Chapter 2
The Time Value of Money

You may have heard the expression, "*A dollar today is worth more than a dollar tomorrow*," which is because a dollar today has more time to accumulate interest. The time value of money deals with this basic idea more broadly, whereby an amount of money at the present time may be worth more than in the future because of its earning potential. In this chapter, we discuss the valuing of money over different time intervals, which includes a study of the present value of future money and the future value of present money. The theory is laid out in a rigorous, detailed, and general framework and accompanied by numerous applications with direct relevance to personal finance.

To be self-contained for readers new to finance, Sections 2.1 to 2.5 introduce our conventions and terminologies associated with time, interest rates, required return rates, total return rates, simple interest, compound interest for integral and nonintegral periods, and generalized compound interest, where the interest rate and compounding period vary. Readers already familiar with these topics should skim those sections for our notational usage. In Section 2.6, we introduce the net present value and internal return rate, including Descartes's Rule of Signs. The theory of annuities is presented in Section 2.7 and includes amortization theory and annuities with varying payments and varying interest rates. Applications of annuity theory to saving, borrowing, equity in a house, sinking funds, the present value of preferred and common stocks, and bond valuation are given in Sections 2.8 to 2.10.

A.O. Petters, X. Dong, *An Introduction to Mathematical Finance with Applications*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-1-4939-3783-7_2

2.1 Time

Before delving into the value of money over time, it is important to be clear about our conventions and notation for time.

Throughout the book, the default unit of time is a year. Unless stated to the contrary, assume that a year consists of 365 calendar days and 252 trading days.[1] When designating time, assume that there is a fixed starting time relative to which the other moments of time are defined. The explicit choice of starting time will depend on the context of the application, but we shall always represent it by 0. Note that the starting time need not be the current time.

We employ the following notation:

$$t_0 = \text{ fixed current time,} \qquad t = \text{ general moment of time.}$$

Note that a general moment of time $t > 0$ simultaneously designates the *number* of years of elapsed time from 0 to the given moment. For example, writing

$$t_0 = \frac{1}{4}, \qquad t = \frac{1}{2}$$

means that the current time is 3 months after the starting time and t is 3 months from now. If October 1, November 1, and December 1 in 2015 mark the times 0, t_1, and t_2, respectively, then

$$t_1 = \frac{1}{12}, \qquad t_2 = \frac{1}{6}.$$

We shall distinguish between an *interval of time*, say, $[t_0, t_f]$, and its *time span* τ, which is the length of the interval:

$$\tau = t_f - t_0 = \text{ number of years from } t_0 \text{ to } t_f, \qquad (t_0 \geq 0).$$

If a time interval is partitioned into equal-length subintervals, then the length of a subinterval is called a *period*. For example, a year has 12 monthly periods and 4 quarterly periods.

We shall employ the following abbreviations:

$$\text{mth} = \text{ month(s),} \qquad \text{yr} = \text{ year(s),} \qquad \text{prd} = \text{ period(s).}$$

In particular, one year is written as "1 yr" and two years as "2 yr."

---

[1] Apart from being mindful of leap years, note that banks may use a 360-day year when computing their charge on loans. Any deviation from a 365-day year will be stated explicitly.

2.2 Interest Rate and Return Rate

2.2.1 *Interest Rate*

You are perhaps most familiar with interest as the rate a bank pays into your savings account (where you lend the bank money) or the rate a bank charges you for a loan (where the bank lends you money). Overall, *interest* is the cost of money. It is the compensation received for lending or investing money. The initial amount of money you lend or borrow is called the *principal* and will be denoted by $\mathtt{F}_0$. Henceforth, assume that money invested—whether in a savings account or in a start-up company—is money lent with the expectation of receiving back more than the amount invested (principal plus interest).

The compensation for lending or the charge for borrowing a principal $\mathtt{F}_0$ is typically expressed as a percent r of $\mathtt{F}_0$ per year:

$$\{\text{compensation or charge per year}\} = r\,\mathtt{F}_0.$$

The percent r is called the *annual interest rate* or the *quoted rate*—e.g., a 5% per annum interest means $r = 0.05$. By default, all interest rates will be on or converted to a per annum basis. For this reason, we sometimes refer to r simply as the *interest rate* rather than the annual interest rate. Interest rates appear in numerous settings—savings accounts, certificates of deposit, credit cards, auto loans, mortgages, treasuries, bonds, etc.

Remark 2.1. Bear in mind that the interest rate used for lending need not equal the interest rate employed for borrowing. However, in later modeling, we shall assume that the two rates are equal (e.g., see page 84). □

We shall also switch freely between expressing r as a percent and decimal. It is possible to have $r > 1$ (interest rate of over 100% per year) or $r < 0$, which can be interpreted as a bank charging you for holding your principal. For simplicity, however, we abide by the following:

Unless stated to the contrary, assume that r is a positive constant.

Though r is constant by default, later in the chapter (e.g., Section 2.5), we shall study models where r varies discretely and continuously with time. When the interest rate r is a function of time, it is common practice to express this as $r(t)$—an abuse of notation that should not cause undue confusion.

Interest rates can, of course, be quoted for any time span (week, month, etc.). For example, an interest rate of 12% per year is mathematically the same as 1% per month. More generally, if we divide a year into k equal-size interest periods, then

$$\text{interest rate per interest period} = \frac{r}{k}.$$

Exact Interest, Ordinary Interest, and Banker's Rule

The *exact time* of a time interval measures the length of the interval in days, but excludes the first day. *Exact interest* is interest computed using 365 days in a year or 366 days for leap years. Credit card companies tend to use exact time and exact interest. *Ordinary interest* is interest calculated using 360 days in a year with 30 days in each month. Banks usually lend using exact time and ordinary interest, which has come to be known as *Banker's Rule*.

2.2.2 Required Return Rate and the Risk-Free Rate

We always assume that when an investor commits her money for a specific period of time, whether to a security, portfolio, or start-up, she expects to be compensated. An investor's *required rate of return* over an investment period is then the interest rate the investor demands as compensation for the following:

- *Opportunity cost:* Since lending prevents an investor from using that money for other investment opportunities, the investor requires compensation for her money being tied up.
- *Inflation:* Since inflation erodes the value of money, the investor requires compensation that covers the impact of inflation.
- *Risk:* Since there is a nonzero probability that earnings promised to the investor will not materialize or that the investor can lose some or all of her money, the investor requires compensation for the risks of the investment.

Unless stated to the contrary, we assume that no compensation to cover taxes and transaction costs is part of a required return rate. It is messy to include these items in an introduction to mathematical finance, not to mention that tax laws and transaction costs change. Readers are referred to Reilly and Brown [16, Chap. 1] for a detailed discussion of the required return rate.

In the absence of inflation and risk, the required return rate is called the *real risk-free rate* and denoted r_{real}. It is a compensation purely for opportunity cost. If there is no risk, but you have inflation and an opportunity cost, then the required return rate is termed the *nominal risk-free rate* or, simply, the *risk-free rate*. When the real risk-free rate is intended as opposed to the risk-free rate, we shall indicate so explicitly.

Notation. Let r denote the risk-free rate.

There is a simple relationship among r_{real}, r, and the inflation rate $\mathtt{i}$. Assume that you invest $\mathtt{F}_0$ in a riskless asset over 1 year. Your required return rate is r, which compensates you for opportunity cost and inflation. Specifically, your compensation for opportunity cost a year from now is $r_{\text{real}}\,\mathtt{F}_0$. However, a year from now, the value of your compensation $r_{\text{real}}\,\mathtt{F}_0$ for opportunity cost will reduce by $\mathtt{i}\,(r_{\text{real}}\,\mathtt{F}_0)$ due to inflation. Furthermore, your initial investment will also reduce in value by $\mathtt{i}\,\mathtt{F}_0$ due to inflation. Your required return rate amount $\mathtt{r}\mathtt{F}_0$ beyond your initial investment should then be

$$\mathtt{r}\mathtt{F}_0 = \mathtt{i}\,\mathtt{F}_0 \;+\; r_{\text{real}}\,\mathtt{F}_0 \;+\; \mathtt{i}\,(r_{\text{real}}\,\mathtt{F}_0).$$

Consequently, we obtain a formula for the real risk-free rate:

$$r_{\text{real}} = \frac{\mathtt{r} - \mathtt{i}}{1 + \mathtt{i}}.$$

A common proxy (i.e., substitute or model) for the risk-free rate r is the coupon rate of a US Treasury. The specific type of US Treasury chosen in applications depends on the time horizon over which an analysis is conducted. In the modeling of derivatives, however, traders typically choose LIBOR as a proxy for r (see Hull [9, p. 74] for more).

When inflation constitutes a major portion of the market risk-free rate r, sometimes r is even called the inflation rate. It is also possible for the inflation rate to be above the market risk-free rate, which, for instance, can be due to the government lowering interest rates to increase liquidity. Hence, one cannot always assume $\mathsf{r} \geq \mathtt{i}$, but would expect it to hold under normal market conditions.

2.2.3 Total Return Rate

Receiving an interest rate of 4% per year on a \$20,000 investment means that over 1 year, say, starting at time 0, you get

$$r\,\mathtt{F}_0 = 0.04 \times \$20{,}000 = \$800.$$

In other words, your investment would grow from \$20,000 to \$20,800 over 1 year. The return rate $R(0,1)$ on your investment over 1 year is the fractional percentage change

$$R(0,1) = \frac{\$20{,}800 - \$20{,}000}{\$20{,}000} = 0.04 = r.$$

If you put $\mathtt{F}(t_0)$ today in an investment that does not pay you any income and the value of the investment at a future time $t_f = t_0 + \tau$ is $\mathtt{F}(t_f)$, then the *total return rate* on your investment from time t_0 to t_f is defined to be

$$R(t_0,t_f) = \frac{\mathtt{F}(t_f) - \mathtt{F}(t_0)}{\mathtt{F}(t_0)}$$

with the *total return amount* defined by $R(t_0,t_f)\mathtt{F}(t_0)$.

The total return rate will not necessarily equal the interest rate. First, on mathematical grounds the interest rate is always positive, while the total return rate can be negative. Second, on financial grounds, the return rate is concerned only with the initial and final values of the investment and so is a performance measure of the investment. However, it is possible to apply an interest rate during each period into which a time span is divided, i.e., the interest rate is involved with the evolution of $\mathtt{F}(t_0)$ to the final value $\mathtt{F}(t_f)$. This will be made explicit when we look at simple and compound interest.

In general, we formalize the total return rate on a per-unit basis and with a cash dividend, i.e., an income.[2] Suppose that your investment has a current per-unit market value of $V(t_0)$, e.g., the price of a stock per share, and per-unit market value $V(t_f)$ at a final time $t_f > t_0$. Assume that the value of an investment at any point in time is nonnegative, i.e., there is no liability:

$$V(t) \geq 0 \qquad\qquad (t \geq 0). \tag{2.1}$$

Assume that the investment pays a per-unit cash dividend of $D(t_0,t_f)$ during the interval $[t_0,t_f)$—e.g., a cash payout per share by a company to shareholders.

Several clarifying remarks are needed about *cash dividends*:

- For simplicity, we do not include any cash dividend at t_f, but tally it as part of the subsequent time interval starting at t_f.[3]
- It is also common practice to assume that $D(t_0,t_f)$ excludes any income such as interest from the cash dividend during $[t_0,t_f)$. This is not a serious concern for sufficiently short investment time intervals. We also exclude complications like share splits and noncash payouts.
- When an investment pays out a cash dividend, it has lost value by the amount of dividend. The market value $V(t_f)$ is then the *ex-dividend* (without dividend) value and the *cum-dividend* (with dividend) value is

  $$V^c(t_f) = V(t_f) + D(t_0,t_f).$$

- In the case of a cash dividend-paying stock, there is actually an *ex-dividend date*, which is the cutoff date to be eligible for a declared cash dividend. It is actually the close of trading on the trading day before the ex-dividend date. The stock is said to be traded *cum-dividend* before the ex-dividend date and *ex-dividend* after that date. For this reason, when modeling, the value of the

[2] A dividend does not have to be in the form of cash. It can be a stock dividend—e.g., a company can pay you additional (typically, fractional) shares for each share of company stock you own.

[3] This bookkeeping for the cash dividend makes it convenient mathematically when considering reinvesting dividends to buy more units of the investment over consecutive time intervals.

stock is adjusted downward by the dividend amount on the ex-dividend date, not on the payment date. Stock price data sets like Yahoo! Finance have a column for adjusted prices, where the adjustments are for cash dividends and stock splits. The ex-dividend dates of stocks typically do not coincide with an exact quarter and are not the same for all companies.

➢ For securities like stocks and bonds, the cash dividends flow in discretely—e.g., quarterly, semiannually, and even annually in some cases. The dividend stream for a sufficiently broad stock index is often modeled as continuous.

Expressing the per-unit total return amount on your investment from t_0 to t_f as a percent $R(t_0,t_f)$ of the initial value $V(t_0)$, we obtain

$$\underbrace{R(t_0,t_f)\,V(t_0)}_{\text{return amount}} = \underbrace{V(t_f) - V(t_0)}_{\text{capital gain}} + \underbrace{D(t_0,t_f)}_{\text{cash dividend}}.$$

The spread $V(t_f) - V(t_0)$ is called a *capital gain*. Note that a negative capital gain is a *capital loss*. Equivalently,

$$R(t_0,t_f) = \underbrace{\frac{V(t_f) - V(t_0)}{V(t_0)}}_{\text{capital-gain return}} + \underbrace{\frac{D(t_0,t_f)}{V(t_0)}}_{\text{dividend yield}} = \frac{V^c(t_f) - V(t_0)}{V(t_0)}. \quad (2.2)$$

This is called the *total rate of return* or *holding-period return* of the investment from t_0 to t_f. We shall often refer to $R(t_0,t_f)$ simply as the *return rate* and at times will even refer to $R(t_0,t_f)$ as the *return* when it is clear from the context that a rate is intended as opposed to the return amount $R(t_0,t_f)\,V(t_0)$. Note that if your ownership in the investment consisted of n units (shares), then the return rate is still given by (2.2) since the numerator and denominator of each term would be multiplied by n and so n would drop out.

Notation. When the return rate depends on the length τ of $[t_0,t_f]$ rather than on the location of $[t_0,t_f]$ on the positive time axis $[0,\infty)$, we set

$$R(t_0,t_f) = R(\tau).$$

The ratio $\frac{D(t_0,t_f)}{V(t_0)}$ in (2.2) is called the *dividend yield* and represents the per-unit cash dividend from the investment as a percent of the initially invested capital $V(t_0)$. Additionally, we refer to the ratio $\frac{V(t_f)}{V(t_0)}$ as the *gross return* from t_0 to t_f.[4] It expresses the final value $V(t_f)$ as a percent of the initial value $V(t_0)$.

[4] Some authors call $\frac{V(t_f)}{V(t_0)}$ the *return rate*, but we shall not abide by that usage.

Example 2.1. Suppose that after 1 year, the return rate on your investment is 50%. Then the gain to you, beyond your initial investment, is 50% of your initial investment. If the return rate is -100%, then you have a complete loss. If the return rate is 200%, then your gain is twice the initial investment, i.e., your initial investment tripled in value over the year. □

Finally, observe that the return rate becomes random if the future value $V(t_f)$ and/or the cash dividend $D(t_0, t_f)$ is random. Almost all the return rates we encounter in this chapter are nonrandom, while all the return rates in Chapter 3 are random.

2.3 Simple Interest

A principal of \$1,000 held for a year at a 12% interest rate has a simple interest of \$120 at the end of 1 year. This amount is the same as adding 12 monthly interests of \$10, each of which is obtained from a monthly interest rate of 1%. For a time span of τ years, if we assume that the interest rate r is applied *only* to the principal $\mathtt{F}_0$, then

$$\text{(simple interest amount earned or owed over } \tau \text{ years)} = r\tau\mathtt{F}_0. \tag{2.3}$$

If an annual simple interest rate is applied over multiple years (or periods) to a principal, then at the end of each year (or period), interest is applied only to the principal and the entire balance is reinvested back into the account. In other words, all interest accrued at the end of each period or year is carried forward without gaining interest. Under *simple interest growth* at rate r, a principal $\mathtt{F}_0$ increases to the following amount at τ years from the present:

$$\mathtt{F}(\tau) = \mathtt{F}_0 + r\tau\mathtt{F}_0 = \left(1 + r\tau\right)\mathtt{F}_0, \tag{2.4}$$

where, by a slight abuse of notation, we write $\mathtt{F}(\tau)$ instead of $\mathtt{F}(t_0 + \tau)$ since the value depends on the length τ of the time interval.

Example 2.2. Suppose that an account has \$700 and pays 4% per annum. Applying a 4% annual simple interest growth to the \$700 for 1 year yields an interest of $0.04 \times \$700 = \$28$ and a total amount accrued of

$$\$700 \; + \; \left(0.04 \times \$700\right) = \$728.$$

To obtain simple interest growth of \$700 over 2 years, we add to the principal a simple interest of $0.04 \times \$700$ at the end of the first year and simple interest of $0.04 \times \$700$ at the end of the second year:

$$\$700 \; + \; \left(0.04 \times \$700\right) \; + \; \left(0.04 \times \$700\right) = \$756,$$

or, equivalently,

$$\$756 = \big(1 + 0.04 \times 2\big)\,\$700. \tag{2.5}$$

Note that a 4% annual simple interest growth applied to \$700 for 2 years is the same as applying 8% per 2 years. □

Investing \$700 under simple interest growth of 4% per annum yields a *future value* of \$756 2 years from now. Conversely, the *present value* of \$756 under 4% annual simple interest discounting is \$700. In general, if at the current time, you invest (or borrow) a principal $\mathtt{F}_0$ under simple interest growth at an interest rate r applied over τ years, then the amount of money you receive (or owe) at the end of the time span is called the *future value* of $\mathtt{F}_0$ and given by

$$\left\{\begin{array}{c}\text{future value of } \mathtt{F}_0 \\ \text{at } \tau \text{ years from now}\end{array}\right\} = \mathtt{F}(\tau) = (1 + r\,\tau)\mathtt{F}_0. \tag{2.6}$$

The principal $\mathtt{F}_0$ is called the *present value* of the future amount $\mathtt{F}(\tau)$:

$$\left\{\begin{array}{l}\text{present value of the amount } \mathtt{F}(\tau), \\ \text{which occurs } \tau \text{ years in the future}\end{array}\right\} = \mathtt{F}_0 = \frac{\mathtt{F}(\tau)}{1 + r\,\tau}. \tag{2.7}$$

The quantity $(1 + r\,\tau)^{-1}$ is called a *discount factor* since it reduces the amount $\mathtt{F}(\tau)$ at the end of the time interval to the amount $\mathtt{F}_0$ at the start of the interval.

In the context of (2.6), we sometimes call the interest rate r the *simple interest growth rate* of the principal $\mathtt{F}_0$, while in the setting of (2.7), we call r the *simple interest discount rate* on the future value $\mathtt{F}(\tau)$. The return rate when $\mathtt{F}_0$ grows under simple interest r over τ years is then

$$R(\tau) = \frac{\mathtt{F}(\tau) - \mathtt{F}_0}{\mathtt{F}_0} = r\,\tau. \tag{2.8}$$

2.4 Compound Interest

We saw above that under simple interest for 2 years, an account with \$700 at 4% per annum will grow to

$$\$700 \;+\; \Big(0.04 \times \$700\Big) = \$728 \tag{2.9}$$

at the end of the first year and to \$756 at the end of second year, after the interest of \$28 for the second year is added. However, there is a way to accumulate more money over the same 2 years using the same simple interest rate. Assume that at the end of the first year, you withdrew the \$728, closed the account, and immediately used the \$728 as principal to open another simple interest account paying the same interest rate. Then a year later, i.e., at the end of the second year, the total you would accrue is

$$\$728 + (0.04 \times \$728) = \$757.12, \tag{2.10}$$

which is greater than the original total of \$756! This type of growth is called *compound interest*. In fact, an annually compounded account earning 4% per annum over 2 years would earn you the latter amount without you needing to engage in the previous inconvenient strategy.

Using (2.9), we can rewrite (2.10) as

$$\$700 \; + \; \Big(0.04 \times \$700\Big) \; + \; \Big(0.04 \times (\$700 + 0.04 \times \$700)\Big) \; = \; \$757.12. \tag{2.11}$$

Equation (2.11) summarizes exactly how the growth process works: annual compounding of a principal of \$700 over 2 years at an interest rate of 4% means that one applies 4% simple interest to \$700 at the end of the first year and then applies 4% simple interest again at the end of the second year to the *entire balance* (principal plus interest) carrying forward from the end of the first year. Rewriting (2.11) as

$$\$757.12 = (1 + 0.04)^2 \times \$700 \tag{2.12}$$

yields the standard form for two annual compounds.

Let us extend (2.12) to a finite number of compoundings. In general, *compound interest* occurs when the time span is divided into multiple periods, and simple interest is applied over each period to the *balance* at the end of the period. *We assume that the entire balance at the end of each period is reinvested back into the total being accrued, i.e., no money is withdrawn and no extra money is added.* For mathematical modeling purposes, we also treat the end of a period as equivalent to the start of the next period.

2.4.1 Compounding: Nonnegative Integer Number of Periods

Assume that an account with an initial amount $\mathtt{F}_0$ (principal) pays an interest rate of r. Divide a year into k interest periods, each of equal length:

$$1 \text{ prd} = \frac{1}{k} \text{ yr}.$$

In a compound interest setting, the end of each period marks when interest is applied to the balance from the start of the period. Consequently, we shall refer to each such period as an *interest period*, *compound interest period* (when being explicit), or a *compounding period*.

> *Unless stated to the contrary, assume that the date when the principal is deposited coincides with the start of an interest period.*

Following the structure of (2.12), we now compute the future value to which the principal F_0 will grow under k-periodic compounding at interest rate r over n interest periods, where n is a nonnegative integer. Since n periods correspond to n/k years, the future value at the end of the nth period is $F(n/k)$. However, in compound interest theory, the emphasis is on the number n of periods over which compounding occurs, rather than the number of years. For this reason, the future value is written as a function of the number of periods as follows:

$$F\left(\frac{n}{k}\right) = F_n.$$

➢ At the end of the first period, apply simple interest to F_0 to obtain the future value F_1 to which F_0 grows over the first period:

$$F_1 = F_0 + \frac{r}{k}F_0 = \left(1+\frac{r}{k}\right)F_0.$$

Now, do not take out any of the money. Instead, reinvest the entire amount F_1 in the account at the end of the first period until the end of the second period.

➢ At the end of the second period, apply simple interest to F_1 to get the future value F_2 to which F_1 grows over the second period:

$$F_2 = F_1 + \frac{r}{k}F_1 = \left(1+\frac{r}{k}\right)^2 F_0.$$

Note that compound interest occurs since interest was added to the whole F_1, yielding interest on the principal F_0 and interest on the interest $(r/k)F_0$. Next, reinvest the entire amount F_2 in the account at the end of the second period until the end of the third period.

➢ Continuing the above process, at the end of the nth period, apply simple interest growth to F_{n-1} to obtain the future value F_n to which F_{n-1} grows over the nth period:

$$F_n = F_{n-1} + \frac{r}{k}F_{n-1} = \left(1+\frac{r}{k}\right)^n F_0, \qquad n = 0,1,2,\ldots.$$

We have established the following: *Under k-periodic compounding over n interest periods at an interest rate r, a principal F_0 will increase to the value F_n at the end of the nth interest period:*

$$F_n = \left(1+\frac{r}{k}\right)^n F_0, \qquad n = 0,1,2,\ldots, \tag{2.13}$$

where $\frac{r}{k}$ is the periodic interest rate. Observe that F_n depends on the size of the time interval over which the compounding occurs. This is because the interest rate is constant for the n periods.

In (2.13), we call $\mathtt{F}_n$ the *future value* of $\mathtt{F}_0$ under k-periodic compounding over n periods at interest rate r. The *present value* of $\mathtt{F}_n$ under the above compounding is defined to be $\mathtt{F}_0$.

Example 2.3. Borrow \$1,000 for a year at 12% interest rate. Applying this interest with monthly compounding yields a balance due of

$$\mathtt{F}_{12} = \left(1 + \frac{0.12}{12}\right)^{12} \$1{,}000 = \$1{,}126.83.$$

The interest owed is then \$1,126.83 − \$1,000 = \$126.83, which is more than the \$120 due when simple interest is applied. □

Example 2.4. (Money's Growth Under Different Compounding Periods) Invest \$1,000 at an interest rate of 7% and consider monthly, weekly, and daily compounding. Determine the future values after 2 years.

Solution. We have $\mathtt{F}_0 = \$1{,}000$, $r = 0.07$, $\tau = 2$, and $k = 12$ (monthly), 52 (weekly), and 365 (daily). The respective number of compounding periods is then 24 (monthly), 104 (weekly), and 730 (daily). By (2.13) on page 23, the future values at the end of 2 years are

$$\begin{aligned} \mathtt{F}_{24} &= \$1{,}000 \times 1.14981 = \$1{,}149.81 && \text{(monthly compounding)} \\ \mathtt{F}_{104} &= \$1{,}000 \times 1.15017 = \$1{,}150.17 && \text{(weekly compounding)} \\ \mathtt{F}_{730} &= \$1{,}000 \times 1.15026 = \$1{,}150.26 && \text{(daily compounding).} \end{aligned}$$

□

2.4.2 Compounding: Nonnegative Real Number of Periods

Suppose a principal of \$10,000,000 undergoes monthly compounding at 10% per annum over a time span of 15.36 mth. What is the principal's value at the end of the time span? First, view 15.36 mth as 15 mth + 0.36 mth. By Equation (2.13), the value at the end of the first 15 months is

$$\mathtt{F}_{15} = \left(1 + \frac{0.1}{12}\right)^{15} \times \$10{,}000{,}000 = \$11{,}325{,}616.82.$$

To how much will $\mathtt{F}_{15}$ grow during the remaining 0.36 mth? For the partial interest period, assume that a bank applies simple interest growth to $\mathtt{F}_{15}$, which yields a total of

$$\widetilde{\mathtt{F}}_{15.36} = \widetilde{\mathtt{F}}_{15+0.36} = \left(1 \;+\; 0.36 \times \frac{0.1}{12}\right) \times \mathtt{F}_{15} = \$11{,}359{,}593.67. \tag{2.14}$$

However, it may concern the reader that compounding occurs over the first 15 mth, but then stops during the remaining 0.36 mth, and is replaced by simple interest growth. We claim that the latter is actually an approximation of the exact mathematical compounding that should be applied during the partial month. We apply *fractional compounding* to $\mathtt{F}_{15}$ during the remaining 0.36 mth, which gives

$$\mathtt{F}_{15.36} = \mathtt{F}_{15+0.36} = \left(1 + \frac{0.1}{12}\right)^{0.36} \times \mathtt{F}_{15} = \$11{,}359{,}503.48. \tag{2.15}$$

In this example, we see that the accrued total in (2.14) is higher by \$90.19 than the total in (2.15) obtained from exact modeling. The bank would be paying more interest if (2.14) is used.

We now present a theoretical basis for (2.15) and the approximation used in (2.14). First, we shall introduce the key defining mathematical property of compound interest as in the treatment by Kellison [10, Sec. 1.5]. For an integral number of interest periods, Equation (2.13) shows that

$$\mathtt{F}_{m+n} = \left(1 + \frac{r}{k}\right)^{m+n} \mathtt{F}_0 = \left(1 + \frac{r}{k}\right)^{m} \left(1 + \frac{r}{k}\right)^{n} \mathtt{F}_0,$$

where m and n are nonnegative integers. We denote the *compound interest growth function over n interest periods* by

$$G(n) = \left(1 + \frac{r}{k}\right)^{n},$$

where

$$G(0) = 1, \qquad G(1) = \left(1 + \frac{r}{k}\right), \qquad G(n) > 1 \quad \text{for } n = 1, 2, \ldots.$$

The inequality $G(n) > 1$ for positive integers n means that the principal will increase for compounding over at least one interest period. The compound interest growth function satisfies:

$$G(m+n) = G(m)G(n). \tag{2.16}$$

In other words, compound interest is such that compounding a principal $\mathtt{F}_0$ over $m+n$ interest periods is the same as compounding $\mathtt{F}_0$ over n interest periods and then compounding the balance at the end of the nth interest period over the remaining m interest periods. Of course, one can interchange m and n. Equation (2.16) embodies the core multiplication property of compound interest.

We then extend (2.16) to a more general defining mathematical property of compound interest, one applicable to a nonintegral number of interest periods. Specifically, for any nonnegative real number x, a principal $\mathtt{F}_0$ is said to grow to the value

$$\mathtt{F}_x = G(x)\mathtt{F}_0$$

by *k-periodic compounding over x interest periods at interest rate r* if the growth function $G(x)$ satisfies the following properties:

$$\begin{aligned}
&G(x+y) = G(x)G(y) \quad \text{for all real numbers } x \geq 0 \text{ and } y \geq 0,\\
&G(0) = 1,\\
&G(1) = \left(1+\frac{r}{k}\right),\\
&G(x) > 1 \quad \text{for all real numbers } x > 0.
\end{aligned} \tag{2.17}$$

The top equation in (2.17) generalizes (2.16) to a nonintegral number of periods. The same intuition carries over from the integral case: compounding a principal $\mathtt{F}_0$ over $x+y$ interest periods to the value

$$\mathtt{F}_{x+y} = G(x+y)\mathtt{F}_0$$

is identical to compounding $\mathtt{F}_0$ for y interest periods to the value $\mathtt{F}_y = G(y)\mathtt{F}_0$ and then compounding $\mathtt{F}_y$ over the remaining x interest periods to the value $G(x)\mathtt{F}_y$. The equation $G(0) = 1$ in (2.17) states that no growth occurs when there is no interest period, while $G(1) = \left(1+\frac{r}{k}\right)$ means that the growth over one interest period is given by simple interest (as we have done all along). Finally, we require the condition $G(x) > 1$ for all $x > 0$ because we assume that compound interest growth increases the principal over a nonzero interest period, even if it is fractional.

Let us now solve for the growth function satisfying (2.17). For mathematical modeling reasons, we shall assume that $G(x)$ is differentiable. Applying a trick similar to the one used in deriving an exponential function, we first consider the derivative of the growth function at x. Using the limit definition of a derivative, we find (Exercise 2.30)

$$G'(x) = G(x)\,G'(0). \tag{2.18}$$

Dividing by $G(x)$, which is allowed since $G(x) > 0$ for all $x \geq 0$, and recalling that $G'(x)/G(x)$ is the derivative of $\ln G(x)$, we obtain

$$\frac{d\ln G(x)}{dx} = G'(0),$$

or, equivalently,

$$d\ln G(x) = G'(0)\,dx.$$

Integrating the equation from 0 to x yields:

$$\ln G(x) - \ln G(0) = G'(0)\, x.$$

But $\ln G(0) = \ln 1 = 0$. Hence:

$$\ln G(x) = G'(0)\, x. \tag{2.19}$$

Equation (2.19) implies:

$$G'(0) = \ln G(1) = \ln\left(1 + \frac{r}{k}\right).$$

Inserting $G'(0)$ back into (2.19), we find:

$$\ln G(x) = x \ln\left(1 + \frac{r}{k}\right) = \ln\left(1 + \frac{r}{k}\right)^x.$$

The binomial series $\left(1 + \frac{r}{k}\right)^x$ with nonintegral x converges for $0 \le r/k < 1$. Exponentiating both sides of the above equation, we obtain the growth function:

$$G(x) = \left(1 + \frac{r}{k}\right)^x.$$

We summarize the result in the following theorem:

Theorem 2.1. *Under k-periodic compound interest at r per annum over a time span of x interest periods, where x is a nonnegative real number, a principal* $\mathtt{F}_0$ *will increase to the following future value at the end of the time span:*

$$\mathtt{F}_x = \left(1 + \frac{r}{k}\right)^x \mathtt{F}_0, \qquad \left(0 \le \frac{r}{k} < 1, \quad x \ge 0\right), \tag{2.20}$$

where $k = 1, 2, \ldots$.

The periodic interest rate $\frac{r}{k}$ in (2.20) is constrained to $0 \le \frac{r}{k} < 1$ to assure convergence of $\mathtt{F}_x$ when the nonnegative real x is not an integer. We do not need this requirement when x is a nonnegative integer. In most applications, we consider $0 < r < k$. For example, under monthly compounding $(k = 12)$, the upper-bound condition expressed in percent means that the compounding interest rate satisfies $r < 1200\%$, which will surely be the case in most applications.

We also call $\mathtt{F}_x$ the *future value* of $\mathtt{F}_0$ at the end of x interest periods from the present and refer to

$$\mathtt{F}_0 = \frac{\mathtt{F}_x}{\left(1 + \frac{r}{k}\right)^x}, \qquad \left(0 \le \frac{r}{k} < 1, \quad x \ge 0\right), \tag{2.21}$$

as the *present value* of $\mathtt{F}_x$. The interest rate r is applied as a growth rate in the future valuing of (2.20) and as a discount rate in the context of (2.21). Since x interest periods is x/k years, the future value $\mathtt{F}_x$ occurs x/k years from now, i.e.,

$$\mathtt{F}_x = \mathtt{F}\left(\frac{x}{k}\right).$$

The number x of interest periods can always be expressed as the sum of an integral number n of interest periods and a fraction ν of an interest period:

$$x = n + \nu,$$

where n is the greatest integer part of x and $0 \leq \nu < 1$. For example, $x = 15.36$ interest periods splits into a sum of $n = 15$ and $\nu = 0.36$ interest periods.

We can then rewrite (2.20) as

$$\mathtt{F}_x = \left(1 + \frac{r}{k}\right)^n \mathtt{F}_\nu = \left(1 + \frac{r}{k}\right)^\nu \mathtt{F}_n, \qquad \left(0 \leq \frac{r}{k} < 1, \quad 0 \leq \nu < 1\right). \quad (2.22)$$

Here $\mathtt{F}_\nu$ is the amount to which $\mathtt{F}_0$ grows over the fraction ν of an interest period, i.e., we have *fractional compounding* during ν mth:

$$\mathtt{F}_\nu = \left(1 + \frac{r}{k}\right)^\nu \mathtt{F}_0.$$

For a proper fractional period, i.e., for $0 < \nu < 1$, the leftmost equality in (2.22) states that the fractionally compounded amount $\mathtt{F}_\nu$ is compounded over n interest periods, and the rightmost equality captures that the accrued amount $\mathtt{F}_n$ is compounded over the fraction ν of an interest period. The left equality applies to settings where the start of the time span does not coincide with the beginning or end of an interest period, while the right equality is for when the end of the time span is not the beginning or end of an interest period.

The rightmost equality in (2.22) also shows that if the interest rate per interest period r/k is sufficiently small, expanding the binomial series yields:

$$\left(1 + \frac{r}{k}\right)^\nu \approx 1 + \nu\frac{r}{k}. \quad (2.23)$$

The amount accrued at the end of x interest periods can then be approximated as follows:

$$\mathtt{F}_x = \left(1 + \frac{r}{k}\right)^\nu \mathtt{F}_n \approx \left(1 + \nu\frac{r}{k}\right) \mathtt{F}_n, \qquad (0 \leq \nu < 1, \quad 0 \leq r/k \ll 1). \quad (2.24)$$

Example 2.5. Returning to the example from the start of this section (page 24), Equation (2.20) shows that a principal of \$10,000,000 compounded monthly at 10% per annum for 15.36 mth will grow to:

$$\mathtt{F}_{15.36} = \left(1 + \frac{0.1}{12}\right)^{15.36} \times \$10{,}000{,}000 = \$11{,}359{,}503.48.$$

Equivalently,

$$F_{15.36} = \$11{,}359{,}503.48 = \left(1 + \frac{0.1}{12}\right)^{0.36} \times F_n,$$

which is the form in (2.22) and the origin of (2.15). Equation (2.14) uses simple interest, rather than fractional compounding, during the remaining 0.36 mth and is justified by (2.24):

$$\widetilde{F}_{15.36} = \left(1 + \nu\frac{r}{k}\right) F_n = \left(1 + 0.36\frac{0.1}{12}\right) \times F_n = \$11{,}359{,}593.67$$

and since $r/k = 0.0083 \ll 1$, we have $\widetilde{F}_{15.36} \approx F_{15.36}$. □

For k-periodic compounding at a constant interest rate r per year, Equation (2.20) yields that a principal F_0 will grow to the following future value over τ years or $k\tau$ periods:

$$F_{k\tau} = \left(1 + \frac{r}{k}\right)^{k\tau} F_0. \tag{2.25}$$

Example 2.6. (Doubling Your Investment) Suppose that you invest F_0 today in an account with k-periodic compounding at r per year. Find a formula for how long it will take you to increase your investment to x_0F_0, where $x_0 > 1$. Does the length of time depend on the initial amount F_0? In particular, how long will it take to double an investment of \$1,000 using 6% per annum with daily compounding? What about \$2,000? Compare with the time it would take using simple interest growth at the same interest rate.

Solution. We want to find how many years τ it will take to have F_0 grow to $F_{k\tau} = x_0F_0$. By (2.25),

$$x_0F_0 = \left(1 + \frac{r}{k}\right)^{k\tau} F_0,$$

which implies that

$$\tau = \frac{\ln x_0}{k \ln\left(1 + \frac{r}{k}\right)} \qquad (r > 0).$$

The time does not depend on the initial F_0.

For $F_0 = \$1{,}000$, $x_0 = 2$, $r = 0.06$, and $k = 365$ (daily), we obtain

$$\tau = \frac{\ln 2}{365 \ln\left(1 + \frac{0.06}{365}\right)} \approx 11.55,$$

so it will take 11.55 years. Since the time span will not depend on the initial investment, we obtain the same answer for \$2,000. For simple interest growth, we have $x_0F_0 = (1 + r\tau)F_0$, which yields $\tau = \dfrac{x_0 - 1}{r} \approx 16.67$. The doubling time is 5.12 years longer. □

2.4.3 Fractional Compounding Versus Simple Interest

Compound interest is constructed by applying simple interest over each interest period to the balance at the start of the interest period. This may suggest that simple interest should then be applied over each (proper) fraction of an interest period to the balance at the start of the fractional interest period (since simple interest adds over different time segments). However, if simple interest is applied over a given fraction of a period, then it does not account for the compounding that has to occur over every fraction of the given fractional interest period. Indeed, a new insight from Theorem 2.1 is that *compounding occurs over every portion of an interest period*. For example, start with a balance $\mathtt{F}_*$ and have it compound for $\frac{1}{3}$ prd. You should not apply simple interest over the $\frac{1}{3}$ prd because the balance $\mathtt{F}_*$ also has to compound over every fraction of the $\frac{1}{3}$ prd. For instance, compounding occurs over the first fourth of the $\frac{1}{3}$ prd and the remaining three-fourths of the period. By (2.20), the correct growth is:

$$\mathtt{F}_{1/3} = \left(1+\frac{r}{k}\right)^{\frac{1}{3}} \mathtt{F}_* = \left(1+\frac{r}{k}\right)^{\frac{3}{4}\left(\frac{1}{3}\right)} \left(1+\frac{r}{k}\right)^{\frac{1}{4}\left(\frac{1}{3}\right)} \mathtt{F}_*.$$

By (2.23), using simple interest over a fraction ν of an interest period would only yield an approximation under the following condition:

$$\mathtt{F}_\nu = \left(1+\frac{r}{k}\right)^{\nu} \mathtt{F}_* \approx \left(1+\nu\frac{r}{k}\right) \mathtt{F}_*, \qquad (0<\nu<1, \quad 0 \le r/k \ll 1).$$

If we do not use simple interest over a fraction of an interest period, then why can we apply simple interest over a whole interest period? The reason is that simple interest over one interest period is equivalent to fractional compounding over the interest period. In fact, decompose an interest period into any two fractional periods, say, ν prd and $(1-\nu)$ prd. Suppose that the balance at the start of the interest period is $\mathtt{F}_*$. Then the balance at the end of the period is:

$$\mathtt{F}_1 = \left(1+\frac{r}{k}\right) \mathtt{F}_* = \left(1+\frac{r}{k}\right)^{1-\nu} \left(1+\frac{r}{k}\right)^{\nu} \mathtt{F}_*.$$

In other words, the simple interest growth of $\mathtt{F}_*$ over 1 prd is the same as fractional compounding of $\mathtt{F}_*$ over ν prd followed by fractional compounding of the accrued amount $\left(1+\frac{r}{k}\right)^{\nu} \mathtt{F}_*$ over the remaining $(1-\nu)$ prd. We could divide an interest period into an arbitrary finite number of subperiods and still obtain that simple interest over one period is fractional compounding over the subperiods:

$$\mathtt{F}_1 = \left(1+\frac{r}{k}\right) \mathtt{F}_* = \left[\prod_{j=1}^{m} \left(1+\frac{r}{k}\right)^{\nu_j}\right] \mathtt{F}_*,$$

where $1 = \nu_1 + \nu_2 + \cdots + \nu_m$ and $0 < \nu_j < 1$ with $j = 1, \ldots, m$.

2.4.4 *Continuous Compounding*

When the number k of compounding periods per year increases without bound, we have continuous compounding. Applying (2.25), the future value under continuous compounding is

$$\mathtt{F}_{\text{cts}} = \lim_{k\to\infty} \mathtt{F}_{k\tau} = \lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{k\tau}\mathtt{F}_0 = \mathtt{F}_0\left(\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{k}\right)^{\tau} = \mathtt{F}_0\,\mathrm{e}^{r\,\tau}, \qquad (2.26)$$

with return rate

$$R(\tau) = \mathrm{e}^{r\,\tau} - 1. \qquad (2.27)$$

Under continuous compounding, \$1 will grow to \$$\mathrm{e}^{r\,t}$ over the time interval $[0,t]$. We can apply the same idea to a security (possibly risky) paying a continuous cash dividend at a constant yield rate q. Suppose that at time 0, you have 1 unit of a security, and as the cash dividend flows in, you continuously buy more units of the security, i.e., the number of units of the security is continuously compounded at rate q. Then 1 unit of the security at time 0 will grow to $\mathrm{e}^{q\,t}$ at time t. Consequently, the cum-dividend value of the security at t is

$$S_t^c = \mathrm{e}^{q\,t}\,S_t, \qquad (2.28)$$

where S_t is the ex-dividend price at t of one unit of the security. *We assume that S_t models the market price at t* since it discounts the cum-dividend price at the dividend yield rate: $S_t = \mathrm{e}^{-q\,t}\,S_t^c$. See the discussion on page 18.

2.5 Generalized Compound Interest

2.5.1 *Varying Interest and Varying Compounding Periods*

This section extends compound interest from a fixed interest rate over a non-negative real number of compounding periods to discretely varying interest rates across compounding intervals of different lengths.

We begin with some needed notation. Suppose that you put the amount $\mathtt{F}_0$ (principal) in an account for a time interval $[t_0, t_f]$, where $t_0 \geq 0$. Assume that each compound interest period is $\frac{1}{k}$ yr. Divide $[t_0, t_f]$ into n subintervals (not necessarily of the same length), say,

$$[t_0,t_1], \quad [t_1,t_2], \quad \ldots, \quad [t_{i-1},t_i], \quad \ldots, \quad [t_{n-1},t_n],$$

where n is a positive integer, $t_n = t_f$. Denote the length of the ith subinterval $[t_{i-1}, t_i]$ by τ_i, which corresponds to the following number of periods:

$$\tau_i \text{ yr} = k\tau_i \text{ prd}, \qquad i = 1, \ldots, n.$$

Suppose that k-periodic compounding at r_i per annum applies during the ith interval $[t_{i-1}, t_i]$ for $i = 1, \ldots, n$.

We now determine a formula for the amount to which the principal $\mathtt{F}_0$ will grow at the future time t_n.

➢ Over the time interval $[t_0, t_1]$, we have k-periodic compounding at interest rate r_1 of the principal $\mathtt{F}_0$. Applying (2.20) on page 27 with $x = k\tau_1$, the value of $\mathtt{F}_0$ grows to the following at time t_1:

$$\mathtt{F}(t_1) = \left(1 + \frac{r_1}{k}\right)^{k\tau_1} \mathtt{F}_0.$$

Reinvest the entire amount $\mathtt{F}_1$ in the account.

➢ Over the next time interval $[t_1, t_2]$, the balance $\mathtt{F}_1$ at time t_1 is k-periodically compounded at rate r_2. By (2.20) with $x = k\tau_2$, the value of $\mathtt{F}(t_1)$ grows to:

$$\mathtt{F}(t_2) = \left(1 + \frac{r_2}{k}\right)^{k\tau_2} \mathtt{F}(t_1) = \left(1 + \frac{r_2}{k}\right)^{k\tau_2} \left(1 + \frac{r_1}{k}\right)^{k\tau_1} \mathtt{F}_0.$$

Reinvest $\mathtt{F}_2$ in the account.

➢ Continuing this process, we find that over the final time interval $[t_{n-1}, t_n]$, the balance $\mathtt{F}(t_{n-1})$ at time t_{n-1} is k-periodically compounded at rate r_n. Again (2.20) yields that $\mathtt{F}(t_{n-1})$ grows to:

$$\mathtt{F}(t_n) = \left(1 + \frac{r_n}{k}\right)^{k\tau_n} \mathtt{F}(t_{n-1}).$$

Explicitly:

$$\mathtt{F}(t_n) = \left(1 + \frac{r_n}{k}\right)^{k\tau_n} \cdots \left(1 + \frac{r_2}{k}\right)^{k\tau_2} \left(1 + \frac{r_1}{k}\right)^{k\tau_1} \mathtt{F}_0. \tag{2.29}$$

Observe $\mathtt{F}(t_n)$ depends on the lengths of the subintervals over which the various interest rates are constant.

We call $\mathtt{F}(t_n)$ the generalized compound interest *future value* of $\mathtt{F}_0$ at time t_n. It is given in product notation as follows:

$$\mathtt{F}(t_n) = \left[\prod_{i=1}^{n} \left(1 + \frac{r_i}{k}\right)^{k\tau_i}\right] \mathtt{F}_0, \qquad \left(0 \le \frac{r_i}{k} < 1\right). \tag{2.30}$$

Here $\mathtt{F}_0$ is termed the *present value* of $\mathtt{F}(t_n)$.

Special Case

Assume that each interval $[t_{i-1}, t_i]$, where $i = 1, \ldots, n$, coincides with a compound interest period. Then the generalized future value (2.30) becomes:

$$\mathtt{F}(t_n) = \left(1 + \frac{r_n}{k}\right) \cdots \left(1 + \frac{r_2}{k}\right)\left(1 + \frac{r_1}{k}\right) \mathtt{F}_0, \tag{2.31}$$

where $\mathtt{F}_0$ is the principal at the initial time t_0. For a constant interest rate amount $r_i = r$, where $i = 1, \ldots, n$, we recover the usual k-periodic compounding formula over n periods or $\frac{n}{k}$ years:

$$\mathtt{F}_n = \mathtt{F}\left(\frac{n}{k}\right) = \left(1 + \frac{r}{k}\right)^n \mathtt{F}_0 = \mathtt{F}_n.$$

Example 2.7. How much will \$1,000 grow after 1.5 years if it is compounding semiannually with annual interest rate 7% applied at the end of the first 6 months, 8% at the end of the first year, and 9% at the end of 1.5 years?

Solution. Use the generalized compound interest formula (2.31) with $k = 2$ (semiannual compounding), $n = 3$ (number of periods), t_0 the current time, $t_3 = t_0 + 1.5$ (future time), $r_1 = 0.07$, $r_2 = 0.08$, and $r_3 = 0.09$. We obtain the following future value:

$$\begin{aligned}\mathtt{F}(t_3) &= \left(1 + \frac{r_3}{k}\right)\left(1 + \frac{r_2}{k}\right)\left(1 + \frac{r_1}{k}\right) \mathtt{F}_0 = 1.045 \times 1.040 \times 1.035 \times \$1{,}000 \\ &= \$1{,}124.84.\end{aligned}$$

□

2.5.2 *APR Versus APY*

We begin by showing how the interest rate r relates to the return rate in the context of compound interest.

At time t_0 invest an amount $\mathtt{F}_0 > 0$ (principal) in an account that grows under k-periodic compounding at interest rate r. Suppose that the account pays no dividend. Let $\mathtt{F}(t_f) > 0$ be the value of the principal at a future time $t_f = t_0 + \tau$. Since a time span of τ years has $k\tau$ periods, Equation (2.20) on page 27 yields that the return rate on the principal $\mathtt{F}_0$ is:

$$R_{\mathrm{CI}}(\tau) = \frac{\mathtt{F}(t_f)}{\mathtt{F}_0} - 1 = \left(1 + \frac{r}{k}\right)^{k\tau} - 1, \tag{2.32}$$

where the subscript CI indicates that the return rate is in the context of compound interest. Note the dependence on the length τ of the time interval $[t_0, t_f]$. For n periods, the return rate becomes:

$$R_{\mathrm{CI}}\left(\frac{n}{k}\right) = \left(1 + \frac{r}{k}\right)^{n} - 1. \tag{2.33}$$

In addition, the interest rate r can be expressed in terms of $R_{\mathrm{CI}}(\tau)$ as follows:

$$r = \frac{(1 + R_{\mathrm{CI}}(\tau))^{\frac{1}{k\tau}} - 1}{1/k}. \tag{2.34}$$

Equation (2.32) also shows that *growing the initial amount $V(t_0)$ to the value $V(t_f)$ under compounding at interest rate r is the same as growing $V(t_0)$ to $V(t_f)$ under simple interest using the return rate $R_{\mathrm{CI}}(\tau)$ over the time span τ*:

$$V(t_f) = \left(1 + R_{\mathrm{CI}}(\tau)\right)\, V(t_0) = \left(1 + \frac{r}{k}\right)^{k\tau} V(t_0).$$

The return rate $R_{\mathrm{CI}}(1)$ over a year is also commonly used. Equation (2.32) yields:

$$R_{\mathrm{CI}}(1) = \left(1 + \frac{r}{k}\right)^{k} - 1, \tag{2.35}$$

which is also called the *annual percentage yield* (APY) or *effective interest rate* and denoted by $R_{\mathrm{CI}}(1) = \mathrm{APY}$. The interest rate r corresponding to $R_{\mathrm{CI}}(1)$ is called the *annual percentage rate* (APR) or *nominal interest rate* and is given by:

$$\mathrm{APR} = \frac{(1 + \mathrm{APY})^{\frac{1}{k}} - 1}{1/k}.$$

The APR should not be confused with the APY, which involves compounding:

$$\mathrm{APY} = \left(1 + \frac{\mathrm{APR}}{k}\right)^{k} - 1.$$

For instance, if you are quoted an APR of 12% per annum on a loan, then the APR arises from a monthly interest rate of $\mathrm{APR}/12 = 1\%$. However, since interest on debt typically involves compounding, the APY gives a true reflection of the interest rate a borrower pays. In this case, the 1% per month interest compounds to an annual percentage yield of

$$\mathrm{APY} = (1 + 0.01)^{12} - 1 = 12.68\%,$$

not 12%. The next example further illustrates the difference.

Example 2.8. If a credit card company quotes only its APR on the card, say, 10.99%, it can cause a consumer to think that after 1 year, the interest amount on a balance of \$2,500 is

$$0.1099 \times \$2{,}500 = \$274.75.$$

However, this is not correct because it assumes simple interest for the year. Most credit cards compound daily or monthly (and may add fees). The true interest rate for a 365-day year with daily compounding is given by the APY:

$$\text{APY} = \left(1 + \frac{0.1099}{365}\right)^{365} - 1 = 11.6148\%.$$

The actual interest amount for the year is then the (effective) return amount:

$$\text{APY} \times \$2{,}500 = 0.116148 \times \$2{,}500 = \$290.37,$$

which is more than the amount \$274.75 naively inferred from an APR of 10.99%. □

2.5.3 Geometric Mean Return Versus Arithmetic Mean Return

An argument essentially the same as the one used to derive (2.35) shows that, given any period return rate R_{prd}, the return rate over a year with compounding at rate R_{prd} per period is given by:

$$R_{\text{ann}} = \left(1 + R_{\text{prd}}\right)^{k} - 1, \tag{2.36}$$

where (as usual) a year is assumed to have k periods. For example, a weekly return rate of 1% annualizes as follows under weekly compounding:

$$R_{\text{ann}} = (1 + 0.01)^{52} - 1 = 67.8\%.$$

We can generalize (2.36) further. First, the return rate (2.33) extends naturally to compound interest with varying interest rates over a time span of n compounding periods, where each period is $\frac{1}{k}$ yr. Assume that the annual interest rates used for the various n consecutive compounding periods are $r_1, \ldots, r_n$, i.e., the interest over the ith period is $\frac{r_i}{k}$. By (2.31) on page 33, the return rate (2.33) generalizes to:

$$R_{\text{CI}}(t_0, t_n) = \frac{\mathtt{F}(t_n)}{\mathtt{F}_0} - 1 = \left(1 + \frac{r_n}{k}\right)\left(1 + \frac{r_{n-1}}{k}\right) \cdots \left(1 + \frac{r_1}{k}\right) - 1. \tag{2.37}$$

Now, assume that you invest $\mathtt{F}_0$ in a nondividend-paying investment that has return rate R_i over the ith period, where $i = 1, \ldots, n$. Explicitly, if V_{i-1} and V_i are the respective values of the investment at the start and end of the ith period, then return rate is

$$R_j^{\text{prd}} = \frac{V_j - V_{j-1}}{V_{j-1}}.$$

Employing arguments similar to those used to derive (2.37), we can extend (2.37) from an ith-period interest rate of $\frac{r_i}{k}$, which is always positive, to the return rate of R_i^{prd}, which is not necessarily positive. In other words, we are generalizing (2.36) to the return rate over n periods by compounding at the respective return rates $R_1^{\text{prd}}, \ldots, R_n^{\text{prd}}$:

$$R_{\text{tot}} \equiv R\left(t_0, t_0 + \frac{n}{k}\right) = \left(1 + R_n^{\text{prd}}\right)\left(1 + R_{n-1}^{\text{prd}}\right) \cdots \left(1 + R_1^{\text{prd}}\right) - 1. \quad (2.38)$$

Note that by (2.1), each factor in the product (2.38) is nonnegative since it is a gross return:

$$1 + R_j^{\text{prd}} = \frac{V_j}{V_{j-1}} \geq 0, \qquad (j = 1, \ldots, n).$$

Equation (2.38) shows that the initial investment $\mathtt{F}_0$ will grow to the following value:

$$\mathtt{F}\left(\frac{n}{k}\right) = (1 + R_{\text{tot}})\, \mathtt{F}_0. \quad (2.39)$$

We remind the reader that in (2.39), we assume you do not withdraw or add any funds to the investment during the n periods. Unless otherwise stated, this is always our assumption when compounding; see Section 2.4.1.

Now, suppose that n periods ago, an investor put $\mathtt{F}_0$ into a nondividend-paying fund and her investment grew by the process in (2.39) to the current value $\mathtt{F}(\frac{n}{k})$. She would now like to forecast the behavior of the fund over the next period using a *single* "mean return rate" x. In other words, we seek a single rate x such that when compounding $\mathtt{F}_0$ using x over each of the past n periods, we obtain the same answer as compounding $\mathtt{F}_0$ using the n return rates $R_1^{\text{prd}}, \ldots, R_n^{\text{prd}}$:

$$\mathtt{F}\left(\frac{n}{k}\right) = (1 + x)^n\, \mathtt{F}_0. \quad (2.40)$$

Comparing (2.39) and (2.40), we see that x must be the *geometric mean return* $\overline{R}_{\text{geom}}^{\text{prd}}$ of $R_1^{\text{prd}}, \ldots, R_k^{\text{prd}}$, namely,

$$\overline{R}_{\text{geom}}^{\text{prd}} = \left[\left(1 + R_n^{\text{prd}}\right)\left(1 + R_{n-1}^{\text{prd}}\right) \cdots \left(1 + R_1^{\text{prd}}\right)\right]^{1/n} - 1.$$

We have:

$$\mathtt{F}\left(\frac{n}{k}\right) = \left(1 + \overline{R}_{\text{geom}}^{\text{prd}}\right)^n \mathtt{F}_0.$$

The geometric mean return relates as follows to the total return rate:

$$R_{\text{tot}} = \left(1 + \overline{R}_{\text{geom}}^{\text{prd}}\right)^n - 1. \quad (2.41)$$

In general, the geometric mean return does not equal the *arithmetic mean return*,

$$\overline{R}_{\text{prd}} = \frac{1}{n}\sum_{j=1}^{n} R_j^{\text{prd}}.$$

In fact,

$$\overline{R}_{\text{geom}}^{\text{prd}} \le \overline{R}_{\text{prd}}.$$

The two means coincide when the period return rates R_j^{prd} are identical for $j = 1, \ldots, n$. The example below illustrates these two means; see Reilly and Brown [16, Sec. 1.2.2] for more.

Example 2.9. (Geometric Mean Return Versus the Arithmetic Mean Return) Suppose that you initially invest \$3,000 in a fund that pays no dividend. Assume that the investment decreases to \$2,000 at the end of 1 year, decreases from \$2,000 to \$1,000 from the end of year 1 to the end of year 2, and increases from \$1,000 to \$3,000 from the end of year 2 to the end of year 3. Then the total return rate on your investment over the 3 years is zero.

Let us compare what the arithmetic and geometric mean returns forecast for the total return rate. The year-to-year return rates over the 3 years are:

$$R(t_0, t_0+1)) = \frac{\$2{,}000}{\$3{,}000} - 1 = -\frac{1}{3}, \qquad R(t_0+1, t_0+2)) = \frac{\$1{,}000}{\$2{,}000} - 1 = -\frac{1}{2},$$
$$R(t_0+2, t_0+3)) = \frac{\$3{,}000}{\$1{,}000} - 1 = 2.$$

The arithmetic mean return is:

$$\overline{R}_{\text{yr}} = \frac{1}{3}\left(-\frac{1}{3} - \frac{1}{2} + 2\right) = \frac{7}{18} = 0.3889,$$

which when compounded annually over the 3 years yields a grossly incorrect return rate:

$$(1 + \overline{R}_{\text{yr}})^3 - 1 = (1 + 0.3889)^3 - 1 = 1.6792 = 168\%!$$

On the other hand, the geometric mean return is:

$$\overline{R}_{\text{geom}}^{\text{yr}} = \left[(1+2)\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\right]^{1/3} - 1 = 0,$$

which gives the correct total return rate:

$$R(t_0, t_0+3) = \left(1 + \overline{R}_{\text{geom}}^{\text{yr}}\right)^3 - 1 = 0\%.$$

This is, of course, an illustration of (2.41).

The arithmetic mean return deviated significantly from the geometric mean because of the high volatility in the yearly return rates. The geometric mean return was better able to capture this dispersion and, hence, produced the correct total return rate. The two measures approximate each other when the return rates do not change significantly from period to period. The geometric mean return is usually employed for longer time horizons, where there is more opportunity for higher volatility. □

2.6 The Net Present Value and Internal Rate of Return

Compound interest can also be applied to develop the notion of a "net present value." This tool helps with deciding whether to partake in a particular investment opportunity. The opportunity can be a project, product line, start-up company, etc. We assume that with an initial capital, the investment opportunity produces net cash flows, i.e., cash inflows minus cash outflows, at different future dates.

> *Unless stated to the contrary, assume that each net cash flow takes taxes into account.*

In addition, when the net cash flow on a particular future date is being estimated, the estimate usually reflects activities over the year leading up to the date. We shall then consider net cash flows on future dates separated by a year. Furthermore, over the time span that an investment opportunity is analyzed for its growth potential, we assume that all the annual net cash flows can be modeled as arising from annual compounding at a constant interest rate. We refer to this constant interest rate as the *compounding growth (annual) rate from investing in the opportunity*.

2.6.1 Present Value and NPV of a Sequence of Net Cash Flows

For concreteness, we shall consider a credible, innovative start-up company. Suppose that the start-up forecasts that, with an initial investment of $250,000, it will generate net cash flows of $155,000 1 year from now, $215,000 2 years from now, and $350,000 3 years from now. The entire initial capital is assumed to be invested to produce these cash flows; e.g., none of the money is put aside in an account unrelated to the company's activities.

An important mathematical function we shall employ is the *present value* $\mathrm{PV}(r)$ of the sequence of net cash flows at an annual discount rate r. Note that in the current context, the present value is expressed as a function of the discount rate r rather than the number of periods n since n will be fixed and r will play a more key role. In our example, we have:

$$\text{PV}(r) = \frac{\$155{,}000}{1+r} + \frac{\$215{,}000}{(1+r)^2} + \frac{\$350{,}000}{(1+r)^3}.$$

Equally important will be the spread between PV(r) and the initial capital. This function is called the *net present value* (NPV) at rate r of the sequence of cash flows and is given by:

$$\text{NPV}(r) = \text{PV}(r) - \$250{,}000.$$

For simplicity, we shall write expressions such as NPV$(r) > 0$*, where it is understood that the "0" represents a zero amount of cash in the currency of the net cash flows.*

Now, an important step in deciding whether to invest in the start-up is to research the marketplace to find the mean compounding growth rate from investing in an alternative opportunity with a similar business profile and risk—e.g., research competitor companies comparable to the start-up in scale, risk, business sector, etc. For illustration, assume that the mean compounding growth rate from investing in an appropriate alternative opportunity is estimated to be:

$$r_{\text{RRR}} = 15\%.$$

We then take r_{RRR} as our *required return rate* for investing in the start-up.

The current market value of the start-up's projected stream of future net cash flows is the present value of these net cash flows discounted at the required return rate of 15%:

$$\begin{aligned} \text{PV}(r_{\text{RRR}}) &= \frac{\$155{,}000}{1+0.15} + \frac{\$215{,}000}{(1+0.15)^2} + \frac{\$350{,}000}{(1+0.15)^3} \\ &= \$134{,}782.61 + \$162{,}570.89 + \$230{,}130.68 \\ &= \$527{,}484.18. \end{aligned} \tag{2.42}$$

It is important to observe that *when determining the present value in (2.42), no assumption is being made about reinvesting the net cash flows* \$155,000, \$215,000, *and* \$350,000.

Equation (2.42) tells us that since the alternative opportunity grows your investment at 15% per annum compounded annually, such an opportunity can generate the start-up's forecasted net cash flows if you invest \$527,484.18 in the opportunity today. To see this, separate the required investment of \$527,484.18 into three parts as follows:

$$\text{PV}(r_{\text{RRR}}) = \$527{,}484.18 = \underbrace{\$134{,}782.61}_{\text{A}_{\text{RRR}}} + \underbrace{\$162{,}570.89}_{\text{B}_{\text{RRR}}} + \underbrace{\$230{,}130.68}_{\text{C}_{\text{RRR}}}.$$

We can then produce the net cash flows by thinking theoretically of the alternative opportunity as growing $\mathtt{A}_{\text{RRR}}$ to a future value of $\text{FV}_{\mathtt{A}_{\text{RRR}}}(1)$ at 1 year out, $\mathtt{B}_{\text{RRR}}$ to a future value of $\text{FV}_{\mathtt{B}_{\text{RRR}}}(2)$ at 2 years out, and $\mathtt{C}_{\text{RRR}}$ to a future value of $\text{FV}_{\mathtt{C}_{\text{RRR}}}(3)$ at 3 years out:

$$\text{FV}_{\mathtt{A}_{\text{RRR}}}(1) = (1 + 0.15) \times \$134{,}782.61 = \$155{,}000$$

$$\text{FV}_{\mathtt{B}_{\text{RRR}}}(2) = (1 + 0.15)^2 \times \$162{,}570.89 = \$215{,}000$$

$$\text{FV}_{\mathtt{C}_{\text{RRR}}}(3) = (1 + 0.15)^3 \times \$230{,}130.68 = \$350{,}000.$$

On the other hand, the credible start-up claims that it can generate the above future net cash flows with an investment today of less than the amount \$527,484.18 required by the alternative opportunity, namely, with an initial investment of only

$$\mathcal{C}_0 = \$250{,}000.$$

Naturally, investors will favor the start-up since the amount $\text{PV}(r_{\text{RRR}})$ required by the alternative opportunity is more expensive than the amount $\mathcal{C}_0$ required by the start-up. In other words, the start-up appears favorable when the net present value at the market required return rate is positive:

$$\text{NPV}(r_{\text{RRR}}) = \text{PV}(r_{\text{RRR}}) - \mathcal{C}_0 > 0.$$

The net present value at the required return rate, namely,

$$\text{NPV}(r_{\text{RRR}}) = \$527{,}484.18 - \$250{,}000 = \$277{,}484.18,$$

then measures how much cheaper (or more expensive, if the difference were negative) it is to invest in the start-up than in the alternative opportunity. Of course, any final decision to invest in a start-up will not rely solely on the NPV, but will be complemented with a detailed analysis of the start-up's business plan, innovative products/services, market environment, management team, etc.

If we had $\text{NPV}(r_{\text{RRR}}) = 0$, i.e., the initial capital required by the start-up to produce the given future net cash flows was the same as that required by the alternative opportunity, then there would be no extra value received from investing in the start-up. In this borderline situation, however, some investors may still invest in the start-up if, for example, it has more long-term promise.

The start-up would not be attractive to investors if $\text{NPV}(r_{\text{RRR}}) < 0$, i.e., if it costs more to receive the same future net cash flows from the start-up than from the alternative opportunity.

2.6.2 *The Internal Return Rate*

Indeed, the start-up can achieve these future net cash flows with less initial capital only if it grows the initial capital at a rate greater than the alternative opportunity's compounding annual growth rate of 15%. The start-up's compounding annual growth rate on the initial capital $\mathcal{C}_0$ is called the *internal rate of return* (IRR) and denoted r_{IRR}. To determine the start-up's IRR, we must find the interest rate r_{IRR} that generates the forecasted net cash flows starting from $\mathcal{C}_0 = \$250,000$:

$$\begin{aligned} \mathcal{C}_1 &= \$155,000 && \text{end of year 1} \\ \mathcal{C}_2 &= \$215,000 && \text{end of year 2} \\ \mathcal{C}_3 &= \$350,000 && \text{end of year 3.} \end{aligned} \tag{2.43}$$

First, separate \$250,000 into three amounts given by the present values of the net cash flows \$155,000, \$215,000, and \$350,000 at the unknown discount rate r_{IRR}. The sum of these individual present values is the present value $\text{PV}(r_{\text{IRR}})$ of the sequence of net cash flows. Explicitly:

$$\$250,000 = \underbrace{\frac{\$155,000}{(1+r_{\text{IRR}})}}_{\texttt{A}_{\text{IRR}}} + \underbrace{\frac{\$215,000}{(1+r_{\text{IRR}})^2}}_{\texttt{B}_{\text{IRR}}} + \underbrace{\frac{\$350,000}{(1+r_{\text{IRR}})^3}}_{\texttt{C}_{\text{IRR}}} = \text{PV}(r_{\text{IRR}}). \tag{2.44}$$

Then the future values at rate r_{IRR} of the three portions of the \$250,000 in (2.44) yield the desired future net cash flows. Specifically, the future value of $\texttt{A}_{\text{IRR}}$ at 1 year out is \$155,000, of $\texttt{B}_{\text{IRR}}$ at 2 years out is \$215,000, and of $\texttt{C}_{\text{IRR}}$ at 3 years out is \$350,000. It suffices then to find the IRR by solving (2.44) for r_{IRR}. Note that (2.44) is equivalent to the vanishing of the net present value at the rate r_{IRR}:

$$\text{NPV}(r_{\text{IRR}}) = \text{PV}(r_{\text{IRR}}) - \$250,000 = 0. \tag{2.45}$$

Employing a software, we find that an approximate solution of (2.44) or, equivalently, (2.45) is:

$$r_{\text{IRR}} = 0.652811.$$

Note that inserting this IRR into (2.44) actually produces \$250,000.04, which, of course, is not the exact value \$250,000 due to the approximate value of r_{IRR}. In other words, decomposing the start-up capital approximately as

$$\$250,000 \approx \underbrace{\$93,779.63}_{\texttt{A}_{\text{IRR}}} + \underbrace{\$78,703.14}_{\texttt{B}_{\text{IRR}}} + \underbrace{\$77,517.27}_{\texttt{C}_{\text{IRR}}} = \text{PV}(0.652811)$$

and future valuing each term by compounding annually at the rate r_{IRR} will yield the desired stream of net cash flows.

The start-up's IRR of 65.2811% compounded annually exceeds the alternative opportunity's compounding annual growth rate of 15%, which makes the start-up favorable. If it turned out that the IRR were 15%, then the start-up's growth rate would be no better than that of the alternative opportunity in the market (borderline case). If, on the other hand, the IRR were less than 15%, the start-up would be unattractive to investors (start-up not favorable).

In our example, we have: $r_{\mathrm{IRR}} > r_{\mathrm{RRR}}$ if and only if $\mathrm{NPV}(r_{\mathrm{RRR}}) > 0$. In other words, the IRR basis for deciding whether to favor the start-up is equivalent, *in this example*, to the choice being based on the NPV. *We have to be careful to not generalize this observation widely.* The example's equivalence of the IRR and NPV criteria is actually based on (2.44) or (2.45) having a unique positive solution and on the net present value being a strictly decreasing function. These two requirements need not hold in general. We address these issues next.

2.6.3 NPV and IRR for General Net Cash Flows

Extend the previous example to a general sequence of net cash flows. Suppose that you are considering a new investment opportunity requiring an initial capital of $\mathcal{C}_0 > 0$ to generate future net cash flows,

$$\mathcal{C}_1, \quad \mathcal{C}_2, \quad \ldots, \quad \mathcal{C}_n,$$

at respective future years $1, 2, \ldots, n$.

Making no assumptions about reinvesting the net cash flows $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$, we see that the present value of this sequence of cash flows at the compound-interest discount rate of r is:

$$\mathrm{PV}(r) = \frac{\mathcal{C}_1}{(1+r)} + \frac{\mathcal{C}_2}{(1+r)^2} + \cdots + \frac{\mathcal{C}_n}{(1+r)^n}, \qquad (r > 0). \tag{2.46}$$

The net present value of the net cash flows is the cost of the alternative investment opportunity minus the cost of the new investment opportunity:

$$\mathrm{NPV}(r) = \mathrm{PV}(r) - \mathcal{C}_0, \qquad (r > 0, \quad \mathcal{C}_0 > 0). \tag{2.47}$$

As before, denote the required return rate of the new investment opportunity by r_{RRR}. Recall that r_{RRR} is the mean compounding (annual) growth rate from investing in an alternative opportunity in the marketplace with business profile and risk similar to the new investment opportunity. *An NPV-based decision-making rule about whether to invest in the new opportunity is as follows*:

➢ If $\mathrm{NPV}(r_{\mathrm{RRR}}) > 0$, then the new investment opportunity is cheaper than the alternative investment and so is favorable.

➢ If $\text{NPV}(r_{\text{RRR}}) < 0$, then the new opportunity is more expensive and not favorable.

➢ If $\text{NPV}(r_{\text{RRR}}) = 0$, then the cost of the new opportunity is the same as the alternative investment and it is borderline whether to invest.

As noted in Section 2.6.1, even with a robust NPV estimate, a real-world business decision about whether to invest in a new opportunity will not use the NPV as the only measure. One has to factor in the business environment, experience of the management team, etc.

An IRR of the new investment is a positive solution, $r = r_{\text{IRR}}$, of the following equation:

$$0 = \text{NPV}(r) = -\mathcal{C}_0 + \frac{\mathcal{C}_1}{(1+r)} + \frac{\mathcal{C}_2}{(1+r)^2} + \cdots + \frac{\mathcal{C}_n}{(1+r)^n}. \tag{2.48}$$

Equation (2.48) is equivalent to a real polynomial, so we are seeking the positive roots of such a real polynomial. Without loss of generality, suppose that the real polynomial has degree k and is of the following form:[5]

$$a_k r^k + a_{k-1} r^{k-1} + \cdots + a_1 r + a_0 = 0, \qquad (a_k \neq 0). \tag{2.49}$$

There is no general formula for the real solutions of (2.49) for all positive integers k.

Perhaps the most cited general result about the number of positive solutions of (2.49) is Descartes's Rule of Signs. Before stating this result, we gather some notation. Let N_+ denote the number of positive solutions of (2.49), where we count the solutions with multiplicity. For example, the polynomial,

$$r^2 - 10r + 25 = (r-5)^2 = 0,$$

has $N_+ = 2$, corresponding to two positive solutions $r = 5$ counted with multiplicity. Let N_{sgn} be the number of sign changes in the ordered sequence of the coefficients in (2.49):

$$a_k, a_{k-1}, \ldots, a_1, a_0. \tag{2.50}$$

Since the zero coefficients do not contribute to a sign change, it suffices to consider the sign changes due to the ordered nonzero coefficients.

Theorem 2.2. (Descartes's Rule of Signs) *The number N_+ of positive solutions of (2.49) is the number N_{sgn} of sign changes of its ordered coefficients in (2.50) or is N_{sgn} minus an even positive integer. Specifically, N_+ equals either N_{sgn}, $N_{\text{sgn}} - 2$, $N_{\text{sgn}} - 4, \ldots, N_{\text{sgn}} - 2(n-1)$, or $N_{\text{sgn}} - 2n$ for some nonnegative integer n.*[6]

[5] If $a_k = 0$, then simply apply the same discussion to the lower degree polynomial.

[6] Using $N_+ \leq N_{\text{sgn}}$, the reason a nonnegative even integer is subtracted from N_{sgn} in the theorem is because N_+ and N_{sgn} have the same parity, i.e., N_+ is even (odd) if and only if N_{sgn} is even (odd). This

Proof. See Meserve [14, p. 156] and Wang [17] for a proof. □

For example, the polynomial equation

$$r^5 - r^2 + r - 1 = 0$$

has three sign changes in its ordered nonzero coefficients: $+1, -1, +1, -1$. By Theorem 2.2, this polynomial equation has either 3 or 1 positive solutions.

The IRR Equation (2.48) is equivalent to a polynomial equation of the form (2.49) with ordered coefficients (2.50). By Theorem 2.2, if these ordered coefficients have one sign change, then there is at most one positive solution. If, in addition, you can prove that the polynomial equation has at least one positive solution, then this solution is the unique positive solution and the desired IRR. In the example of the start-up, Equation (2.45) is equivalent to a cubic equation:

$$p(r) = -250{,}000\,r^3 - 595{,}000\,r^2 - 225{,}000\,r + 470{,}000 = 0. \tag{2.51}$$

There is one sign change, so there is at most one positive solution. Since $p(r) > 0$ at $r = 0$ and $p(r) \to -\infty$ as $r \to \infty$, its graph must cross the positive r-axis, which means that $p(r)$ must have at least one positive solution. Hence, the cubic equation has a unique, positive solution, which is the desired r_{IRR}. Using a software, we found the approximate positive solution to be $r_{\text{IRR}} = 0.652811$.

We also observed that for the required return rate of $r_{\text{RRR}} = 15\%$, the IRR criterion to favor the start-up, namely, $r_{\text{IRR}} > r_{\text{RRR}}$, is equivalent to the NPV criterion of $\text{NPV}(r_{\text{RRR}}) > 0$. This is not true in general, but holds in the example because the function $\text{NPV}(r)$ in (2.45) is strictly decreasing. The next result shows when the situation of the example holds.

Theorem 2.3.

1) *Suppose that all the future net cash flows are positive. Then* $\text{NPV}(r)$ *is a strictly decreasing function of r and, if there is an* $r = r_{\text{IRR}}$, *then* r_{IRR} *is the only IRR.*[7]
2) *If there is an* r_{IRR} *and* $\text{NPV}(r)$ *is strictly decreasing, then the IRR and NPV decision-making criteria are equivalent:*

   a) $r_{\text{IRR}} > r_{\text{RRR}}$ *if and only if* $\text{NPV}(r_{\text{RRR}}) > 0$.
   b) $r_{\text{IRR}} = r_{\text{RRR}}$ *if and only if* $\text{NPV}(r_{\text{RRR}}) = 0$.
   c) $r_{\text{IRR}} < r_{\text{RRR}}$ *if and only if* $\text{NPV}(r_{\text{RRR}}) < 0$.

Proof.

1) Since $C_i > 0$ for $i = 1, \ldots, n$, the derivative of the NPV function satisfies:

---

implies $N_{\text{sgn}} - N_+$ is a nonnegative even number, i.e., $N_+ = N_{\text{sgn}} - \text{even}$. In particular, N_+ is either $N_{\text{sgn}}, N_{\text{sgn}} - 2, \ldots, N_{\text{sgn}} - 2(n-1)$, or $N_{\text{sgn}} - 2n$ for some nonnegative integer n.

[7] By definition, we assume $r_{\text{IRR}} > 0$.

$$\frac{d}{dr}\text{NPV}(r) = -\frac{\mathcal{C}_1}{(1+r)^2} - 2\frac{\mathcal{C}_2}{(1+r)^3} - \cdots - n\frac{\mathcal{C}_n}{(1+r)^{n+1}} < 0.$$

Consequently, the function NPV(r) is strictly decreasing and, hence, if the graph of NPV(r) crosses the positive r-axis, i.e., there is an IRR, the graph will do so only once, namely, at a unique value r_{IRR}.

2) We are given that an IRR exists: $r_{\text{IRR}} > 0$. Since NPV(r) is strictly decreasing, we have $r_2 > r_1 > 0$ if and only if $\text{NPV}(r_1) > \text{NPV}(r_2)$. For part a) choose $r_2 = r_{\text{IRR}}$, $r_1 = r_{\text{RRR}}$, and observe that $\text{NPV}(r_{\text{IRR}}) = 0$. For part c) choose $r_2 = r_{\text{RRR}}$, $r_1 = r_{\text{IRR}}$. Part b) holds since $\text{NPV}(r_{\text{IRR}}) = 0$. □

It is important to note that in a real-world decision-making setting, the NPV and IRR criteria have aspects not explicitly spelled out in Theorem 2.3. For example, consider the start-up we have been exploring:

$\mathcal{C}_0$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	r_{RRR}	NPV(r_{RRR})	r_{IRR}
\$250,000	\$155,000	\$215,000	\$350,000	15%	\$277,484.18	65.28%

The table shows that with an initial investment of $\mathcal{C}_0$, the start-up is attractive because it has:

➢ Positive future net cash flows $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ that are nontrivial as a percent of the initial capital and are nontrivially increasing. In particular, $\mathcal{C}_1$ is more than half the initial capital, $\mathcal{C}_2$ is about 86% of the initial capital, and $\mathcal{C}_3$ is 140% of the initial capital. Moreover, the net cash flow increases by about 39% from year 1 to 2 and about 63% from year 2 to 3.

➢ A nontrivially positive NPV value of $\text{NPV}(r_{\text{RRR}}) = \$277{,}484.18$, which makes the start-up much cheaper to invest in than a comparable alternative opportunity by more than the initial capital.

➢ A quite large compounding growth rate of $r_{\text{IRR}} = 65.28\%$ compared to the required return rate of $r_{\text{RRR}} = 15\%$, i.e., the r_{IRR} is more than four times r_{RRR}.

No IRR and Multiple IRRs

The discussion so far assumes a unique IRR. However, complications already arise in the simple case of net cash flows over 2 years ($n = 2$), where the IRR Equation (2.48) becomes a quadratic in r:

$$\underbrace{-\mathcal{C}_0}_{a}\, r^2 + \underbrace{(\mathcal{C}_1 - 2\mathcal{C}_0)}_{b}\, r + \underbrace{(-\mathcal{C}_0 + \mathcal{C}_1 + \mathcal{C}_2)}_{c} = 0, \qquad (\mathcal{C}_0 > 0). \tag{2.52}$$

For example, if the net cash flows are $\mathcal{C}_1 = 2\mathcal{C}_0$ and $\mathcal{C}_2 = -2\mathcal{C}_0$, then the quadratic reduces to one with no real solution: $r^2 + 1 = 0$. *In this case, there is no IRR*.

It is also possible to have multiple IRRs. The quadratic (2.52) has two positive solutions, say, r_1 and r_2, if and only if the following positivity conditions hold:

$$\begin{aligned} 0 &< b^2 - 4ac = \mathcal{C}_1^2 + 4\mathcal{C}_0\mathcal{C}_2 \\ 0 &< r_1 + r_2 = -\frac{b}{a} = \frac{\mathcal{C}_1 - 2\mathcal{C}_0}{\mathcal{C}_0} \\ 0 &< r_1 r_2 = \frac{c}{a} = \frac{\mathcal{C}_0 - \mathcal{C}_1 - \mathcal{C}_2}{\mathcal{C}_0}. \end{aligned}$$

These positivity conditions are equivalent to

$$2\mathcal{C}_0 < \mathcal{C}_1, \qquad \frac{-\mathcal{C}_1^2}{4\mathcal{C}_0} < \mathcal{C}_2 < \mathcal{C}_0 - \mathcal{C}_1, \qquad\qquad (\mathcal{C}_0 > 0).$$

Choosing $\mathcal{C}_0 = \$10{,}000$, $\mathcal{C}_1 = \$25{,}000$, and $\mathcal{C}_2 = -\$15{,}620$, we obtain two IRRs:

$$r_1^{\mathrm{IRR}} = 22.76\%, \qquad r_2^{\mathrm{IRR}} = 27.24\%.$$

In the case of multiple IRRs, it is possible to construct a modified IRR. Nonetheless, in practice it is simplest to work with the NPV when there is no IRR or multiple IRRs. Additionally, in cases where usage of the NPV and/or IRR are unclear, it may be wise to hold off from making an investment decision.

Readers are referred to Bodie, Kane, and Marcus [1, Chaps. 5, 6] for a detailed practical discussion of the uses of the NPV, IRR, and other tools in investment decision-making.

2.7 Annuity Theory

In this section, we continue our study of cash flow sequences by considering annuities. An *annuity* is a series of payments made at equal time periods with interest. Examples of annuities are the payment sequences of Social Security funds, pensions, car loans, credit card debt, and mortgages. We shall study annuities with identical payments and a constant interest rate and then generalize them to payments and interest rates that vary discretely in time.

For simplicity, we shall explicitly indicate when the annuity payments vary and, by default, abide by the following:

> *Unless stated to the contrary, assume that each annuity payment is the same amount.*

The *term* of an annuity is the time from the start of the first payment period to the end of the last payment period. For an *ordinary annuity*, payments occur at the end of each time period. When the payments occur at the start of each period, we have an *annuity due*, which will not be treated in the text; see Guthrie and Lemon [8] and Muksian [15] for an introduction.

An ordinary annuity is called *simple* if, at the end of each payment period, both a payment and the simple interest on the balance from the beginning of the payment period are applied. Note that the entire balance from the previous period is reinvested. Hence, for a simple ordinary annuity, the total accrued at the end of a payment period has the following form:

$$\begin{aligned}(\text{total accrued}) = (\text{payment}) + (\text{previous balance})\\ +(\text{simple interest on previous balance}).\end{aligned} \tag{2.53}$$

Here "previous balance" refers to the balance from the end of the previous payment period, which recall we treat mathematically the same as the start of the current period. Since the simple interest applied to a previous balance will yield interest on the principal and interest on the interest, we obtain compound interest naturally.

> *Unless stated to the contrary, assume that all loans are simple ordinary annuities.*

2.7.1 Future and Present Values of Simple Ordinary Annuities

Future Value of a Simple Ordinary Annuity

The *future value* of a simple ordinary annuity is the amount to which the sequence of payments of the annuity will grow, taking into account appreciation due to periodic compounding. We shall see that the annuity's future value is the sum of the end-of-term future values of the individual payments of the annuity.

Consider a simple ordinary annuity based on k-periodic compounding at interest rate r. This divides each year into k equal-length payment periods. Assume that each payment is the same amount $\mathcal{P}$ and the annuity has a term of n periods, where n is a positive integer. The total accrued at the end of the ith period will be denoted by $\mathcal{S}_i$. We shall apply (2.53) to obtain an expression for the total amount $\mathcal{S}_n$ accrued over the n periods:

- ➢ At the end of the first payment period, a payment $\mathcal{P}$ is made. Since there is no balance from the beginning of this period, the total accrued at the end of the first period is:

$$\mathcal{S}_1 = \mathcal{P}.$$

  Reinvest the entire amount $\mathcal{S}_1$ in the annuity.

➢ At the end of the second period, the payment is $\mathcal{P}$, the previous balance is $\mathcal{S}_1$, and the simple interest earned on the entire reinvested amount $\mathcal{S}_1$ is $(r/k)\mathcal{S}_1$. The total accrued at the end of the second period is then:

$$\mathcal{S}_2 = \mathcal{P} + \mathcal{S}_1 + \frac{r}{k}\mathcal{S}_1 = \mathcal{P} + \left(1 + \frac{r}{k}\right)\mathcal{P}.$$

Reinvest the entire amount $\mathcal{S}_2$ in the annuity.

➢ At the end of the 3rd period, the payment is $\mathcal{P}$, the previous balance is $\mathcal{S}_2$, and the simple interest earned on $\mathcal{S}_2$ is $(r/k)\mathcal{S}_2$. The total accrued is:

$$\mathcal{S}_3 = \mathcal{P} + \mathcal{S}_2 + \frac{r}{k}\mathcal{S}_2 = \mathcal{P} + \left(1 + \frac{r}{k}\right)\mathcal{P} + \left(1 + \frac{r}{k}\right)^2\mathcal{P}.$$

Reinvest $\mathcal{S}_3$.

➢ Continuing the above process, at the end of the nth period, the payment is $\mathcal{P}$, the previous balance is $\mathcal{S}_{n-1}$, and the simple interest earned on $\mathcal{S}_{n-1}$ is $(r/k)\mathcal{S}_{n-1}$. The total accrued at the end of the nth period is:

$$\mathcal{S}_n = \mathcal{P} + \mathcal{S}_{n-1} + \frac{r}{k}\mathcal{S}_{n-1}$$

or

$$\mathcal{S}_n = \mathcal{P} + \left(1 + \frac{r}{k}\right)\mathcal{P} + \left(1 + \frac{r}{k}\right)^2\mathcal{P} + \cdots + \left(1 + \frac{r}{k}\right)^{n-1}\mathcal{P}. \tag{2.54}$$

Equation (2.54) shows that *the future value of a simple ordinary annuity is the sum of each of the payments future valued to the end of the annuity.* To see this, in (2.54) the future values of these payments are shown from right to left. Explicitly, the 1st payment $\mathcal{P}$ is at the end of the first period, so its future value at the end of term (i.e., end of the nth period) is $(1 + r/k)^{n-1}\mathcal{P}$. The 2nd payment $\mathcal{P}$ is at the end of the second period, which has a future value at the end of the term of $(1 + r/k)^{n-2}\mathcal{P}$. The $(n-2)$nd payment $\mathcal{P}$ has an end-of-term future value of $(1 + r/k)^2\mathcal{P}$, and the $(n-1)$st payment $\mathcal{P}$ has $(1 + r/k)\mathcal{P}$. The nth payment $\mathcal{P}$ is at the end of the term so it equals its end-of-term future value. By (2.54), the sum of these future values is $\mathcal{S}_n$.

The right-hand side of (2.54) has a simpler expression. Applying the geometric sum,

$$a + ax + \cdots + ax^{m-1} = \left(\frac{1 - x^m}{1 - x}\right)a, \qquad (m \geq 1, \quad x \neq 1), \tag{2.55}$$

with $a = \mathcal{P}$, $x = 1 + r/k \neq 1$ (since $r > 0$), and $m = n \geq 1$, we obtain:

$$\mathcal{S}_n = \frac{\left[(1 + \frac{r}{k})^n - 1\right]}{r/k}\mathcal{P} \qquad (r > 0, \quad n \geq 1).$$

Remark 2.2. In actuarial science, the future value $\mathcal{S}_n$ is denoted by $s_{\overline{n}|}$ (pronounced "s angle n"). □

We summarize the result of the above analysis in the following theorem.

Theorem 2.4. *At the end of n periods, the future value of the simple ordinary annuity with payments $\mathcal{P}$ and k-periodic compounding at r per annum is:*

$$\mathcal{S}_n = \frac{\left[(1+\frac{r}{k})^n - 1\right]}{r/k}\mathcal{P} \qquad (r > 0, \quad n = 1,2,\dots). \tag{2.56}$$

Let us consider how $\mathcal{S}_n$ behaves as a function of $r \geq 0$. Intuitively, we expect that as the interest rate r increases, the total $\mathcal{S}_n$ accumulated after n periods should increase. Of course, this is not true for $n = 1$.[8] However, for $n = 2,3,\dots$, Equation (2.54) readily yields:

$$\frac{d\mathcal{S}_n}{dr} = \frac{\mathcal{P}}{k} + 2\left(1+\frac{r}{k}\right)\frac{\mathcal{P}}{k} + \cdots + (n-1)\left(1+\frac{r}{k}\right)^{n-2}\frac{\mathcal{P}}{k} > 0.$$

It follows that *for $n \geq 2$, the total amount $\mathcal{S}_n$ accrued over n periods increases as r increases.* Additionally, for $n = 2$ we have

$$\frac{d^2\mathcal{S}_2}{dr^2} = 0.$$

However, if $n = 3,4,\dots$, then

$$\frac{d^2\mathcal{S}_n}{dr^2} = 2\frac{\mathcal{P}}{k^2} + \cdots + (n-1)(n-2)\left(1+\frac{r}{k}\right)^{n-3}\frac{\mathcal{P}}{k^2} > 0.$$

Hence, *for $n \geq 3$, the total amount $\mathcal{S}_n$ accumulated over n periods accelerates*[9] *in value as the interest rate r increases.*

Present Value of a Simple Ordinary Annuity

The *present value*, denoted by $\mathcal{A}_n$, of a simple ordinary annuity is the amount needed today, taking interest into account, in order to be able to pay the amount $\mathcal{P}$ at the end of each period for a total of n periods. In particular, an interest per period of r/k is applied at the end of each period to the balance from the start of that period. The funds are, of course, exhausted by the end of the last period. For example, if $\mathcal{A}_n$ is a loan, then it would be paid off completely at the end of the nth period. The total payout would be $n\mathcal{P}$.

[8] If there is only one period, then $\mathcal{S}_1 = \mathcal{P}$ (constant) for all r since the principal is added only at the end of the first period, but the first interest payment occurs at the end of the second period.

[9] That is, $\mathcal{S}_n$ is concave up as a function of r (it has an increasing slope).

Let us determine a formula for $\mathcal{A}_n$. The first payment $\mathcal{P}$ will be made at the end of the first period. The present value of that future payment $\mathcal{P}$ is then $\dfrac{\mathcal{P}}{(1+r/k)}$. This is the amount needed at the start of the annuity's term in order for the amount to grow to $\mathcal{P}$ after one period. The present value of the second payment $\mathcal{P}$ is $\dfrac{\mathcal{P}}{(1+r/k)^2}$ since after two compoundings, i.e., at the end of the second period, it grows to $\mathcal{P}$. Consequently, at the start of the annuity, the individual present values of the n payments are given in sequential order as follows:

$$\frac{\mathcal{P}}{(1+\frac{r}{k})}, \quad \frac{\mathcal{P}}{(1+\frac{r}{k})^2}, \quad \dots, \quad \frac{\mathcal{P}}{(1+\frac{r}{k})^n}.$$

Then $\mathcal{A}_n$ is the sum of the present values of all payments:

$$\mathcal{A}_n = \frac{\mathcal{P}}{(1+\frac{r}{k})} + \frac{\mathcal{P}}{(1+\frac{r}{k})^2} + \cdots + \frac{\mathcal{P}}{(1+\frac{r}{k})^n}. \tag{2.57}$$

Equation (2.57) can be expressed as:

$$\begin{aligned}
\mathcal{A}_n &= \left(1+\frac{r}{k}\right)^{-1}\mathcal{P} + \left(1+\frac{r}{k}\right)^{-2}\mathcal{P} + \cdots + \left(1+\frac{r}{k}\right)^{-n}\mathcal{P} \\
&= \left(1+\frac{r}{k}\right)^{-1}\left[1+\left(1+\frac{r}{k}\right)^{-1} + \left(1+\frac{r}{k}\right)^{-2} + \cdots \right.\\
&\qquad \left. + \left(1+\frac{r}{k}\right)^{-(n-1)}\right]\mathcal{P} \\
&= \frac{(1+\frac{r}{k})^{-1}\left[1-(1+\frac{r}{k})^{-n}\right]\mathcal{P}}{1-(1+r/k)^{-1}}.
\end{aligned}$$

The last equality above follows from the geometric series (2.55) with $a = \mathcal{P}$ and $x = (1+r/k)^{-1}$ and $m = n$. Further simplification yields:

Theorem 2.5. *The present value of a simple ordinary annuity over n periods and with payments $\mathcal{P}$ and k-periodic compounding at r per annum is:*

$$\mathcal{A}_n = \frac{\left[1-(1+\frac{r}{k})^{-n}\right]}{\frac{r}{k}}\mathcal{P} \qquad (r>0, \quad n = 1,2,\dots). \tag{2.58}$$

Theorem 2.5 gives a formula for the amount needed today at interest rate r in order to be able to pay out the amount $\mathcal{P}$ each period for n periods.

Remark 2.3. The present value $\mathcal{A}_n$ is usually denoted by $a_{\overline{n}|}$ and the discount factor $(1+r/k)^{-1}$ by ν in actuarial science. □

The Total Number of Periods as a Function of the Payment per Period

We know intuitively that increasing the per-period payments of a loan will shorten the time it takes to pay off the loan. In fact, we can solve (2.58) for an exact formula relating the number n of periods to payoff in terms of the inputs $\mathcal{P}$, $\mathcal{A}_n$ (loan amount), r, and k:

$$n = -\frac{\ln\left(1 - \frac{(r/k)\,\mathcal{A}_n}{\mathcal{P}}\right)}{\ln\left(1 + \frac{r}{k}\right)}. \tag{2.59}$$

To understand how n varies with $\mathcal{P}$, we can fix $\mathcal{A}_n$, r, and k and treat n formally as a function of $\mathcal{P}$ given by (2.59). This treatment, of course, will lead to noninteger values of n, which we round off to find the approximate integer value. *For general values of $\mathcal{A}_n > 0$, $r > 0$, and k (nonnegative integer), as $\mathcal{P}$ increases, the total number of periods n strictly decreases and the rate of decrease slows down.* In other words, the quantity n as a function of $\mathcal{P}$ is convex, i.e., $n(\mathcal{P})$ is everywhere concave up. Explicitly, though the function $n(\mathcal{P})$ is a strictly decreasing function, it has an increasing slope:

$$\frac{dn}{d\mathcal{P}} = -\frac{(r/k)\,\mathcal{A}_n}{\left(1 - \frac{(r/k)\,\mathcal{A}_n}{\mathcal{P}}\right)\mathcal{P}^2 \ln\left(1 + \frac{r}{k}\right)} < 0 \qquad (r > 0)$$

$$\frac{d^2 n}{d\mathcal{P}^2} = \frac{\left(2 - \frac{(r/k)\,\mathcal{A}_n}{\mathcal{P}}\right)(r/k)\,\mathcal{A}_n}{\left(1 - \frac{(r/k)\mathcal{A}_n}{\mathcal{P}}\right)^2 \mathcal{P}^3 \ln\left(1 + \frac{r}{k}\right)} > 0.$$

Here we used $\ln\left(1 + \frac{r}{k}\right) > 0$ (since $r > 0$) and employed (2.58) to conclude that

$$1 - \frac{(r/k)\,\mathcal{A}_n}{\mathcal{P}} = \left(1 + \frac{r}{k}\right)^{-n} > 0.$$

Note that the quantity $(r/k)\mathcal{A}_n$ is the (simple) interest on the loan at the end of the first period. An example of n as a function of the per-period payment $\mathcal{P}$ is shown in Figure 2.1.

Present Value of a Perpetuity

A *perpetuity* is a sequence of cash flows that continues indefinitely. Though a perpetuity has no future value, a simple-ordinary-annuity perpetuity has a present value given by the following geometric series:

$$\mathcal{A}_\infty = \lim_{n\to\infty} \mathcal{A}_n = \lim_{n\to\infty} \frac{\left[1 - \left(1 + \frac{r}{k}\right)^{-n}\right]}{\frac{r}{k}}\mathcal{P} = \frac{\mathcal{P}}{\frac{r}{k}} \qquad (r > 0), \tag{2.60}$$

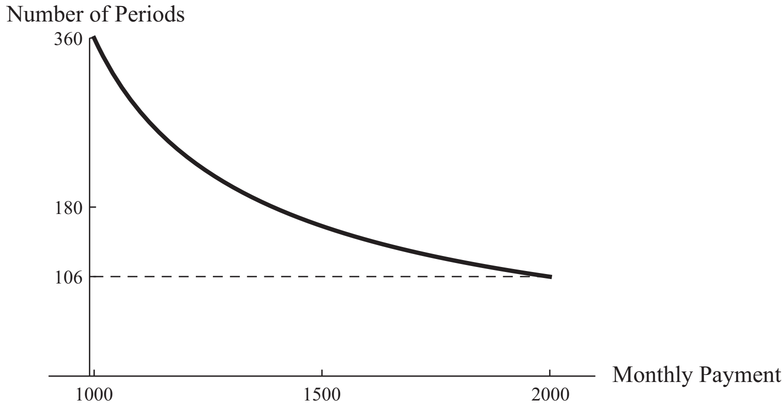

Fig. 2.1 The graph shows the total number of periods n as a function of monthly payments $\mathcal{P}$ for a loan of $\mathcal{A}_n = \$162{,}412$ at 6.25% per annum compounded monthly ($k = 12$). The loan is paid off in about 30 years (or 360 months) if the monthly payment is \$1,000. Doubling the payments yields a payoff time of 8 years and 10 months (or 106 months), which is much less than half of the time for a \$1,000 monthly payment.

which is an immediate consequence of (2.58). See Section 2.9.1 on page 64 for a *growing perpetuity*, i.e., one where the payments $\mathcal{P}$ increase at a certain rate.

Relating Future and Present Values of a Simple Ordinary Annuity

If you put aside the amount $\mathcal{A}_n$ today and have it grow by k-periodic compounding with interest rate r, then after n periods, the initial amount will grow to $\mathcal{S}_n$. In other words, the initial amount $\mathcal{A}_n$ is the present value of the future amount $\mathcal{S}_n$ under periodic compounding. To see this, note that by (2.56) and (2.58), we obtain:

$$\begin{aligned}\frac{\mathcal{S}_n}{\left(1+\frac{r}{k}\right)^n} &= \frac{\left[\left(1+\frac{r}{k}\right)^n - 1\right]\mathcal{P}}{\frac{r}{k}}\left(1+\frac{r}{k}\right)^{-n} \\ &= \frac{\left[1-\left(1+\frac{r}{k}\right)^{-n}\right]}{\frac{r}{k}}\mathcal{P} \\ &= \mathcal{A}_n, \end{aligned} \tag{2.61}$$

where $r > 0$. Equation (2.61) shows that an equivalent way of determining the future value of a simple ordinary annuity is to take the present value of the sequence of payments and then take the future value of that present value.

2.7.2 Amortization Theory

Amortization is the reducing of a given loan amount (the principal) through a series of payments over a fixed time span whereby one accounts explicitly for the portion of each payment that goes toward the principal and the portion toward the interest owed on the loan. The most common amortization is through a *mortgage*, which is a loan where the borrower (mortgagor) gives the lender (mortgagee) a lien on property as security for the repayment of the loan. The mortgagor has use of the property and the lien is removed when the obligation is fully paid. A mortgage usually involves real estate.[10]

What happens if you are amortizing a debt with equal periodic payments and at some point decide to pay off the remainder of the debt in one lump-sum payment? This occurs each time a house with an outstanding mortgage is sold. How much of each periodic payment is used for interest and how much is used to reduce the unpaid balance of the principal? This issue is also important because the interest part of the payment may be tax deductible (as it is in the USA). In order to answer these questions, we must take a close look at the mathematical structure of amortization.

A loan is paid off with interest through the full sequence of its stipulated minimal payments. We then model the amount of a loan by the present value of the entire sequence of its required future payments. Specifically, we model the loan amount by the present value of a simple ordinary annuity. The initial amount of the loan is $\mathcal{A}_n$ (principal balance), the payment at the end of each period is $\mathcal{P}$, and the loan is for n periods. Each period is $(1/k)$th of a year and the annual interest rate is r, i.e., the interest applied at the end of each period is r/k.

Unpaid Principal Balances

We determine the unpaid balance on the principal at the end of each period of the loan.

For notational simplicity, define

$$\mathrm{y} \equiv 1 + \frac{r}{k}, \qquad (r > 0).$$

Then by (2.58), each end-of-period payment can be expressed as:

$$\mathcal{P} = \frac{(\mathrm{y}-1)}{1-\mathrm{y}^{-n}}\mathcal{A}_n = \frac{\mathrm{y}^n(\mathrm{y}-1)}{\mathrm{y}^n - 1}\mathcal{A}_n. \tag{2.62}$$

[10] While a typical mortgage is a loan used to buy a fixed asset like a house or land, which also secures the loan, a mortgage used to buy movable property such as a mobile home or operational equipment that acts as security for the loan is called a *chattel mortgage* or *secured transaction*.

Denote the initial amount of the loan by:

$$\mathcal{B}_0 = \mathcal{A}_n.$$

Then the unpaid principal balances at the end of the different periods are given as follows:

- ➢ At the end of the first period, an interest $(r/k)\mathcal{A}_n$ is added to the starting balance $\mathcal{A}_n$ and a payout/withdrawal of $\mathcal{P}$ is made. The unpaid principal balance at the end of the first period is:

$$\mathcal{B}_1 = \mathcal{A}_n + \frac{r}{k}\mathcal{A}_n - \mathcal{P} = \mathrm{y}\mathcal{A}_n - \mathcal{P}.$$

- ➢ At the end of the second period, an interest $(r/k)\mathcal{B}_1$ is added to the balance $\mathcal{B}_1$ from the start of the second period and then a payout/withdrawal of $\mathcal{P}$ is made. The unpaid principal balance at the end of the second period is:

$$\mathcal{B}_2 = \mathcal{B}_1 + \frac{r}{k}\mathcal{B}_1 - \mathcal{P} = \mathrm{y}\mathcal{B}_1 - \mathcal{P} = \mathrm{y}(\mathrm{y}\mathcal{A}_n - \mathcal{P}) - \mathcal{P} = \mathrm{y}^2\mathcal{A}_n - (1+\mathrm{y})\mathcal{P}.$$

- ➢ At the end of the 3rd period, an interest $(r/k)\mathcal{B}_2$ is added to the balance $\mathcal{B}_2$ from the start of the 3rd period and then a payout/withdrawal of $\mathcal{P}$ is made. The unpaid principal balance at the end of the 3rd period is

$$\begin{aligned}\mathcal{B}_3 &= \mathcal{B}_2 + \frac{r}{k}\mathcal{B}_2 - \mathcal{P} = \mathrm{y}\mathcal{B}_2 - \mathcal{P}\\ &= \mathrm{y}\left[\mathrm{y}^2\mathcal{A}_n - (1+\mathrm{y})\mathcal{P}\right] - \mathcal{P}\\ &= \mathrm{y}^3\mathcal{A}_n - (1+\mathrm{y}+\mathrm{y}^2)\mathcal{P}.\end{aligned}$$

- ➢ Continuing the above process, at the end of the ℓth period, an interest $(r/k)\mathcal{B}_{\ell-1}$ is added to the balance $\mathcal{B}_{\ell-1}$ from the start of the ℓth period and then a payout/withdrawal of $\mathcal{P}$ is made. The unpaid principal balance at the end of the ℓth period is:

$$\begin{aligned}\mathcal{B}_\ell &= \mathcal{B}_{\ell-1} + \frac{r}{k}\mathcal{B}_{\ell-1} - \mathcal{P}\\ &= \mathrm{y}^\ell\mathcal{A}_n - (1+\mathrm{y}+\mathrm{y}^2+\cdots+\mathrm{y}^{\ell-1})\mathcal{P}\\ &= \mathrm{y}^\ell\mathcal{A}_n - \frac{1-\mathrm{y}^\ell}{1-\mathrm{y}}\mathcal{P}\\ &= \mathrm{y}^\ell\mathcal{A}_n + \frac{(1-\mathrm{y}^\ell)}{(\mathrm{y}-1)}\frac{\mathrm{y}^n(\mathrm{y}-1)}{(\mathrm{y}^n-1)}\mathcal{A}_n, \qquad [by\ (2.62)]\\ &= \frac{\mathrm{y}^\ell(\mathrm{y}^n-1)+\mathrm{y}^n(1-\mathrm{y}^\ell)}{\mathrm{y}^n-1}\mathcal{A}_n\\ &= \frac{\mathrm{y}^n-\mathrm{y}^\ell}{\mathrm{y}^n-1}\mathcal{A}_n \qquad (\ell = 1,2,\ldots,n),\end{aligned}$$

where $\mathcal{B}_0 = \mathcal{A}_n$. Hence:

Theorem 2.6. *The unpaid principal balance at the end of the ℓth period is given in terms of $\mathcal{A}_n$, $r > 0$, and k by:*

$$\mathcal{B}_\ell = \frac{\left(1+\frac{r}{k}\right)^n - \left(1+\frac{r}{k}\right)^\ell}{\left(1+\frac{r}{k}\right)^n - 1}\mathcal{A}_n, \tag{2.63}$$

where $n = 1,2,\ldots$ and $\ell = 0,1,2,\ldots,n$.

The unpaid principal balance at the end of the loan's term is $\mathcal{B}_n = 0$.

Amount of Per-Period Payment Toward Interest and Unpaid Balance

At the end of each period, a portion of the payment $\mathcal{P}$ is used toward interest on the loan, the other portion toward reduction of the loan's unpaid principal balance.

Notation. Let:

$\mathcal{I}_\ell =$ the portion of the payment $\mathcal{P}$ at the end of ℓth period that is applied toward interest on the loan (i.e., the interest payment at the end of period ℓ).

$\mathfrak{P}_\ell =$ the portion of payment $\mathcal{P}$ at the end of the ℓth period that is applied toward the unpaid principal balance of the loan.

We now express $\mathcal{I}_\ell$ and $\mathfrak{P}_\ell$ in terms of $\mathcal{A}_n$ and r. The interest payment at the end of period ℓ is

$$\mathcal{I}_\ell = \left(\frac{r}{k}\right)\mathcal{B}_{\ell-1} = \left(\frac{r}{k}\right)\frac{\left[\left(1+\frac{r}{k}\right)^n - \left(1+\frac{r}{k}\right)^{\ell-1}\right]}{\left(1+\frac{r}{k}\right)^n - 1}\mathcal{A}_n, \tag{2.64}$$

where $r > 0$, $n = 1,2,\ldots$ and $\ell = 1,2,\ldots,n$.

For the payment $\mathfrak{P}_\ell$ toward the principal, Equations (2.62) and (2.64) yield:

$$\begin{aligned}
\mathfrak{P}_\ell &= \mathcal{P} - \mathcal{I}_\ell \\
&= \frac{(y-1)\,y^n}{y^n-1}\mathcal{A}_n - \frac{(y-1)\left[y^n - y^{\ell-1}\right]}{y^n-1}\mathcal{A}_n \\
&= \frac{(y-1)y^{\ell-1}}{y^n-1}\mathcal{A}_n \\
&= \left(\frac{r}{k}\right)\frac{\left(1+\frac{r}{k}\right)^{\ell-1}}{\left(1+\frac{r}{k}\right)^n - 1}\mathcal{A}_n.
\end{aligned} \tag{2.65}$$

As a check, we show that the payments, $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n$, toward the principal add up to the total loan amount $\mathcal{A}_n$:

$$\sum_{\ell=1}^{n} \mathfrak{P}_\ell = \sum_{\ell=1}^{n} \frac{(y-1)y^{\ell-1}}{y^n-1}\mathcal{A}_n = \mathcal{A}_n \frac{y-1}{y^n-1} \sum_{\ell=1}^{n} y^{\ell-1}$$
$$= \mathcal{A}_n \frac{y-1}{y^n-1} \sum_{\ell=0}^{n-1} y^{\ell} = \mathcal{A}_n \frac{y-1}{y^n-1} \frac{y^n-1}{y-1}$$
$$= \mathcal{A}_n.$$

Therefore:

Theorem 2.7. *The total interest paid during a loan with n periods and k-periodic compounding at interest rate r is then:*

$$\sum_{\ell=1}^{n} \mathcal{I}_\ell = \sum_{\ell=1}^{n} (\mathcal{P} - \mathfrak{P}_\ell) = n\mathcal{P} - \mathcal{A}_n, \tag{2.66}$$

where $n = 1, 2, \ldots$.

Note that $n\mathcal{P}$ is the total amount paid into the loan over the life of the loan and $n\mathcal{P} - \mathcal{A}_n$ is the total cost of the loan.

Remark 2.4. If you receive a loan today for the amount $\mathcal{A}_n$ at fixed interest rate r, fixed payment $\mathcal{P}$ per period, and a term of n periods, then the sum $n\mathcal{P}$ of all your future payments adds money at different future times without present or future valuing them. In fact, the present value of all the future payments is the loan amount $\mathcal{A}_n$ and the future value is $\mathcal{S}_n$, neither of which is $n\mathcal{P}$. The meaning of $n\mathcal{P}$ is the amount you would, in principle, have to pay the lender today if immediately after receiving the loan you want to pay the loan off, but the lender penalizes you by requiring you to pay the principal $\mathcal{A}_n$ plus the total interest for the full term of the loan. Of course, this is merely theoretical since the majority of loans would not have such a drastic penalty. □

2.7.3 Annuities with Varying Payments and Interest Rates

Applying essentially the same arguments used to establish the future value annuity Equation (2.54), we can generalize to a simple ordinary annuity with a sequence of varying payments, $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, and respective varying interest rates, $r_1, r_2, \ldots, r_n$, over n interest periods that coincide with the payment periods. We assume k-periodic compounding. *The payment $\mathcal{P}_\ell$ occurs at the end of the ℓth period, and the interest r_ℓ is applied at the end of the ℓth period to the balance from the start of the ℓth interest period, where* $\ell = 1, \ldots, n$. Assume that there is no balance at the start of the first period.

Future Value of a Generalized Simple Ordinary Annuity

The pattern for the future value of a simple ordinary annuity generalized to varying payments and varying interest rates emerges as follows:

- At the end of the first payment period, a payment $\mathcal{P}_1$ is made. Because there is no balance from the beginning of this period, the total accrued at the end of the first period is:
$$\mathcal{S}_1 = \mathcal{P}_1.$$
Reinvest $\mathcal{S}_1$ in the annuity.
- At the end of the second period, the payment is $\mathcal{P}_2$, the previous balance is $\mathcal{S}_1$, and the simple interest earned on $\mathcal{S}_1$ is $(r_2/k)\mathcal{S}_1$. The total accrued is:
$$\mathcal{S}_2 = \mathcal{P}_2 + \mathcal{S}_1 + \frac{r_2}{k}\mathcal{S}_1 = \mathcal{P}_2 + \left(1 + \frac{r_2}{k}\right)\mathcal{P}_1.$$
Reinvest $\mathcal{S}_2$ in the annuity.
- At the end of the 3rd period, the payment is $\mathcal{P}_3$, the previous balance is $\mathcal{S}_2$, and the simple interest earned on $\mathcal{S}_2$ is $(r_3/k)\mathcal{S}_2$. The total accrued is:
$$\mathcal{S}_3 = \mathcal{P}_3 + \mathcal{S}_2 + \frac{r_3}{k}\mathcal{S}_2 = \mathcal{P}_3 + \left(1 + \frac{r_3}{k}\right)\mathcal{P}_2 + \left(1 + \frac{r_3}{k}\right)\left(1 + \frac{r_2}{k}\right)\mathcal{P}_1.$$
Reinvest $\mathcal{S}_3$ in the annuity.
- Continuing the above process, at the end of the nth period, the payment is $\mathcal{P}_n$, the previous balance is $\mathcal{S}_{n-1}$, and the simple interest earned on $\mathcal{S}_{n-1}$ is $(r_n/k)\mathcal{S}_{n-1}$. The total accrued is
$$\mathcal{S}_n = \mathcal{P}_n + \mathcal{S}_{n-1} + \frac{r_n}{k}\mathcal{S}_{n-1}$$
or
$$\begin{aligned}\mathcal{S}_n = \mathcal{P}_n &+ \left(1 + \frac{r_n}{k}\right)\mathcal{P}_{n-1} + \left(1 + \frac{r_n}{k}\right)\left(1 + \frac{r_{n-1}}{k}\right)\mathcal{P}_{n-2} + \\ &\cdots + \left(1 + \frac{r_n}{k}\right)\left(1 + \frac{r_{n-1}}{k}\right)\cdots\left(1 + \frac{r_2}{k}\right)\mathcal{P}_1.\end{aligned} \tag{2.67}$$

Observe that, by letting $r_{n+1} = 0$ and rewriting (2.67) as

$$\begin{aligned}\mathcal{S}_n = &\left(1 + \frac{r_{n+1}}{k}\right)\mathcal{P}_n + \left(1 + \frac{r_{n+1}}{k}\right)\left(1 + \frac{r_n}{k}\right)\mathcal{P}_{n-1} \\ &+ \left(1 + \frac{r_{n+1}}{k}\right)\left(1 + \frac{r_n}{k}\right)\left(1 + \frac{r_{n-1}}{k}\right)\mathcal{P}_{n-2} \\ &\cdots + \left(1 + \frac{r_{n+1}}{k}\right)\left(1 + \frac{r_n}{k}\right)\left(1 + \frac{r_{n-1}}{k}\right)\cdots\left(1 + \frac{r_2}{k}\right)\mathcal{P}_1,\end{aligned}$$

we see that (2.67) can be expressed more compactly as follows:

Theorem 2.8. *The future value at the end of n payment periods, which coincide with the interest periods, of the simple ordinary annuity with payments* $\mathcal{P}_1, \dots, \mathcal{P}_n$ *and k-periodic compounding at respective interest rates* $r_2, \dots, r_n$ *during the consecutive interest periods is:*

$$\mathcal{S}_n = \sum_{\ell=0}^{n-1} \left[\prod_{j=0}^{\ell} \left(1 + \frac{r_{n+1-j}}{k} \right) \right] \mathcal{P}_{n-\ell}, \qquad (n = 1, 2, \dots), \tag{2.68}$$

where $r_i > 0$ *for* $i = 2, \dots, n$ *and* $r_{n+1} = 0$.

When $r_i = r$ for $i = 2, \dots, n$, and $\mathcal{P}_i = \mathcal{P}$ for $i = 1, \dots, n$, Equation (2.68) recovers (2.54) on page 48.

An application of (2.68) to sinking funds is given in Section 2.8.3.

Present Value of a Generalized Simple Ordinary Annuity

Similarly, the present value Equation (2.57) generalizes naturally to the case of a sequence of payments, $\mathcal{P}_1, \dots, \mathcal{P}_n$, and interest rates $r_1, \dots, r_n$. Here the amount $\mathcal{P}_i$ is paid at the end of the ith period, and the interest r_i is applied at the end of the ith period to the balance from the end of the $(i-1)$st period.

When simple interest at rate r_1 is applied at the end of the first period to the initial amount $\mathcal{P}_1 (1 + r_1/k)^{-1}$, we obtain the first payment $\mathcal{P}_1$. Applying compound interest with rates r_1 and r_2 at the end of the first and second periods, respectively, to the initial amount $\mathcal{P}_2 (1 + r_1/k)^{-1}(1 + r_2/k)^{-1}$ yields the second payment $\mathcal{P}_2$. Continuing this process gives the initial amount that will grow to the nth payment $\mathcal{P}_n$. These initial amounts are the present values of the sequence of payments under compound interest at different rates. Summing all the present values gives the following *present value* for the generalized annuity:

$$\mathcal{A}_n = \frac{\mathcal{P}_1}{\left(1 + \frac{r_1}{k}\right)} + \frac{\mathcal{P}_2}{\left(1 + \frac{r_1}{k}\right)\left(1 + \frac{r_2}{k}\right)} + \cdots + \frac{\mathcal{P}_n}{\left(1 + \frac{r_1}{k}\right)\left(1 + \frac{r_2}{k}\right) \cdots \left(1 + \frac{r_n}{k}\right)}, \tag{2.69}$$

or, more compactly,

$$\mathcal{A}_n = \sum_{\ell=1}^{n} \left[\frac{\mathcal{P}_\ell}{\prod_{j=1}^{\ell} \left(1 + \frac{r_j}{k}\right)} \right], \tag{2.70}$$

where $n = 1, 2, \dots$ and $r_i > 0$ for $i = 1, \dots, n$. In the special case where $r_i = r$ and $\mathcal{P}_i = \mathcal{P}$ for $i = 1, \dots, n$, Equation (2.70) yields (2.57) on page 50.

Applications of (2.69) to the dividend discount model and bond pricing are given, respectively, in Section 2.9.1 and 2.10 (see page 68).

Relating Future and Present Values of a Generalized Simple Ordinary Annuity

We now show that $\mathcal{S}_n$ in (2.67) is the generalized future value of $\mathcal{A}_n$ in (2.69) under the generalized periodic compounding in (2.31) (see page 33). Using Equation (2.67), direct calculation shows that

$$\begin{aligned}
&\frac{\mathcal{S}_n}{\left(1+\frac{r_1}{k}\right)\left(1+\frac{r_2}{k}\right)\cdots\left(1+\frac{r_{n-1}}{k}\right)\left(1+\frac{r_n}{k}\right)} \\
&= \frac{\mathcal{P}_n}{\left(1+\frac{r_1}{k}\right)\left(1+\frac{r_2}{k}\right)\cdots\left(1+\frac{r_n}{k}\right)} + \cdots + \frac{\mathcal{P}_2}{\left(1+\frac{r_1}{k}\right)\left(1+\frac{r_2}{k}\right)} + \frac{\mathcal{P}_1}{\left(1+\frac{r_1}{k}\right)} \\
&= \mathcal{A}_n.
\end{aligned}$$

It immediately follows that the relationship between the future and present values in Equation (2.61) generalizes to

$$\mathcal{A}_n = \frac{\mathcal{S}_n}{\prod_{j=1}^{n}\left(1+\frac{r_j}{k}\right)}, \tag{2.71}$$

where $n = 1, 2, \ldots$ and $r_i > 0$ for $i = 1, \ldots, n$.

2.8 Applications of Annuities

2.8.1 Saving, Borrowing, and Spending

Example 2.10. (Saving During College) A prospective college student plans to deposit \$25 every month in an "untouchable" savings account, starting the first of July of the year she enters college until the last deposit on the thirtieth of June of her graduating year. Assume that she secured a fixed interest rate of 2.25% annually. Assume that the account compounds monthly.

a) Using this average interest rate, estimate how much she would have on July 1st of her graduating year.

Solution. Use the future value $\mathcal{S}_n$ in (2.56). We have $k = 12$ for monthly compounding and since the period is 4 years, we have $n = 4 \times 12 = 48$ periods, $r = 0.0225$, and $\mathcal{P} = \$25$. By (2.56),

$$\mathcal{S}_{48} = \frac{\left[\left(1 + \frac{0.0225}{12}\right)^{48} - 1\right]}{\frac{0.0225}{12}} \times \$25 = \$1{,}254.43.$$

b) If her target is to have at least $1,300 on July 1st of her graduating year, determine the minimum required interest rate.

Solution. Solving the equation,

$$\$1{,}300 = \mathcal{S}_{48} = \frac{\left[\left(1 + \frac{r}{12}\right)^{48} - 1\right]}{\frac{r}{12}} \times \$25,$$

implicitly for r (use a software package), we obtain the smallest interest rate to be $r = 4.04\%$. Note that this is the smallest value of r that works since $\mathcal{S}_n$ is a strictly increasing function of r for natural numbers $n \geq 2$ (see page 49).

□

Example 2.11. (Saving for Retirement) Suppose that you open a retirement fund at the start of a month and you deposit $200 at the end of each month. If the fund pays 4% per annum compounded monthly, how much would you accumulate at the end of 25 years?

Solution. This problem deals with the future value $\mathcal{S}_n$ in (2.56). For monthly compounding ($k = 12$), we have $n = 25 \times 12 = 300$ periods, $r = 0.04$, and $\mathcal{P} = \$200$. Equation (2.56) then yields the following future value:

$$\mathcal{S}_{300} = \frac{\left[\left(1 + \frac{0.04}{12}\right)^{300} - 1\right]}{\frac{0.04}{12}} \times \$200 = 514.13 \times \$200 = \$102{,}826.$$

□

Example 2.12. (Total Paid on Loan) A relative is considering a 20-year loan of $150,000 with an interest rate of 8% compounded monthly. Assuming you hold the loan the entire term and make the minimum payment at the end of each month, what is the total amount you pay into the loan?

Solution. We have $\mathcal{A}_n = \$150{,}000$, $k = 12$, $r = 0.08$, and $n = 20 \times 12 = 240$ periods, so by the present value annuity formula, we obtain the minimum monthly payment:

$$\mathcal{P} = \frac{(r/k)\,\mathcal{A}_n}{1 - \left(1 + \frac{r}{k}\right)^{-n}} = \$1{,}254.66.$$

Since there are 240 months, the total paid is: $240 \times \$1{,}254.66 = \$301{,}118.40$.

□

Example 2.13. (Paying Off Debt) Suppose that you borrow \$100,000 at an annual interest rate of 6% with monthly compounding. For an ordinary annuity based on this compounding, what is your minimum payment per month to pay off the loan in 10 years?

Solution. The problem requires the present value $\mathcal{A}_n$. Since $k = 12$, there are $10 \times 12 = 120$ periods. Using $\mathcal{A}_{120} = \$100{,}000$ and $\frac{r}{k} = \frac{0.06}{12} = 0.005$, we get:

$$\mathcal{P} = \frac{(0.005) \times \$100{,}000}{1 - (1.005)^{-120}} = \$1{,}110.21.$$

□

Example 2.14. (How Much Loan Can You Afford) Suppose that you can pay \$1,495 per month for the next 15 years. What is the largest loan you can afford at 6.25% per annum with monthly compounding?

Solution. Assume that the first payment is made 1 month from now. We have $n = 15 \times 12 = 180$ periods (months), $\mathcal{P} = \$1{,}495$, and $r = 0.0625$. The maximum loan you can afford is:

$$\mathcal{A}_n = \frac{\left[1 - \left(1 + \frac{0.0625}{12}\right)^{-180}\right]}{\frac{0.0625}{12}} \times \$1{,}495 = \$174{,}359.71.$$

□

Example 2.15. (Living Off a Lump Sum) Suppose that you inherited \$300,000 and invested it in an account with an annual interest rate of 7% compounded monthly. For an ordinary annuity based on this compounding, if you want your inheritance to last 20 years, what is the maximum fixed amount you can spend from the account per month?

Solution. Using the present value annuity formula with $\mathcal{A}_n = \$300{,}000, k = 12$, $r = 0.07$, $n = 20 \times 12 = 240$ periods, we obtain:

$$\mathcal{P} = \frac{(r/k)\,\mathcal{A}_n}{1 - \left(1 + \frac{r}{k}\right)^{-n}} = \frac{(0.07/12) \times \$300{,}000}{1 - (1 + \frac{0.07}{12})^{-240}} = \$2{,}325.90.$$

□

2.8.2 Equity in a House

Example 2.16. (House Equity) A couple bought their house 11 years ago for \$225,000 and put down 10% on the house. On the balance, they took out a

15-year mortgage at 5.75% per annum with monthly compounding. The current *net market value* of the house is its current market value minus all costs in selling the house today. Suppose that the current net market value is now \$350,000 and the couple wants to sell their house.

a) How much equity (to the nearest dollar) is in the house today? *Equity* in a house is defined as:

$$\text{equity} = (\text{current net market value}) - (\text{unpaid loan balance}).$$

Solution. The couple puts down 10% or \$22,500 at the start, so the mortgage is for $\mathcal{A}_n = \$225{,}000 - \$22{,}500 = \$202{,}500$. Since $n = 15 \times 12 = 180$, $r = 0.0575$, $k = 12$, and $\ell = 132$, Equation (2.63) yields the unpaid balance at the end of the 132nd month:

$$\mathcal{B}_{132} = \frac{\left(1+\frac{r}{k}\right)^n - \left(1+\frac{r}{k}\right)^\ell}{\left(1+\frac{r}{k}\right)^n - 1}\mathcal{A}_n = \$71{,}952.87.$$

Hence, the equity is: $\$350{,}000 - \mathcal{B}_{132} = \$278{,}047.13$.

b) What are the 1st and 132nd interest payments?

Solution. We have $\mathcal{I}_1 = \frac{r}{k}\mathcal{B}_0 = \$970.31$ and $\mathcal{I}_{132} = \frac{r}{k}\mathcal{B}_{131} = \$351.15$.

□

2.8.3 *Sinking Funds*

A *sinking fund* is an account into which one (individual or company) regularly deposits money in order to cover an obligation or debt that will come due at a known future date.

Example 2.17. (Saving for College Tuition) When a child was born in 2011, her parents decided to invest in her college education. This was motivated by a forecast that 4 years of in-state tuition at an average public college will be about \$96,000 when she will attend college. Suppose that the parents want to accumulate that amount by their child's 17th birthday. They open a sinking fund into which they make a deposit on each birthday of the child up to the 17th birthday. Assume that the first deposit is for the amount $\mathcal{P}$ and thereafter the parents increase the deposited amount by 4% annually. Suppose that the bank where they have the sinking fund pays a fixed 5.5% per annum compounded annually. What should the minimum annual deposits be in order for the amount in the fund to reach at least \$96,000 after her 17th deposit?

Solution. This example applies generalized future value annuity formula in (2.68), i.e.,

$$\mathcal{S}_n = \sum_{\ell=0}^{n-1} \left[\prod_{j=0}^{\ell} \left(1 + \frac{r_{n+1-j}}{k}\right) \right] \mathcal{P}_{n-\ell}, \qquad n = 1,2,3\dots,$$

where $r_{n+1} = 0$. Note that r_1 does not appear in the formula since no interest is paid at the end of the first period (because the first deposit is not made at the start of the first period, but at the end of the first period).

We have $n = 17$, $\mathcal{S}_{17} = \$96{,}000$, $r_2 = \cdots = r_{17} = 0.055$, and $k = 1$. The product in the sum above then becomes:

$$\prod_{j=0}^{\ell} \left(1 + \frac{r_{n+1-j}}{k}\right) = \left(1 + \frac{r_n}{k}\right)\left(1 + \frac{r_{n-1}}{k}\right) \cdots \left(1 + \frac{r_{n-(\ell-1)}}{k}\right) = (1.055)^{\ell},$$

where $\ell = 0,1,2,\dots,n-1$.

Let us determine the deposits $\mathcal{P}_1,\dots,\mathcal{P}_n$. The deposit $\mathcal{P}_1 = \mathcal{P}$ is made on the first birthday. On the second birthday, it is increased by 4% to $\mathcal{P}_2 = \mathcal{P}_1 + 0.04\mathcal{P}_1 = (1.04)\mathcal{P}$. On the 3rd birthday, the deposit is $\mathcal{P}_3 = \mathcal{P}_2 + 0.04\mathcal{P}_2 = (1.04)^2\mathcal{P}$. For $j = 1,\dots,n$, the deposit on the jth birthday is then $\mathcal{P}_j = (1.04)^{j-1}\mathcal{P}$. It follows: $\mathcal{P}_{n-\ell} = (1.04)^{16-\ell}\mathcal{P}$.

The target amount for the sinking fund can then be expressed as:

$$\begin{aligned} \$96{,}000 &= \mathcal{P} \times \sum_{\ell=0}^{16} (1.055)^{\ell} (1.04)^{16-\ell} = \mathcal{P} \times (1.04)^{16} \times \sum_{\ell=0}^{16} \left(\frac{1.055}{1.04}\right)^{\ell} \\ &= \mathcal{P} \times 1.87298 \times 19.1104 \;=\; 35.7934\,\mathcal{P}. \end{aligned}$$

This yields $\mathcal{P} = \$2{,}682.06$, which is the first deposit. Hence, for $j = 1,\dots,17$, the minimum deposit on the jth birthday must be: $\mathcal{P}_j = (1.04)^{j-1} \times \$2{,}682.06$, which has values

$$\mathcal{P}_1 = \$2{,}682.06, \quad \mathcal{P}_2 = \$2{,}789.34, \quad \dots, \quad \mathcal{P}_{16} = \$4{,}830.24, \quad \mathcal{P}_{17} = \$5{,}023.45.$$

□

2.9 Applications to Stock Valuation

This section applies the theory of annuities to determining the present values of preferred and common stocks. The main tool is the dividend discount model. A stochastic model for the future value of a stock will be taken up in a later chapter.

2.9.1 The Dividend Discount Model

The *dividend discount model* (DDM) was pioneered by Williams [18] (1938) and Gordon [7] (1959). *The fundamental hypothesis of the DDM is that, if a stock is held for n years, then its current value is the present value of the sequence of its expected future cash dividends through n years plus the present value of the stock's expected price in n years*.

A stock has no maturity date and so is a security in perpetuity. Suppose that the stock pays a dividend and the (annual) required return rate of the stock is $\Bbbk$.[11] Assume that you will hold the stock for n years. Let $\mathcal{D}_0$ be the current cash dividend, i.e., the total cash dividend per share over the previous year. Suppose that all future cash dividends are expected to grow at a constant annual rate g, which we assume is less than the required return rate ($\Bbbk > g$). Let $\mathcal{D}(i)$ denote the expected cash dividend per share for the interval from the present time to i years out, where $i = 1,\ldots,n$. Then the expected sequence of future cash dividends per share for years 1 through n is:

$$\mathcal{D}(1) = (1+g)\,\mathcal{D}_0, \qquad \mathcal{D}(2) = (1+g)^2\,\mathcal{D}_0, \qquad \ldots, \qquad \mathcal{D}(n) = (1+g)^n\,\mathcal{D}_0.$$

The share value $S_0^{(n)}$ of the stock today is the present value of the expected dividend cash flows and the expected price of the stock n years from now:

$$S_0^{(n)} = \frac{\mathcal{D}(1)}{1+\Bbbk} + \frac{\mathcal{D}(2)}{(1+\Bbbk)^2} + \cdots + \frac{\mathcal{D}(n)}{(1+\Bbbk)^n} + \frac{\mathcal{T}_n}{(1+\Bbbk)^n}, \qquad (\Bbbk > 0), \quad (2.72)$$

where $\mathcal{T}_n$ is the *terminal price*, i.e., the expected price of the stock in n years. Note that (2.72) is a special case of the generalized present value equation (2.69) (page 58) with $k = 1,\ r_i = \Bbbk,\ \mathcal{P}_i = \mathcal{D}(i)$ for $i = 1,\ldots,n-1$, and $\mathcal{P}_n = \mathcal{D}(n) + \mathcal{T}_n$.

Now, if you hold the stock in perpetuity (indefinitely) rather than for n years, then there is no terminal price, and the stock's present share price becomes:

$$S_0 = \lim_{n\to\infty} S_0^{(n)} = \mathcal{D}_0 \sum_{\ell=1}^{\infty} \left(\frac{1+g}{1+\Bbbk}\right)^{\ell}, \qquad (\Bbbk > g > 0).$$

Since $\frac{1+g}{1+\Bbbk} < 1$ due to $\Bbbk > g$, the geometric series yields

$$\frac{1}{1-\left(\frac{1+g}{1+\Bbbk}\right)} = \sum_{\ell=0}^{\infty} \left(\frac{1+g}{1+\Bbbk}\right)^{\ell} = 1 + \sum_{\ell=1}^{\infty} \left(\frac{1+g}{1+\Bbbk}\right)^{\ell}.$$

Hence, the present share price of the stock becomes:

[11] Recall that the marketplace is assumed to be in equilibrium, which allows for the required return rate of the stock to be estimated using the CAPM model; see Chapter 4 for an introduction.

$$S_0 = \frac{(1+g)\,\mathcal{D}_0}{\Bbbk - g} = \frac{\mathcal{D}(1)}{\Bbbk - g}, \qquad (\Bbbk > g > 0). \tag{2.73}$$

Equation (2.73) is called the *Gordon growth model*. This is an example of a *growing perpetuity*, i.e., a perpetuity with payments that increase each period.

The Gordon growth model generalizes naturally to allow for k compoundings per year through the replacements $g \to g/k$ and $\Bbbk \to \Bbbk/k$:

$$S_0 = \frac{\left(1+\frac{g}{k}\right)\mathcal{D}_0}{\frac{\Bbbk}{k} - \frac{g}{k}} = \frac{\mathcal{D}(1)}{\frac{\Bbbk}{k} - \frac{g}{k}} \qquad (\Bbbk > g > 0),$$

where $\mathcal{D}(j) = \left(1+\frac{g}{k}\right)^j \mathcal{D}_0$.

2.9.2 Present Value of Preferred and Common Stocks

A preferred stock grants its holder ownership in a corporation, but no voting rights, and a claim on assets in the event of bankruptcy that comes before any claim of the common stock holders. Preferred stocks are considered fixed-income securities because they promise to pay a fixed cash dividend, which has priority over any cash dividends paid to common stock holders. Since the future cash dividends are fixed and expected to be paid indefinitely, the value of a preferred stock is then obtained from Equation (2.73) using $g = 0$:

$$S_0 = \frac{\mathcal{D}_0}{\Bbbk} \qquad (\Bbbk > 0) \tag{2.74}$$

Example 2.18. (Preferred Stocks) Suppose that a preferred stock has a fixed total annual cash dividend per share of \$2.50. Assume an annual required return rate of 13% for the stock. How much should you pay for the preferred stock?

Solution. We apply Equation (2.74) with $\mathcal{D}_0 = \$2.50$ and $\Bbbk = 0.13$. The current share price of the preferred stock is: $S_0 = \frac{\mathcal{D}_0}{\Bbbk} = \frac{\$2.50}{0.13} = \$19.23$.

□

Common stocks do not have a promise to pay cash dividends. Nonetheless, if a common stock currently pays no cash dividends, there is still investor expectation that earnings are being reinvested in the company to create growth which will lead to cash dividends in the future. Due to the uncertainty of future cash dividends for common stocks, we shall model their valuation under certain assumptions about the expected cash dividends.

Example 2.19. (Common Stocks) Suppose that the total cash dividend of a stock last year was \$2.75 per share and dividends are expected to increase at 3% per annum. If the annual required return rate is 10%, find the share price of the stock today.

Solution. By (2.73), the price is: $S_0 = \frac{(1+g)\mathcal{D}_0}{\mathfrak{k}-g} = \frac{(1+0.03)\times\$2.75}{(0.1-0.03)} = \$40.46.$

□

2.10 Applications to Bond Valuation

The US bond market is vast—much bigger than its stock market. As measured at the end of 2012 in terms of capitalizations, the US bond market was twice as big as the US stock market for domestic companies.[12] As with other fixed-income financial investments, the price of a bond is the present value of its cash flow. We shall explore how to value bonds.

2.10.1 Bond Terminologies

A *bond* is a contract between an issuer (bond seller) and a lender (bondholder) legally binding the issuer to repay the lender a specified fixed amount at maturity and a series of interest payments during the life of the bond. In essence, a bond is an IOU.[13] The specific terms for a bond's duration, interest payment, etc. are described fully in the contract (indenture). The funds raised by bond issues are used for capital expenditures, operations, corporate takeovers, public projects, etc.

Bonds are usually redeemed on the maturity date. However, some bonds have the option to be *callable*,[14] i.e., such bonds give the issuer the right, but not the obligation, to redeem (call) the bond prior to the maturity date. There are also bonds with the option to be *convertible*, i.e., the bondholder has the right to exchange the bond for a different security (e.g., shares of common stock), and have a prescribed variable interest rate or even deferred interest. To avoid confusion about which types of bonds are intended, we assume

> *Unless stated to the contrary, all bonds are without options, i.e., they are noncallable, nonconvertible, etc., and have a fixed interest paid every 6 months.*

[12] http://www.learnbonds.com/how-big-is-the-bond-market/

[13] IOU is an abbreviation for "I owe you."

[14] Most corporate bonds are callable. Also, the US Treasury has not issued callable bonds since 1985.

Although both bonds and stocks are securities of a company, they are different in the sense that bondholders are creditors of the company, whereas stockholders are owners of the company. The cash flows from a company's bonds are more reliable than those from its stocks since the company has a legal obligation to repay its bondholders. Sometimes even when a company becomes insolvent, its bondholders may still get back some compensation, while compensation is not guaranteed for its common stockholders.

We now list and discuss some basic terminologies and features of bonds:

➢ The *issue date* of a bond is the date on which the bond issuer receives the loan from the lender and from which the lender is entitled to receive interest from the issuer.

➢ The *maturity value* $\mathcal{M}$ (also known as the *par value*, *face value*, *principal*) of a bond is the unit of the amount borrowed at the time it was issued. It is traditionally in units of $1,000, but municipal bonds are usually sold in units of $5,000.

➢ There are two main markets for bonds: *primary market*, where bonds are sold for the first time to institutional investors, and *secondary market*, where the resale of bonds taking place after their initial offering is open to the public, though individual investors will need to have a brokerage account to transact trades. Bonds selling at their maturity value are called *par bonds*. In the secondary market, bonds are traded at prices that are typically different from the maturity value. If a bond sells at a market price above (respectively, below) its maturity value, then it is called a *premium bond* (respectively, *discount bond*).

Remark 2.5. The primary bond market is essentially an institutional market. In practice, the US Treasury uses an auction process to sell treasury bills, notes, and bonds in the primary market, whereas the pricing of newly issued corporate bonds is negotiated between the corporation (or its representative) on the one hand and investment bankers and large institutional investors on the other. □

➢ The *maturity date* is the date on which the bond issuer must repay the lender the bond's maturity value. Note that callable bonds have features which allow for the principal to be repaid before the maturity date.

➢ The *term to maturity*, or simply *maturity*, of a bond is the length of the time interval between the issue date and the maturity date.

Bonds can be classified into three groups: *short term*, *intermediate term* and *long term* according to maturities of, respectively, 1–5 years, 5–12 years, and greater than 12 years.

- A *coupon payment* $\mathcal{C}$, or simply *coupon*, is an interest payment of a bond.

  Most bonds have a fixed coupon that does not change during the life of the bond and is paid at regular time intervals, usually semi-annually.

  If a bond does not pay a coupon during its life, it is called a *zero-coupon bond*. To compensate for no coupon payments, such bonds are issued at a deep discount from their value at maturity.[15] For this reason, they are also called *discount bonds* or *deep discount bonds*. Zero-coupon bonds are similar to US government savings bonds[16] in concept and have significant theoretical value.

- The *current yield* indicates the yield of a security based on its current market value. The *current yield of a bond*, denoted by r, is determined by the formula below:

$$r = \frac{\text{annual coupon amount}}{\text{current bond price}}. \tag{2.75}$$

- The *coupon rate* or *interest rate*, denoted by r_C, is defined by the current yield when the bond price is equal to its maturity value. That is:

$$r_C = \frac{\text{annual coupon amount}}{\text{maturity value}}.$$

- A bond's *yield to maturity* (YTM), denoted by r_Y, is the marketplace's annual required return rate of the bond held to maturity and whose future coupon payments are reinvested at the same rate. Equivalently, a bond's YTM equates the present value of the bond's future cash flows to the bond's current market price:

$$\text{current bond price} = \sum_{\ell=1}^{n-1} \frac{\mathcal{C}}{(1+\frac{r_Y}{k})^{\ell}} + \frac{\mathcal{C}+\mathcal{M}}{(1+\frac{r_Y}{k})^{n}}, \tag{2.76}$$

  where n is the number of coupon payments remaining on the bond and k is the number of coupon payments per annum (typically, $k = 2$). Equation (2.76) is obtained by applying Equation (2.69) on page 58 with $\mathcal{P}_i = \mathcal{C}$ for $i = 1,2,\ldots,n-1$, $\mathcal{P}_n = \mathcal{C} + \mathcal{M}$, and $r_i = r_Y$ for $i = 1,2,\ldots,n$.

  Denote the current bond price and $\frac{r_Y}{k}$ in (2.76) by $B(n)$ and $\hat{r}_Y$, respectively. Then:

$$B(n) = \frac{\mathcal{M}}{(1+\hat{r}_Y)^n} + \sum_{\ell=1}^{n} \frac{\mathcal{C}}{(1+\hat{r}_Y)^{\ell}}. \tag{2.77}$$

[15] For example, such a bond might be issued at a 50% discount from its maturity value.

[16] A savings bond offers a fixed rate of interest over a fixed period of time, but cannot be traded after being purchased.

By (2.58) on page 50, Equation (2.77) is equivalent to

$$B(n) = \frac{\mathcal{M}}{(1+\hat{r}_Y)^n} + \frac{\left((1+\hat{r}_Y)^n - 1\right)\mathcal{C}}{(1+\hat{r}_Y)^n\,\hat{r}_Y} \qquad (\hat{r}_Y > 0). \tag{2.78}$$

➢ Finally, the following relationships among yield to maturity (r_Y), current yield (r), and coupon rate (r_C) hold (Exercise 2.35):

1. A bond trades at par iff $r_Y = r = r_C$ (see Proposition 2.1).
2. A bond trades as a discount bond iff $r_Y > r > r_C$.
3. A bond trades as a premium bond iff $r_Y < r < r_C$.

Remark 2.6. Generally, evaluating financial investment performance can be a complicated task as there are different measures to be applied to serve different purposes. Yield is a measure of an investment income that an investor receives annually. As a bond investor, if you just want to hold on to your bond until its maturity, the coupon rate is the only measure that matters. However, if you need to sell your bond before maturity, you have to adopt the current yield as a measure. Yield to maturity is a measure that enables you to compare different bonds by taking the effect of compound interest into consideration under the assumption that all the coupon payments are reinvested at the same rate and you hold the bond to maturity.[17] □

2.10.2 Bond Prices Versus Interest Rates and Yield to Maturity

Bond Price with YTM at the Coupon Rate

For a bond being traded after it was originally issued, we expect intuitively that when the YTM is at the coupon rate, then the market value of the bond should be its maturity value. The following proposition confirms that intuitive result and its converse:

Proposition 2.1. *Suppose that a bond has n coupon payments remaining. The market price of the bond equals its maturity value exactly when its coupon rate is the YTM:*

$$r_Y = r_C \quad \text{if and only if} \quad B(n) = \mathcal{M}. \tag{2.79}$$

[17] It is worth noting that comparing different bonds by their percentage change in price is often misleading since the significance is not the same for an identical percentage price change of bonds with different interest rates. Also, it is important to realize that reinvesting all the coupon payments at the same rate is rather difficult if not impossible in practice.

Proof. If $\hat{r}_C = \hat{r}_Y$, where $\hat{r}_C = r_C/2$ and $\hat{r}_Y = r_Y/2$, then $\mathcal{C} = \hat{r}_C \mathcal{M} = \hat{r}_Y \mathcal{M}$ and the bond valuation Equation (2.78) becomes:

$$B(n) = \frac{\mathcal{M}}{(1+\hat{r}_Y)^n} + \frac{\left((1+\hat{r}_Y)^n - 1\right)\mathcal{M}}{(1+\hat{r}_Y)^n} = \frac{\mathcal{M}}{(1+\hat{r}_Y)^n}\left(1 + (1+\hat{r}_Y)^n - 1\right)$$
$$= \mathcal{M}.$$

Conversely, if $B(n) = \mathcal{M}$, then (since $\mathcal{C} = \hat{r}_C \mathcal{M}$) Equation (2.78) reduces to:

$$1 = \frac{1}{(1+\hat{r}_Y)^n} + \frac{\left((1+\hat{r}_Y)^n - 1\right)\hat{r}_C}{(1+\hat{r}_Y)^n\,\hat{r}_Y}.$$

Multiplying through by $(1+\hat{r}_Y)^n$ yields: $(1+\hat{r}_Y)^n - 1 = \frac{\hat{r}_C((1+\hat{r}_Y)^n-1)}{\hat{r}_Y}$. Hence, $\hat{r}_C = \hat{r}_Y$. □

Bond Prices Move in Opposite Direction to Interest Rates

The interest rate probably has the single largest impact on the prices of all bonds. The following three examples are related to each other and illustrate the relationship between bond prices on the one hand and interest rates and YTM on the other.

Example 2.20. Suppose that a 30-year bond with an annual 3% coupon rate payable semiannually was issued by the US Treasury on the first trading day of 2013. If the maturity value is \$1,000, what is the semiannual coupon amount?

Solution. Solve for the semiannual coupon amount $\mathcal{C}$ from the equation

$$3\% = \frac{2\mathcal{C}}{\$1{,}000}$$

to obtain $\mathcal{C} = \$15$. Assume, for simplicity, that the bond was sold in the primary market at its maturity value. Then by Proposition 2.1, the YTM equals 3%.

□

In the next two examples, the bond in Example 2.20 will be referred to as "the first bond."

Example 2.21. Since the Feds kept interest rates artificially low in 2013, doubling the interest rate in 10 years from 2013 is not an unreasonable speculation. Suppose that another 30-year bond with an annual 6% coupon rate payable semiannually will be issued by the Treasury on the first trading day of 2023. What will be the price of the first bond at the time of the second bond initial offering?

Solution. For the simplicity of our argument, we assume almost no intraday bond price fluctuations on the issue date of the second bond. Let

$$\begin{aligned} t_0 &= \text{the issue date of the second bond,} \\ r_1 &= \text{the (current) yield of the first bond at } t_0, \\ B_1 &= \text{the price of the first bond at } t_0. \end{aligned}$$

Since no investors will buy a bond with 3% annual yield when they have the choice to purchase a bond of the same type with 6% annual yield, we have $r_1 = 6\%$. In other words, the current yield of the first bond will be forced to approach 6% on the issue date of the second bond under the law of supply and demand. To speculate on the price of the first bond, we apply (2.75) and solve for B_1 from the equation, $r_1 = 6\% = \frac{2\times\$15}{B_1}$, to obtain $B_1 = \$500$. *Observe that when the interest rate rises from* 3% *to* 6%, *the first bond's price will fall from* \$1,000 *to* \$500.

□

Example 2.22. Suppose that you will purchase the first bond on the first trading day of 2023 at the price \$500 and hold it to the maturity date of the first trading day of 2043. What will be the yield to maturity?

Solution. We need to solve the bond Equation (2.78) for r_Y, which in our setting is[18]

$$B_1 = \frac{((1+\frac{r_Y}{k})^n - 1)\,\mathcal{C}}{\frac{r_Y}{k}(1+\frac{r_Y}{k})^n} + \frac{\mathcal{M}}{(1+\frac{r_Y}{k})^n}.$$

Using $B_1 = \$500$, $\mathcal{C} = \$15$, $\mathcal{M} = \$1{,}000$, $k = 2$ (semiannual compounding), and $n = 40$ (number of coupon payments remaining), we obtain $r_Y = 8.084\%$. Indeed, *when the price of a bond drops from* \$1,000 *to* \$500, *the yield to maturity rises from* 3% *to* 8.084%. Note that we assumed the bond was sold in the primary market at its maturity value.

□

We now establish, in general, the observation at the end of the solution of Example 2.22. Take the first and second derivatives of the bond's present value (2.77):

$$\frac{dB(n)}{d\hat{r}_Y} = -n\frac{\mathcal{M}}{(1+\hat{r}_Y)^{n+1}} - \mathcal{C}\sum_{\ell=1}^{n}\frac{\ell}{(1+\hat{r}_Y)^{\ell+1}} \;<\; 0$$

and

[18] As before, there is no general analytical solution r_Y for every n. In most applications, we can only estimate r_Y numerically using a software.

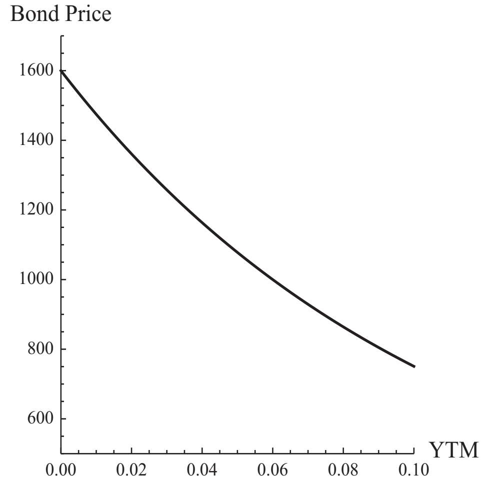

Fig. 2.2 The price of a bond is a strictly decreasing, concave-up function of the bond's YTM. The graph illustrates this for a bond with $1,000 maturity value and 6% coupon rate. Note that when the YTM is 6%, the bond's price is its maturity value.

$$\frac{d^2B(n)}{d\hat{r}_Y^2} = n(n+1)\frac{\mathcal{M}}{(1+\hat{r}_Y)^{n+2}} + \mathcal{C}\sum_{\ell=1}^{n}\frac{\ell(\ell+1)}{(1+\hat{r}_Y)^{\ell+2}} > 0.$$

In other words, the bond's price is not only strictly decreasing as the yield increases, but has a convex graph, i.e., the graph is everywhere concave up (increasing slope). Figure 2.2 depicts this property for a bond with $1,000 maturity value and 6% annual coupon rate.

2.11 Exercises

2.11.1 Conceptual Exercises

2.1. A physicist summed up the growth rate of an initial sum of money held over a fixed time span as follows: "If simple interest is applied during the time span, then the initial sum will grow with uniform (constant) velocity as the interest rate increases. If periodic compound interest is applied, then the growth of the initial sum will accelerate as interest increases." Do you agree with this interpretation? Justify your answer.

2.2. Theorem 2.1 on page 27 yields that k-periodic compounding of a principal $\mathcal{F}_0$ at r per annum over a time span of τ years consisting of x interest periods gives a future value,

$$\mathcal{F}_x = \left(1+\frac{r}{k}\right)^x \mathcal{F}_0,$$

where x is a nonnegative real number and $0 \leq \frac{r}{k} < 1$ for $k = 1, 2, \ldots$. Does $\mathcal{F}_x$ increase or decrease as k increases indefinitely? Justify your answer.

2.3. Suppose you purchase a lottery ticket for \$2. What is your return rate if you lose? What if you win \$200 million? Express your answer as a percentage.

2.4. Consider an investment that promises a fixed sequence of future cash dividends. Briefly explain why an increase in the required return rate on the investment would decrease the current value of the investment.

2.5. Explain what the following is stating financially about the start-up: "A start-up's NPV at 30% is \$35,000."

2.6. A friend borrows \$1,000 from a lender that gives him the loan as a simple ordinary annuity at a fixed interest rate over 2 years with a payment of \$100 per month. If your friend carries the loan to its full term, then he will have to pay more than the amount of the loan in just interest. True or false? Justify your answer.

2.7. A loan with a fixed payment of \$1,000 per month for 5 years has the stipulation that you will have to pay all the interest due on the loan even if you pay the loan off early. If immediately after you receive the loan, you want to pay it off, how much do you have to pay the lender?

2.8. How would you modify the interpretation of the noncallable bond pricing formula (2.78) on page 69 to obtain the current price of a callable bond, i.e., a bond where the issuer has the right, but not the obligation, to redeem (in practice, cancel) the bond before maturity? Use a single call date, i.e., a date when the issuer can redeem the bond before maturity. Compare the price of a callable bond with a noncallable one. Corporations issue callable bonds because if interest rates go down, they can call their bonds and refinance their debt at a lower interest rate.

2.9. How would you modify the interpretation of the noncallable bond pricing formula (2.78) on page 69 to obtain the current price of a puttable bond, i.e., a bond where the investor has the right, but not the obligation, to redeem the bond before maturity? Use a single put date, i.e., a date on which the investor can redeem the bond before maturity. Compare the price of a puttable bond with a noncallable one. Investors buy puttable bonds because if interest rates increase, they can sell back their original bonds at the put value and invest the proceeds in a higher interest rate bond.

2.11.2 Application Exercises

2.10. Consider a principal $\mathcal{F}_0$ that is held for n_{exact} days during a non-leap year at the simple interest rate r. By what percent is the simple interest amount using Banker's Rule greater than the simple interest amount employing exact time and exact interest?

2.11. (Selling or Buying a Loan) On November 12, 2007, a borrower closes on a loan for $176,000 at 6.25% per annum compounded daily. Repayment of the loan's maturity value (principal plus interest) is due in full on April 15, 2008. Suppose that the fine print of the original loan stipulated that the lender can sell the loan on the condition that the interest rate and maturity date remain the same. The lender sells the loan to another lender on January 5, 2008. The new lender agrees to purchase the debt for the present value of the maturity value at 10% per annum compounded daily. Assume that interest compounds daily and the borrower does not default on the loan. Use Banker's Rule when solving the following:

a) What is the maturity value of the loan?
b) What will the first lender receive for selling the loan? Is any profit made by the first lender?
c) What profit will the second lender make on the loan's maturity date if the conditions of the original loan are unchanged?
d) Though the original interest rate and maturity date are unchanged, the second lender is not prevented from reissuing the loan with a new start date set as the loan's purchase date and with the new loan's principal set as the value of the loan on the purchase date. Does the second lender make more profit by resetting the loan in this way? Explain.

2.12. For an interest rate of 4% per year, compare the future value 2 years from now to which $10,000 increases under daily compounding versus continuous compounding. Assume 365 days per year and express your answer as a fractional-difference percentage of the daily compounding case.

2.13. Suppose that at the start of college, you have $1,000 to invest and would like for it to grow to $1,250 at the end of your senior year through monthly compounding. Determine the general formula for the interest rate required for the growth and then compute the interest rate.

2.14. Assume that college tuition is currently 30 times its cost 15 years ago. Assuming annual compounding, what is the interest rate r that gives the rate of increase in tuition?

2.15. How much should you have today in an account with monthly compounding and annual interest rate of 4% to receive $1,000 per month forever?

2.16. (Equity in a House) A couple purchased a house 7 years ago for $375,000. The house was financed by paying 20% down and signing a 30-year mortgage at 6.5% on the unpaid balance. The net market value of the house is now $400,000. Assume that the couple wishes to sell the house.

a) How much equity (to the nearest dollar) does the family have in the house now, after making 84 monthly payments?
b) Find the first interest payment $\mathcal{I}_1$ and the 84th interest payment $\mathcal{I}_{84}$.

2.17. (Social Security Benefits) We present a simplified problem to illustrate Social Security benefits. A college graduate begins work at age 22. She has an annual income of $\$70,000$ until retirement (a simplification), pays 12.4% of this income into Social Security each year, and retires at age 65 with Social Security benefits of $\$20,000$ annually. How long must she live before the present value of these benefits equals the present value of her annual contributions? In other words, how long must she live after retirement to get back the full value of her contributions to Social Security? Will she get the entire value? Assume a discount rate of 4% per year, no change in her salary, and that all payments and benefits occur at the end of each year.

2.18. (Worker's Compensation) The usual legal settlement for an industrial accident is the present value of the employee's lifetime earnings. If you expect to work for 10 more years, make $\$70,000$ a year in the next 2 years, and get a raise of $\$5,000$ every 2 years, what would be your settlement? Assume an annual discount rate of 4% in the first 5 years and 6% in the second 5 years, and that your paycheck is received at the end of each year.

2.19. (Bonds) Suppose that you bought a 30-year bond with 4% annual coupon rate. You wish to sell that bond at a later date when the remaining life of the bond is 2.5 years and the current YTM of your bond has declined to 2%.

a) What is the fair value, as determined by the present value method, of the bond at the time of your sale?
b) How much would you earn if you purchased the bond for $\$1,000$, sold it at the fair value, and did not reinvest the coupon payments?

2.20. (Bonds) Bonds are generally quoted as a percentage of their face value. A bond selling at 99.2% of its face value is quoted as 99.2. The following information for a treasury bond was provided by the WSJ market data center on December 4, 2013:

Maturity	Coupon	Current price	Previous price	Change	Yield
11/30/20	2.000	99.20	99.00	0.203	2.123

The coupon column refers to the annual coupon rate. Verify that the last column indicates YTM.

Purchasing a House

The remaining Application Exercises deal with purchasing a house. Assume that you are currently renting an apartment for $1,040 per month and you have been considering buying a house. You have saved $10,000 toward a down payment for the house.

A salesperson informs you that he has a new house for sale, where the house and land were independently appraised at $200,000, but are being sold by the builder at a discount price of $185,000. The builder wants to get rid of the property quickly because the house is the last one to be sold in the development and the builder is moving on to construction of a new development.

The salesperson connects you with his in-house lender, to whom you give details about your income and grant permission to review your credit and eligibility for a loan. You inform her that you are prepared to make a down payment of $10,000 toward the house if necessary. She gets back to you with good news that, if you put $8,100 toward the house, then they can give you a 30-year loan for the balance of $176,900 at 6.25% per annum (compounded monthly). Note that lenders require the house to appraise at or above the purchase price; otherwise, they may reject the loan or require more down payment. The lender computes the monthly mortgage payment at $1,089.20. She informs you that the remaining $1,900 of your $10,000 can be used toward costs associated with the final evaluation of the physical property and the closing of the purchase (property inspector fee, termite inspector fee, official survey, attorney fees, etc.). The builder agrees to pay for costs beyond your $1,900 and make necessary repairs you identify during the period you have to inspect the property (the due diligence period).

Hearing the news about your qualification for the loan, the salesperson asks you how much rent you are now paying. When you inform him that you pay $1,040 per month, he quickly points out that it would be a mere extra $50 per month for you to meet the mortgage payments. He emphasizes that it is better to own than to rent, especially if the mortgage is just a bit more than your current rent.

You are thrilled! After the excitement subsides, however, you decide to run the numbers yourself to make sure you get a clear understanding of what you are getting into financially.[19] The problems in this project help guide you through some of this analysis.

2.21. Show that the monthly loan payment on the unpaid principal balance of $176,900 is $1,089.20.

2.22. In addition to closing fees paid to settle the loan, there are expenses beyond the monthly mortgage payments.

First, since your deposit was less than 20% of the purchase price, you are required to take out a private mortgage insurance (PMI) to protect the lender

[19] Mortgages on a house are generally modeled as simple ordinary annuities by lenders.

if you default on the loan. The PMI typically lasts until the unpaid principal balance of the mortgage is paid down to 80% of the original value of the house, where the house's original value is the lesser of the purchase price and the official appraised value of the house used in closing the sale. Note that the bank may also require your payment history to be in good standing (e.g., no late payments in the past year or two) before removing PMI. Of course, if the value of the house increases nontrivially, you may be able to remove the PMI earlier. Suppose that the PMI is $141.52 per month.

Second, along with PMI, you have to pay for hazard insurance to cover unplanned damages to the house due to fire, smoke, wind, etc. Assume that the hazard insurance is $36.50 per month.

Third, you have to pay property taxes to the tax district (e.g., county and city) where the house is located. The property (i.e., house and land) will be valued within your tax district, which is a valuation that is separate from the appraisal done when purchasing the house. The resulting tax district's valuation is the taxable value of the house and is the amount to which the property tax rate will be applied. Suppose that the annual property tax rate is 1.3% and the taxable value of the property is $189,986. For this project, the taxable property value is less than the appraised value (i.e., $200,000) used for the purchase. Sometimes, however, the taxable value can be higher which was not uncommon in the aftermath of the 2008 mortgage crisis.

The PMI, hazard insurance, and property tax payments are in addition to the monthly loan payment, and all together they form a single payment you make to the lender. The lender or a company hired by the lender manages these payments by taking out the portion for the loan payment (principal plus interest) and depositing the rest into an escrow account, which is used to pay the annual insurance premiums and property taxes on behalf of the borrower.

Finally, assume that the property is in a housing development that comes with a mandatory Homeowners Association (HOA) fee. The HOA fee is used to maintain the grounds, roads, etc. in the development. If you do not pay the fee, the HOA can foreclose on your property. Assume an HOA fee of $100 per month.

a) What is the estimated total monthly PITI, i.e., the minimum monthly payment covering the principal, interest, taxes, and (hazard) insurance?
b) Identify two other mandatory house expenses that are outside of the PITI payment and other basic house costs like utilities and repairs. Do exclude costs like groceries, tuition, medical expenses, etc., which are more associated with running a home. What is the minimum monthly cost of the house during the first year if you now include these two mandatory house expenses and PITI? Which of these housing costs will likely increase in the future?

c) What is your opinion about the salesperson's pitch about the cost of renting versus buying a house?

2.23. Fill out the amortization schedule below, which is for the first 5 months of the loan.

Payment #	**Payment** ($\mathcal{P}$)	**Principal** ($\mathfrak{P}_\ell$)	**Interest** ($\mathcal{I}_\ell$)	**Bal.** ($\mathcal{B}_\ell$)
1	1,089.20	167.85	921.35	176,732.15
2	1,089.20			
3	1,089.20			
4	1,089.20			
5	1,089.20			

2.24. Are there discrepancies in the above amortization table? If so, explain how to remove them mathematically.

For the remaining problems, note that only the payments toward principal and interest (PI) are relevant to the loan's balance. Costs associated with property taxes, hazard insurance, PMI, HOA, etc. are separate expenses and do not impact the balance of the loan. Such costs are typically not included in the loan's cost.

2.25. Using a software, compute the numbered payment at which the unpaid balance on the loan will first dip below 80% of the original value of the house. Roughly how many years and months does it take to reach that balance? If the value of the house has not decreased below its original value at that point in time, you would stop paying PMI henceforth.

2.26. Determine the total amount you would pay into the mortgage, excluding escrow payments, if you make only the minimum payment over the full 30 years. What is the total cost of the mortgage? Is it more than the mortgage?

2.27. Estimate the number of years and months it would take to pay off the mortgage if you double your monthly payments.

2.28. Estimate the total you would pay into the mortgage if you double your monthly payments. What is the total cost of the mortgage for doubled payments? Is it more than the mortgage?

2.11.3 Theoretical Exercises

2.29. Suppose that an initial capital $\mathcal{F}_0$ grows to an amount $\mathcal{F}(\tau)$ over a time span τ. A mathematician modeling the growth observes that for all time spans

x and y, the accumulated amount $\mathcal{F}(x)$ is a differentiable function satisfying the following:

$$\mathcal{F}(x+h) = \mathcal{F}(x) + \mathcal{F}(h) - \mathcal{F}_0, \qquad \mathcal{F}(0) = \mathcal{F}_0, \qquad \frac{d\mathcal{F}}{dx}(0) = r\,\mathcal{F}_0,$$

where $r \geq 0$. Determine the type of growth model, i.e., find $\mathcal{F}(x)$.

2.30. Derive Equation (2.18) on page 26: $G'(x) = G(x)\,G'(0)$.

2.31. (Capital After Spending, Inflation, and Interest) Consider the following setup:

- Begin with an initial capital $C(0)$ in an interest-bearing account and let $C(n)$ be the remaining capital at the end of the nth year.
- Assume an interest rate r is applied at the end of each year to the capital remaining on that date.
- At the end of the first year, assume that an amount S was spent from $C(0)$ on goods and services, and money will be spent on similar goods and services in each of the subsequent years.
- Suppose that the amount spent at the end of any specific year is the total amount spent by the end of the first year increased in subsequent years at the annual inflation rate $\mathtt{i}$ compounding annually until the end of the specified year. Assume that $r > \mathtt{i}$ since investors are not interested in a market interest rate that is below the inflation rate.

a) Show that the total capital at the end of the $(n+1)$st year can be expressed recursively as follows in terms of the capital at the end of the previous year, taking into account spending, inflation, and interest growth:

$$C(n+1) = (1+r)\,[C(n) - (1+\mathtt{i})^n S]. \tag{2.80}$$

b) Use induction to show that

$$C(n) = (1+r)^n \left[C_0 - \frac{1+r}{r-\mathtt{i}} S\right] + \left(\frac{1+r}{r-\mathtt{i}}\right)(1+\mathtt{i})^n\, S.$$

2.32. Suppose that after this year, your grandmother will receive regular payments from a retirement fund, but she has to choose between two options for how to receive the payments during $n+1$ years. She does not plan to spend any of the money until after the $n+1$ years. Assume that she will save all the disbursements in an account that accrues the payments as a simple ordinary annuity with k-periodic compounding at interest rate r (e.g., each payment date coincides with an interest date).

The payment start date will differ for the two plans, but both payment options will have the last payment at the start of the last interest period during the $(n+1)$st year. Your job is to help her choose between the two options.

a) **(A General Future Value Formula)** The current problem determines a general formula that incorporates the future value of the payments into your grandmother's account. Suppose that regular payments of $\mathcal{P}$ into an interest-bearing account form a simple ordinary annuity with k-periodic compounding at interest rate r. Assume that the account receives the first payment at the end of the first interest period of a certain year and the last payment at the end of the Nth interest period going forward, with no payment at the end of the $(N+1)$st interest period. Show that the amount accrued in the account at the end of the $(N+1)$st period is:

$$FV \equiv \left[\frac{(1+r/k)^{N+1} - (1+r/k)}{r/k}\right] \mathcal{P}, \tag{2.81}$$

where N is the total number of payments into the account.

b) We now explore the future values associated with the following two plans for receiving payment.

   i. *Plan A.* Assume that Plan A begins officially at the start of next year with payments of A starting at the end of the first interest period of next year. Show that the total amount she would accrue at the end of the $(n+1)$st year is:

$$\mathrm{FV}_A \equiv \left[\frac{(1+r/k)^{(n+1)k} - (1+r/k)}{r/k}\right] A.$$

   ii. *Plan B.* Under Plan B, your grandmother receives payments of B with the choice of officially starting at the beginning of the $(q+1)$st year after Plan A starts and the first payment disbursing at the end of the first interest period of the official starting year. Show that the total amount she would accrue by the end of the $(n+1)$st year is

$$\mathrm{FV}_B \equiv \left[\frac{(1+r/k)^{[(n+1)-q]k} - (1+r/k)}{r/k}\right] B,$$

   where $q = 1, 2, \ldots$. Note that for $q = 0$, the two options coincide.

c) **(Choosing Between Plans A and B)** Naturally, since Plan B starts out later than Plan A and both have the same last-payment date, the payment amount of Plan B has to be higher than that of Plan A, i.e., $B > A$. Suppose that the account's interest rate exceeds a threshold as follows:

$$r > k\left[(B/A)^{1/q} - 1\right].$$

   i. Show that there is no n such that the amounts accrued under both options are equal by the end of the $(n+1)$st year.
   ii. Show that Plan A is superior to Plan B, i.e., prove $\mathrm{FV}_A > \mathrm{FV}_B$.

2.33. (Relating Present and Future Values of a Generalized Annuity) Using

$$\mathcal{S}_n = \sum_{\ell=0}^{n-1} \left[\prod_{j=0}^{\ell} \left(1 + \frac{r_{n+1-j}}{k} \right) \right] \mathcal{P}_{n-\ell},$$

verify the formula

$$\mathcal{A}_n = \frac{\mathcal{S}_n}{\prod_{j=1}^{n} \left(1 + \frac{r_j}{k} \right)},$$

where $n = 1, 2, \ldots, r_j > 0$ for $j = 1, \ldots, n$, and $r_{n+1} = 0$.

2.34. (Bonds) Given a coupon bond described by Equation (2.76) on page 68, find the future value at maturity of the bond's cash flow.

2.35. (Bonds) Show that for a coupon bond, its yield to maturity (r_Y), current yield (r), and coupon rate (r_C) have the following relationships:

a) A bond trades at a discount if and only if $r_Y > r > r_C$.
b) A bond trades at a premium if and only if $r_Y < r < r_C$.

References

[1] Bodie, Z., Kane, A., Marcus, A.: Investments, 9th edn. McGraw-Hill Irwin, New York (2011)
[2] Brealey, R., Myers, S., and Allen, F.: Principles of Corporate Finance. McGraw-Hill Irwin, New York (2011)
[3] Brown, S., Kritzman, M.: Quantitative Methods for Financial Analysis. Dow Jones-Irwin, Homewood (1990)
[4] Chaplinsky, S., Doherty, P., Schill, M.: Methods of evaluating mergers and acquisitions. Note Number UVA-F-1274. University of Virginia Darden Business Publishing (2000)
[5] Choudhry, M.: Fixed-Income Securities and Derivatives Handbook. Bloomberg Press, New York (2005)
[6] Davis, M.: The Math of Money. Copernicus Books, New York (2001)
[7] Gordon, M.J.: Dividends, earnings and stock prices. Rev. Econ. Stat. **41**, 99 (1959)
[8] Guthrie, G, Lemon, L.: Mathematics of Interest Rates and Finance. Prentice Hall, Upper Saddle River (2004)
[9] Hull, J.: Options, Futures, and Other Derivatives, 7th edn. Pearson Prentice Hall, Upper Saddle River (2009)
[10] Kellison, S.: The Theory of Interest, 2nd edn. Irwin McGraw-Hill, Boston (1991)

[11] Koller, T., Goedhart, M., Wessels, D.: Valuation: Measuring and Managing the Value of Companies. Wiley, Hoboken (2010)
[12] L.E.K. Consulting, LLC: Discounted Cash Flow Valuation Primer. L.E.K. Consulting, Chicago (2003)
[13] Lovelock, D., Mendel, M., Wright, A.: An Introduction to the Mathematics of Money. Springer, New York (2007)
[14] Meserve, B.: Fundamental Concepts of Algebra. Dover, New York (1981)
[15] Muksian, R.: Mathematics of Interest Rates, Insurance, Social Security, and Pensions. Prentice Hall, Upper Saddle River (2003)
[16] Reilly, F., Brown, K.: Investment Analysis and Portfolio Management. South-Western Cengage Learning, Mason (2009)
[17] Wang, X.: A simple proof of Descartes's Rule of Signs. Am. Math. Mon. **111**, 525 (2004)
[18] Williams, J.B.: The Theory of Investment Value. Harvard University Press, Cambridge (1938). Reprinted in 1997 by Fraser Publishing

http://www.springer.com/978-1-4939-3781-3

An Introduction to Mathematical Finance with Applications
Understanding and Building Financial Intuition
Petters, A.O.; Dong, X.
2016, XVII, 483 p. 52 illus., 12 illus. in color., Hardcover
ISBN: 978-1-4939-3781-3