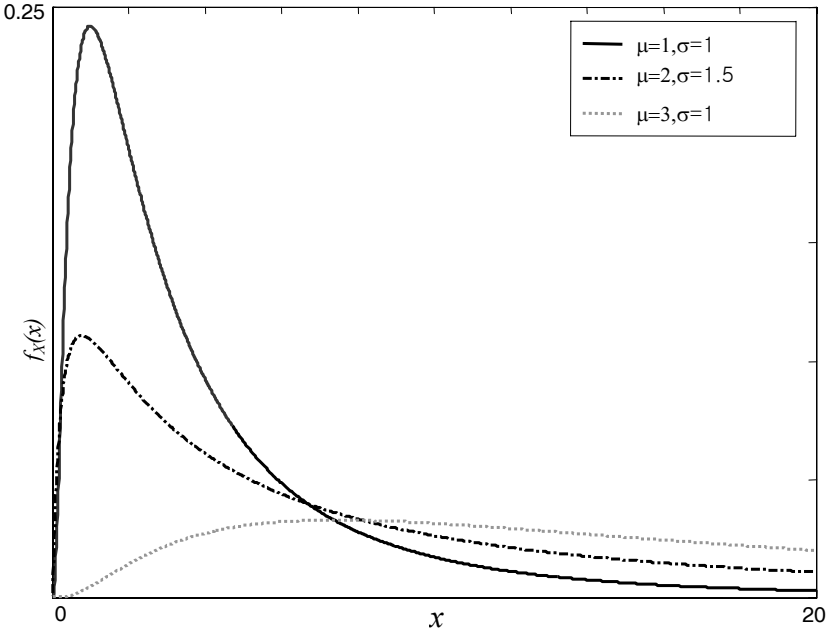


Figure 2.5. Lognormal Density Function

The CDF of the lognormal distribution is given by

$$F_X(x) = \frac{1}{\sigma_Y\sqrt{2\pi}}\int_0^{\pi}\frac{1}{x}\exp\left[-\frac{1}{2}\left(\frac{(\ln x-\mu_Y)^2}{2\sigma_Y^2}\right)\right] \tag{2.46}$$

Example 2.4

If ln X has a Gaussian distribution, what is the distribution of X?

Solution:

This problem is the proof of the derived density of the lognormal distribution. Let $Y = \ln X$. Then $X = e^Y$. From Equation 2.12 and Equation 2.32, the mean of X is

$$\mu_X = E[X] = E[e^Y] = \frac{1}{\sigma_Y\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp[y]\exp\left[-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]dy$$

$$= \left[\frac{1}{\sigma_Y\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left(\frac{y-(\mu_Y+\sigma_Y^2)}{\sigma_Y}\right)^2\right\}dy\right]\exp(\mu_Y+\frac{1}{2}\sigma_Y^2)$$

Since the quantity inside the bracket of the above equation is the unit area of the Gaussian density function $\sim N(\mu_Y+\sigma_Y^2, \sigma_Y)$, we have

$$\mu_X = \exp(\mu_Y+\frac{1}{2}\sigma_Y^2) \;\rightarrow\; \mu_Y = \ln\mu_X - \frac{1}{2}\sigma_Y^2$$

This result is the same as Equation 2.45.

Similarly,

$$E[X^2] = \frac{1}{\sigma_Y\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp[2y]\exp\left[-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]dy$$

$$= \left[\frac{1}{\sigma_Y\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left(\frac{y-(\mu_Y+2\sigma_Y^2)}{\sigma_Y}\right)^2\right\}dy\right]\exp[2(\mu_Y+\sigma_Y^2)]$$

$$= \exp[2(\mu_Y+\sigma_Y^2)]$$

From Equation 2.19, the variance of X is

$$Var[X] = \sigma_X^2 = \exp[2(\mu_Y + \sigma_Y^2)] - \exp[2(\mu_Y + \frac{1}{2}\sigma_Y^2)]$$
$$= \mu_X^2(\exp(\sigma_Y^2) - 1)$$

Thus, we obtain

$$\sigma_Y^2 = \ln\left[\left(\frac{\sigma_X}{\mu_X}\right)^2 + 1\right]$$

From the given condition

$$\left|\frac{dy}{dx}\right| = \frac{1}{x}$$

According to Equation 2.4,

$$f_X(x) = \frac{1}{\sqrt{2\pi}x\sigma_Y}\exp\left[-\frac{1}{2}\left(\frac{\ln x - \mu_Y}{\sigma_Y}\right)^2\right]$$

Therefore, if ln X is normal, the random variable X has a lognormal distribution.

Gamma Distribution

The *gamma distribution* (Figure 2.6) consists of the gamma function, a mathematical function defined in terms of an integral. This distribution is important because it allows us to define two families of random variables, the exponential and chi-square, which are used extensively in applied engineering and statistics.

The density function associated with the gamma distribution is defined by

$$f_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta},\ 0 \le x < \infty \tag{2.47}$$

where the parameters α and β satisfy $\alpha > 0$ and $\beta > 0$, and the gamma function is $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$.

Let X be a gamma random variable with parameters α and β. Then the mean and variance for X are given by

$$E[X] = \mu = \alpha\beta,\ V[X] = \sigma^2 = \alpha\beta^2 \tag{2.48}$$

The gamma CDF is

$$F_X(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-x/\beta} dt \tag{2.49}$$

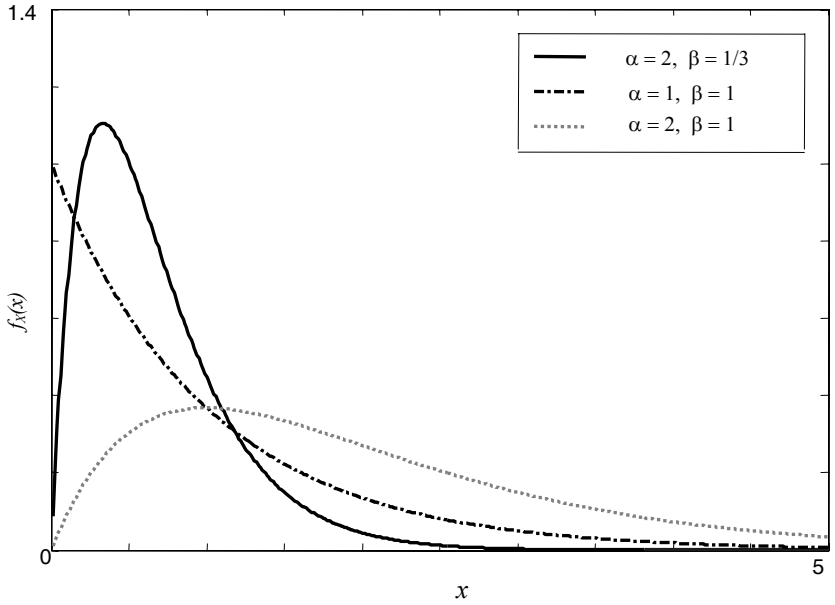


Figure 2.6. Gamma Density Functions

Extreme Value Distribution

The *extreme value distribution* is used to represent the maximum or minimum of a number of samples of various distributions. There are three types of extreme value distributions, namely Type I, Type II, and Type III. The *Type I extreme value distribution*, also referred to as the *Gumbel distribution*, is the distribution of the maximum or minimum of a number of samples of normally distributed data.

The density function of the Type I extreme value distribution is defined by

$$f_X(x) = \alpha \exp\left[-\exp\left(-\alpha(x-u)\right)\right]\exp\left[-\alpha(x-u)\right], \quad -\infty < x < \infty,\ \alpha > 0 \tag{2.50}$$

where α and u are scale and location parameters, respectively.
The CDF of the extreme value distribution is given by

$$F_X(x) = \exp\left[-\exp\left(-\alpha(x-u)\right)\right] \tag{2.51}$$

Due to the functional form of Equation 2.51, it is also referred to as a doubly exponential distribution. Similar to the relationship between the Gaussian distribution and lognormal distribution, the *Type II extreme value distribution*, also referred to as the *Frechet distribution*, can be derived by using parameters $u = ln\ v$, $\alpha = k$ in the Type I distribution. The PDF of the Type II extreme value distribution is

$$f_X(x) = \frac{k}{v}\left(\frac{v}{x}\right)^{k+1} \exp\left[-\left(\frac{v}{x}\right)^k\right],\ 0 \le x < \infty,\ k \ge 2 \tag{2.52}$$

The corresponding CDF is

$$F_X(x) = \exp\left[-\left(\frac{v}{x}\right)^k\right] \tag{2.53}$$

The density functions of the Type I and Type II extreme value distributions are shown in Figure 2.7. The following subsection will discuss the last type of the extreme value distribution, the *Type III extreme value distribution*, also known as the *Weibull distribution*.

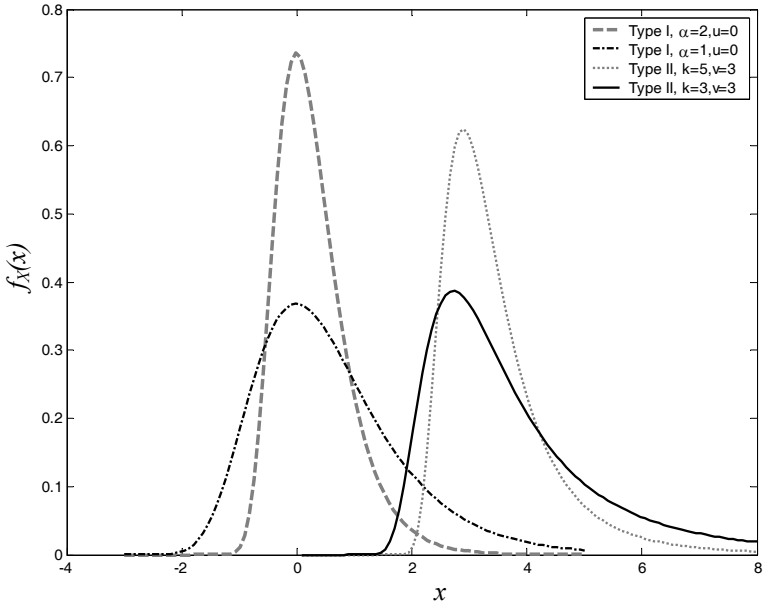


Figure 2.7. Type I and Type II Extreme Value Density Functions

Weibull Distribution

The Weibull distribution (Figure 2.8), also referred to as the Type III extreme value distribution, is well suited for describing the *weakest link* phenomena, or a situation where there are competing flaws contributing to failure. It is often used to describe fatigue, fracture of brittle materials, and strength in composites. The distribution of wind speeds at a given location on Earth can also be described with this distribution.

The probability density function is

$$f_X(x) = \frac{\alpha x^{\alpha-1}}{\beta^{\alpha}} \exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right], \; x \geq 0, \; \alpha > 0, \; \beta > 0 \tag{2.54}$$

and the CDF is

$$F_X(x) = 1 - \exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right], \; x > 0 \tag{2.55}$$

Every location is characterized by a particular shape and scale parameter. This is a two-parameter family, α and β. The moments in terms of the parameters are

$$E(X^n) = \beta^n \Gamma\left(\frac{n}{\alpha} + 1\right) \tag{2.56}$$

where $\Gamma(.)$ is the gamma function.

The mean and coefficient of variation are

$$\mu_X = \beta \; \Gamma\left(\frac{1}{\alpha} + 1\right) \tag{2.57}$$

$$COV_X = \left[\frac{\Gamma\left(\frac{2}{\alpha} + 1\right)}{\Gamma^2\left(\frac{1}{\alpha} + 1\right)} - 1\right]^{0.5} \tag{2.58}$$

The mean and standard deviation are complicated functions of the parameters α and β. However, the following simplified parameters, which provide very good accuracy over the range that is of interest to engineers, are recommended in [2]:

$$\alpha = COV_X^{-1.08}, \quad \beta = \frac{\mu_X}{\Gamma\left(\frac{1}{\alpha} + 1\right)} \tag{2.59}$$

To illustrate the use of the Weibull distribution in the weakest link phenomenon, suppose there is a chain with N links, each of which has a random strength X. The strength of the chain y based on the weakest link is

$$P(X > y) = P(x_1 > y \cap x_2 > y \cap x_3 > y \cap ... \cap x_N > y) \tag{2.60}$$

If all the link strengths are independent, then

$$P(X > y) = P(x_1 > y)P(x_2 > y)P(x_3 > y)...P(x_N > y) \tag{2.61}$$

Furthermore, if all of the links follow the same strength distribution, a single CDF can express all the probabilities on the right side:

$$P(x_i < y) = 1 - P(x_i \le y) = 1 - F_X(y) \tag{2.62}$$

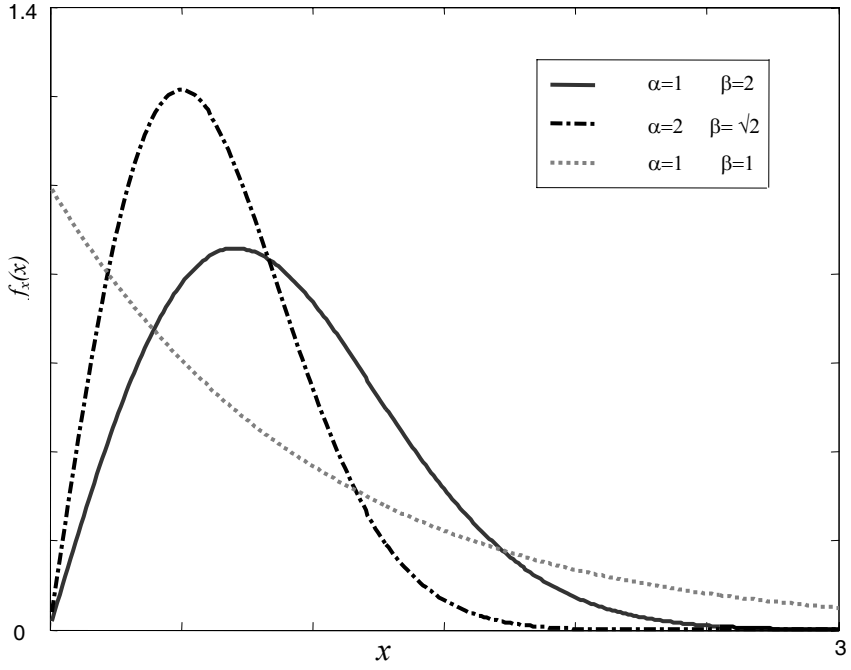


Figure 2.8. Weibull Density Function

So, the CDF for the chain can be expressed as

$$F_Y(y) = 1 - P(y < x_i) = 1 - [1 - F_X(y)]^N \tag{2.63}$$

Assuming the link strengths follow a Weibull distribution, Equation 2.55 can be substituted into the above expression:

$$F_Y(y) = 1 - \left[\exp\left(\frac{-y}{\beta} \right)^{\alpha} \right]^N = 1 - \exp\left[-N \left(\frac{y}{\beta} \right)^{\alpha} \right] \tag{2.64}$$

And so the strength of the entire chain is governed by the Weibull distribution:

$$F_Y(y) = 1 - \left[\exp\left(\frac{-y}{\beta'} \right)^{\alpha} \right] \tag{2.65}$$

with a scale parameter of

$$\beta' = N^{-\frac{1}{\alpha}} \beta \tag{2.66}$$

Example 2.5

A chain consists of welded links, each of which has Weibull-strength-distribution parameters of $\alpha = 5$ and $\beta = 5000$ N. Find (a) the mean strength of a link, (b) the mean strength of a 50-link and a 100-link chain, and (c) the load at which there is a 1% probability of failure for both chains.

Solution:

(a) From Equation 2.57, the mean strength is

$$\mu_X = 5000\Gamma(1/5 + 1) = 5000(0.918) = 4590 \text{ N}$$

(b) From Equation 2.66

For a 50-link chain,

$$\beta' = 50^{-1/5}(5000) = 2287 \text{ N}$$
$$\mu_Y = 2287\,\Gamma(1.2) = 2099 \text{ N}$$

For a 100-link chain,

$$\beta' = 100^{-1/5}(5000) = 1991 \text{ N}$$
$$\mu_Y = 1991\,\Gamma(1.2) = 1827 \text{ N}$$

(c) From Equation 2.65

$$0.01 = 1 - \exp\left(-\left[\frac{y}{\beta'}\right]^{\alpha}\right) \rightarrow \ln(0.99) = -\left[\frac{y}{\beta'}\right]^{\alpha}$$
$$y = \beta'[\ln(1/0.99)]^{1/\alpha}$$
$$y_{50} = 2287(0.3985) = 911 \text{ N}$$
$$y_{100} = 1991(0.5521) = 793 \text{ N}$$

Exponential Distribution

The *exponential distribution* is a special case of the Weibull distribution for $\alpha = 1$.

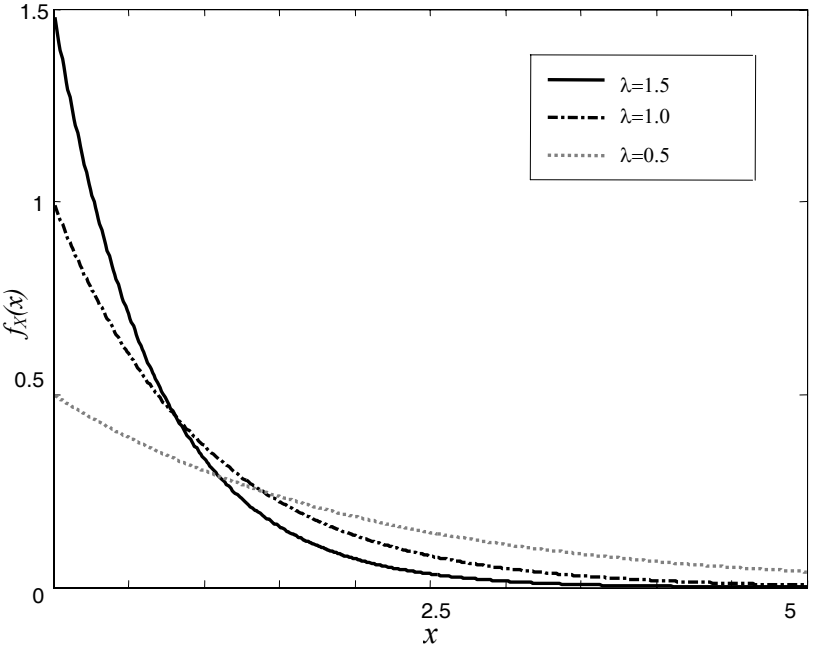


Figure 2.9. Exponential Density Function

The PDF is

$$f_X(x) = \lambda \exp[-\lambda x],\ 0 \le x < \infty \tag{2.67}$$

The CDF is given by

$$F_X(x) = 1 - \exp[-\lambda x],\ 0 \le x < \infty \tag{2.68}$$

The moments in terms of the parameter λ are

$$\mu_X = \frac{1}{\lambda},\ \sigma_X^2 = \frac{1}{\lambda^2} \tag{2.69}$$

The exponential distribution is commonly used in reliability analysis. As shown in Figure 2.9, this distribution is well suited to represent the long flat portion of the *bathtub curve*, which is the phenomenon that probability failures are usually high early in the lifecycle, low in the middle, and rise strongly toward the end.

Example 2.6

If x_1 and x_2 are normally distributed with mean = 0 and standard deviation = 1, sketch the PDFs or histograms of (a) $x_1 + x_2$, (b) $\exp(x_1)$, (c) $x_1^2 + x_2^2$, and (d) $\sqrt{x_1^2 + x_2^2}$ by using numerical tools (*i.e.*, MATLAB® or Mathematica).

Solution:

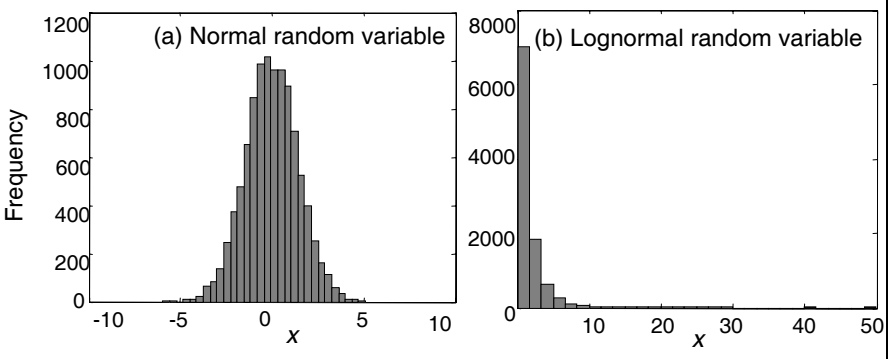


Figure 2.10. Probability Density Functions

When x_1 and x_2 are normally distributed, $x_1 + x_2$ is also normally distributed (Figure 2.10a), as mentioned earlier. This is also true for $ax_1 + bx_2$, where a and b are arbitrary constants. Furthermore, $\exp(x_1)$ is a lognormal random variable

(Figure 2.10b), and $x_1^2 + x_2^2$ is known as a Chi-squared random variable with two degrees of freedom (Figure 2.10c). And, $\sqrt{x_1^2 + x_2^2}$ is known as a Rayleigh random variable (Figure 2.10d). The properties of these distributions can be found in [6].

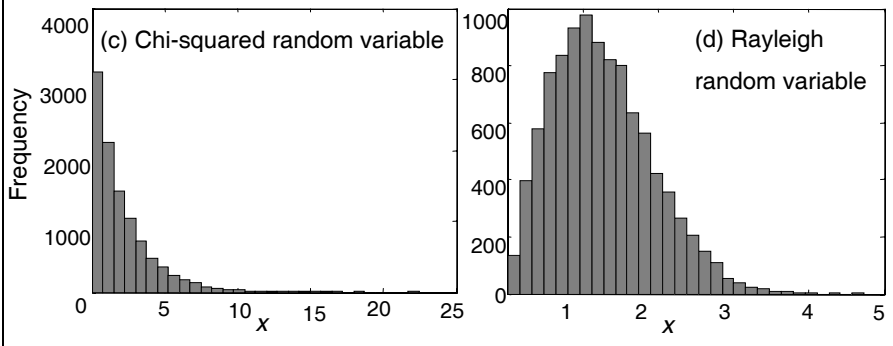


Figure 2.10. Probability Density Functions (contd.)

2.2 Random Field

A practical engineering structure has uncertainties in loads, geometry, material properties, manufacturing processes, and operational environments. These uncertainties fluctuate over space and time domains, and the responses of the structure are accordingly affected by these parameters. The estimation and representation of spatial- and time-variant data have become extremely important to realistic simulations that incorporate uncertainty analysis. The mathematical model of the variability, parameterized by the correlation between different locations, can be characterized by means of random field. This section briefly reviews the concept of the random field.

2.2.1 Random Field and Its Discretization

A *random field* is a random function of one or more variables. Many distributed properties in structural problems are random. For example, structural mechanical problems involve random fields, such as loads and stiffness properties. Efficient and realistic representation of the inputs will facilitate accurate estimations of random responses' statistics. Therefore, engineers should be able to handle these random field inputs and assess the allowable bounds for corresponding random responses, *i.e.*, stress and deflections, to determine the safety of structures. However, traditional deterministic analysis, such as the finite element method, uses a single design point, considering it sufficient to represent the response (Figure 2.11a). This simulation of a single design point is inadequate and unrealistic when

characterizing systems under varying loads and material properties. For instance, in studying the response of an aircraft to gust loads, we cannot cover all types of gusts and speeds in a single simulation.

The mathematical model of the spatial variability, parameterized by the correlation between different locations, can be characterized by means of random field. Generally, the terminologies of a random field and a *random process* are used interchangeably in the literature, but the random field treats multidimensional variations, while the random process is used for a single coordinate, usually time [12], [13]. The basic idea of the random process is that the outcome of each experiment is a function over an interval of the domain rather than a single value (Figure 2.11b).

Thus, analysis of the random process is a realistic approach that can produce a whole design space instead of just a one-point result. The resulting function, which is generated for all the points ($\omega_1,\ldots,\ \omega_n$) in the sample space Ω, is known as a *realization* of a random process, and the collection of realizations is referred to as an *ensemble* [10],[12]. When we consider a set of samples in the interval, [t_0 t_n] (Figure 2.11b), the joint probability distributions of n random variables X can specify the particular random process. Thus, the moments of the random process $X(t)$ can be defined by similar formulas in accordance with the definition of the moments of the random variable.

The mean of the random process $X(t)$ is

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x\ f_X(x,t)dx \tag{2.70}$$

the *autocovariance* is

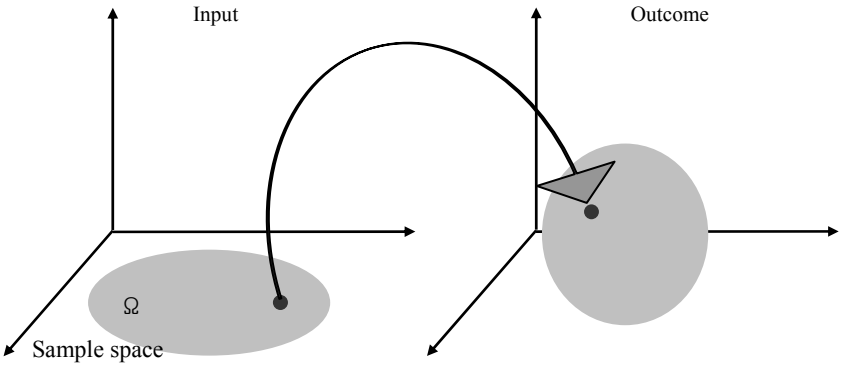
$$\begin{aligned} C_{XX}(t_1,t_2) &= E[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-\mu_X(t_1))(x_2-\mu_X(t_2))f_{XX}(x_1,x_2;t_1,t_2)dx_1dx_2 \end{aligned} \tag{2.71}$$

and the *autocorrelation* is

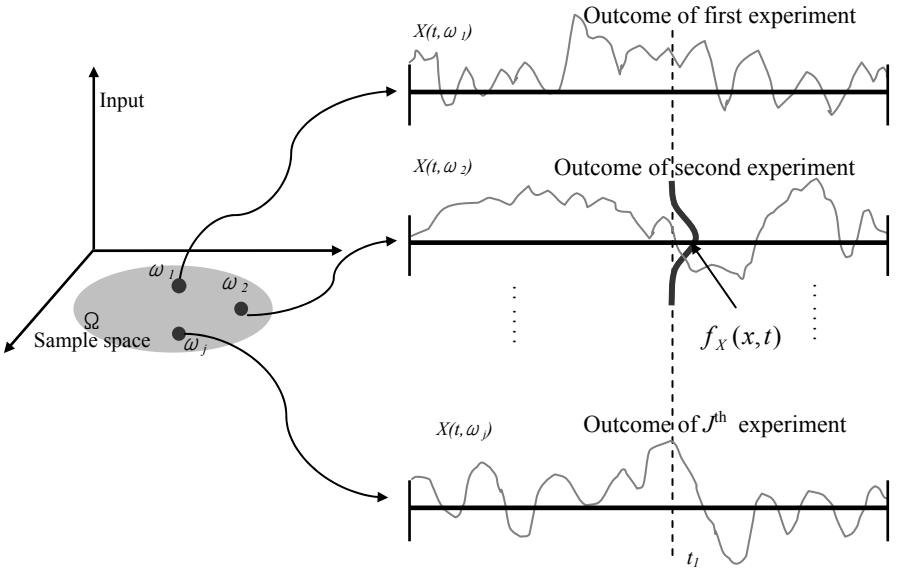
$$\begin{aligned} R_{XX}(t_1,t_2) &= E[X(t_1)X(t_2)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1x_2f_{XX}(x_1,x_2;t_1,t_2)dx_1dx_2 \end{aligned} \tag{2.72}$$

The autocorrelation function describes the correlation between all realizations at points t_1 and t_2. The prefix "auto" indicates that the integrand is composed of the same function at two points. Thus, the *cross-covariance* indicates a second-moment of two different functions. In particular, if the PDF $f_X(x,t)$ of the random process $X(t)$ is independent of t, namely $f_X(x,t) = f_X(x)$, then the process is referred to as a *stationary* process; otherwise, it is called a *nonstationary* process. Accordingly, all the moments of the stationary process are independent of t. If only the mean and the autocorrelation function of a random process are independent of t, then the process is said to be a *weakly stationary*. This process is a special case of

homogenous processes that shows some symmetry in the domain. If the ensemble average of a stationary process is equal to the corresponding time average, the process is called *ergodic*. Figure 2.12 shows the classification of random processes. As seen in the figure, all ergodic processes are stationary, but not all stationary process are ergodic.



(a) Deterministic Concept



(b) Random Process Concept

Figure 2.11. Deterministic and Random Process Concepts

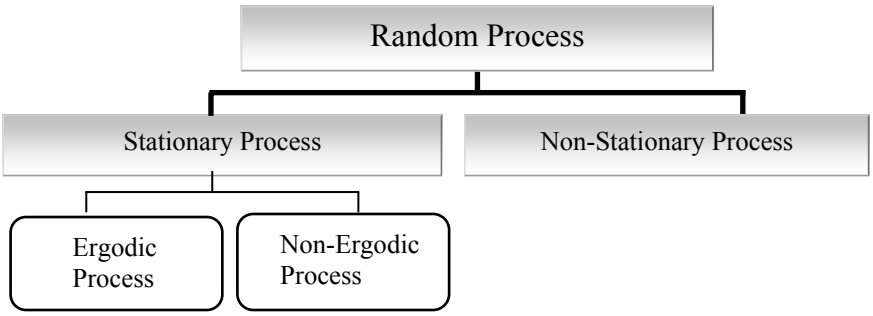


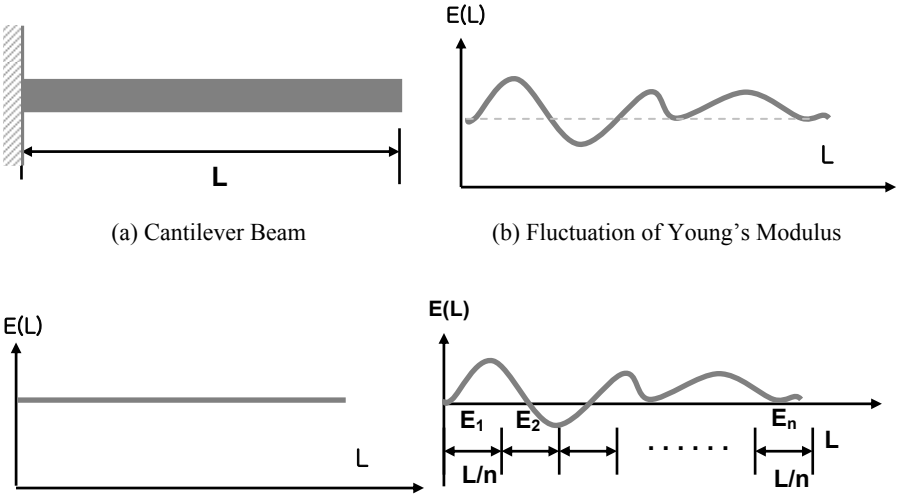
Figure 2.12. Classification of a Random Process

As previously stated, the random process can be thought of as a random function. Because manipulating random variables is easier than using the random function directly, a series of deterministic functions with random coefficients is frequently used to replace the random function. After its discretization, the continuous random process is an indexed set of an infinite number of random variables. Various methods, including the use of orthogonal polynomials and Taylor series representations, have been devised to replace random functions with random variables, depending on how the functions and the random coefficients are chosen. In the *Taylor series* representation [7],

$$U(x) \approx U_0 + U_1 x + U_2 x^2 + \ldots + U_n x^n \tag{2.73}$$

where $U(x)$ is a random function, and U_i are random variables with distributions determined by the distributions of $U(x)$. The deficiency of the Taylor series is that it requires many terms to get accurate results at points far from the origin. Thus, the use of orthogonal polynomials, which have a constant accuracy over the whole valid range of the approximation, is more suitable for the estimation of large fluctuations over domains. After first discussing the fundamentals of random field discretization, we provide complete details about the orthogonal polynomials, including the polynomial chaos expansion and the Karhunen-Loeve expansion [5].

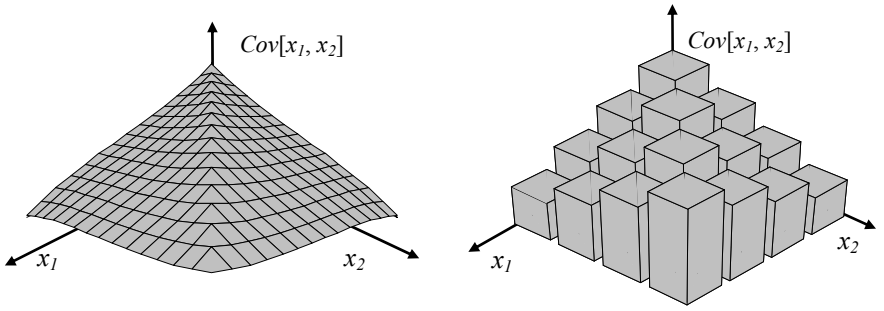
Consider a simple cantilever beam, as illustrated in Figure 2.13a, with its Young's modulus, E, fluctuating over the length of the beam (Figure 2.13b). Obviously, the fluctuation of the Young's modulus should be considered in the analysis process. To do this, the *random field discretization* is used to describe the spatial variability of the stochastic structural properties over the structure. First, the randomness of the Young's modulus can be split into two parts, the mean part (Figure 2.13c) and the fluctuation part (Figure 2.13d), in order to reduce the bias and to better facilitate analysis. The discretization of the random field is similar to the finite element discretization of structures. In the discretization procedure, the particular value of E_n is assumed to have the same value for the entire n^{th} segment, and its accuracy depends on the size of the segments. After the discretization procedure, the random field can be replaced by a set of correlated random variables.



(a) Cantilever Beam (b) Fluctuation of Young's Modulus

(c) Mean Part (d) Fluctuation Part

Figure 2.13. Random Field Discretization



(a) Continuous Covariance (b) Discretized Covariance

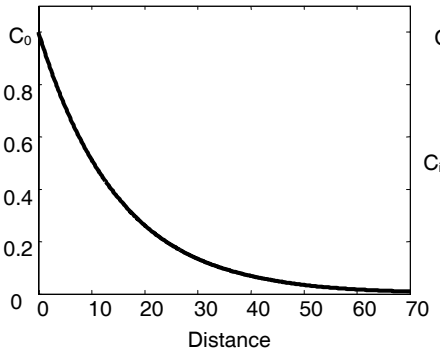
Figure 2.14. Discretization of Covariance

Several methods have been suggested to produce the random field discretization [8]. The utilization of orthogonal polynomials specifically will be discussed in Chapters 5 and 6. In order to properly discretize the random fields, the characterization and representation of the statistical correlation of each random variable (*i.e.*, E_n) are critical. Since the most widely used characterizations of the random fields are the first- and second-moment characterizations, the random-field discretization involves the discretization of its covariance function. The degree of correlation between the random process at nearby points can be specified by covariance functions. Figure 2.14b shows an illustration of the approximate covariance function for four elements along each direction in the finite element method. The structural properties of each element are modeled as random variables so that they have correspondingly different covariance values. Increasing the

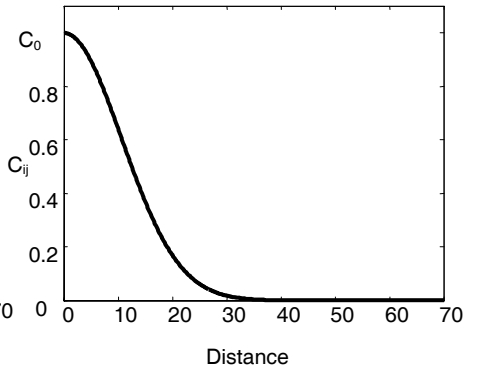
number of elements facilitates an accurate approximation of the actual covariance (Figure 2.14a). Thus, to ensure accurate analysis results, the engineer should consider the stochastic and modeling complexities of the problem before determining the size and number of elements.

2.2.2 Covariance Function

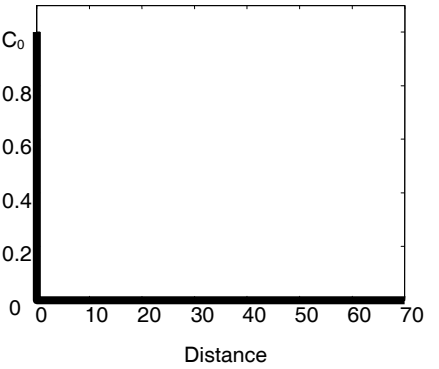
The degree of correlation between the random process at nearby points can be specified by *covariance functions*. If the variability of the random field is entirely random, the covariance function will decay asymptotically to zero. The points close together yield high correlation, and the lag points, vice versa. The following descriptions explain three well-known models of covariance functions, and the corresponding covariance functions are shown in Figure 2.15.



(a) Exponential Model



(b) Gaussian Model



(c) Nugget-effect Model

Figure 2.15. Covariance Functions

Exponential Model

The *exponential models* (Figure 2.15a) drop exponentially with increasing distance:

$$C_{ij} = C_0 \exp\left[-\left|\frac{x_{ij}}{l}\right|\right] \tag{2.74}$$

where C_0 is a variance, x_{ij} is the distance between two points (x_i and x_j), and l is the correlation length of the field. The correlation length indicates how quickly the covariance falls off. If the distance between two separated points is greater than the correlation length, the points are, statistically, nearly independent.

Gaussian Model

The *Gaussian model* (Figure 2.15b), also called the *squared exponential model*, is a commonly-used covariance function in random field analysis:

$$C_{ij} = C_0 \exp\left[-\left(\frac{x_{ij}}{l}\right)^2\right] \tag{2.75}$$

In contrast to the exponential model, the covariance in the Gaussian model remains flat at the origin. After the inflection point, it decays exponentially. Thus, this model provides a smoother stochastic process than the exponential model.

Nugget-effect Model

The *nugget-effect model*, also called the *delta-correlated model*, (Figure 2.15c) is an appropriate tool for discontinuous systems, which abruptly change values from one location to another:

$$C_{ij} = \begin{cases} C_0 & for \ \left|x_{ij}\right| = 0 \\ 0 & for \ \left|x_{ij}\right| > 0 \end{cases} \tag{2.76}$$

This model describes purely uncorrelated variation in population density, which is an equivalent concept to white noise in signal processing, and shows contributions to variability without spatial continuity.

The eigenfunctions and eigenvalues are extracted from the covariance function (C_{ij}) so that the collection of eigenfunctions represents the random fields. The extraction procedure of eigenfunctions and eigenvalues is known as *spectral decomposition* in continuous systems, and it is also known as the orthogonal transform or the Karhunen-Loeve transform. Chapters 5 and 6 describe the details

of the Karhunen-Loeve transform and the new procedures developed for representing the random field.

Example 2.7

Suppose we have four points on a line, x = [1, 2, 3, 4]. (a) When a Gaussian covariance is given with a variance $C_0 = 1$, find a correlation length (l) which yields C_{14} or $C_{41} = 0.65$, (b) Use the correlation length obtained in step (a) to construct a Gaussian covariance matrix.

Solution:

(a) From Equation 2.75

$$l = \left|x_{ij}\right| / \sqrt{-\ln(C_{ij}/C_0)}, \qquad \therefore \; l = \left|4-1\right| / \sqrt{-\ln(0.65)} = 4.5708$$

(b) The distance matrix is

$$x_{ij} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

Again, from Equation 2.75 with $l = 4.5708$, $C_0 = 1$

$$C_{ij} = \begin{bmatrix} 1 & 0.95 & 0.83 & 0.65 \\ 0.95 & 1 & 0.95 & 0.83 \\ 0.83 & 0.95 & 1 & 0.95 \\ 0.65 & 0.83 & 0.95 & 1 \end{bmatrix}$$

Note: Before proceeding, it is necessary to check the positive definiteness of the covariance matrix (all positive eigenvalues). If the matrix is not positive definite, there are several possible reasons: a) linear dependency – some of the covariances are linear function of other covariances, b) small sample size − the small sample size yields mere sampling fluctuations which cause the non-positive definite matrix, and c) missing data or input data error.

2.3 Fitting Regression Models

Regression analysis is the investigation of the functional relationship between two or more variables. Some specific aspects related to the proposed framework of

Chapters 5 and 6 are presented in this section. More complete details of the regression procedures are available in [9].

2.3.1 Linear Regression Procedure

In many situations, two or more variables are inherently related, and the investigation of the functional relationship between these measured variables (*independent variables*) and predicted variables (*dependent variables*) is the basic idea behind regression analysis. There are two types of regression analysis: linear and nonlinear. If the relation of the dependent and independent variable is assumed to be a linear function of some parameters, the regression model is called *linear*; otherwise, it is referred to as *nonlinear*. The earliest form of regression analysis for linear problems was studied by Gauss and Legendere in the 17th century. Their method, known as the least squares method, is a technique which minimizes the sum of squares of the residuals (differences between fitted function and given data) to find a best fit.

Consider the linear regression model:

$$y(x) = \beta_0 + \beta_1 f_1(x) + ... + \beta_k f_k(x) + \varepsilon \tag{2.77}$$

where β_i, $i = 0,1,2,...,k$, are the *regression coefficients* and ε, the error of the model equation, is assumed to be normally distributed with mean zero and variance σ_e^2.

Equation 2.77 can be written in matrix notation for n sample values of x and y as

$$Y = X\hat{\beta} + e \tag{2.78}$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} X = \begin{bmatrix} 1 & f_1(x_1) & f_2(x_1) \dots & f_k(x_1) \\ 1 & f_1(x_2) & f_2(x_2) \dots & f_k(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & f_1(x_n) & f_2(x_n) \dots & f_k(x_n) \end{bmatrix} \hat{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} and\ e = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

Generally, the method of least squares is used to obtain the regression coefficients:

$$\hat{\beta} = (X^T X)^{-1} X^T Y \tag{2.79}$$

The fitted model and the residuals are

$$\hat{Y} = X\hat{\beta} \text{ and } e = Y - \hat{Y} \tag{2.80}$$

The covariance matrix of $\hat{\beta}$ is

$$Cov(\hat{\beta}) = E\{(\hat{\beta} - E[\hat{\beta}])(\hat{\beta} - E[\hat{\beta}])^T\} = \sigma_e^2 (X^T X)^{-1} \tag{2.81}$$

where *E(.)* denotes the expected value.

The *total sum of squares* (Equation 2.82), *regression sum of squares* (Equation 2.83), and *error* (*residual*) *sum of squares* (Equation 2.84) are given as

$$SS_t = Y^T Y \tag{2.82}$$

$$SS_r = \hat{Y}^T \hat{Y} = \hat{\beta}^T X^T Y \tag{2.83}$$

$$SS_e = e^T e \text{ or } SS_e = SS_t - SS_r \tag{2.84}$$

SS_t is the total variation in y, SS_r is the variation due to regression and SS_e is the part of the variation in y that cannot be explained by the regression. The smaller SS_e is as a fraction of SS_t is, the better is the quality of the regression model.

2.3.2 Linear Regression with Polynomial Fit

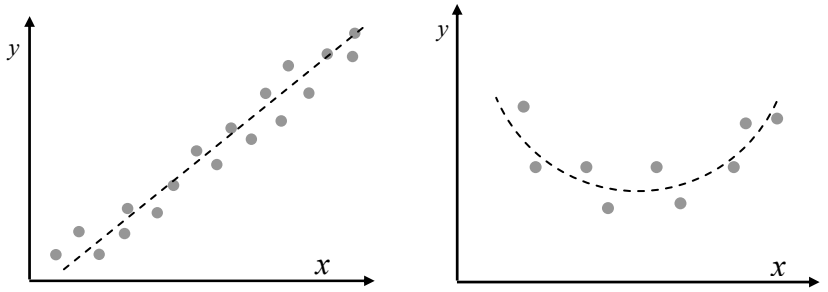
The linear model implicitly assumes that a plot of the dependent versus independent variables lies on a straight line before the addition with some random noise (Figure 2.16a). However, appropriate perception of the nonlinearity will provide better accuracy for quantifying nonlinear relationships (Figure 2.16b). The current section discusses the fitting of polynomials to data in some detail, and the next section presents statistical tests to help determine the appropriate regression model to use.

An understanding of polynomial approximation provides the foundation for most numerical analysis since polynomials are frequently used to approximate numerical solutions. In addition, solutions of many physical problems resemble polynomials, and in turn, polynomial approximation sometimes produces exact answers. Generally, the polynomial provides nonlinear relationships between response variables and explanatory variables. The response can be measured with fitted coefficients and selected ordered polynomials, and fitting errors. Another usage of polynomials is to approximate probability density or distribution functions.

Similar to the previous linear regression case (Equation 2.77), the linear regression of polynomial model of a one-dimensional case can be written as

$$y(x) = \beta_0 p_0(x) + \beta_1 p_1(x) + ... + \beta_m p_m(x) + \varepsilon \tag{2.85}$$

where the degree of $p_i(x)$ is $i = 0,\ldots,m$, and the polynomial approximation is said to be of order m for this case. The simplest polynomial model is the monomials of x^m (*i.e.*, $p_0(x) = 1$, $p_1(x) = x$, …, $p_m(x) = x^m$).



(a) Linear relationship (b) Nonlinear relationship

Figure 2.16. Appropriate Regression Models

This polynomial regression model can be solved in precisely the same manner as in the previous section (Equation 2.79), and the X matrix can be written as

$$X = \begin{bmatrix} p_0(x_1) & p_1(x_1) & p_2(x_1)\ldots & p_m(x_1) \\ p_0(x_2) & p_1(x_2) & p_2(x_2)\ldots & p_m(x_2) \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \\ p_0(x_n) & p_1(x_n) & p_2(x_n)\ldots & p_m(x_n) \end{bmatrix} \tag{2.86}$$

For the linear regression of polynomial model of the monomials of x^m, the columns of X can be nearly collinear. This causes an ill-conditioned problem, because negative values of x produce negative values for all odd powers, while positive values of x produce large positive values for the entire function. Hence, small changes in $p(x)$ lead to relatively large changes in the coefficients, β_i. More satisfactory solutions can be obtained when orthogonal polynomials (*i.e.*, Hermite, Laguerre, and Legendre) are used, reducing the effect of the ill-conditioned problem.

An important issue is to determine an appropriate order of polynomials. We can use high-order polynomials to closely fit representative sets of data, but the high-order model is generally less accurate in the area between selected sampling points. A relatively low-order polynomial fit to a large number of sampling points, provides a smooth interpolation that better fits the entire domain. The following section discusses the related topics of determining an appropriate order and checking the model adequacy of the regression.

2.3.3 ANOVA and Other Statistical Tests

Testing significance of regression involves *ANalysis Of VAriance* (ANOVA). ANOVA can determine the significant contributors of the model and can estimate the lack of fit and the confidence interval on the mean response. The test procedure is usually summarized in an analysis of variance table such as Table 2.2. In Table

2.2, n and k indicate the number of sample values or observations, and the number of treatments or regressors, respectively. When we omit the mean effect (β_o), the degrees of freedom of the total should be n-1 and the source of variance should be labeled as "Total, corrected" in Table 2.2. However, in the literature, the label "Total" is also used for the case of n-1 degrees of freedom. The test statistic F_o in Table 2.2 contributes to the significance test of the regression model. If the observed value of F_o is larger than the F-statistic, $F_o > F_{\alpha,1,df_e}$, then the coefficient is judged to have a significant effect on the regression model. The F-statistic has two parameters in this case, denoted by α and df_e. The df_e is the degrees of freedom of the residual, and α indicates the $100^{th}(1-\alpha)$ percentile of the F distribution. The percentage of the F distribution for specific degrees of freedom can be calculated and tabulated (Appendix D).

Plots of the residuals, e versus the corresponding fitted values $\hat{Y}$, or the observed values, Y versus $\hat{Y}$, are good measures for determining model adequacy. These graphical plots and other statistical tests (e.g., normal probability plot [9]) yield the residual analysis, which can detect model inadequacies with little additional effort. Visual inspections of residuals are preferable to understand certain characteristics of the regression results. Analysts can easily construct the plots, which organize the data to reveal useful information. Example patterns of residual plots, including satisfactory, funnel, double bow, and nonlinear cases, are available in [4] and [9]. Abnormality of the residual plots indicates that the selected model is inadequate or that an error exists in the analysis. When the residual analysis detects these common types of model inadequacies, the analysts require or consider extra terms in the regression model (e.g., higher order or interaction terms).

Table 2.2. Analysis of Variance for Significance of Regression

Source of variance	Sum of squares	Degrees of freedom (df)	Mean square	F_0
Regression	SS_r	$df_r = k$	$MS_r = SS_r / df_r$	MS_r/MS_e
Residual	SS_e	$df_e = n-k$	$MS_e = SS_e / df_e$	
Total	SS_t	$df_t = n$		

To select the appropriate order of approximation polynomials, we can proceed using either of two strategies. One approach is a *forward selection procedure*, which involves increasing the polynomial order until the highest-order term is nonsignificant according to the significance test. The other approach is the *backward elimination procedure*, which fits a response model using the highest-order term and then deletes terms one at a time. Thus, the α value of F-statistics can indicate the acceptance and rejection levels of the regressors. Typically, the α values of 0.05 and 0.10 are common choices for both the acceptance and rejection levels, but these values can be adjusted according to the analyst's experience. Some researchers prefer to set a larger value for the rejection level α than for the

acceptance level α to protect the rejection of regressors that are already admitted. Alternative methods that involve R^2, s^2, and C_p statistics select the best regression equations, and further discussions of their uses and advantages can be found in [4].

Example 2.8

Suppose we have this data:

x	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6
y	24	20	10	13	12	6	5	1	1	0

(a) Fit the linear model ($y = \beta_0 + \beta_1 x + \varepsilon$) to the data above, and compute ANOVA.
(b) Fit the polynomial model ($y = \beta_0 + \beta_1 x + \beta_2 x^2 + \varepsilon$) to the data above, compute ANOVA, and check for the significance of the nonlinear term.

Solution:

In matrix notation, the coefficients can be obtained from Equation 2.79:

$$[X^T X] = \begin{bmatrix} 10 & 17 \\ 17 & 32.2 \end{bmatrix}, \; [X^T X]^{-1} = \frac{1}{33}\begin{bmatrix} 32.2 & -17 \\ -17 & 10 \end{bmatrix},$$

$$[X^T Y] = \begin{bmatrix} 92 \\ 114 \end{bmatrix}, \; \hat{\beta} = [X^T X]^{-1} X^T Y = \begin{bmatrix} 31.04 \\ -12.85 \end{bmatrix}$$

$$\hat{y} = [\, 20.76 \;\; 18.19 \;\; 15.62 \;\; 13.05 \;\; 10.48 \;\; 7.91 \;\; 5.34 \;\; 2.77 \;\; 0.20 \;\; -2.36]^T$$

From Equation 2.82 ~ 2.84,

$$SS_t = Y^T Y = 1452.0$$

$$SS_r = \hat{Y}^T \hat{Y} = \hat{\beta}^T X^T Y = 1391.16$$

$$SS_e = e^T e = 60.84$$

ANOVA is obtained from Table 2.2:

Source of variance	Sum of squares	Degrees of freedom (df)	Mean square	F_0
Regression	1391.16	2	695.58	91.40
Residual	60.84	8	7.61	
Total	1452.0	10		

The same procedure can be applied to the polynomial regression:

$$[X^T X] = \begin{bmatrix} 10 & 17 & 32.2 \\ 17 & 32.2 & 65.96 \\ 32.2 & 65.96 & 142.68 \end{bmatrix}, \quad [X^T Y] = \begin{bmatrix} 92 \\ 114 \\ 156 \end{bmatrix}$$

From Equation 2.79,

$$\hat{\beta} = [X^T X]^{-1} X^T Y = \begin{bmatrix} 42.96 \\ -28.68 \\ 4.66 \end{bmatrix}$$

$$\hat{y} = [22.99 \;\; 18.94 \;\; 15.25 \;\; 11.94 \;\; 8.99 \;\; 6.43 \;\; 4.23 \;\; 2.40 \;\; 0.95 \;\; -1.14]^T$$

From Equation 2.82 ~ 2.84,

$$SS_t = Y^T Y = 1452.0$$
$$SS_r = \hat{Y}^T \hat{Y} = \hat{\beta}^T X^T Y = 1409.35$$
$$SS_e = e^T e = 42.65$$

To check for the significance of the added model term, x^2, we need to break up the regression sum of squares (SS_r) into the component:

SS_{x^2} = SS_r of the added/current model - SS_r of the reduced/previous model
= 1409.35 - 1391.16 = 18.19

From Table 2.2

Source of variance	Sum of squares	Degrees of freedom (df)	Mean square	F_0
Regression	1409.35	3	469.78	77.14
$\beta_0 + \beta_1 x$	1391.16	2	695.58	114.22
$\beta_2 x^2$	18.19	1	18.19	2.99
Residual	42.65	7	6.09	
Total	1452.0	10		

Since 5% and 10% points of the F distribution (Appendix D) are $F_{.05,1,7} = 5.59$ and $F_{.10,1,7} = 3.59$, respectively, the coefficient, β_2, of the nonlinear term is not significant. Thus, we can assume that the exploration of higher-order models (third, fourth, etc.) are not necessary, and their effects are negligible.

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