# Contents

1 Introduction ................................................. 1  
  1.1 Stochastic Models and Metastability ....................... 1  
  1.2 Timescales and Slow–Fast Systems ......................... 6  
  1.3 Examples ................................................. 8  
  1.4 Reader’s Guide ........................................... 13  
Bibliographic Comments ....................................... 15  

2 Deterministic Slow–Fast Systems .......................... 17  
  2.1 Slow Manifolds ............................................. 18  
  2.1.1 Definitions and Examples ............................. 18  
  2.1.2 Convergence towards a Stable Slow Manifold .......... 22  
  2.1.3 Geometric Singular Perturbation Theory ............... 24  
  2.2 Dynamic Bifurcations ..................................... 27  
  2.2.1 Centre-Manifold Reduction ............................. 27  
  2.2.2 Saddle-Node Bifurcation .............................. 28  
  2.2.3 Symmetric Pitchfork Bifurcation and Bifurcation Delay 34  
  2.2.4 How to Obtain Scaling Laws ........................... 37  
  2.2.5 Hopf Bifurcation and Bifurcation Delay ............... 43  
  2.3 Periodic Orbits and Averaging ............................ 45  
  2.3.1 Convergence towards a Stable Periodic Orbit ......... 45  
  2.3.2 Invariant Manifolds ................................... 47  
Bibliographic Comments ....................................... 48  

3 One-Dimensional Slowly Time-Dependent Systems ......... 51  
  3.1 Stable Equilibrium Branches .............................. 53  
  3.1.1 Linear Case ........................................... 56  
  3.1.2 Nonlinear Case ....................................... 62  
  3.1.3 Moment Estimates .................................... 66  
  3.2 Unstable Equilibrium Branches ............................ 68  
  3.2.1 Diffusion-Dominated Escape ........................... 71  
  3.2.2 Drift-Dominated Escape ............................... 78
3.3 Saddle–Node Bifurcation ........................................ 84
  3.3.1 Before the Jump ..................................... 87
  3.3.2 Strong-Noise Regime .................................. 90
  3.3.3 Weak-Noise Regime .................................. 96
3.4 Symmetric Pitchfork Bifurcation ............................. 97
  3.4.1 Before the Bifurcation ................................ 99
  3.4.2 Leaving the Unstable Branch .......................... 101
  3.4.3 Reaching a Stable Branch ............................. 103
3.5 Other One-Dimensional Bifurcations .......................... 105
  3.5.1 Transcritical Bifurcation ............................ 105
  3.5.2 Asymmetric Pitchfork Bifurcation ..................... 108
Bibliographic Comments .......................................... 110

4 Stochastic Resonance ........................................ 111
  4.1 The Phenomenon of Stochastic Resonance .................. 112
    4.1.1 Origin and Qualitative Description .................. 112
    4.1.2 Spectral-Theoretic Results ........................ 116
    4.1.3 Large-Deviation Results ............................ 124
    4.1.4 Residence-Time Distributions ......................... 126
  4.2 Stochastic Synchronisation: Sample-Paths Approach ...... 132
    4.2.1 Avoided Transcritical Bifurcation ................ 132
    4.2.2 Weak-Noise Regime ................................. 135
    4.2.3 Synchronisation Regime ............................ 138
    4.2.4 Symmetric Case ................................... 139
Bibliographic Comments .......................................... 141

5 Multi-Dimensional Slow–Fast Systems ......................... 143
  5.1 Slow Manifolds ......................................... 144
    5.1.1 Concentration of Sample Paths ..................... 145
    5.1.2 Proof of Theorem 5.1.6 ............................ 151
    5.1.3 Reduction to Slow Variables ........................ 164
    5.1.4 Refined Concentration Results ...................... 166
  5.2 Periodic Orbits ......................................... 172
    5.2.1 Dynamics near a Fixed Periodic Orbit ............... 172
    5.2.2 Dynamics near a Slowly Varying Periodic Orbit .... 175
  5.3 Bifurcations ........................................... 178
    5.3.1 Concentration Results and Reduction ............... 178
    5.3.2 Hopf Bifurcation ................................. 185
Bibliographic Comments .......................................... 190

6 Applications .............................................. 193
  6.1 Nonlinear Oscillators .................................. 194
    6.1.1 The Overdamped Langevin Equation .................. 194
    6.1.2 The van der Pol Oscillator ........................ 196
  6.2 Simple Climate Models .................................. 199
## Contents

6.2.1 The North-Atlantic Thermohaline Circulation .......... 200  
6.2.2 Ice Ages and Dansgaard–Oeschger Events .......... 204  
6.3 Neural Dynamics ........................................ 207  
6.3.1 Excitability ...................................... 209  
6.3.2 Bursting ......................................... 212  
6.4 Models from Solid-State Physics ........................... 214  
6.4.1 Ferromagnets and Hysteresis .......................... 214  
6.4.2 Josephson Junctions .................................. 219  

A A Brief Introduction to Stochastic Differential Equations . 223  
A.1 Brownian Motion ........................................ 223  
A.2 Stochastic Integrals ...................................... 225  
A.3 Strong Solutions ........................................ 229  
A.4 Semigroups and Generators ............................... 230  
A.5 Large Deviations ....................................... 232  
A.6 The Exit Problem ....................................... 234  
Bibliographic Comments ..................................... 236  

B Some Useful Inequalities ................................... 239  
B.1 Doob’s Submartingale Inequality and a Bernstein Inequality .. 239  
B.2 Using Tail Estimates ..................................... 240  
B.3 Comparison Lemma ...................................... 241  
B.4 Reflection Principle ...................................... 242  

C First-Passage Times for Gaussian Processes ................. 243  
C.1 First Passage through a Curved Boundary ................ 243  
C.2 Small-Ball Probabilities for Brownian Motion .......... 247  
Bibliographic Comments ..................................... 248  

References .................................................... 249  

List of Symbols and Acronyms ................................. 263  

Index .......................................................... 271