Chapter 2
A Gallery of Discrete Volumes

Few things are harder to put up with than a good example.

Mark Twain (1835–1910)

A unifying theme of this book is the study of the number of integer points in polytopes, where the polytopes live in a real Euclidean space $\mathbb{R}^d$. The integer points $\mathbb{Z}^d$ form a lattice in $\mathbb{R}^d$, and we often call the integer points **lattice points**. This chapter carries us through concrete instances of lattice-point enumeration in various integral and rational polytopes. There is a tremendous amount of research taking place along these lines, even as the reader is looking at these pages.

2.1 The Language of Polytopes

A polytope in dimension 1 is a closed interval; the number of integer points in $\left[\frac{a}{b}, \frac{c}{d}\right]$ is easily seen to be $\left\lfloor \frac{c}{d} \right\rfloor - \left\lfloor \frac{a-1}{b} \right\rfloor$ (Exercise 2.1; here we assume that $a, b, c, d \in \mathbb{Z}$ with $\frac{a}{b} < \frac{c}{d}$). A 2-dimensional convex polytope is a **convex polygon**: a compact convex subset of $\mathbb{R}^2$ bounded by a simple closed curve that is made up of finitely many line segments.

In general dimension d, a **convex polytope** is the convex hull of finitely many points in $\mathbb{R}^d$. To be precise, for a finite point set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subset \mathbb{R}^d$, the polytope $\mathcal{P}$ is the smallest convex set containing those points; that is,

$$\mathcal{P} = \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n : \text{ all } \lambda_k \geq 0 \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

This definition is called the **vertex description** of $\mathcal{P}$, and we use the notation

$$\mathcal{P} = \operatorname{conv}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\},$$

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the convex hull of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. In particular, a polytope is a *closed* subset of $\mathbb{R}^d$. Many polytopes that we will study, however, are not defined in this way, but rather as bounded intersections of finitely many half-spaces and hyperplanes. One example is the polytope $\mathcal{P}$ defined by (1.4) in Chapter 1. (A set, bounded or not, that can be described as the intersection of finitely many half-spaces and hyperplanes is a **polyhedron**.) This **hyperplane description** of a polytope is, in fact, equivalent to the vertex description. The fact that every polytope has both a vertex and a hyperplane description is highly nontrivial, both algorithmically and conceptually. We carefully work out a proof in Appendix A.

The **dimension** of a polytope $\mathcal{P}$ is the dimension of the affine space

$$\operatorname{span}\mathcal{P} := \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x}, \mathbf{y} \in \mathcal{P}, \lambda \in \mathbb{R}\}$$

spanned by $\mathcal{P}$. If $\mathcal{P}$ has dimension d, we use the notation $\dim \mathcal{P} = d$ and call $\mathcal{P}$ a d-polytope. Note that $\mathcal{P} \subset \mathbb{R}^d$ does not necessarily have dimension d. For example, the polytope $\mathcal{P}$ defined by (1.4) has dimension $d-1$.

For a convex polytope $\mathcal{P} \subset \mathbb{R}^d$, we say that the hyperplane $H = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\right\}$ is a **supporting hyperplane** of $\mathcal{P}$ if $\mathcal{P}$ lies entirely on one side of H, that is,

$$\mathcal{P} \subset \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \le b\right\} \qquad \text{or} \qquad \mathcal{P} \subset \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \ge b\right\}.$$

A **face** of $\mathcal{P}$ is a set of the form $\mathcal{P} \cap H$, where H is a supporting hyperplane of $\mathcal{P}$. Note that $\mathcal{P}$ itself is a face of $\mathcal{P}$, corresponding to the **degenerate hyperplane** $\mathbb{R}^d$,[1] and the empty set $\varnothing$ is a face of $\mathcal{P}$, corresponding to a hyperplane that does not meet $\mathcal{P}$. The $(d-1)$-dimensional faces are called **facets**, the 1-dimensional faces **edges**, and the 0-dimensional faces **vertices** of $\mathcal{P}$. Vertices are the “extreme points” of a polytope.

A convex d-polytope has at least $d+1$ vertices. A convex d-polytope with exactly $d+1$ vertices is called a d-**simplex**. Every 1-dimensional convex polytope is a 1-simplex, namely, a line segment. The 2-dimensional simplices are the triangles, the 3-dimensional simplices the tetrahedra.

A convex polytope $\mathcal{P}$ is called **integral** if all of its vertices have integer coordinates,[2] and $\mathcal{P}$ is called **rational** if all of its vertices have rational coordinates.

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[1] In the remainder of the book, we will reserve the term *hyperplane* for nondegenerate hyperplanes, i.e., sets of the form $\left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\right\}$, where not all of the entries of $\mathbf{a}$ are zero.

[2] Integral polytopes are also called **lattice polytopes**.

2.2 The Unit Cube

As a warmup example, we begin with the **unit d-cube** $\square := [0,1]^d$, which simultaneously offers simple geometry and an endless fountain of research questions. The vertex description of $\square$ is given by the set of 2^d vertices $\{(x_1, x_2, \ldots, x_d) : \text{ all } x_k = 0 \text{ or } 1\}$. The hyperplane description is

$$\square = \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : 0 \le x_k \le 1 \text{ for all } k = 1, 2, \ldots, d \right\}.$$

Thus, there are the $2d$ bounding hyperplanes $x_1 = 0$, $x_1 = 1$, $x_2 = 0$, $x_2 = 1$, $\ldots$, $x_d = 0$, $x_d = 1$.

We now compute the discrete volume of an integer dilate of $\square$. That is, we seek the number of integer points $t\,\square \cap \mathbb{Z}^d$ for all $t \in \mathbb{Z}_{>0}$. Here $t\mathcal{P}$ denotes the dilated polytope

$$\{(tx_1, tx_2, \ldots, tx_d) : (x_1, x_2, \ldots, x_d) \in \mathcal{P}\},$$

for a polytope $\mathcal{P}$. What is the discrete volume of $\square$? We dilate by the positive integer t, as depicted in Figure 2.1, and count:

$$\#\left(t\,\square \cap \mathbb{Z}^d\right) = \#\left([0,t]^d \cap \mathbb{Z}^d\right) = (t+1)^d.$$

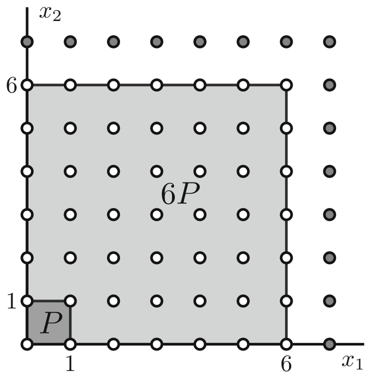

Fig. 2.1 The 6^{th} dilate of $\square$ in dimension 2.

We generally denote the **lattice-point enumerator** for the t^{th} dilate of $\mathcal{P} \subset \mathbb{R}^d$ by

$$L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right),$$

a useful object that we also call the **discrete volume** of $\mathcal{P}$. We may also think of leaving $\mathcal{P}$ fixed and shrinking the integer lattice:

$$L_{\mathcal{P}}(t) = \#\left(\mathcal{P} \cap \frac{1}{t}\,\mathbb{Z}^d\right).$$

With this convention, $L_{\square}(t) = (t+1)^d$, a polynomial in the integer variable t. Notice that the coefficients of this polynomial are the **binomial coefficients** $\binom{d}{k}$, defined through

$$\binom{m}{n} := \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} \tag{2.1}$$

for $m \in \mathbb{C}, n \in \mathbb{Z}_{>0}$.

What about the *interior* $\square^\circ$ of the cube? The number of interior integer points in $t\,\square^\circ$ is

$$L_{\square^\circ}(t) = \#\left(t\,\square^\circ \cap \mathbb{Z}^d\right) = \#\left((0,t)^d \cap \mathbb{Z}^d\right) = (t-1)^d.$$

Notice that this polynomial equals $(-1)^d L_{\square}(-t)$, the evaluation of the polynomial $L_{\square}(t)$ at negative integers, up to a sign.

We now introduce another important tool for analyzing a polytope $\mathcal{P}$, namely the generating function of $L_{\mathcal{P}}$:

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \ge 1} L_{\mathcal{P}}(t)\, z^t.$$

This generating function is also called the **Ehrhart series** of $\mathcal{P}$.

In our case, the Ehrhart series of $\mathcal{P} = \square$ takes on a special form. To illustrate, we define the **Eulerian number** $A(d,k)$ through[3]

$$\sum_{j \ge 0} j^d\, z^j = \frac{\sum_{k=0}^{d} A(d,k)\, z^k}{(1-z)^{d+1}}. \tag{2.2}$$

It is not hard to see that the polynomial $\sum_{k=1}^{d} A(d,k)\, z^k$ is the numerator of the rational function

$$\left(z\frac{d}{dz}\right)^d \left(\frac{1}{1-z}\right) = \underbrace{z\frac{d}{dz}\cdots z\frac{d}{dz}}_{d \text{ times}}\left(\frac{1}{1-z}\right).$$

The Eulerian numbers have many fascinating properties, including

$$\begin{aligned} A(d,k) &= A(d, d+1-k), \\ A(d,k) &= (d-k+1)\, A(d-1,k-1) + k\, A(d-1,k), \\ \sum_{k=0}^{d} A(d,k) &= d!, \end{aligned} \tag{2.3}$$

[3] There are two slightly conflicting definitions of *Eulerian numbers* in the literature: sometimes, they are defined through $\sum_{j\ge 0}(j+1)^d\, z^j = \frac{\sum_{k=0}^{d} A(d,k) z^k}{(1-z)^{d+1}}$ instead of (2.2).

$$A(d,k) = \sum_{j=0}^{k} (-1)^j \binom{d+1}{j} (k-j)^d .$$

The first few Eulerian numbers $A(d,k)$ for $0 \le k \le d$ are

$d = 0$: 1
$d = 1$: 0 1
$d = 2$: 0 1 1
$d = 3$: 0 1 4 1
$d = 4$: 0 1 11 11 1
$d = 5$: 0 1 26 66 26 1
$d = 6$: 0 1 57 302 302 57 1

(see also [1, Sequence A008292]).

With this definition, we can now express the Ehrhart series of $\square$ in terms of Eulerian numbers:

$$\begin{aligned}\mathrm{Ehr}_{\square}(z) &= 1 + \sum_{t \ge 1} (t+1)^d z^t = \sum_{t \ge 0} (t+1)^d z^t = \frac{1}{z} \sum_{t \ge 1} t^d z^t \\ &= \frac{\sum_{k=1}^{d} A(d,k)\, z^{k-1}}{(1-z)^{d+1}} .\end{aligned}$$

To summarize, we have proved the following theorem.

Theorem 2.1. *Let $\square$ be the unit d-cube.*

(a) *The lattice-point enumerator of $\square$ is the polynomial*

$$L_{\square}(t) = (t+1)^d = \sum_{k=0}^{d} \binom{d}{k} t^k .$$

(b) *Its evaluation at negative integers yields the relation*

$$(-1)^d L_{\square}(-t) = L_{\square^\circ}(t) .$$

(c) *The Ehrhart series of $\square$ is* $\mathrm{Ehr}_{\square}(z) = \frac{\sum_{k=1}^{d} A(d,k) z^{k-1}}{(1-z)^{d+1}}$. $\square$

2.3 The Standard Simplex

The **standard simplex** Δ in dimension d is the convex hull of the $d+1$ points $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ and the origin; here $\mathbf{e}_j$ is the unit vector $(0, \dots, 1, \dots, 0)$, with a 1 in the j^{th} position. Figure 2.2 shows Δ for $d = 3$. On the other hand, Δ can also be realized by its hyperplane description, namely

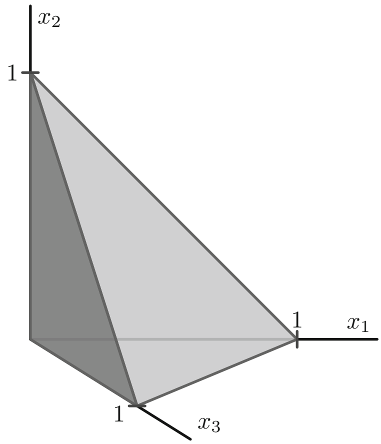

Fig. 2.2 The standard simplex Δ in dimension 3.

$$\Delta = \left\{ (x_1, x_2 \dots, x_d) \in \mathbb{R}^d : \, x_1 + x_2 + \dots + x_d \le 1 \text{ and all } x_k \ge 0 \right\}.$$

In the case of the standard simplex, the dilate $t\Delta$ is now given by

$$t\Delta = \left\{ (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : \, x_1 + x_2 + \dots + x_d \le t \text{ and all } x_k \ge 0 \right\}.$$

To compute the discrete volume of Δ, we would like to use the methods developed in Chapter 1, but there is an extra twist. The counting functions in Chapter 1 were defined by equalities, whereas the standard simplex is defined by an *inequality*. We are trying to count all integer solutions $(m_1, m_2, \dots, m_d) \in \mathbb{Z}_{\ge 0}^d$ to

$$m_1 + m_2 + \dots + m_d \le t\,. \tag{2.4}$$

To translate this inequality in d variables into an equality in $d+1$ variables, we introduce a *slack variable* $m_{d+1} \in \mathbb{Z}_{\ge 0}$, which picks up the difference between the right-hand and left-hand sides of (2.4). So the number of solutions $(m_1, m_2, \dots, m_d) \in \mathbb{Z}_{\ge 0}^d$ to (2.4) equals the number of solutions $(m_1, m_2, \dots, m_{d+1}) \in \mathbb{Z}_{\ge 0}^{d+1}$ to

$$m_1 + m_2 + \dots + m_{d+1} = t\,.$$

Now the methods of Chapter 1 apply:

$$\begin{aligned}\#\left(t\Delta \cap \mathbb{Z}^d\right) &= \operatorname{const}\left(\left(\sum_{m_1 \geq 0} z^{m_1}\right)\left(\sum_{m_2 \geq 0} z^{m_2}\right)\cdots\left(\sum_{m_{d+1} \geq 0} z^{m_{d+1}}\right) z^{-t}\right)\\ &= \operatorname{const}\left(\frac{1}{(1-z)^{d+1} z^t}\right). \end{aligned} \tag{2.5}$$

In contrast with Chapter 1, we do not require a partial fraction expansion but simply use the **binomial series**

$$\frac{1}{(1-z)^{d+1}} = \sum_{k \geq 0} \binom{d+k}{d} z^k \tag{2.6}$$

for $d \geq 0$. The constant-term identity (2.5) requires us to find the coefficient of z^t in the binomial series (2.6), which is $\binom{d+t}{d}$. Hence the discrete volume of Δ is given by $L_\Delta(t) = \binom{d+t}{d}$, a polynomial in the integer variable t of degree d. Incidentally, the coefficients of this polynomial function in t have an alternative life in traditional combinatorics:

$$L_\Delta(t) = \frac{1}{d!} \sum_{k=0}^{d} (-1)^{d-k} \operatorname{stirl}(d+1, k+1)\, t^k,$$

where $\operatorname{stirl}(n, j)$ is the *Stirling number of the first kind* (see Exercise 2.11). We also notice that (2.6) is, by definition, the Ehrhart series of Δ.

Let's repeat this computation for the *interior* Δ° of the standard d-simplex. Now we introduce a slack variable $m_{d+1} > 0$, so that strict inequality is forced:

$$\begin{aligned} L_{\Delta^\circ}(t) &= \#\left\{(m_1, m_2, \ldots, m_d) \in \mathbb{Z}_{>0}^d : m_1 + m_2 + \cdots + m_d < t\right\}\\ &= \#\left\{(m_1, m_2, \ldots, m_{d+1}) \in \mathbb{Z}_{>0}^{d+1} : m_1 + m_2 + \cdots + m_{d+1} = t\right\}. \end{aligned}$$

Now

$$\begin{aligned} L_{\Delta^\circ}(t) &= \operatorname{const}\left(\left(\sum_{m_1 > 0} z^{m_1}\right)\left(\sum_{m_2 > 0} z^{m_2}\right)\cdots\left(\sum_{m_{d+1} > 0} z^{m_{d+1}}\right) z^{-t}\right)\\ &= \operatorname{const}\left(\left(\frac{z}{1-z}\right)^{d+1} z^{-t}\right)\\ &= \operatorname{const}\left(z^{d+1-t} \sum_{k \geq 0} \binom{d+k}{d} z^k\right) = \binom{t-1}{d}. \end{aligned}$$

It is a fun exercise to prove that

$$(-1)^d \binom{d-t}{d} = \binom{t-1}{d} \tag{2.7}$$

(see Exercise 2.10). We have arrived at our destination:

Theorem 2.2. *Let Δ be the standard d-simplex.*

(a) *The lattice-point enumerator of Δ is the polynomial $L_\Delta(t) = \binom{d+t}{d}$.*
(b) *Its evaluation at negative integers yields $(-1)^d L_\Delta(-t) = L_{\Delta^\circ}(t)$.*
(c) *The Ehrhart series of Δ is $\mathrm{Ehr}_\Delta(z) = \frac{1}{(1-z)^{d+1}}$.* □

2.4 The Bernoulli Polynomials as Lattice-Point Enumerators of Pyramids

There is a fascinating connection between the Bernoulli polynomials and certain pyramids over unit cubes. The **Bernoulli polynomials** $B_k(x)$ are defined through the generating function

$$\frac{z\,e^{xz}}{e^z - 1} = \sum_{k \geq 0} \frac{B_k(x)}{k!}\, z^k \tag{2.8}$$

and are ubiquitous in the study of the Riemann zeta function, among other objects; they are named after Jacob Bernoulli (1654–1705).[4] The Bernoulli polynomials will play a prominent role in Chapter 12 in the context of Euler–Maclaurin summation. The first few Bernoulli polynomials are

$$\begin{aligned}
B_0(x) &= 1\,,\\
B_1(x) &= x - \frac{1}{2}\,,\\
B_2(x) &= x^2 - x + \frac{1}{6}\,,\\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x\,,\\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}\,,\\
B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x\,,\\
B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}\,.
\end{aligned}$$

The **Bernoulli numbers** are $B_k := B_k(0)$ (see also [1, Sequences A000367 & A002445]) and have the generating function

$$\frac{z}{e^z - 1} = \sum_{k \geq 0} \frac{B_k}{k!}\, z^k.$$

[4] For more information about Bernoulli, see
http://www-history.mcs.st-and.ac.uk/Mathematicians/Bernoulli_Jacob.html.

Lemma 2.3. *For integers $d \ge 1$ and $n \ge 2$,*

$$\sum_{k=0}^{n-1} k^{d-1} = \frac{1}{d}\left(B_d(n) - B_d\right).$$

Proof. We play with the generating function of $\frac{B_d(n)-B_d}{d!}$:

$$\sum_{d\ge 0} \frac{B_d(n) - B_d}{d!}\, z^d = z\,\frac{e^{nz}-1}{e^z-1} = z\sum_{k=0}^{n-1} e^{kz} = z\sum_{k=0}^{n-1}\sum_{j\ge 0}\frac{(kz)^j}{j!}$$
$$= \sum_{j\ge 0}\left(\sum_{k=0}^{n-1} k^j\right)\frac{z^{j+1}}{j!} = \sum_{j\ge 1}\left(\sum_{k=0}^{n-1} k^{j-1}\right)\frac{z^j}{(j-1)!}.$$

Now compare coefficients on both sides. $\square$

Consider a $(d-1)$-dimensional unit cube embedded into $\mathbb{R}^d$ and form a d-dimensional pyramid by adjoining one more vertex at $(0,0,\ldots,0,1)$, as depicted in Figure 2.3. More precisely, this geometric object has the following hyperplane description:

$$\mathcal{P} = \left\{(x_1,x_2,\ldots,x_d) \in \mathbb{R}^d : 0 \le x_1, x_2, \ldots, x_{d-1} \le 1 - x_d \le 1\right\}. \tag{2.9}$$

By definition, $\mathcal{P}$ is contained in the unit d-cube; in fact, its vertices are a subset of the vertices of the d-cube.

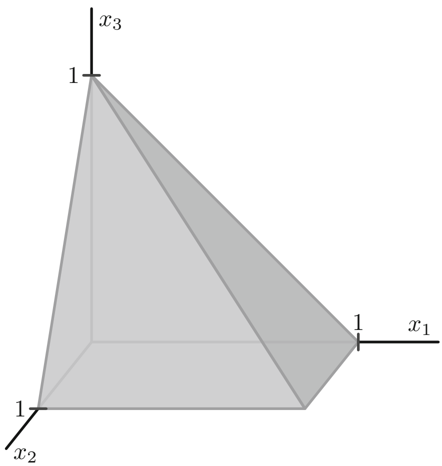

Fig. 2.3 The pyramid $\mathcal{P}$ in dimension 3.

We now count lattice points in integer dilates of $\mathcal{P}$. This number equals

$$\#\left\{(m_1,m_2,\ldots,m_d) \in \mathbb{Z}^d : 0 \le m_k \le t - m_d \le t \text{ for } k = 1,2,\ldots,d-1\right\}.$$

In this case, we just count the solutions to $0 \le m_k \le t - m_d \le t$ directly: once we choose the integer m_d (between 0 and t), we have $t - m_d + 1$ independent choices for each of the integers $m_1, m_2, \dots, m_{d-1}$. Hence

$$L_{\mathcal{P}}(t) = \sum_{m_d=0}^{t} (t - m_d + 1)^{d-1} = \sum_{k=1}^{t+1} k^{d-1} = \frac{1}{d} \left(B_d(t+2) - B_d \right), \qquad (2.10)$$

by Lemma 2.3. (Here we need to require $d \ge 2$.) This is, naturally, a polynomial in t.

We now turn our attention to the number of *interior* lattice points in $\mathcal{P}$:

$$L_{\mathcal{P}^\circ}(t) = \# \left\{ (m_1, m_2, \dots, m_d) \in \mathbb{Z}^d : \begin{array}{l} 0 < m_k < t - m_d < t \\ \text{for all } k = 1, 2, \dots, d-1 \end{array} \right\}.$$

By a similar counting argument,

$$L_{\mathcal{P}^\circ}(t) = \sum_{m_d=1}^{t-1} (t - m_d - 1)^{d-1} = \sum_{k=0}^{t-2} k^{d-1} = \frac{1}{d} \left(B_d(t-1) - B_d \right).$$

Incidentally, the Bernoulli polynomials are known (Exercise 2.15) to have the symmetry

$$B_d(1-x) = (-1)^d B_d(x). \qquad (2.11)$$

This identity coupled with the fact (Exercise 2.16) that

$$B_d = 0 \text{ for all odd } d \ge 3 \qquad (2.12)$$

gives the relation

$$\begin{aligned} L_{\mathcal{P}}(-t) &= \frac{1}{d} \left(B_d(-t+2) - B_d \right) = \frac{1}{d} \left(B_d \left(1 - (t-1) \right) - B_d \right) \\ &= (-1)^d \frac{1}{d} \left(B_d(t-1) - B_d \right) = (-1)^d \, L_{\mathcal{P}^\circ}(t). \end{aligned}$$

Next we compute the Ehrhart series of $\mathcal{P}$. We can actually do this in somewhat greater generality. Namely, for a polytope $\mathcal{Q} \subset \mathbb{R}^{d-1}$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$, define $\operatorname{Pyr}(\mathcal{Q})$, the **pyramid over** $\mathcal{Q}$, as the convex hull of $(\mathbf{v}_1, 0), (\mathbf{v}_2, 0), \dots, (\mathbf{v}_m, 0), (0, \dots, 0, 1)$ in $\mathbb{R}^d$. In our example above, the d-polytope $\mathcal{P}$ is equal to $\operatorname{Pyr}(\square)$ for the unit $(d-1)$-cube $\square$. The number of integer points in $t \operatorname{Pyr}(\mathcal{Q})$ is, by construction,

$$L_{\operatorname{Pyr}(\mathcal{Q})}(t) = 1 + L_{\mathcal{Q}}(1) + L_{\mathcal{Q}}(2) + \cdots + L_{\mathcal{Q}}(t) = 1 + \sum_{j=1}^{t} L_{\mathcal{Q}}(j),$$

because in $t \operatorname{Pyr}(\mathcal{Q})$, there is one lattice point with x_d-coordinate t, we have $L_{\mathcal{Q}}(1)$ lattice points with x_d-coordinate $t-1$, there are $L_{\mathcal{Q}}(2)$ lattice points with x_d-coordinate $t-2$, etc., up to $L_\mathcal{Q}(t)$ lattice points with $x_d = 0$. Figure 2.4 shows an instance for a pyramid over a square.

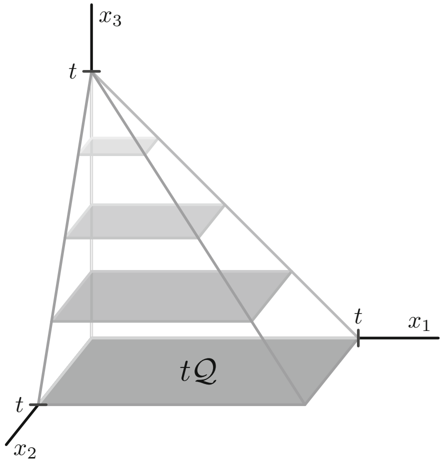

Fig. 2.4 Counting the lattice points in $t\,\mathrm{Pyr}(\mathcal{Q})$.

This identity for $L_{\mathrm{Pyr}(\mathcal{Q})}(t)$ allows us to compute the Ehrhart series of $\mathrm{Pyr}(\mathcal{Q})$ from the Ehrhart series of $\mathcal{Q}$:

Theorem 2.4. $\mathrm{Ehr}_{\mathrm{Pyr}(\mathcal{Q})}(z) = \dfrac{\mathrm{Ehr}_\mathcal{Q}(z)}{1-z}$.

Proof.

$$\begin{aligned}
\mathrm{Ehr}_{\mathrm{Pyr}(\mathcal{Q})}(z) &= 1 + \sum_{t\geq 1} L_{\mathrm{Pyr}(\mathcal{Q})}(t)\, z^t = 1 + \sum_{t\geq 1}\left(1 + \sum_{j=1}^{t} L_\mathcal{Q}(j)\right) z^t \\
&= \sum_{t\geq 0} z^t + \sum_{t\geq 1}\sum_{j=1}^{t} L_\mathcal{Q}(j)\, z^t = \frac{1}{1-z} + \sum_{j\geq 1} L_\mathcal{Q}(j) \sum_{t\geq j} z^t \\
&= \frac{1}{1-z} + \sum_{j\geq 1} L_\mathcal{Q}(j) \frac{z^j}{1-z} = \frac{1+\sum_{j\geq 1} L_\mathcal{Q}(j)\, z^j}{1-z} . \qquad \square
\end{aligned}$$

Our pyramid $\mathcal{P}$ that began this section is a pyramid over the unit $(d-1)$-cube, and so

$$\mathrm{Ehr}_\mathcal{P}(z) = \frac{1}{1-z}\,\frac{\sum_{k=1}^{d-1} A\,(d-1,k)\, z^{k-1}}{(1-z)^d} = \frac{\sum_{k=1}^{d-1} A\,(d-1,k)\, z^{k-1}}{(1-z)^{d+1}} . \quad (2.13)$$

Let's summarize what we have proved for the pyramid over the unit cube.

Theorem 2.5. *Given $d \geq 2$, let $\mathcal{P}$ be the d-pyramid*

$$\mathcal{P} = \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \, 0 \leq x_1, x_2, \ldots, x_{d-1} \leq 1 - x_d \leq 1 \right\}.$$

(a) *The lattice-point enumerator of $\mathcal{P}$ is the polynomial*

$$L_{\mathcal{P}}(t) = \frac{1}{d} \left(B_d(t+2) - B_d \right).$$

(b) *Its evaluation at negative integers yields $(-1)^d L_{\mathcal{P}}(-t) = L_{\mathcal{P}^\circ}(t)$.*

(c) *The Ehrhart series of $\mathcal{P}$ is $\operatorname{Ehr}_{\mathcal{P}}(z) = \frac{\sum_{k=1}^{d-1} A(d-1,k) z^{k-1}}{(1-z)^{d+1}}$.* □

Patterns are emerging...

2.5 The Lattice-Point Enumerators of the Cross-Polytopes

Consider the **cross-polytope** $\diamond$ in $\mathbb{R}^d$ given by the hyperplane description

$$\diamond := \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \, |x_1| + |x_2| + \cdots + |x_d| \leq 1 \right\}. \tag{2.14}$$

Figure 2.5 shows the 3-dimensional instance of $\diamond$, an octahedron. The vertices of $\diamond$ are $(\pm 1, 0, \ldots, 0), (0, \pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)$.

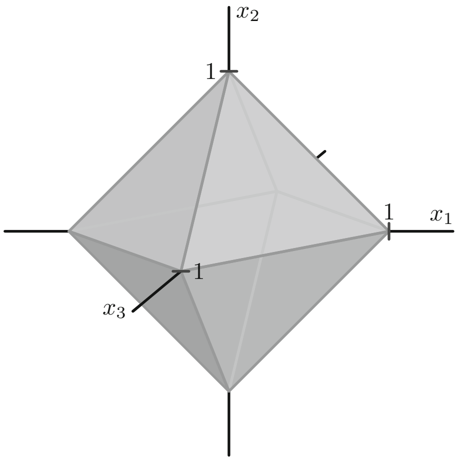

Fig. 2.5 The cross-polytope $\diamond$ in dimension 3.

To compute the discrete volume of $\diamond$, we use a process similar to that of Section 2.4. Namely, for a $(d-1)$-polytope $\mathcal{Q}$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$

such that the origin is in $\mathcal{Q}$, we define $\operatorname{BiPyr}(\mathcal{Q})$, the **bipyramid over** $\mathcal{Q}$, as the convex hull of

$$(\mathbf{v}_1, 0), (\mathbf{v}_2, 0), \dots, (\mathbf{v}_m, 0), (0, \dots, 0, 1), \quad \text{and} \quad (0, \dots, 0, -1).$$

In our example above, the d-dimensional cross-polytope is the bipyramid over the $(d-1)$-dimensional cross-polytope. The number of integer points in $t \operatorname{BiPyr}(\mathcal{Q})$ is, by construction,

$$\begin{aligned} L_{\operatorname{BiPyr}(\mathcal{Q})}(t) &= 2 + 2L_{\mathcal{Q}}(1) + 2L_{\mathcal{Q}}(2) + \cdots + 2L_{\mathcal{Q}}(t-1) + L_{\mathcal{Q}}(t) \\ &= 2 + 2\sum_{j=1}^{t-1} L_{\mathcal{Q}}(j) + L_{\mathcal{Q}}(t). \end{aligned}$$

This identity allows us to compute the Ehrhart series of $\operatorname{BiPyr}(\mathcal{Q})$ from the Ehrhart series of $\mathcal{Q}$, in a manner similar to the proof of Theorem 2.4. We leave the proof of the following result as Exercise 2.23.

Theorem 2.6. *If $\mathcal{Q}$ contains the origin, then* $\operatorname{Ehr}_{\operatorname{BiPyr}(\mathcal{Q})}(z) = \frac{1+z}{1-z} \operatorname{Ehr}_{\mathcal{Q}}(z)$. □

This theorem allows us to compute the Ehrhart series of $\diamond$ effortlessly: The cross-polytope $\diamond$ in dimension 0 is the origin, with Ehrhart series $\frac{1}{1-z}$. The higher-dimensional cross-polytopes can be computed recursively through Theorem 2.6 as

$$\operatorname{Ehr}_{\diamond}(z) = \frac{(1+z)^d}{(1-z)^{d+1}}.$$

Since $\operatorname{Ehr}_{\diamond}(z) = 1 + \sum_{t \ge 1} L_{\diamond}(t)\, z^t$, we can retrieve $L_{\diamond}(t)$ by expanding $\operatorname{Ehr}_{\diamond}(z)$ into its power series at $z = 0$:

$$\begin{aligned} \operatorname{Ehr}_{\diamond}(z) &= \frac{(1+z)^d}{(1-z)^{d+1}} = \frac{\sum_{k=0}^{d} \binom{d}{k} z^k}{(1-z)^{d+1}} \\ &= \sum_{k=0}^{d} \binom{d}{k} z^k \sum_{t \ge 0} \binom{t+d}{d} z^t = \sum_{k=0}^{d} \binom{d}{k} \sum_{t \ge k} \binom{t-k+d}{d} z^t \\ &= \sum_{k=0}^{d} \binom{d}{k} \sum_{t \ge 0} \binom{t-k+d}{d} z^t. \end{aligned}$$

In the last step, we used the fact that $\binom{t-k+d}{d} = 0$ for $0 \le t < k$. But then

$$1 + \sum_{t \ge 1} L_{\diamond}(t)\, z^t = \sum_{t \ge 0} \sum_{k=0}^{d} \binom{d}{k} \binom{t-k+d}{d} z^t,$$

and hence $L_{\diamond}(t) = \sum_{k=0}^{d} \binom{d}{k} \binom{t-k+d}{d}$ for all $t \ge 1$.

We finish this section by counting the *interior* lattice points in $t\Diamond$. We begin by noticing, since t is an integer, that

$$\begin{aligned} L_{\Diamond^\circ}(t) &= \#\left\{(m_1, m_2, \dots, m_d) \in \mathbb{Z}^d : |m_1| + |m_2| + \cdots + |m_d| < t\right\} \\ &= \#\left\{(m_1, m_2, \dots, m_d) \in \mathbb{Z}^d : |m_1| + |m_2| + \cdots + |m_d| \le t-1\right\} \\ &= L_\Diamond(t-1) \,. \end{aligned}$$

On the other hand, we can use (2.7):

$$\begin{aligned} L_\Diamond(-t) &= \sum_{k=0}^{d} \binom{d}{k} \binom{-t-k+d}{d} \\ &= \sum_{k=0}^{d} \binom{d}{k} (-1)^d \binom{t-1+k}{d} \\ &= (-1)^d \sum_{k=0}^{d} \binom{d}{d-k} \binom{t-1+d-k}{d} \\ &= (-1)^d L_\Diamond(t-1) \,. \end{aligned}$$

Comparing the last two computations, we see that $(-1)^d L_\Diamond(-t) = L_{\Diamond^\circ}(t)$. Let's summarize:

Theorem 2.7. *Let $\Diamond$ be the cross-polytope in $\mathbb{R}^d$.*

(a) *The lattice-point enumerator of $\Diamond$ is the polynomial*

$$L_\Diamond(t) = \sum_{k=0}^{d} \binom{d}{k} \binom{t-k+d}{d} .$$

(b) *Its evaluation at negative integers yields $(-1)^d L_\Diamond(-t) = L_{\Diamond^\circ}(t)$.*

(c) *The Ehrhart series of $\mathcal{P}$ is $\mathrm{Ehr}_\Diamond(z) = \frac{(1+z)^d}{(1-z)^{d+1}}$.* □

2.6 Pick's Theorem

Returning to basic concepts, we now give a complete account of $L_\mathcal{P}$ for all integral convex polygons $\mathcal{P}$ in $\mathbb{R}^2$. Denote the number of integer points inside the polygon $\mathcal{P}$ by I, and the number of integer points on the boundary of $\mathcal{P}$ by B. The following result, called *Pick's theorem* in honor of its discoverer Georg Alexander Pick (1859–1942), presents the astonishing fact that the area A of $\mathcal{P}$ can be computed simply by counting lattice points:

Theorem 2.8 (Pick's theorem). *For an integral convex polygon,*

$$A = I + \frac{1}{2}B - 1\,.$$

Proof. We begin by proving that Pick's identity has an additive character: we can decompose $\mathcal{P}$ into the union of two integral polygons $\mathcal{P}_1$ and $\mathcal{P}_2$ by joining two vertices of $\mathcal{P}$ with a line segment, as shown in Figure 2.6.

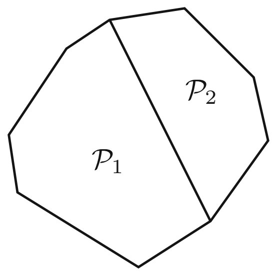

Fig. 2.6 Decomposition of a polygon into two.

We claim that the validity of Pick's identity for $\mathcal{P}$ follows from the validity of Pick's identity for $\mathcal{P}_1$ and $\mathcal{P}_2$. Denote the area, number of interior lattice points, and number of boundary lattice points of $\mathcal{P}_k$ by A_k, I_k, and B_k, respectively, for $k = 1, 2$. Clearly,

$$A = A_1 + A_2\,.$$

Furthermore, if we denote the number of lattice points on the edge common to $\mathcal{P}_1$ and $\mathcal{P}_2$ by L, then

$$I = I_1 + I_2 + L - 2 \qquad \text{and} \qquad B = B_1 + B_2 - 2L + 2\,.$$

Thus

$$\begin{aligned} I + \frac{1}{2}B - 1 &= I_1 + I_2 + L - 2 + \frac{1}{2}B_1 + \frac{1}{2}B_2 - L + 1 - 1 \\ &= I_1 + \frac{1}{2}B_1 - 1 + I_2 + \frac{1}{2}B_2 - 1\,. \end{aligned}$$

This proves the claim. Note that our proof also shows that the validity of Pick's identity for $\mathcal{P}_1$ follows from the validity of Pick's identity for $\mathcal{P}$ and $\mathcal{P}_2$.

Now, every convex polygon can be decomposed into triangles that share a common vertex, as illustrated in Figure 2.7. Hence it suffices to prove Pick's theorem for triangles. Further simplifying the picture, we can embed every integral triangle into an integral rectangle, as suggested by Figure 2.8.

This reduces the proof of Pick's theorem to proving the theorem for integral rectangles whose edges are parallel to the coordinate axes, and for rectangular

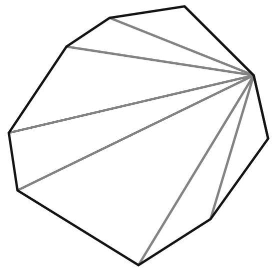

Fig. 2.7 Triangulation of a polygon.

triangles two of whose edges are parallel to the coordinate axes. These two cases are left to the reader as Exercise 2.25. $\square$

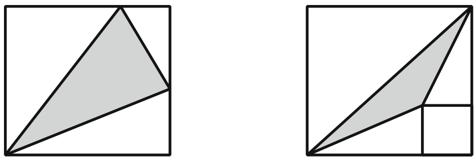

Fig. 2.8 Embedding a triangle in a rectangle.

Pick's theorem allows us to count not only the lattice points strictly inside the polygon $\mathcal{P}$, but also the total number of lattice points contained in $\mathcal{P}$, because this number is

$$I + B = A - \frac{1}{2}B + 1 + B = A + \frac{1}{2}B + 1 \,. \tag{2.15}$$

From this identity, it is now easy to describe the lattice-point enumerator $L_{\mathcal{P}}$:

Theorem 2.9. *Suppose $\mathcal{P}$ is an integral convex polygon with area A and B lattice points on its boundary.*

(a) *The lattice-point enumerator of $\mathcal{P}$ is the polynomial*

$$L_{\mathcal{P}}(t) = A\,t^2 + \frac{1}{2}B\,t + 1 \,.$$

(b) *Its evaluation at negative integers yields the relation*

$$L_{\mathcal{P}}(-t) = L_{\mathcal{P}^\circ}(t) \,.$$

(c) *The Ehrhart series of $\mathcal{P}$ is*

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{\left(A - \frac{B}{2} + 1\right) z^2 + \left(A + \frac{B}{2} - 2\right) z + 1}{(1-z)^3} \,.$$

Note that in the numerator of the Ehrhart series, the coefficient of z^2 is $L_{\mathcal{P}^\circ}(1)$, and the coefficient of z is $L_{\mathcal{P}}(1) - 3$.

Proof. Statement (a) will follow from (2.15) if we can prove that the area of $t\mathcal{P}$ is At^2 and that the number of boundary points on $t\mathcal{P}$ is Bt, which is the content of Exercise 2.26. Statement (b) follows with $L_{\mathcal{P}^\circ}(t) = L_{\mathcal{P}}(t) - Bt$. Finally, the Ehrhart series is

$$\begin{aligned}\mathrm{Ehr}_{\mathcal{P}}(z) &= 1 + \sum_{t \ge 1} L_{\mathcal{P}}(t)\, z^t \\ &= \sum_{t \ge 0} \left(A\, t^2 + \frac{B}{2}\, t + 1 \right) z^t \\ &= A\, \frac{z^2 + z}{(1-z)^3} + \frac{B}{2}\, \frac{z}{(1-z)^2} + \frac{1}{1-z} \\ &= \frac{\left(A - \frac{B}{2} + 1\right) z^2 + \left(A + \frac{B}{2} - 2\right) z + 1}{(1-z)^3}\,. \end{aligned}$$

□

2.7 Polygons with Rational Vertices

In this section we will establish formulas for the number of integer points in a *rational* convex polygon and its integral dilates.

A natural first step is to fix a triangulation of the polygon $\mathcal{P}$, which reduces our problem to that of counting integer points in rational *triangles*. However, this procedure merits some remarks. After counting lattice points in the triangles, we need to reassemble the triangles to form the polygon. But then we need to take care of the overcounting on line segments (where the triangles meet). Computing the number of lattice points on rational line segments is considerably easier than enumerating lattice points in 2-dimensional regions; however, it is still nontrivial (see Theorem 1.5).

After triangulating $\mathcal{P}$, we can further simplify the picture by embedding an arbitrary rational triangle in a rational rectangle, as in Figure 2.8. To compute lattice points in a triangle, we can first count the points in a rectangle with edges parallel to the coordinate axes, and then subtract the number of points in three right triangles, each with two edges are parallel to the axes, and possibly another rectangle, as shown in Figure 2.8. Since rectangles are easy to deal with (see Exercise 2.2), the problem reduces to finding a formula for a right triangle two of whose edges are parallel to the coordinate axes.

We now adjust and expand our generating-function machinery to these right triangles. Such a triangle $\mathcal{T}$ is a subset of $\mathbb{R}^2$ consisting of all points (x, y) satisfying

$$x \ge \frac{a}{d}\,, \ y \ge \frac{b}{d}\,, \ ex + fy \le r$$

for some integers a, b, d, e, f, r (with $ea + fb \le rd$; otherwise, the triangle would be empty). Because the lattice-point count is invariant under horizontal and vertical integer translations and under flipping about the x- or y-axis, we may assume that $a, b, d, e, f, r \ge 0$ and $a, b < d$. (One should meditate about this fact for a minute.) Thus we arrive at the triangle $\mathcal{T}$ depicted in Figure 2.9.

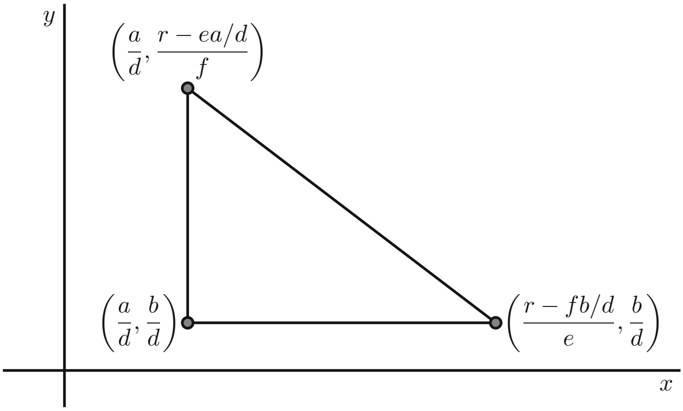

Fig. 2.9 A right rational triangle.

To make our life a little easier, let's assume for the moment that e and f are relatively prime; we will deal with the general case in the exercises. So let

$$\mathcal{T} = \left\{ (x, y) \in \mathbb{R}^2 : \, x \ge \frac{a}{d} \, , \; y \ge \frac{b}{d} \, , \; ex + fy \le r \right\} . \tag{2.16}$$

To derive a formula for

$$L_{\mathcal{T}}(t) = \# \left\{ (m, n) \in \mathbb{Z}^2 : \, m \ge \frac{ta}{d} \, , \; n \ge \frac{tb}{d} \, , \; em + fn \le tr \right\} ,$$

we want to use methods similar to those in Chapter 1. As in Section 2.3, we introduce a slack variable s:

$$\begin{aligned} L_{\mathcal{T}}(t) &= \# \left\{ (m, n) \in \mathbb{Z}^2 : \, m \ge \frac{ta}{d} \, , \; n \ge \frac{tb}{d} \, , \; em + fn \le tr \right\} \\ &= \# \left\{ (m, n, s) \in \mathbb{Z}^3 : \, m \ge \frac{ta}{d} \, , \; n \ge \frac{tb}{d} \, , \; s \ge 0, \; em + fn + s = tr \right\} . \end{aligned}$$

This counting function can now, as earlier, be interpreted as the coefficient of z^{tr} in the function

$$\left(\sum_{m \geq \frac{ta}{d}} z^{em} \right) \left(\sum_{n \geq \frac{tb}{d}} z^{fn} \right) \left(\sum_{s \geq 0} z^{s} \right) .$$

Here the subscript (e.g., $m \geq \frac{ta}{d}$) under a summation sign means *sum over all integers satisfying this condition.* For example, in the first sum, we begin with the least integer greater than or equal to $\frac{ta}{d}$, which is denoted by $\left\lceil \frac{ta}{d} \right\rceil$ (and is equal to $\left\lfloor \frac{ta-1}{d} \right\rfloor + 1$ by Exercise 1.4(j)). Hence the above generating function can be rewritten as

$$\begin{aligned} \left(\sum_{m \geq \lceil \frac{ta}{d} \rceil} z^{em} \right) \left(\sum_{n \geq \lceil \frac{tb}{d} \rceil} z^{fn} \right) \left(\sum_{s \geq 0} z^{s} \right) &= \frac{z^{\lceil \frac{ta}{d} \rceil e}}{1-z^e} \frac{z^{\lceil \frac{tb}{d} \rceil f}}{1-z^f} \frac{1}{1-z} \\ &= \frac{z^{u+v}}{(1-z^e)(1-z^f)(1-z)} , \end{aligned} \tag{2.17}$$

where we have introduced, for ease of notation,

$$u := \left\lceil \frac{ta}{d} \right\rceil e \qquad \text{and} \qquad v := \left\lceil \frac{tb}{d} \right\rceil f . \tag{2.18}$$

To extract the coefficient of z^{tr} of our generating function (2.17), we use familiar methods. As usual, we shift this coefficient to a constant term:

$$\begin{aligned} L_{\mathcal{T}}(t) &= \operatorname{const} \left(\frac{z^{u+v-tr}}{(1-z^e)(1-z^f)(1-z)} \right) \\ &= \operatorname{const} \left(\frac{1}{(1-z^e)(1-z^f)(1-z) z^{tr-u-v}} \right) . \end{aligned}$$

Before we apply the partial fraction machinery to this function, we should make sure that it is indeed a proper rational function, that is, that the total degree satisfies

$$u + v - tr - e - f - 1 < 0 \tag{2.19}$$

(see Exercise 2.33). Then we expand into partial fractions (here we are using our assumption that e and f do not have any common factors):

$$\begin{aligned} &\frac{1}{(1-z^e)(1-z^f)(1-z) z^{tr-u-v}} \\ &\quad = \sum_{j=1}^{e-1} \frac{A_j}{z-\xi_e^j} + \sum_{j=1}^{f-1} \frac{B_j}{z-\xi_f^j} + \sum_{k=1}^{3} \frac{C_k}{(z-1)^k} + \sum_{k=1}^{tr-u-v} \frac{D_k}{z^k} . \end{aligned} \tag{2.20}$$

As we have seen numerous times before, the coefficients D_k do not contribute to the constant term, so that we obtain

$$L_{\mathcal{T}}(t) = -\sum_{j=1}^{e-1} \frac{A_j}{\xi_e^j} - \sum_{l=1}^{f-1} \frac{B_l}{\xi_f^l} - C_1 + C_2 - C_3 \,. \tag{2.21}$$

We invite the reader to compute the coefficients appearing in this formula (Exercise 2.34):

$$\begin{aligned}
A_j &= -\frac{\xi_e^{j(v-tr+1)}}{e\left(1-\xi_e^{jf}\right)\left(1-\xi_e^j\right)}\,, \\
B_l &= -\frac{\xi_f^{l(u-tr+1)}}{f\left(1-\xi_f^{le}\right)\left(1-\xi_f^l\right)}\,, \\
C_1 &= -\frac{(u+v-tr)^2}{2ef} + \frac{u+v-tr}{2}\left(-\frac{1}{ef}+\frac{1}{e}+\frac{1}{f}\right) + \frac{1}{4}\left(\frac{1}{e}+\frac{1}{f}-1\right) \\
&\quad - \frac{1}{12}\left(\frac{e}{f}+\frac{1}{ef}+\frac{f}{e}\right), \\
C_2 &= -\frac{u+v-tr+1}{ef} + \frac{1}{2e} + \frac{1}{2f}\,, \\
C_3 &= -\frac{1}{ef}\,.
\end{aligned} \tag{2.22}$$

Putting these ingredients into (2.21) yields the following formula for our lattice-point count.

Theorem 2.10. *For the rectangular rational triangle $\mathcal{T}$ given by* (2.16)*, where e and f are relatively prime,*

$$\begin{aligned}
L_{\mathcal{T}}(t) = \frac{1}{2ef}(tr-u-v)^2 &+ \frac{1}{2}(tr-u-v)\left(\frac{1}{e}+\frac{1}{f}+\frac{1}{ef}\right) \\
&+ \frac{1}{4}\left(1+\frac{1}{e}+\frac{1}{f}\right) + \frac{1}{12}\left(\frac{e}{f}+\frac{f}{e}+\frac{1}{ef}\right) \\
&+ \frac{1}{e}\sum_{j=1}^{e-1}\frac{\xi_e^{j(v-tr)}}{\left(1-\xi_e^{jf}\right)\left(1-\xi_e^j\right)} + \frac{1}{f}\sum_{l=1}^{f-1}\frac{\xi_f^{l(u-tr)}}{\left(1-\xi_f^{le}\right)\left(1-\xi_f^l\right)}\,. \qquad \square
\end{aligned}$$

This identity can be rephrased in terms of the Fourier–Dedekind sum that we introduced in (1.13):

$$\begin{aligned}
L_{\mathcal{T}}(t) = \frac{1}{2ef}(tr-u-v)^2 &+ \frac{1}{2}(tr-u-v)\left(\frac{1}{e}+\frac{1}{f}+\frac{1}{ef}\right) \\
&+ \frac{1}{4}\left(1+\frac{1}{e}+\frac{1}{f}\right) + \frac{1}{12}\left(\frac{e}{f}+\frac{f}{e}+\frac{1}{ef}\right) \\
&+ s_{v-tr}(f,1;e) + s_{u-tr}(e,1;f)\,.
\end{aligned}$$

The general formula for $L_{\mathcal{T}}$—not assuming that e and f are relatively prime—is the content of Exercise 2.36.

Let's pause for a moment and study the nature of $L_{\mathcal{T}}$ as a function of t. Aside from the last two finite sums (which will be put in the spotlight in Chapter 8) and the appearance of u and v, the function $L_{\mathcal{T}}$ is a quadratic polynomial in t. And in those two sums, t appears only in the exponent of roots of unity, namely as the exponent of ξ_e and ξ_f. As a function of t, ξ_e^t is *periodic* with period e, and similarly, ξ_f^t is periodic with period f. We should also remember that u and v are functions of t; but they can be easily written in terms of the fractional-part function, which again gives rise to periodic functions in t. So $L_{\mathcal{T}}(t)$ is a (quadratic) "polynomial" in t, whose coefficients are periodic functions in t. This is reminiscent of the counting functions of Chapter 1, which showed a similar periodic-polynomial behavior. Inspired by both examples, we define a **quasipolynomial** Q as an expression of the form

$$Q(t) = c_n(t)\ t^n + \cdots + c_1(t)\ t + c_0(t)\,,$$

where $c_0, \ldots, c_n$ are periodic functions in t. The **degree** of Q is n,[5] and the least common period of $c_0, \ldots, c_n$ is the **period** of Q. Alternatively, for a quasipolynomial Q, there exist a positive integer k and polynomials $p_0, p_1, \ldots, p_{k-1}$ such that

$$Q(t) = \begin{cases} p_0(t) & \text{if } t \equiv 0 \bmod k, \\ p_1(t) & \text{if } t \equiv 1 \bmod k, \\ \quad\vdots & \\ p_{k-1}(t) & \text{if } t \equiv k-1 \bmod k. \end{cases}$$

The minimal such k is the period of Q, and for this minimal k, the polynomials $p_0, p_1, \ldots, p_{k-1}$ are the **constituents** of Q.

By the triangulation and embedding-in-a-box arguments that began this section, we can now state a general structural result for rational polygons.

Theorem 2.11. *Let $\mathcal{P}$ be any rational polygon. Then $L_{\mathcal{P}}(t)$ is a quasipolynomial of degree 2. Its leading coefficient is the area of $\mathcal{P}$ (in particular, it is a constant).*

We have the technology at this point to study the period of L_P; we let the reader enjoy the ensuing details (see Exercise 2.37).

Proof. By Exercises 2.2 and 2.36 (the general form of Theorem 2.10), the theorem holds for rational rectangles and right triangles whose edges are parallel to the axes. Now use the additivity of both degree-2 quasipolynomials and areas along with Theorem 1.5. $\square$

[5] Here we tacitly assume that c_n is not the zero function.

2.8 Euler's Generating Function for General Rational Polytopes

By now, we have computed several instances of counting functions by setting up a generating function that fits the particular problem in which we are interested. In this section, we set up such a generating function for the lattice-point enumerator of an arbitrary rational polytope. Such a polytope is given by its hyperplane description as an intersection of half-spaces and hyperplanes. The half-spaces are algebraically given by linear inequalities, the hyperplanes by linear equations. If the polytope is rational, we can choose the coefficients of these inequalities and equations to be integers (Exercise 2.7). To unify both descriptions, we can introduce slack variables to turn the half-space inequalities into equalities. Furthermore, by translating our polytope into the nonnegative orthant (we can always shift a polytope by an integer vector without changing the lattice-point count), we may assume that all points in the polytope have nonnegative coordinates. In summary, after a harmless integer translation, we can describe every rational polytope $\mathcal{P}$ as

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \, \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\} \tag{2.23}$$

for some integral matrix $\mathbf{A} \in \mathbb{Z}^{m \times d}$ and some integer vector $\mathbf{b} \in \mathbb{Z}^m$. (Note that d is not necessarily the dimension of $\mathcal{P}$.) To describe the t^{th} dilate of $\mathcal{P}$, we simply scale a point $\mathbf{x} \in \mathcal{P}$ by $\frac{1}{t}$, or alternatively, multiply $\mathbf{b}$ by t:

$$t\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \, \mathbf{A}\,\frac{\mathbf{x}}{t} = \mathbf{b} \right\} = \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \, \mathbf{A}\,\mathbf{x} = t\mathbf{b} \right\}.$$

Hence the lattice-point enumerator of $\mathcal{P}$ is the counting function

$$L_{\mathcal{P}}(t) = \# \left\{ \mathbf{x} \in \mathbb{Z}^d_{\geq 0} : \, \mathbf{A}\,\mathbf{x} = t\mathbf{b} \right\}. \tag{2.24}$$

Example 2.12. Suppose $\mathcal{P}$ is the quadrilateral with vertices $(0,0)$, $(2,0)$, $(1,1)$, and $\left(0, \frac{3}{2}\right)$ pictured in Figure 2.10. The half-space-inequality description of $\mathcal{P}$ is

$$\mathcal{P} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \, x_1, x_2 \geq 0, \, \begin{array}{c} x_1 + 2x_2 \leq 3, \\ x_1 + x_2 \leq 2 \end{array} \right\}.$$

Thus,

$$\begin{aligned} L_{\mathcal{P}}(t) &= \# \left\{ (x_1, x_2) \in \mathbb{Z}^2 : \, x_1, x_2 \geq 0, \, \begin{array}{c} x_1 + 2x_2 \leq 3t, \\ x_1 + x_2 \leq 2t \end{array} \right\} \\ &= \# \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : \, x_1, x_2, x_3, x_4 \geq 0, \, \begin{array}{c} x_1 + 2x_2 + x_3 = 3t, \\ x_1 + x_2 + x_4 = 2t \end{array} \right\} \\ &= \# \left\{ \mathbf{x} \in \mathbb{Z}^4_{\geq 0} : \, \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3t \\ 2t \end{pmatrix} \right\}. \end{aligned}$$

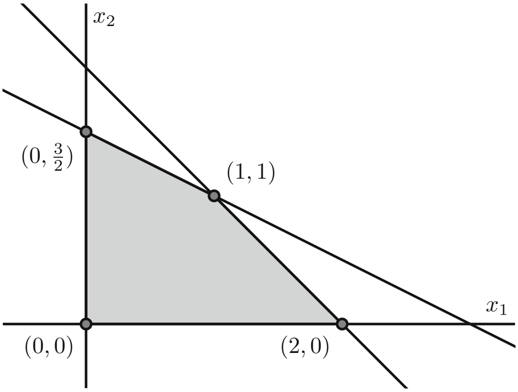

Fig. 2.10 The quadrilateral $\mathcal{P}$ from Example 2.12.

Using the ideas from Sections 1.3, 1.5, 2.3, and 2.7, we now construct a generating function for this counting function. In those previous sections, the lattice-point enumerator could be described with only *one* nontrivial linear equation, whereas now we have a system of such linear constraints. However, we can use the same approach of encoding the linear equation into geometric series; we just need more than one variable. When we expand the function

$$f(z_1, z_2) := \frac{1}{(1 - z_1 z_2)(1 - z_1^2 z_2)(1 - z_1)(1 - z_2) \, z_1^{3t} z_2^{2t}}$$

into geometric series,

$$\begin{aligned} f(z_1, z_2) &= \\ &= \left(\sum_{n_1 \ge 0} (z_1 z_2)^{n_1} \right) \left(\sum_{n_2 \ge 0} \left(z_1^2 z_2\right)^{n_2} \right) \left(\sum_{n_3 \ge 0} z_1^{n_3} \right) \left(\sum_{n_4 \ge 0} z_2^{n_4} \right) \frac{1}{z_1^{3t} z_2^{2t}} \\ &= \sum_{n_1, \dots, n_4 \ge 0} z_1^{n_1 + 2n_2 + n_3 - 3t} z_2^{n_1 + n_2 + n_4 - 2t} . \end{aligned}$$

When we compute the constant term (in both z_1 and z_2), we are counting solutions $(n_1, n_2, n_3, n_4) \in \mathbb{Z}_{\ge 0}^4$ to

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} = \begin{pmatrix} 3t \\ 2t \end{pmatrix},$$

that is, the constant term of $f(z_1, z_2)$ counts the integer points in $\mathcal{P}$:

$$L_{\mathcal{P}}(t) = \operatorname{const} \frac{1}{(1 - z_1 z_2)(1 - z_1^2 z_2)(1 - z_1)(1 - z_2) \, z_1^{3t} z_2^{2t}} .$$

We invite the reader to actually compute this constant term (Exercise 2.38). It turns out to be

$$\frac{7}{4}\,t^2 + \frac{5}{2}\,t + \frac{7+(-1)^t}{8}\,. \qquad \square$$

Returning to the general case of a polytope $\mathcal{P}$ given by (2.23), we denote the columns of $\mathbf{A}$ by $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_d$. Let $\mathbf{z} = (z_1, z_2, \dots, z_m)$ and expand the function

$$\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})\,(1-\mathbf{z}^{\mathbf{c}_2})\cdots(1-\mathbf{z}^{\mathbf{c}_d})\,\mathbf{z}^{t\mathbf{b}}} \tag{2.25}$$

in terms of geometric series:

$$\left(\sum_{n_1\ge 0}\mathbf{z}^{n_1\mathbf{c}_1}\right)\left(\sum_{n_2\ge 0}\mathbf{z}^{n_2\mathbf{c}_2}\right)\cdots\left(\sum_{n_d\ge 0}\mathbf{z}^{n_d\mathbf{c}_d}\right)\frac{1}{\mathbf{z}^{t\mathbf{b}}}\,.$$

Here we use the abbreviating notation $\mathbf{z}^{\mathbf{a}} := z_1^{a_1} z_2^{a_2}\cdots z_m^{a_m}$ for the vectors $\mathbf{z} = (z_1, z_2, \dots, z_m) \in \mathbb{C}^m$ and $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbb{Z}^m$. In multiplying out everything, a typical exponent will look like

$$n_1\mathbf{c}_1 + n_2\mathbf{c}_2 + \cdots + n_d\mathbf{c}_d - t\mathbf{b} = \mathbf{A}\mathbf{n} - t\mathbf{b}\,,$$

where $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d_{\ge 0}$. That is, if we take the constant term of our generating function (2.25), we are counting integer vectors $\mathbf{n} \in \mathbb{Z}^d_{\ge 0}$ satisfying

$$\mathbf{A}\mathbf{n} - t\mathbf{b} = \mathbf{0}\,, \qquad \text{that is,} \qquad \mathbf{A}\mathbf{n} = t\mathbf{b}\,.$$

So this constant term will pick up exactly the number of lattice points $\mathbf{n} \in \mathbb{Z}^d_{\ge 0}$ in $t\mathcal{P}$:

Theorem 2.13 (Euler's generating function). *Suppose the rational polytope $\mathcal{P}$ is given by* (2.23)*. Then the lattice-point enumerator of $\mathcal{P}$ can be computed as follows:*

$$L_{\mathcal{P}}(t) = \operatorname{const}\left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})\,(1-\mathbf{z}^{\mathbf{c}_2})\cdots(1-\mathbf{z}^{\mathbf{c}_d})\,\mathbf{z}^{t\mathbf{b}}}\right). \qquad \square$$

We finish this section with rephrasing this constant-term identity in terms of Ehrhart series.

Corollary 2.14. *Suppose the rational polytope $\mathcal{P}$ is given by* (2.23)*. Then the Ehrhart series of $\mathcal{P}$ can be computed as*

$$\operatorname{Ehr}_{\mathcal{P}}(x) = \operatorname{const}\left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})\,(1-\mathbf{z}^{\mathbf{c}_2})\cdots(1-\mathbf{z}^{\mathbf{c}_d})\left(1-\frac{x}{\mathbf{z}^{\mathbf{b}}}\right)}\right).$$

Proof. By Theorem 2.13,

$$\begin{aligned}
\mathrm{Ehr}_{\mathcal{P}}(x) &= \sum_{t\geq 0} \operatorname{const}\left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})(1-\mathbf{z}^{\mathbf{c}_2})\cdots(1-\mathbf{z}^{\mathbf{c}_d})\,\mathbf{z}^{t\mathbf{b}}}\right)x^t \\
&= \operatorname{const}\left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})(1-\mathbf{z}^{\mathbf{c}_2})\cdots(1-\mathbf{z}^{\mathbf{c}_d})}\sum_{t\geq 0}\frac{x^t}{\mathbf{z}^{t\mathbf{b}}}\right) \\
&= \operatorname{const}\left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})(1-\mathbf{z}^{\mathbf{c}_2})\cdots(1-\mathbf{z}^{\mathbf{c}_d})}\,\frac{1}{1-\frac{x}{\mathbf{z}^{\mathbf{b}}}}\right). \qquad \square
\end{aligned}$$

Notes

1. Convex polytopes are beautiful objects with a rich history and interesting theory, which we have only glimpsed here. For good introductions to polytopes, we recommend [68,126,258]. Polytopes appear in a vast range of current research areas, including Gröbner bases and commutative algebra [236], combinatorial optimization [95,213], integral geometry [149], K-theory [73], and geometry of numbers [220].

2. The distinction between the vertex and hyperplane description of a convex polytope leads to an interesting algorithmic question; namely, how quickly can we retrieve the first piece of data from the second and vice versa [213,258]?

3. Ehrhart series are named after Eugène Ehrhart (1906–2000),[6] who proved the main structure theorems which we will see in Chapter 3. The Ehrhart series of a polytope is an example of a *Hilbert–Poincaré series*. These series appear in the study of *graded algebras* (see, for example, [134,229]); in the Ehrhart case, this algebra lives in $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_d^{\pm 1}, z_{d+1}]$ and is generated by the monomials $\mathbf{z}^{\mathbf{m}}$, where $\mathbf{m}$ ranges over all integer points in cone($\mathcal{P}$), the cone over $\mathcal{P}$, which we will define in Chapter 3. Ehrhart series also appear in the context of *toric varieties*, a vast and fruitful subject [91,116].

4. The Eulerian numbers $A(d,k)$ are named after Leonhard Euler (1707–1783)[7] and arise naturally in the statistics of permutations: $A(d,k)$ counts permutations of $\{1,2,\dots,d\}$ with $k-1$ ascents. For more on $A(d,k)$, see [87, Section 6.5] and [139]; for more connections between $A(d,k)$ and Ehrhart theory, see [32].

5. The pyramids of Section 2.4 have an interpretation as *order polytopes* [230]. A curious fact about the lattice-point enumerators of these pyramids is that they have arbitrarily large real roots as the dimension grows [36].

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[6] For more information about Ehrhart, see http://icps.u-strasbg.fr/∼clauss/Ehrhart.html.

[7] For more information about Euler, see http://www-history.mcs.st-andrews.ac.uk/Biographies/Euler.html.

6. The counting function $L_\diamond$ for the cross-polytope can, incidentally, also be written as

$$\sum_{k=0}^{\min(d,t)} 2^k \binom{d}{k} \binom{t}{k} .$$

In particular, $L_\diamond$ is symmetric in d and t. The cross-polytope counting functions bear a connection to Laguerre polynomials, the d-dimensional harmonic oscillator, and the Riemann hypothesis. This connection appeared in [75], where Daniel Bump, Kwok–Kwong Choi, Pär Kurlberg, and Jeffrey Vaaler also found a curious fact about the roots of the polynomials $L_\diamond$: they all have real part $-\frac{1}{2}$ (an instance of a *local Riemann hypothesis*). This fact was proved independently by Peter Kirschenhofer, Attila Pethő, and Robert Tichy [148]; see also the notes in Chapter 4.

7. Theorem 2.8 marks the beginning of the general study of lattice-point enumeration in polytopes. Its amazingly simple statement was discovered by Georg Alexander Pick (1859–1942)[8] in 1899 [190]. Pick's theorem holds also for a nonconvex polygon, provided its boundary forms a simple curve. In Chapter 14, we prove a generalization of Pick's theorem that includes nonconvex curves.

8. The results of Section 2.7 appeared in [43]. We will see in Chapter 8 that the finite sums over roots of unity can be rephrased in terms of *Dedekind–Rademacher sums*, which—as we will also see in Chapter 8—can be computed very quickly. The theorems of Section 2.7 will then imply that the discrete volume of every rational polygon can be computed efficiently.

9. If we replace $t\mathbf{b}$ in (2.24) by a variable integer vector $\mathbf{v}$, the counting function

$$f(\mathbf{v}) = \#\left\{\mathbf{x} \in \mathbb{Z}^d_{\geq 0} : \mathbf{A}\,\mathbf{x} = \mathbf{v}\right\}$$

is called a *vector partition function*: it counts partitions of the vector $\mathbf{v}$ in terms of the columns of $\mathbf{A}$. Vector partition functions are the multivariate analogues of our lattice-point enumerators $L_\mathcal{P}(t)$. They have many interesting properties and give rise to intriguing open questions [30, 58, 90, 235, 240].

10. While Leonhard Euler most likely did not think of lattice-point enumeration in the sense of Ehrhart, we attribute Theorem 2.13 to him, since he certainly worked with generating functions of this type, probably thinking of them as vector partition functions. Percy MacMahon (1854–1929)[9] developed powerful machinery for manipulating multivariate generating functions [167]; his viewpoint and motivation came from integer partitions, but his work

[8] For more information about Pick, see
http://www-history.mcs.st-andrews.ac.uk/Biographies/Pick.html.

[9] For more information about MacMahon, see
http://www-history.mcs.st-andrews.ac.uk/Biographies/MacMahon.html.

can be applied to more general linear-constraint settings, such as vector partition functions. The potential of Euler's generating function for Ehrhart polynomials was already realized by Ehrhart [109, 111]. Several modern approaches to computing Ehrhart polynomials are based on Theorem 2.13 (see, for example, [29, 67, 159]).

Exercises

2.1. ♣ Fix positive integers a, b, c, d such that $\gcd(a,b) = \gcd(c,d) = 1$ and $\frac{a}{b} < \frac{c}{d}$, and let $\mathcal{P}$ be the interval $\left[\frac{a}{b}, \frac{c}{d}\right]$ (so $\mathcal{P}$ is a 1-dimensional rational convex polytope). Compute $L_{\mathcal{P}}(t) = \#\left(t\mathcal{P} \cap \mathbb{Z}\right)$ and $L_{\mathcal{P}^\circ}(t)$ and show directly that $L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}^\circ}(t)$ are quasipolynomials with period $\operatorname{lcm}(b,d)$ that satisfy

$$L_{\mathcal{P}^\circ}(-t) = -L_{\mathcal{P}}(t)\,.$$

(*Hint:* Exercise 1.4(i).)

2.2. ♣ Fix positive rational numbers a_1, b_1, a_2, b_2 and let $\mathcal{R}$ be the rectangle with vertices (a_1, b_1), (a_2, b_1), (a_2, b_2), and (a_1, b_2). Compute $L_{\mathcal{R}}(t)$ and $\operatorname{Ehr}_{\mathcal{R}}(z)$.

2.3. Fix positive integers a and b, and let $\mathcal{T}$ be a triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$.

(a) Compute $L_{\mathcal{T}}(t)$ and $\operatorname{Ehr}_{\mathcal{T}}(z)$.
(b) Use (a) to derive the following formula for the greatest common divisor of a and b:

$$\gcd(a,b) = 2\sum_{k=1}^{b-1}\left\lfloor \frac{ka}{b} \right\rfloor + a + b - ab\,.$$

(*Hint:* Exercise 1.12.)

2.4. Prove that for two polytopes $\mathcal{P} \subset \mathbb{R}^m$ and $\mathcal{Q} \subset \mathbb{R}^n$,

$$\#\left((\mathcal{P} \times \mathcal{Q}) \cap \mathbb{Z}^{m+n}\right) = \#\left(\mathcal{P} \cap \mathbb{Z}^m\right) \cdot \#\left(\mathcal{Q} \cap \mathbb{Z}^n\right).$$

Hence, $L_{\mathcal{P}\times\mathcal{Q}}(t) = L_{\mathcal{P}}(t)\, L_{\mathcal{Q}}(t)$.

2.5. Prove that if $\mathcal{F}$ is a face of $\mathcal{P}$ and $\mathcal{G}$ is a face of $\mathcal{F}$, then $\mathcal{G}$ is also a face of $\mathcal{P}$. (That is, the face relation is transitive.)

2.6. ♣ Suppose Δ is a d-simplex with vertices $V = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}\}$. Prove that for every nonempty subset $W \subseteq V$, $\operatorname{conv} W$ is a face of Δ, and conversely, that every face of Δ is of the form $\operatorname{conv} W$ for some $W \subseteq V$. Conclude the following corollaries from this characterization of the faces of a simplex:

(a) A face of a simplex is again a simplex.

(b) The intersection of two faces of a simplex Δ is again a face of Δ.

2.7. ♣ Prove that a rational convex polytope can be described by a system of linear inequalities and equations with *integral* coefficients.

2.8. ♣ Prove the properties (2.3) of the Eulerian numbers for all integers $1 \le k \le d$, namely:

(a) $A(d,k) = A(d, d+1-k)$
(b) $A(d,k) = (d-k+1)\, A(d-1, k-1) + k\, A(d-1,k)$
(c) $\displaystyle\sum_{k=0}^{d} A(d,k) = d!$
(d) $\displaystyle A(d,k) = \sum_{j=0}^{k} (-1)^j \binom{d+1}{j} (k-j)^d.$

2.9. ♣ Prove (2.6); namely, for $d \ge 0$, $\frac{1}{(1-z)^{d+1}} = \sum_{k \ge 0} \binom{d+k}{d} z^k$.

2.10. ♣ Prove (2.7): For $t, k \in \mathbb{Z}$ and $d \in \mathbb{Z}_{>0}$,

$$(-1)^d \binom{-t+k}{d} = \binom{t+d-1-k}{d}.$$

2.11. The **Stirling numbers of the first kind**, $\operatorname{stirl}(n,m)$, are defined through the finite generating function

$$x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} \operatorname{stirl}(n,m)\, x^m.$$

(See also [1, Sequence A008275].) Prove that

$$\frac{1}{d!} \sum_{k=0}^{d} (-1)^{d-k} \operatorname{stirl}(d+1, k+1)\, t^k = \binom{d+t}{d},$$

the lattice-point enumerator for the standard d-simplex.

2.12. Give a direct proof that the number of solutions $(m_1, m_2, \dots, m_{d+1}) \in \mathbb{Z}_{\ge 0}^{d+1}$ to $m_1 + m_2 + \cdots + m_{d+1} = t$ equals $\binom{d+t}{d}$. (*Hint:* think of t objects lined up and separated by d walls.)

2.13. Compute $L_{\mathcal{P}}(t)$, where $\mathcal{P}$ is the regular tetrahedron with vertices $(0,0,0), (1,1,0), (1,0,1), (0,1,1)$.

2.14. ♣ Prove that the power series

$$\sum_{k \ge 0} \frac{B_k}{k!}\, z^k$$

that defines the Bernoulli numbers has radius of convergence 2π.

2.15. ♣ Prove (2.11); namely, $B_d(1-x) = (-1)^d B_d(x)$.

2.16. ♣ Prove (2.12); namely, $B_d = 0$ for all odd $d \geq 3$.

2.17. Show that for each positive integer n,

$$n\, x^{n-1} = \sum_{k=1}^{n} \binom{n}{k} B_{n-k}(x) \, .$$

This gives us a change of basis for the polynomials of degree $\leq n$, allowing us to represent every polynomial as a sum of Bernoulli polynomials.

2.18. As a complement to the previous exercise, show that we also have a change of basis in the other direction. Namely, we can represent a single Bernoulli polynomial in terms of the monomials as follows:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k\, x^{n-k}.$$

2.19. Show that for all positive integers m, n and for all $x \in \mathbb{R}$,

$$\frac{1}{m} \sum_{k=0}^{m-1} B_n \left(x + \frac{k}{m} \right) = m^{-n} B_n(mx) \, .$$

(This is a *Hecke-operator*-type identity, originally found by Joseph Ludwig Raabe in 1851.)

2.20. Show that $B_n(x+1) - B_n(x) = n\, x^{n-1}$.

2.21. An alternative way to define the Bernoulli polynomials is to give elementary properties that uniquely characterize them. Show that the following three properties uniquely determine the Bernoulli polynomials, as defined in the text by (2.8):

(a) $B_0(x) = 1$.
(b) $\frac{dB_n(x)}{dx} = n\, B_{n-1}(x)$, for all $n \geq 1$.
(c) $\int_0^1 B_n(x)\, dx = 0$, for all $n \geq 1$.

2.22. Use (2.13) to derive the following identity, which expresses a Bernoulli polynomial in terms of Eulerian numbers and binomial coefficients:

$$\frac{1}{d} \left(B_d(t+2) - B_d \right) = A\,(d-1, d-1) \binom{t+d-2}{d} + A\,(d-1, d-2) \binom{t+d-3}{d} + \cdots + A\,(d-1, 1) \binom{t}{d} \, .$$

2.23. ♣ Prove Theorem 2.6: $\mathrm{Ehr}_{\mathrm{BiPyr}(\mathcal{Q})}(z) = \frac{1+z}{1-z} \, \mathrm{Ehr}_{\mathcal{Q}}(z)$.

2.24. A **Delannoy path** is a path through lattice points in the plane with steps $(1,0)$, $(0,1)$, and $(1,1)$ (i.e., "east," "north," and "northeast"). Find a recurrence for the number $D(m,n)$ of Delannoy paths from the origin to the point (m,n), and use it to compute the two-variable generating function

$$\sum_{m\ge 0}\sum_{n\ge 0} D(m,n)\, z^m w^n = \frac{1}{1-z-w-zw}\,.$$

Conclude from this generating function that the Ehrhart polynomial $L_\diamond(t)$ of the d-dimensional cross-polytope equals $D(t,d)$. (*Hint:* start with the d^{th} derivative of the generating function of $D(t,d)$ with respect to w.)

2.25. ♣ Let $\mathcal{R}$ be an integral rectangle whose edges are parallel to the coordinate axes, and let $\mathcal{T}$ be a rectangular triangle two of whose edges are parallel to the coordinate axes. Show that Pick's theorem holds for $\mathcal{R}$ and $\mathcal{T}$.

2.26. ♣ Suppose $\mathcal{P}$ is an integral polygon with area A and with B lattice points on its boundary. Show that the area of $t\mathcal{P}$ is At^2, and the number of boundary points on $t\mathcal{P}$ is Bt. (*Hint:* Exercise 1.12.)

2.27. Let $\mathcal{P}$ be the self-intersecting polygon defined by the line segments $[(0,0),(4,2)]$, $[(4,2),(4,0)]$, $[(4,0),(0,2)]$, and $[(0,2),(0,0)]$. Show that Pick's theorem does not hold for $\mathcal{P}$.

2.28. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are integral polygons, and that $\mathcal{Q}$ lies entirely inside $\mathcal{P}$. Then the area bounded by the boundaries of $\mathcal{P}$ and $\mathcal{Q}$, denoted by $\mathcal{P}-\mathcal{Q}$, is a "doubly connected polygon." Find and prove the analogue of Pick's theorem for $\mathcal{P}-\mathcal{Q}$. Generalize your formula to a polygon with n "holes" (instead of one).

2.29. Show that every convex integral polygon with more than four vertices must have an interior lattice point.

2.30. Consider the rhombus

$$\mathcal{R} = \{(x,y) :\, a|x| + b|y| \le ab\}\,,$$

where a and b are fixed positive integers. Find a formula for $L_{\mathcal{R}}(t)$.

2.31. We define the n^{th} **Farey sequence** to be the sequence, in order from smallest to largest, of all the rational numbers $\frac{a}{b}$ in the interval $[0,1]$ such that a and b are coprime and $b \le n$. For instance, the sixth Farey sequence is $\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}$.

(a) For two consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$ in a Farey sequence, prove that $bc - ad = 1$.

(b) For three consecutive fractions $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ in a Farey sequence, show that $\frac{c}{d} = \frac{a+e}{b+f}$.

2.32. Let $\lceil x \rceil$ denote the smallest integer greater than or equal to x. Prove that for all positive integers a and b,

$$a + (-1)^b \sum_{m=0}^{a} (-1)^{\left\lceil \frac{bm}{a} \right\rceil} \equiv b + (-1)^a \sum_{n=0}^{b} (-1)^{\left\lceil \frac{an}{b} \right\rceil} \bmod 4 .$$

(*Hint:* This is a variation of Exercise 1.6. One way to obtain this identity is by counting lattice points in a certain triangle, keeping track only of the parity.)

2.33. ♣ Verify (2.19).

2.34. ♣ Compute the partial fraction coefficients (2.22).

2.35. ♣ Let a, b be positive integers. Show that

$$\frac{1}{1-z^{ab}} = -\frac{\xi_a^k}{ab}\left(z - \xi_a^k\right)^{-1} + \frac{ab-1}{2ab} + \text{terms with positive powers of } \left(z - \xi_a^k\right).$$

2.36. ♣ Let $\mathcal{T}$ be given by (2.16), and let $c = \gcd(e, f)$. Prove that

$$\begin{aligned} L_{\mathcal{T}}(t) = {} & \frac{1}{2ef}\left(tr - u - v\right)^2 + \frac{1}{2}\left(tr - u - v\right)\left(\frac{1}{e} + \frac{1}{f} + \frac{1}{ef}\right) \\ & + \frac{1}{4}\left(1 + \frac{1}{e} + \frac{1}{f}\right) + \frac{1}{12}\left(\frac{e}{f} + \frac{f}{e} + \frac{1}{ef}\right) \\ & + \left(\frac{1}{2e} + \frac{1}{2f} - \frac{u+v-tr}{ef}\right)\sum_{k=1}^{c-1}\frac{\xi_c^{-ktr}}{1-\xi_c^k} - \frac{1}{ef}\sum_{k=1}^{c-1}\frac{\xi_c^{k(-tr+1)}}{(1-\xi_c^k)^2} \\ & + \frac{1}{e}\sum_{\substack{j=1 \\ \frac{e}{c} \nmid j}}^{e-1}\frac{\xi_e^{j(v-tr)}}{\left(1-\xi_e^{jf}\right)\left(1-\xi_e^j\right)} + \frac{1}{f}\sum_{\substack{l=1 \\ \frac{f}{c} \nmid l}}^{f-1}\frac{\xi_f^{l(u-tr)}}{\left(1-\xi_f^{le}\right)\left(1-\xi_f^l\right)} . \end{aligned}$$

2.37. Let $\mathcal{P}$ be a rational polygon, and let d be the least common multiple of the denominators of the vertices of $\mathcal{P}$. Prove directly (using Exercise 2.36) that the period of $L_{\mathcal{P}}$ divides d.

2.38. ♣ Finish the calculation in Example 2.12, that is, compute

$$\operatorname{const} \frac{1}{(1-z_1z_2)\left(1-z_1^2z_2\right)(1-z_1)(1-z_2)\, z_1^{3t} z_2^{2t}} .$$

2.39. Compute the vector partition function of the quadrilateral given in Example 2.12, that is, compute the counting function

$$f\left(v_1, v_2\right) := \#\left\{\mathbf{x} \in \mathbb{Z}_{\geq 0}^4 : \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right\}$$

for $v_1, v_2 \in \mathbb{Z}$. (This function depends on the relationship between v_1 and v_2.)

2.40. Search on the Internet for the program `polymake` [118]. You can download it for free. Experiment.

Open Problems

2.41. Choose $d+1$ of the 2^d vertices of the unit d-cube $\square$, and let Δ be the simplex defined by their convex hull.

(a) Which choice of vertices maximizes $\operatorname{vol} \Delta$?
(b) What is the maximum volume of such a Δ?

2.42. Find classes of integral d-polytopes $(\mathcal{P}_d)_{d \geq 1}$ for which $L_{\mathcal{P}_d}(t)$ is symmetric in d and t. (The standard simplices Δ and the cross-polytopes $\diamond$ form two such classes.)

2.43. We mentioned already in the notes that all the roots of the polynomials $L_\diamond$ have real part $-\frac{1}{2}$; see [75, 148]. Find other classes of polytopes whose lattice-point enumerator exhibits such special behavior.

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