Preface

> Engineering is like dancing; you don't learn it in a darkened lecture hall watching slides: you learn it by getting out on the dance floor and having your toes stepped on.
> —Professor Jack Alford (1920–2006), cofounder of the Engineering Clinic at Harvey Mudd College, who hired the author in 1971 as an assistant professor. The same can be said for doing definite integrals.

To really appreciate this book, one dedicated to the arcane art of calculating definite integrals, it is necessary (although perhaps it is *not* sufficient) that you be the sort of person who finds the following question fascinating, one right up there in a fierce battle with a hot cup of coffee and a sugar donut for first place on the list of sinful pleasures:

without actually calculating x, show that

if $x + \frac{1}{x} = 1$ it then follows that $x^7 + \frac{1}{x^7} = 1$.

Okay, I know what many (but, I hope, not *you*) are thinking at being confronted with a question like this: of what Earthly significance could such a problem possibly have? Well, *none* as far as I know, but its fascination (or not) for *you* provides (I think) excellent psychological insight into whether or not *you* should spend time and/or good money on this book. If the problem leaves someone confused, puzzled, or indifferent (maybe all three) then my advice to them would be to put this book down and to look instead for a good mystery novel, the latest Lincoln biography (there seems to be a new one every year—what could *possibly* be left unsaid?), or perhaps a vegetarian cookbook.

But, if your pen is already out and scrawled pages of calculations are beginning to pile-up on your desk, then by gosh you *are* just the sort of person for whom I wrote this book. (If, after valiant effort you are still stumped but nonetheless just *have* to see how to do it—or if your pen simply ran dry—an analysis is provided at the end of the book.) More specifically, I've written with three distinct types of readers in mind: (1) physics/engineering/math students in their undergraduate years; (2) professors looking for interesting lecture material; and (3) nonacademic professionals looking for a 'good technical read.'

There are two possible concerns associated with calculating definite integrals that we should address with no delay. First, do *real* mathematicians actually do that sort of thing? Isn't mere *computation* the dirty business (best done out of sight, in the shadows of back-alleys so as not to irreparably damage the young minds of impressionable youths) of grease-covered engineers with leaky pens in their shirts, or of geeky physicists in rumbled pants and chalk dust on their noses? Isn't it in the deep, clear ocean of analytical proofs and theorems where we find *real* mathematicians, swimming like powerful, sleek seals? As an engineer, myself, I find that attitude just a bit elitist, and so I am pleased to point to the pleasure in computation that many of the great mathematicians enjoyed, from Newton to the present day.

Let me give you two examples of that. First, the reputation of the greatest English mathematician of the first half of the twentieth century, G.H. Hardy (1877–1947), partially rests on his phenomenal skill at doing definite integrals. (Hardy appears in numerous places in this book.) And second, the hero of this book (Riemann) is best known today for (besides his integral) his formulation of the greatest unsolved problem in mathematics, about which I'll tell you lots more at the end of the book. But after his death, when his private notes on that very problem were studied, it was found that imbedded in all the deep theoretical stuff was a calculation of $\sqrt{2}$. To 38 decimal places!

The other concern I occasionally hear strikes me as just plain crazy; the complaint that there is no end to definite integrals. (This should, instead, be a cause for *joy*.) You can fiddle with integrands, and with upper and lower limits, in an uncountable infinity of ways,[1] goes the grumbling, so what's the point of calculating definite integrals since you can't possibly do them all? I hope writing this concern out in words is sufficient to make clear its ludicrous nature. We can never do all possible definite integrals, so why bother doing any? Well, what's next—you can't possibly add together all possible pairs of the real numbers, so why bother learning to add? Like I said—that's nuts!

What makes doing the specific integrals in this book of value aren't the specific answers we'll obtain, but rather the tricks (excuse me, the *methods*) we'll use in obtaining those answers; methods you may be able to use in evaluating the integrals you will encounter in the future in your own work. Many of the integrals I'll show you do have important uses in mathematical physics and engineering, but others are included just because they look, at first sight, to be so damn tough that it's a real kick to see how they simply crumble away when attacked with *the right trick*.

From the above you've probably gathered that I've written this book in a light-hearted manner (that's code for 'this is *not* a rigorous math textbook'). I am not going to be terribly concerned, for example, with proving the uniform convergence of *anything*, and if you don't know what that means don't worry about it because I'm not going to worry about it, either. It's not that issues of rigor aren't

[1] You'll see, in the next chapter, that with a suitable change of variable we can transform any integral into an integral from 0 to ∞, or from 1 to ∞, or from 0 to 1. So things aren't *quite* so bad as I've made them out.

important—*they are*—but not for us, here. When, after grinding through a long, convoluted sequence of manipulations to arrive at what we *think* is the value for some definite integral, I'll then simply unleash a wonderful MATLAB numerical integration command (*quad*)—short for *quadrature*—and we'll *calculate* the value. If our theoretical answer says it's $\sqrt{\pi} = 1.772453\ldots$ and *quad* says it's -9.3, we'll of course suspect that *somewhere* in all our calculations we just maybe fell off a cliff! If, however, *quad* says it is 1.77246, well then, that's good enough for me and on we'll go, happy with success and flushed with pleasure, to the next problem.

Having said that, I would be less than honest if I don't admit, right now, that such happiness *could* be delusional. Consider, for example, the following counter-example to this book's operational philosophy. Suppose you have used a computer software package to show the following:

$$\int_0^\infty \cos(x)\frac{\sin(4x)}{x}dx = 1.57079632679\ldots,$$

$$\int_0^\infty \cos(x)\cos\left(\frac{x}{2}\right)\frac{\sin(4x)}{x}dx = 1.57079632679\ldots,$$

$$\int_0^\infty \cos(x)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{3}\right)\frac{\sin(4x)}{x}dx = 1.57079632679\ldots,$$

and so on, all the way out to

$$\int_0^\infty \cos(x)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{3}\right)\ldots\cos\left(\frac{x}{30}\right)\frac{\sin(4x)}{x}dx = 1.57079632679\ldots.$$

One would have to be blind (as well as totally lacking in imagination) not to immediately suspect two things:

$$\text{the consistent value of } 1.57079\ldots\text{is actually } \frac{\pi}{2}, \tag{1}$$

and

$$\int_0^\infty \left\{\prod_{k=1}^{n}\cos\left(\frac{x}{k}\right)\right\}\frac{\sin(4x)}{x}dx = \frac{\pi}{2} \text{ for } \textit{all} \text{ n}. \tag{2}$$

This is exciting! But then you run the very next case, with $n = 31$, and the computer returns an answer of

$$\int_0^\infty \cos(x)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{3}\right)\ldots\cos\left(\frac{x}{30}\right)\cos\left(\frac{x}{31}\right)\frac{\sin(4x)}{x}dx = 1.57079632533\ldots.$$

In this book I would dismiss the deviation (notice those last three digits!) as round-off error—*and I would be wrong*! It's *not* round-off error and, despite the highly suggestive numerical computations, the supposed identity "for all n" is simply *not true*. It's 'almost' true, but in math 'almost' doesn't cut-it.[2]

That's the sort of nightmarish thing that makes mathematicians feel obligated to clearly state any assumptions they make and, if they be really pure, to show that these assumptions are valid before going forward with an analysis. I will not be so constrained here and, despite the previous example of how badly things can go wrong, I'll assume just about anything that's convenient at the moment (short of something *really* absurd, like $1 + 1 = 3$), deferring the moment of truth to when we 'check' a theoretical result with MATLAB. A true mathematician would feel shame (perhaps even thinking that a state of moral degeneracy had been entered) if they should adopt such a cavalier attitude. I, on the other hand, will be immune to such soul-crushing doubts. Still, remain aware that we *will* be taking some risks.

So I *will* admit, again, that violation of one or more of the conditions that rigorous analyses have established *can* lead to disaster. Additional humorous examples of this disconcerting event can be found in a paper[3] by a mathematician with a sense of humor. The paper opens with this provocative line: "Browsing through an integral table on a dull Sunday afternoon [don't you often do the very same thing?] some time ago, I came across four divergent trigonometric integrals. I wondered how those divergent integrals [with *incorrect* finite values] ended up in a respectable table." A couple of sentences later the author writes "We have no intent to defame either the well-known mathematician who made the original error [the rightfully famous French genius Augustin-Louis Cauchy (1789–1857) that you'll get to know when we get to contour integration], or the editors of the otherwise fine tables in which the integrals appear. We all make mistakes and we're not out to point the finger at anyone . . ."

And if we *do* fall off a cliff, well, so what? Nobody need know. We'll just quietly gather-up our pages of faulty analysis, rip them into pieces, and toss the whole rotten mess into the fireplace. Our mathematical sins will be just between us and God (who is well known for being forgiving).

Avoiding a computer is not necessarily a help, however. Here's a specific example of what I mean by that. In a classic of its genre,[4] Murray Spiegel (late professor of mathematics at Rensselaer Polytechnic Institute) asks readers to show that

[2] For an informative discussion of the fascinating mathematics behind these calculations, see Nick Lord, "An Amusing Sequence of Trigonometrical Integrals," *The Mathematical Gazette*, July 2007, pp. 281–285.

[3] Erik Talvila, "Some Divergent Trigonometric Integrals," *The American Mathematical Monthly*, May 2001, pp. 432–436.

[4] M. R. Spiegel, *Outline of Theory and Problems of Complex Variables with an Introduction to Conformal Mapping and Its Applications*, Schaum 1964, p. 198 (problem 91).

$$\int_0^\infty \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi\ln(2)}{2}$$

which equals 1.088793 . . . . One can only wonder at how many students struggled (and for how long) to do this, as the given answer is incorrect. Later in this book, in (5.1.3), we'll do this integral correctly, but a use of *quad* (*not* available to Spiegel in 1964) quickly shows the numerical value is actually the *significantly* greater 1.4603 . . . . At the end of this Preface I'll show you two examples (including Spiegel's integral) of this helpful use of *quad*.

Our use of *quad* does prompt the question of *why*, if we can always calculate the value of any definite integral to as many decimal digits we wish, do we even care about finding exact expressions for these integrals? This is really a philosophical issue, and I think it gets to the mysterious interconnectedness of mathematics—how seemingly unrelated concepts can turnout to actually be intimately related. The expressions we'll find for many of the definite integrals evaluated in this book will involve such familiar numbers as ln(2) and π, and other numbers that are not so well known, like *Catalan's constant* (usually written as G) after the French mathematician Eugène Catalan (1814–1894). The common thread that stitches these and other numbers together is that all can be written as infinite series that can, in turn, be written as definite integrals:

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \int\limits_0^1 \frac{2x}{1+x^2}\,dx = 0.693147\ldots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \int\limits_0^1 \frac{1}{1+x^2}\,dx = 0.785398\ldots$$

$$G = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \ldots = \int_1^\infty \frac{\ln(x)}{1+x^2}\,dx = 0.9159655\ldots.$$

And surely it is understanding at a far deeper level to know that the famous Fresnel integrals $\int_0^\infty \cos(x^2)dx$ and $\int_0^\infty \sin(x^2)dx$ are *exactly* equal to $\frac{1}{2}\sqrt{\frac{\pi}{2}}$, compared to knowing only that they are 'pretty near' 0.6267.

In 2004 a wonderful book, very much like this one in spirit, was published by two mathematicians, and so I hope my cavalier words will appear to be appalling ones only to the most rigid of hard-core purists. That book, *Irresistible Integrals* (Cambridge University Press) by the late George Boras, and Victor Moll at Tulane University, is not quite as willing as this one is to return to the devil-may-care, eighteenth century mathematics of Euler's day, but I strongly suspect the authors were often tempted. Their subtitle gave them away: *Symbolics, Analysis and*

Experiments [particularly notice this word!] *in the Evaluation of Integrals*. Being mathematicians, their technical will-power was stronger than is my puny electrical engineer's dedication to rigor, but every now and then even they could not totally suppress their sheer pleasure at doing definite integrals.

And then 3 years later, in another book coauthored by Moll, we find a statement of philosophy that exactly mirrors my own (and that of this book): "Given an interesting identity buried in a long and complicated paper on an unfamiliar subject, which would give you more confidence in its correctness: staring at the proof, or confirming computationally that it is correct to 10,000 decimal places?"[5] That book and *Irresistible Integrals* are really fun math books to read.

Irresistible Integrals is different from this one, though, in that Boras and Moll wrote for a more mathematically sophisticated audience than I have, assuming a level of knowledge equivalent to that of a junior/senior college math major. They also use *Mathematica* much more than I use MATLAB. I, on the other hand, have assumed far less, just what a good student would know—with one BIG exception—after the first year of high school AP calculus, plus just a bit of exposure to the concept of a differential equation. That big exception is contour integration, which Boras and Moll avoided in their book because "not all [math majors] (we fear, few) study complex analysis."

Now *that*, I have to say, caught me by surprise. For a modern undergraduate math major not to have ever had a course in complex analysis seems to me to be shocking. As an electrical engineering major, *50 years ago*, I took complex analysis up through contour integration (from Stanford's math department) at the start of my junior year using R.V. Churchill's famous book *Complex Variables and Applications*. (I still have my beat-up, coffee-stained copy.) I think contour integration is just too beautiful and powerful to be left out of this book but, recognizing that my assumed reader may not have prior knowledge of complex *analysis*, all the integrals done in this book by contour integration are gathered together in their own chapter at the end of the book. Further, in that chapter I've included a 'crash mini-course' in the theoretical complex analysis required to understand the technique (assuming only that the reader has already encountered complex *numbers* and their manipulation).

Irresistible Integrals contains many beautiful results, but a significant fraction of them is presented mostly as 'sketches,' with the derivation details (often presenting substantial challenges) left up to the reader. In this book *every* result is *fully* derived. Indeed, there are results here that are not in the Boras and Moll book, such as the famous integral first worked out in 1697 by the Swiss mathematician John Bernoulli (1667–1748), a result that so fascinated him he called it his "series mirabili" ("marvelous series"):

[5] *Experimental Mathematics in Action*, A. K. Peters 2007, pp. 4–5. Our calculations here with *quad* won't be to 10,000 decimal places, but the idea is the same.

$$\int_0^1 x^x\,dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \ldots = 0.78343\ldots$$

or its variant

$$\int_0^1 x^{-x}\,dx = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \ldots = 1.29128\ldots.$$

Also derived here are the equally exotic integrals

$$\int_0^1 x^{x^2}\,dx = 1 - \frac{1}{3^2} + \frac{1}{5^3} - \frac{1}{7^4} + \frac{1}{9^5} - \ldots = 0.89648\ldots$$

and

$$\int_0^1 x^{\sqrt{x}}\,dx = 1 - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{4}\right)^3 - \left(\frac{2}{5}\right)^4 + \left(\frac{2}{6}\right)^5 - \ldots = 0.65858\ldots.$$

I don't believe either of these last two integrals has appeared in *any* book before now.

One famous integral that is also not in *Irresistible Integrals* is particularly interesting, in that it *seemingly* (I'll explain this in just a bit) owed its evaluation to a mathematician at Tulane University, Professor Moll's home institution. The then head of Tulane's math department, Professor Herbert Buchanan (1881–1974), opened a 1936 paper[6] with the following words: "In the consideration of a research problem in quantum mechanics, Professor J.C. Morris of Princeton University recently encountered the integral

$$I = \int_0^\infty \frac{x^3}{e^x - 1}\,dx.$$

Since the integral does not yield to any ordinary methods of attack, Professor Morris asked the author to evaluate it [Joseph Chandler Morris (1902–1970) was a graduate of Tulane who did his PhD in physics at Princeton; later he was head of the physics department, and then a Vice-President, at Tulane]." Professor Buchanan then showed that the integral is equal to an infinite series that sums to 6.49 . . . , and just after arriving at that value he wrote "It had been found from other considerations [the details of which are not mentioned, but which I'm guessing were the results of either numerical calculations or even, perhaps, of *physics* experiments

[6] H. E. Buchanan, "On a Certain Integral Arising in Quantum Mechanics," *National Mathematics Magazine*, April 1936, pp. 247–248. I treated this integral, in a way different from Buchanan, in my book *Mrs. Perkins's Electric Quilt*, Princeton 2009, pp. 100–102, and in Chap. 5 we'll derive it in yet a different way, as a special case of (5.3.4).

done at Princeton by Morris] that the integral should have a value between 6.3 and 6.9. Thus the value above [$6.4939\ldots = \frac{\pi^4}{15}$] furnishes a theoretical verification of experimental results."

So here we have an important definite integral apparently 'discovered' by a physicist and solved by a mathematician. In fact, as you'll learn in Chap. 5, Buchanan was *not* the first to do this integral; it had been evaluated by Riemann in 1859, long before 1936. Nonetheless, this is a nice illustration of the fruitful coexistence and positive interaction of experiment and theory, and it is perfectly aligned with the approach I took while writing this book.

There is one more way this book differs from *Irresistible Integrals*, that reflects my background as an engineer rather than as a professional mathematician. I have, all through the book, made an effort to bring into many of the discussions a variety of physical applications, from such diverse fields as radio theory and theoretical mechanics. In all such cases, however, math plays a central role. So, for example, when the topic of elliptic integrals comes up (at the end of Chap. 6), I do so in the context of a famous *physics* problem. The origin of that problem is due, however, not to a physicist but to a nineteenth century *mathematician*.

Let me close this Preface on the same note that opened it. Despite all the math in it, this book has been written in the spirit of 'let's have fun.' That's the same attitude Hardy had when, in 1926, he replied to a plea for help from a young undergraduate at Trinity College, Cambridge. That year, while he was still a teenager, H.S.M. Coxeter (1907–2003) had undertaken a study of various four-dimensional shapes. His investigations had *suggested* to him ("by a geometrical consideration and verified graphically") several quite spectacular definite integrals, like[7]

$$\int_0^{\pi/2} \cos^{-1}\left\{\frac{\cos(x)}{1 + 2\cos(x)}\right\} dx = \frac{5\pi^2}{24}.$$

In a letter to the *Mathematical Gazette* he asked if any reader of the journal could show him how to *derive* such an integral (we'll calculate the above so-called *Coxeter's integral* later, in the longest derivation in this book). Coxeter went on to become one of the world's great geometers and, as he wrote decades later in the Preface to his 1968 book *Twelve Geometric Essays*, "I can still recall the thrill of receiving [solutions from Hardy] during my second month as a freshman at Cambridge." Accompanying Hardy's solutions was a note scribbled in a margin

[7] MATLAB's *quad* says this integral is $2.0561677\ldots$, which agrees quite nicely with $\frac{5\pi^2}{24} = 2.0561675\ldots$. The code syntax is: *quad(@(x)acos(cos(x)./(1+2*cos(x))),0,pi/2)*.

For the integral I showed you earlier, from Spiegel's book, the *quad* code is (I've used $1e6 = 10^6$ for the upper limit of infinity): *quad(@(x)log(1+x)./(1+x.^2),0,1e6)*. Most of the integrals in this book are one-dimensional but, for those times that we will encounter higher dimensional integrals there is *dblquad* and *triplequad*, and MATLAB's companion, the Symbolic Math Toolbox and its command *int* (for 'integrate'), can do them, too. The syntax for those cases will be explained when we first encounter multidimensional integrals.

declaring that "I tried very hard not to spend time on your integrals, but to me the challenge of a definite integral is irresistible."[8]

If you share Hardy's (and my) fascination for definite integrals, then this is a book for you. Still, despite my admiration for Hardy's near magical talent for integrals, I don't think he was *always* correct. I write that nearly blasphemous statement because, in addition to Boras and Moll, another bountiful source of integrals is *A Treatise on the Integral Calculus*, a massive two-volume work of nearly 1,900 pages by the English educator Joseph Edwards (1854–1931). Although now long out-of-print, both volumes are on the Web as Google scans and available for free download. In an April 1922 review in *Nature* that stops just short of being a sneer, Hardy made it quite clear that he did not like Edwards' work ("Mr. Edwards's book may serve to remind us that the early nineteenth century is not yet dead," and "it cannot be treated as a serious contribution to analysis"). Finally admitting that there is *some* good in the book, even then Hardy couldn't resist tossing a cream pie in Edwards' face with his last sentence: "The book, in short, may be useful to a sufficiently sophisticated teacher, provided he is careful not to allow it to pass into his pupil's hands." Well, I disagree. I found Edwards' *Treatise* to be a terrific read, a treasure chest absolutely *stuffed* with mathematical gems.

You'll find some of them in this book. Also included are dozens of challenge problems, with complete, detailed solutions at the back of the book if you get stuck. Enjoy!

Durham, NH Paul J. Nahin

[8] And so we see where Boras and Moll got the title of their book. Several years ago, in my book *Dr. Euler's Fabulous Formula* (Princeton 2006, 2011), I gave another example of Hardy's fascination with definite integrals: see that book's Section 5.7, "Hardy and Schuster, and their optical integral," pp. 263–274. There I wrote "displaying an unevaluated definite integral to Hardy was very much like waving a red flag in front of a bull." Later in this book I'll show you a 'first principles' derivation of the optical integral (Hardy's far more sophisticated derivation uses Fourier transforms).

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A Collection of Sneaky Tricks, Sly Substitutions, and Numerous Other Stupendously Clever, Awesomely Wicked, and Devilishly Seductive Maneuvers for Computing Nearly 200 Perplexing Definite Integrals From Physics, Engineering, and Mathematics (Plus 60 Challenge Problems with Complete, Detailed Solutions)
Nahin, P.J.
2015, XXIII, 412 p. 34 illus., 1 illus. in color., Softcover
ISBN: 978-1-4939-1276-6