# Preface

This book was devised with the purpose of presenting a new method, developed in a series of papers, and intended to propose a more efficient numerical approach to solving partial integrodifferential equations. The latter appear in mathematical finance if, e.g., one takes into account jumps in the underlying spot price when pricing options written on this underlying.

The idea of the method was originated during a talk with my cousen-uncle Prof. Yakov Pesin in late 2007. We spoke about local and nonlocal operators, and at that time, my physics background pushed from the bottom of my memory a representation of the shift operator as an exponential. That was applied to the jump integral to transform it from a nonlocal operator to a local one.

This idea was then further exploited, and the first results were presented at the Global Derivatives and Trading conference in 2009 in joint work with Peter Carr. Later, we coauthored two more papers on this subject.

It turned out that the proposed method of pseudodifferential operators (MPsDO) is sufficiently general, and once the characteristic function of some Lévy process is known in closed form, the jump integral can be written in the form of a local pseudodifferential operator. The remaining work was to construct an efficient finite difference scheme to solve the evolutionary equation in which this pseudodifferential operator was an evolutionary operator. This scheme should be at least of the second order of approximation in all temporal and spatial directions, be unconditionally stable, and preserve nonnegativity of the solution.

Of course, as stated, this is a very ambitions goal. By that time, I was aware of M-matrices and the role that they play in the theory of stability of finite difference schemes. However, that didn't help much in this particular case. Fortunately, a bit later I discovered for myself an extension of an M-matrix called an EM-matrix. Application of this theory to constructing finite difference schemes demonstrated considerable power, and it was taken as the main tool in all future extensions of the MPsDO theory.

Since MPsDO is new as applied to Lévy processes in finance and was described just in a set of papers, I felt the necessity to collect all the results in one place, also arranging them together with the major facts from the modern theory of finite difference schemes, the theory of M-matrices and EM-matrices, etc. Also, it would be useful to provide some typical examples of problems that could be efficiently solved using this method. That was the motivation for this book finally to be written.

In the literature, there are many books about finite difference schemes and applications of the finite difference method to solving various PDEs and PIDEs. A majority of them use a fairly traditional approach to introducing finite differences and the ways to construct various schemes. Therefore, to give the reader a smooth introduction to the subject and make the book self-contained, Chapter 1 provides a short overview of such an approach. Then Chapter 2 presents a different (and more general) modern view based on the theory of operators. This approach includes formal exponential operator solutions of linear evolutionary equations and a theory of Padé approximations. Our main requirements for a good finite difference scheme are then presented and translated into some mathematical statements that are widely used throughout this book. Another important notion, that of operator splitting, is also introduced in this chapter, which is our basic tool in solving multidimensional PDEs and PIDEs. We provide some general facts about the splitting techniques known in the literature, and also describe our own contribution to construction of the splitting scheme for parabolic equations with mixed derivatives. The latter has some nice properties. Namely, it is unconditionally stable, preserves the nonnegativity of a solution, is of second order of approximation in space and time, and can be solved with linear complexity in each spatial dimension. Also, this scheme eliminates the necessity to use the Rannacher scheme at the first few temporal steps, which is usually done in the literature for better stability. In an appendix, we also outline some high-order compact finite difference schemes constructed for pricing American options.

In Chapter 3, the basic facts from the theory of M-matrices and EM-matrixces are provided as a short introduction to this theory, since it is not so well known in financial mathematics. We also prove all the main theorems and lemmas necessary in subsequent chapters for constructing finite difference solutions of PIDEs under some Lévy models.

Part II presents the theoretical core of MPsDO. In Chapter 4, we provide a short introduction to Lévy processes. The next chapter considers some basic examples of transforming a typical PIDE of option pricing theory into a pseudoparabolic equation. Chapter 6 modifies this approach by first adding a generality through the use of the shift operator and building a connection of the resulting pseudodifferential equation with the characteristic function of the underlying stochastic process. Construction of concrete finite difference schemes using this method is then presented in detail for many popular jump models, which include jump-diffusion and pure jump models. Chapter 7 further extends the idea of MPsDO by applying it to the solution of the forward Kolmogorov equation.

In Part III, we present a multidimensional version of MPsDO that is used to solve some typical problems in computational finance. This includes a structural default model with jumps, a local stochastic volatility model with stochastic interest rates and jumps, and also the stochastic skew model of Carr and Wu. We also add extra complexity to the traditional statement of these problems by taking into account jumps in each stochastic component while all jumps are fully correlated, and we show how this setting can be efficiently addressed within the framework of MPsDO. Various numerical results including some unusual ones are also presented and discussed.

Overall, the book could be potentially helpful for readers who want to become familiar with the modern finite difference theory being used for solving various applied problems of mathematical finance. From the mathematical point of view, the level of detail is closer to what one finds in the applied mathematics literature than in that of abstract or theoretical mathematics.

Part I could be the subject matter for a half-semester course on the contemporary finite difference approach in a master's or doctoral program in computational finance or financial engineering or even applied mathematics. In the latter case, for readers with no knowledge of finance, a short explanation of important financial terms and notions used in this book is given in a glossary. This book can also be used as a training course for practitioners who want to extend their knowledge of modern tools of computational finance.

Part I and II together with the second chapter of Part III could be used as a full semester course on the same subject. I used to teach some topics of this book as a part of my special course on computational finance at the Tandon School of Engineering at NYU in 2009–2015. I thank all my students for their questions, comments, and remarks.

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