Chapter 2
‘Easy’ Integrals

2.1 Six ‘Easy’ Warm-Ups

You should always be alert, when confronted by a definite integral, for the happy possibility that although the integral might look ‘interesting’ (that is, hard!) just *maybe* it will still yield to a direct, frontal attack. The first six integrals in this chapter are in that category. If $a$ and $b$ are positive constants, calculate:

(2.1.a) \[ \int_{1}^{\infty} \frac{1}{(x + a)\sqrt{x - 1}} \, dx \]

and

(2.1.b) \[ \int_{0}^{\infty} \ln \left( 1 + \frac{a^2}{x^2} \right) \, dx \]

and

(2.1.c) \[ \int_{0}^{\infty} \frac{\ln(x)}{x^2 + b^2} \, dx \]

and

(2.1.d) \[ \int_{0}^{\infty} \frac{1}{1 + e^{ax}} \, dx. \]
Finally, calculate

\[(2.1.e) \quad \int_1^\infty \frac{1}{\sqrt{x + x^2}} \, dx\]

and

\[(2.1.f) \quad \int_1^\infty \frac{dx}{\cosh(x)}\]

For \((2.1.a)\) make the change of variable \(x - 1 = t^2\) and so

\[\frac{dx}{dt} = 2t\]

or

\[dx = 2t \, dt = 2\sqrt{x - 1} \, dt.\]

Since

\[x = 1 + t^2\]

then we have

\[\int_1^\infty \frac{1}{(x + a)\sqrt{x - 1}} \, dx = \int_0^\infty \frac{2\sqrt{x - 1}}{(1 + t^2 + a)\sqrt{x - 1}} \, dt = 2\int_0^\infty \frac{dt}{(a + 1) + t^2}.\]

We immediately recognize this last integral as being of the form

\[\int \frac{dt}{c^2 + t^2} = \frac{1}{c} \tan^{-1} \left( \frac{t}{c} \right)\]

and so

\[\int_1^\infty \frac{1}{(x + a)\sqrt{x - 1}} \, dx = 2 \left\{ \frac{1}{\sqrt{a + 1}} \tan^{-1} \left( \frac{t}{\sqrt{a + 1}} \right) \right\}^\infty_0 = \frac{2}{\sqrt{a + 1}} \tan^{-1}(\infty) = \frac{2}{\sqrt{a + 1}} \left( \frac{\pi}{2} \right)\]
which gives us

\[ f_1^\infty \frac{1}{(x+a)\sqrt{x-1}} \, dx = \frac{\pi}{\sqrt{a+1}}. \]

As a check, for \( a = 99 \) we have the value of the integral equal to \( \frac{\pi}{10} \approx 3.1415 \ldots \), while \( \text{quad says} \quad \text{quad} (\text{quad}(x)I_1/(x+99), x, \text{sqrt}(x-1)), I, Ie5) = 0.30784 \ldots \) Notice that the upper limit of infinity has been replaced with the finite (but ‘large’) number \( 10^5 \).

For (2.1.b) integration-by-parts will do the job. That is, we’ll use

\[
\int_0^\infty u \, dv = (uv)\bigg|_0^\infty - \int_0^\infty v \, du,
\]

where

\[ u = \ln\left(1 + \frac{a^2}{x^2}\right) \]

and \( dv = dx \). Then, \( v = x \) and

\[ du = \left(-\frac{2a^2}{x}\right)\left(\frac{1}{x^2 + a^2}\right) \, dx. \]

So,

\[
\int_0^\infty \ln\left(1 + \frac{a^2}{x^2}\right) \, dx = \left\{ x \ln\left(1 + \frac{a^2}{x^2}\right) \right\}\bigg|_0^\infty - \int_0^\infty x \left(-\frac{2a^2}{x}\right) \left(\frac{1}{x^2 + a^2}\right) \, dx \\
= 2a^2 \int_0^\infty \frac{dx}{x^2 + a^2} = 2a^2 \left\{ \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \right\}\bigg|_0^\infty = 2a \tan^{-1}(\infty)
\]

and thus\(^1\)

\(^1\)In this derivation we’ve assumed \( \lim_{x \to 0} x \ln\left(1 + \frac{a^2}{x^2}\right) = \lim_{x \to \infty} x \ln\left(1 + \frac{a^2}{x^2}\right) = 0. \) To see that these two assumptions are correct, recall the power series expansion for the log function when \( p \approx 0: \ln(1 + p) = p - \frac{1}{2} p^2 + \frac{1}{3} p^3 - \cdots \). So, with \( p = \frac{x^2}{a^2} \) (which \( \to 0 \) as \( x \to \infty \)), we have \( x \ln\left(1 + \frac{x^2}{a^2}\right) = x \left[ \frac{x^2}{2a^2} - \frac{1}{2} \left(\frac{x^2}{a^2}\right)^2 + \frac{1}{3} \left(\frac{x^2}{a^2}\right)^3 - \cdots \right] = \frac{x^2}{a^2} - \frac{1}{2} \frac{x^4}{a^4} + \cdots \) which \( \to 0 \) as \( x \to \infty \). On the other hand, as \( x \to 0 \) we have \( x \ln\left(1 + \frac{x^2}{a^2}\right) \approx x \ln\left(\frac{x^2}{a^2}\right) = x \ln(a^2) - x \ln(x^2) = x \ln(a^2) - 2x \ln(x) \) and both of these terms go to zero as \( x \) goes to zero (the first term is obvious, and in the second term \( x \) vanishes faster than \( \ln(x) \) blows-up).
As a check, for $a = 10$ we have the value of the integral equal to 31.415926..., while \textit{quad} says \textit{quad(@(x)log(1+(100./x.^2)),0,1000)} = 31.31593...

For (2.1.c) let $x = \frac{1}{t}$ and so $dx = -\frac{1}{t^2} dt$. Our integral then becomes

$$I = \int_{0}^{\infty} \frac{\ln(x)}{x^2 + b^2} dx = \int_{0}^{\infty} \frac{\ln(\frac{1}{t})}{\frac{1}{t^2} + b^2} \left( -\frac{1}{t^2} \right) dt = -\int_{0}^{\infty} \frac{\ln(\frac{1}{t})}{1 + b^2 t^2} dt$$

Let $s = bt$ (and so $dt = \frac{1}{b} ds$). Then,

$$I = -\int_{0}^{\infty} \frac{\ln(t)}{b^2 t^2 + 1} dt = -\int_{0}^{\infty} \ln(\frac{s}{b}) \left( \frac{1}{b} \right) ds$$

$$= \frac{1}{b} \left[ -\int_{0}^{\infty} \frac{\ln(s)}{s^2 + 1} ds + \int_{0}^{\infty} \frac{\ln(b)}{s^2 + 1} ds \right]$$

or, as the first integral in the brackets is zero—we showed this in (1.5.1)—then we have

$$I = \frac{\ln(b)}{b} \int_{0}^{\infty} \frac{1}{s^2 + 1} ds = \frac{\ln(b)}{b} \left( \tan^{-1}(s) \right)_{0}^{\infty}$$

and thus

$$\int_{0}^{\infty} \frac{\ln(x)}{x^2 + b^2} dx = \frac{\pi}{2b} \ln(b) .$$

Notice that this reduces to zero (as it should) when $b = 1$. For $b = 2$ our formula says the integral is equal to $\frac{\pi}{4} \ln(2) = 0.544396...$ and MATLAB agrees: \textit{quad(@(x)log(x)./(4+x.^2),0,10000)} = 0.543365...

For (2.1.d) a simple substitution is all we need. Letting $u = e^{ax}$ (and so $du = ae^{ax}$ and thus $dx = \frac{du}{ae^{ax}} = \frac{du}{a u}$) we have
For a \( a = \pi \), for example, the integral’s value is 0.220635 \ldots, and in agreement we have \( \text{quad}(\pi I.\,(1+exp(pi*x)),0,1000) = 0.220635 \ldots \).

For (2.1.e), consider the indefinite integral

\[
\int \frac{dx}{x + x^m} = \int \frac{x^{-m}}{x^{1-m} + 1} \, dx.
\]

Notice that

\[
\frac{d}{dx} \ln(x^{1-m} + 1) = \frac{(1-m)x^{1-m-1}}{x^{1-m} + 1} = \frac{(1-m)x^{-m}}{x^{1-m} + 1}
\]

and so

\[
\int \frac{dx}{x + x^m} = \frac{1}{1-m} \ln(x^{1-m} + 1) + C
\]

where \( C \) is an arbitrary constant of integration. Thus, with \( m = \sqrt{2} \), we have

\[
\int_{\sqrt{2}}^{\infty} \frac{dx}{x + x^{\sqrt{2}}} = \frac{1}{1-\sqrt{2}} \left\{ \ln\left(x^{1-\sqrt{2}} + 1\right) \right\}_{\sqrt{2}}^{\infty}
\]

or, because \( 1 - \sqrt{2} < 0 \) and so \( \lim_{x \to \infty} x^{1-\sqrt{2}} = 0 \), a bit of elementary complex number arithmetic gives us our answer:

\[
\int_{\sqrt{2}}^{\infty} \frac{dx}{x + x^{\sqrt{2}}} = (1 + \sqrt{2}) \ln\left\{ 1 + 2^{\sqrt{2}(1-\sqrt{2})} \right\}.
\]

The expression on the right is 1.506332\ldots, while \( \text{quad} \) says the value of the integral is \( \text{quad}(\pi I.\,(1+x*x.sqrt(2)),sqrt(2),1e5) = 1.48592\ldots \).

For (2.1.f) let \( t = e^x \) and so \( \frac{dt}{dx} = e^x \) or \( dx = \frac{dt}{e^x} \).
Then,

\[
\int_{-\infty}^{\infty} \frac{dx}{\cosh(x)} = \int_{-\infty}^{\infty} \frac{dx}{\frac{e^x + e^{-x}}{2}} = 2 \int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = 2 \int_{0}^{\infty} \frac{1}{t} \frac{dt}{t + \frac{1}{t}}
\]

\[
= 2 \int_{0}^{\infty} \frac{1}{t^2 + 1} dt = 2 \tan^{-1}(t) \bigg|_{0}^{\infty} = 2 \left( \frac{\pi}{2} \right)
\]

and so

(2.1.6) \[\int_{-\infty}^{\infty} \frac{dx}{\cosh(x)} = \pi.\]

MATLAB agrees, as \(\text{quad}(@\left(x\right)1./\cosh\left(x\right),-20,20) = 3.1415929\ldots\)

### 2.2 A New Trick

The next four examples illustrate an often powerful trick for calculating definite integrals, that of ‘flipping’ the integration variable’s ‘direction.’ That is, if \(x\) goes from 0 to \(\pi\), try changing to \(y = \pi - x\). This may seem almost trivial, but it often works! With this idea in mind, let’s calculate

(2.2.a) \[\int_{0}^{\pi/2} \frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)} + \sqrt{\cos(x)}} \, dx\]

and

(2.2.b) \[\int_{0}^{\pi} \frac{x \sin(x)}{1 + \cos^2(x)} \, dx\]

and

(2.2.c) \[\int_{0}^{\pi/2} \frac{\sin^2(x)}{\sin(x) + \cos(x)} \, dx\]

and

(2.2.d) \[\int_{0}^{1} \frac{\ln(x + 1)}{x^2 + 1} \, dx.\]
For (2.2.a), make the substitution \( x = \frac{\pi}{2} - y \). Then, \( dx = -dy \) and

\[
I = \int_{\pi/2}^{0} \frac{\sqrt{\sin \left( \frac{\pi}{2} - y \right)}}{\sqrt{\sin \left( \frac{\pi}{2} - y \right)} + \sqrt{\cos \left( \frac{\pi}{2} - y \right)}} (-dy) = \int_{0}^{\pi/2} \frac{\sqrt{\cos (y)}}{\sqrt{\cos (y)} + \sqrt{\sin (y)}} \, dy.
\]

Adding this expression to the original \( I \) (and changing the dummy variable of integration variable \( y \) back to \( x \)) gives

\[
2I = \int_{0}^{\pi/2} \frac{\sqrt{\sin (x)} + \sqrt{\cos (x)}}{\sqrt{\sin (x)} + \sqrt{\cos (x)}} \, dx = \int_{0}^{\pi/2} \, dx = \frac{\pi}{2},
\]

and so

\[
2I = \int_{0}^{\pi/2} \frac{\sqrt{\sin (x)} + \sqrt{\cos (x)}}{\sqrt{\sin (x)} + \sqrt{\cos (x)}} \, dx = \int_{0}^{\pi/2} \, dx = \frac{\pi}{2},
\]

or,

\[
(2.2.1) \quad \int_{0}^{\pi/2} \frac{\sqrt{\sin (x)}}{\sqrt{\sin (x)} + \sqrt{\cos (x)}} \, dx = \frac{\pi}{4}.
\]

This says the integral is equal to 0.785398..., and \textit{quad} agrees: \textit{quad}(@(x)sqrt(sin(x))/(sqrt(sin(x))+sqrt(cos(x))),0,pi/2) = 0.785398... .

In (2.2.b) make the substitution \( y = \pi - x \) (and so \( dx = -dy \)). Then,

\[
I = \int_{\pi}^{0} \frac{(\pi - y) \sin (\pi - y)}{1 + \cos^2(\pi - y)} (-dy)
\]

\[
= \int_{0}^{\pi} \frac{(\pi - y) \{ \sin (\pi) \cos (y) - \cos (\pi) \sin (y) \}}{1 + \{ \cos (\pi) \cos (y) + \sin (\pi) \sin (y) \}^2} \, dy.
\]

or,

\[
I = \int_{0}^{\pi} \frac{(\pi - y) \sin (y)}{1 + \cos^2(y)} \, dy = \pi \int_{0}^{\pi} \frac{\sin (y)}{1 + \cos^2(y)} \, dy - \int_{0}^{\pi} \frac{y \sin (y)}{1 + \cos^2(y)} \, dy.
\]

That is,

\[
I = \pi \int_{0}^{\pi} \frac{\sin (x)}{1 + \cos^2(x)} \, dx - I
\]

or,

\[
I = \frac{\pi}{2} \int_{0}^{\pi} \frac{\sin (x)}{1 + \cos^2(x)} \, dx.
\]
Now, let \( u = \cos(x) \) and so \( \frac{du}{dx} = -\sin(x) \) (that is, \( dx = -du/\sin(x) \)). Thus,

\[
I = \frac{\pi}{2} \int_{-1}^{1} \frac{\sin(x)}{1 + u^2} \left( -\frac{du}{\sin(x)} \right) = -\frac{\pi}{2} \int_{-1}^{1} \frac{du}{1 + u^2} = \frac{\pi}{2} \int_{-1}^{1} \frac{du}{1 + u^2} = \frac{\pi}{2} \left\{ \tan^{-1}(u) \right\} \bigg|_{-1}^{1} \\
= \frac{\pi}{2} \left[ \tan^{-1}(1) - \tan^{-1}(-1) \right] = \frac{\pi}{2} \left[ \frac{\pi}{4} + \frac{\pi}{4} \right] = \frac{\pi^2}{4}.
\]

So,

\[
(2.2.2) \quad \int_{0}^{\pi} \frac{x \sin(x)}{1 + \cos^2(x)} \, dx = \frac{\pi^2}{4}.
\]

This is equal to 2.4674 . . ., and quad agrees as \( \text{quad}(@((x.*\sin(x))./(1+(\cos(x).^2)))@0,\pi) = 2.4674 \) . . .

For (2.2.c), make the substitution \( x = \frac{\pi}{2} - y \). Then, \( dx = -dy \) and

\[
I = \int_{0}^{\pi/2} \frac{\sin^2 \left( \frac{\pi}{2} - y \right)}{\sin \left( \frac{\pi}{2} - y \right) + \cos \left( \frac{\pi}{2} - y \right)} \, (-dy).
\]

Since
\[
\sin \left( \frac{\pi}{2} - y \right) = \sin \left( \frac{\pi}{2} \right) \cos(y) - \cos \left( \frac{\pi}{2} \right) \sin(y) = \cos(y)
\]

and
\[
\cos \left( \frac{\pi}{2} - y \right) = \cos \left( \frac{\pi}{2} \right) \cos(y) + \sin \left( \frac{\pi}{2} \right) \sin(y) = \sin(y),
\]

we have

\[
I = \int_{0}^{\pi/2} \frac{\cos^2(y)}{\cos(y) + \sin(y)} \, dy
\]

and so, changing the dummy variable of integration back to \( x \),

\[
2I = \int_{0}^{\pi/2} \frac{\sin^2(x) + \cos^2(x)}{\cos(x) + \sin(x)} \, dx = \int_{0}^{\pi/2} \frac{1}{\cos(x) + \sin(x)} \, dx.
\]
Now, change variable to

\[ z = \tan \left( \frac{x}{2} \right). \]

Then we have

\[ \frac{dz}{dx} = \frac{1}{2} \frac{\cos^2 \left( \frac{x}{2} \right) + \frac{1}{2} \sin^2 \left( \frac{x}{2} \right)}{\cos^2 \left( \frac{x}{2} \right)} = \frac{1}{2} \frac{\frac{1}{2} \cos^2 \left( \frac{x}{2} \right)}{1 + \tan^2 \left( \frac{x}{2} \right)} = \frac{1}{2} \left[ 1 + \tan^2 \left( \frac{x}{2} \right) \right] \]

or,

\[ \frac{dz}{dx} = \frac{1 + z^2}{2} \]

and so

\[ dx = \frac{2}{1 + z^2} \, dz. \]

From the double-angle formulas from trigonometry we can write

\[ \sin(x) = 2 \sin \left( \frac{x}{2} \right) \cos \left( \frac{x}{2} \right) = 2 \frac{\sin \left( \frac{x}{2} \right)}{\cos \left( \frac{x}{2} \right)} \cos^2 \left( \frac{x}{2} \right) \]

and so

\[ \sin(x) = 2 \tan \left( \frac{x}{2} \right) \frac{1}{1 + \tan^2 \left( \frac{x}{2} \right)} = \frac{2z}{1 + z^2}, \]

as well as

\[ \cos(x) = \cos^2 \left( \frac{x}{2} \right) - \sin^2 \left( \frac{x}{2} \right) = \cos^2 \left( \frac{x}{2} \right) \left[ 1 - \frac{\sin^2 \left( \frac{x}{2} \right)}{\cos^2 \left( \frac{x}{2} \right)} \right] \]

\[ = \frac{1}{1 + \tan^2 \left( \frac{x}{2} \right)} \left[ 1 - \frac{\tan^2 \left( \frac{x}{2} \right)}{1 + \tan^2 \left( \frac{x}{2} \right)} \right] = \frac{1}{1 + \tan^2 \left( \frac{x}{2} \right)} \left[ 1 - \tan^2 \left( \frac{x}{2} \right) \right] = \frac{1 - z^2}{1 + z^2}. \]
Thus,

\[
2I = \int_0^1 \frac{1}{2z + \frac{1 - z^2}{1 + z^2}} \left( \frac{2}{1 + z^2} \right) dz = 2 \int_0^1 \frac{dz}{1 + 2z - z^2} = 2 \int_0^1 \frac{dz}{2 - [z^2 - 2z + 1]}
\]

\[
= 2 \int_0^1 \frac{dz}{2 - (z - 1)^2}.
\]

Next, writing the integrand as a partial fraction expansion we have

\[
2I = \frac{2}{2\sqrt{2}} \int_0^1 \left\{ \frac{1}{\sqrt{2} - (z - 1)} + \frac{1}{\sqrt{2} + (z - 1)} \right\} dz = \frac{2}{2\sqrt{2}} \int_0^1 \left\{ \frac{1}{1 + \sqrt{2} - z} + \frac{1}{-1 + \sqrt{2} + z} \right\} dz
\]

\[
= \frac{2}{2\sqrt{2}} \left[ \int_0^1 \frac{dz}{z + \sqrt{2} - 1} - \int_0^1 \frac{dz}{z - 1 - \sqrt{2}} \right].
\]

Letting \( u = z + \sqrt{2} - 1 \) in the first integral in the brackets gives us

\[
\int_0^1 \frac{dz}{z + \sqrt{2} - 1} = \int_{\sqrt{2} - 1}^{\sqrt{2}} \frac{du}{u} = \ln(u)|_{\sqrt{2} - 1}^{\sqrt{2}} = \ln \left( \frac{\sqrt{2}}{\sqrt{2} - 1} \right).
\]

Letting \( u = z - 1 - \sqrt{2} \) in the second integral in the brackets gives us

\[
\int_0^1 \frac{dz}{z - 1 - \sqrt{2}} = \int_{-1 - \sqrt{2}}^{-\sqrt{2}} \frac{du}{u} = \ln(u)|_{-1 - \sqrt{2}}^{-\sqrt{2}} = \ln \left( \frac{\sqrt{2}}{1 + \sqrt{2}} \right).
\]

Thus,

\[
2I = \frac{2}{2\sqrt{2}} \left[ \ln \left( \frac{\sqrt{2}}{\sqrt{2} - 1} \right) - \ln \left( \frac{\sqrt{2}}{1 + \sqrt{2}} \right) \right] = \frac{2}{2\sqrt{2}} \ln \left( \frac{\sqrt{2}}{\sqrt{2} - 1} \cdot \frac{1 + \sqrt{2}}{\sqrt{2}} \right)
\]

\[
= \frac{1}{\sqrt{2}} \ln \left( \frac{1 + \sqrt{2}}{\sqrt{2} - 1} \right) - \frac{1}{\sqrt{2}} \ln \left( \frac{1 + \sqrt{2}}{\sqrt{2} - 1} \cdot \frac{\sqrt{2} + 1}{\sqrt{2}} \right)
\]

\[
= \frac{1}{\sqrt{2}} \ln \left( \frac{\sqrt{2} + 1 + 2 + \sqrt{2}}{2 - 1} \right) = \frac{1}{\sqrt{2}} \ln \left( 3 + 2\sqrt{2} \right)
\]

or,

\[
(2.2.3) \quad \int_0^{\pi/2} \frac{\sin^2(x)}{\sin(x) + \cos(x)} \, dx = \frac{1}{2\sqrt{2}} \ln \left( 3 + 2\sqrt{2} \right).
\]
The value of the integral is 0.623225…, and MATLAB agrees as \( \text{quad(@sin}^2(x)/\text{sin}(x)+\text{cos}(x)),0,\pi/2) = 0.623225 \ldots \)

In (2.2.d) let

\[
x = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}
\]

and so

\[
\frac{dx}{d\theta} = \frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\theta)} = 1 + \tan^2(\theta)
\]

and thus we have \( dx = (1 + \tan^2(\theta)) \, d\theta \) and so our integral is

\[
\begin{align*}
I &= \int_0^{\pi/4} \ln(x + 1) \, dx = \int_0^{\pi/4} \frac{\ln\{\tan(\theta) + 1\}}{1 + \tan^2(\theta)} \{1 + \tan^2(\theta)\} \, d\theta \\
&= \int_0^{\pi/4} \ln\{\tan(\theta) + 1\} \, d\theta.
\end{align*}
\]

Now, make the change of variable that ‘flips’ the direction of integration, that is, \( u = \frac{\pi}{4} - \theta \) (and so \( du = -d\theta \)). Then,

\[
I = \int_{\pi/4}^0 \ln\{\tan(\pi/4 - u) + 1\} \, (-du) = \int_0^{\pi/4} \ln\{\tan(\pi/4 - u) + 1\} \, du
\]

or, changing back to \( \theta \) as the dummy variable of integration,

\[
I = \int_0^{\pi/4} \ln\{\tan(\pi/4 - \theta) + 1\} \, d\theta.
\]

Next, recall the identity

\[
\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha) \tan(\beta)}.
\]

With \( \alpha = \frac{\pi}{4} \) and \( \beta = \theta \) we have

\[
\tan(\frac{\pi}{4} - \theta) = \frac{\tan(\frac{\pi}{4}) - \tan(\theta)}{1 + \tan(\frac{\pi}{4}) \tan(\theta)} = \frac{1 - \tan(\theta)}{1 + \tan(\theta)}
\]
and so
\[
\begin{align*}
I &= \int_{0}^{\pi/4} \ln \left\{ \frac{1 - \tan(\theta)}{1 + \tan(\theta)} + 1 \right\} \, d\theta = \int_{0}^{\pi/4} \ln \left\{ \frac{2}{1 + \tan(\theta)} \right\} \, d\theta \\
&= \int_{0}^{\pi/4} \ln(2) \, d\theta - \int_{0}^{\pi/4} \ln(1 + \tan(\theta)) \, d\theta.
\end{align*}
\]

But the last integral is I, and so
\[
I = \frac{\pi}{4} \ln(2) - I.
\]
or, at last (and using \(x\) as the dummy variable of integration),
\[
(2.2.4) \quad \int_{0}^{1} \frac{\ln(x + 1)}{x^2 + 1} \, dx = \int_{0}^{\pi/4} \ln(1 + \tan(x)) \, dx = \frac{\pi}{8} \ln(2).
\]

This integral is often called Serret’s integral, after the French mathematician Joseph Serret (1819–1885) who did it in 1844. Our result says the two above integrals are both equal to 0.27219826... and quad agrees, as quad(@(x)log(tan(x)+1),0,pi/4) = 0.27219826... and quad(@(x)log(x+1)./(x.^2+1),0,1) = 0.27219823...

If we make the change of variable \(x = \frac{t}{a}\), we can generalize this result as follows.

Since \(dx = \frac{1}{a} \, dt\), then
\[
\frac{\pi}{8} \ln(2) = \int_{0}^{1} \frac{\ln(x + 1)}{x^2 + 1} \, dx = \int_{0}^{a} \frac{\ln \left( \frac{t + 1}{a} \right)}{\left( \frac{t}{a} \right)^2 + 1} \left( \frac{1}{a} \right) \, dt = a \int_{0}^{a} \frac{\ln(t + a) - \ln(a)}{t^2 + a^2} \, dt
\]
\[
= a \left\{ \int_{0}^{a} \frac{\ln(t + a)}{t^2 + a^2} \, dt - \ln(a) \left[ \frac{1}{a} \tan^{-1} \left( \frac{t}{a} \right) \right]_{0}^{a} \right\}
\]
\[
= a \left\{ \int_{0}^{a} \frac{\ln(t + a)}{t^2 + a^2} \, dt - \frac{1}{a} \ln(a) \left[ \tan^{-1}(1) \right] \right\}
\]
\[
= a \left\{ \int_{0}^{a} \frac{\ln(t + a)}{t^2 + a^2} \, dt - \frac{\pi}{4a} \ln(a) \right\} = a \int_{0}^{a} \frac{\ln(t + a)}{t^2 + a^2} \, dt - \frac{\pi}{4} \ln(a).
\]

Thus,
\[
\int_{0}^{a} \frac{\ln(t + a)}{t^2 + a^2} \, dt = \frac{\pi}{8a} \ln(2) + \frac{\pi}{4a} \ln(a) = \frac{\pi}{8a} \ln(2) + \frac{2\pi}{8a} \ln(a)
\]
or, finally, changing the dummy variable of integration back to $x$,

\[
\int_0^a \ln(x + a) \frac{dx}{x^2 + a^2} = \frac{\pi}{8a} \ln(2a^2).
\]

### 2.3 Two Old Tricks, Plus a New One

The following integral, with *arbitrarily many and different* quadratic factors in the denominator of the integrand, may at first glance look impossibly difficult:

\[
\int_0^\infty \frac{dx}{(x^2 + a_1^2)(x^2 + a_2^2)(x^2 + a_3^2) \ldots (x^2 + a_n^2)}, \quad \text{with all of the } a_i \neq 0 \text{ and different.}
\]

Not so, as I’ll now show you. Let’s start by writing the integrand in partial fraction form (look back at how we got (2.2.3)), as

\[
\frac{1}{(x^2 + a_1^2)(x^2 + a_2^2)(x^2 + a_3^2) \ldots (x^2 + a_n^2)} = \frac{c_1}{x^2 + a_1^2} + \frac{c_2}{x^2 + a_2^2} + \frac{c_3}{x^2 + a_3^2} + \ldots + \frac{c_n}{x^2 + a_n^2}
\]

where the $c$’s are all constants.\(^2\) Fix your attention on any one of the terms on the right, say the $k$-th one. Then, multiplying through by $x^2 + a_k^2$, we have

\[
\frac{x^2 + a_k^2}{(x^2 + a_1^2)(x^2 + a_2^2)(x^2 + a_3^2) \ldots (x^2 + a_k^2) \ldots (x^2 + a_n^2)} = \frac{c_1(x^2 + a_k^2)}{x^2 + a_1^2} + \frac{c_2(x^2 + a_k^2)}{x^2 + a_2^2} + \ldots + c_k + \ldots + \frac{c_n(x^2 + a_k^2)}{x^2 + a_n^2}.
\]

Thus, cancelling the $x^2 + a_k^2$ factor in the numerator and denominator on the left in the above expression, and setting $x$ to the particular value of $i a_k$ where $i = \sqrt{-1}$, we have\(^3\)

\(^2\)Writing the partial fraction expansion this way is where the assumption that all the $a_i$ are different comes into play. If any of the $a_i$ appears multiple times, then the correct partial fraction expansion of the integrand is not as I’ve written it.

\(^3\)There are two points to be clear on at this point. First, since we are working with an identity it must be true for all values of $x$, and I’ve just picked a particularly convenient one. Second, if the use of an imaginary $x$ bothers you, just remember the philosophical spirit of this book—*anything* (well, *almost* anything) goes, and we’ll check our result when we get to the end!
\[
\frac{1}{(-a_k^2 + a_1^2)(-a_k^2 + a_2^2)(-a_k^2 + a_3^2) \ldots (-a_k^2 + a_n^2)} = \prod_{j=1, j \neq k}^{n} \frac{1}{(a_j^2 - a_k^2)} = c_k
\]

where only the \(c_k\) term survives on the right since all the other terms in the partial fraction expansion end-up with a zero in their numerator. So, our original partial fraction expansion is just

\[
\frac{1}{(x^2 + a_1^2)(x^2 + a_2^2)(x^2 + a_3^2) \ldots (x^2 + a_n^2)} = \sum_{k=1}^{n} \frac{c_k}{x^2 + a_k^2}
\]

and thus

\[
\int_{0}^{\infty} \frac{dx}{(x^2 + a_1^2)(x^2 + a_2^2)(x^2 + a_3^2) \ldots (x^2 + a_n^2)} = \sum_{k=1}^{n} c_k \int_{0}^{\infty} \frac{dx}{x^2 + a_k^2}
\]

\[
= \sum_{k=1}^{n} c_k \left[ \tan^{-1} \left( \frac{x}{a_k} \right) \right]_{0}^{\infty} = \sum_{k=1}^{n} c_k \left( \frac{\pi}{2} \right)
\]

or, at last, with all the \(a_i \neq 0\),

\[
\int_{0}^{\infty} \frac{dx}{(x^2 + a_1^2)(x^2 + a_2^2)(x^2 + a_3^2) \ldots (x^2 + a_n^2)} = \left( \frac{n}{2} \right) \sum_{k=1}^{n} \frac{c_k}{a_k},
\]

where \(c_k = \frac{1}{\prod_{j=1, j \neq k}^{n} (a_j^2 - a_k^2)}\) and \(a_1 \neq a_j \) if \(j \neq 1\).

For example, suppose \(a_1^2 = 1, a_2^2 = 4, \) and \(a_3^2 = 9\). Then, \(a_1 = 1, a_2 = 2, \) and \(a_3 = 3\). This gives the values of the c’s as

\[
c_1 = \frac{1}{(4 - 1)(9 - 1)} = \frac{1}{24}, c_2 = \frac{1}{(1 - 4)(9 - 4)} = \frac{1}{15}, c_3 = \frac{1}{(1 - 9)(4 - 9)} = \frac{1}{40}.
\]

Then the value of the integral is

\[
\left( \frac{\pi}{2} \right) \left[ \frac{1}{1} - \frac{1}{3} + \frac{1}{30} \right] = \left( \frac{\pi}{2} \right) \left[ \frac{1}{24} - \frac{1}{30} + \frac{1}{120} \right] = \left( \frac{\pi}{2} \right) \left[ \frac{2}{120} \right] = \frac{\pi}{120} = 0.0261799 \ldots
\]

and quad agrees, as \(quad(@(x)int((x.^2+1).*((x.^2+4).*((x.^2+9)))*0.100) = 0.02617989 \ldots \).

Here’s another example of the use of a partial fraction expansion to evaluate an integral. Here I’ll do

\[
\int_{0}^{\infty} \frac{dx}{x^4 + 2x^2 \cosh(2\alpha) + 1}
\]
where $\alpha$ is an arbitrary constant. Writing the hyperbolic cosine in the denominator of the integrand out in exponential form, we have

$$x^4 + 2x^2\cosh(2\alpha) + 1 = x^4 + 2x^2 \left[ \frac{e^{2\alpha} + e^{-2\alpha}}{2} \right] + 1 = x^4 + x^2 e^{2\alpha} + x^2 e^{-2\alpha} + 1 = (x^2 + e^{2\alpha})(x^2 + e^{-2\alpha}).$$

So, we can write the integrand in the following partial fraction form (with $A$ and $B$ as constants):

$$\frac{1}{x^4 + 2x^2\cosh(2\alpha) + 1} = \frac{A}{x^2 + e^{2\alpha}} + \frac{B}{x^2 + e^{-2\alpha}}.$$

That is,

$$(A + B)x^2 + Ae^{-2\alpha} + Be^{2\alpha} = 1$$

which, since there is no $x^2$ term on the right, immediately tells us that $A = -B$. Thus,

$$-Be^{-2\alpha} + Be^{2\alpha} = 1$$

and so the constant $B$ is given by

$$B = \frac{1}{e^{2\alpha} - e^{-2\alpha}}.$$

Therefore,

$$\int_0^\infty \frac{dx}{x^4 + 2x^2\cosh(2\alpha) + 1} = \frac{1}{e^{2\alpha} - e^{-2\alpha}} \left[ \int_0^\infty \frac{dx}{x^2 + e^{-2\alpha}} - \int_0^\infty \frac{dx}{x^2 + e^{2\alpha}} \right]$$

$$= \frac{1}{e^{2\alpha} - e^{-2\alpha}} \left[ \frac{1}{e^{-\alpha}} \tan^{-1} \left( \frac{x}{e^{-\alpha}} \right) - \frac{1}{e^{\alpha}} \tan^{-1} \left( \frac{x}{e^{\alpha}} \right) \right]_0^\infty$$

$$= \frac{1}{e^{2\alpha} - e^{-2\alpha}} \left[ e^{\alpha} \frac{\pi}{2} - e^{-\alpha} \frac{\pi}{2} \right] = \frac{\pi}{2} \frac{e^{\alpha} - e^{-\alpha}}{e^{2\alpha} - e^{-2\alpha}}$$

$$= \frac{\pi}{2} \frac{1}{(e^{\alpha} + e^{-\alpha})(e^{\alpha} - e^{-\alpha})} = \frac{\pi}{2} \frac{1}{2 \left( e^{\alpha} + e^{-\alpha} \right)}.$$
or, finally,

\[
\int_0^\infty \frac{dx}{x^4 + 2x^2 \cosh(2\alpha) + 1} = \frac{\pi}{4 \cosh(\alpha)}.
\]

For \(\alpha = 1\), for example, this integral equals 0.5089806 \ldots and \textit{quad} agrees, as \textit{quad}(@(x)1./(x.^4+2*cos(2)*x.^2+1),0,1000) = 0.5089809 \ldots.

Back in Chap. 1 (Sect. 1.5) I showed you how the ‘evenness’ or ‘oddness’ of an integrand (if one of those two properties is present) can be of great help in transforming a ‘hard’ integral into an ‘easy’ one. As a more sophisticated example of this than was the example in Chap. 1, let’s calculate the value of

\[
\int_0^\infty \frac{dx}{x^4 + 2x^2 \cos(2\alpha) + 1}
\]

where, as before, \(\alpha\) is an arbitrary constant. This may superficially look a lot like the integral we just finished but, as you’ll soon see, there is a really big difference in how we’ll do this new one.

We start by making the change of variable \(y = \frac{1}{x}\) (and so \(\frac{dy}{dx} = -\frac{1}{x^2}\) or \(dx = -\frac{1}{y^2}\) \(dy\)). Then,

\[
I = \int_0^\infty \frac{dx}{x^4 + 2x^2 \cos(2\alpha) + 1}
\]

\[
= \int_0^\infty \frac{\frac{1}{y^2} \; dy}{\frac{1}{y^4} + 2\frac{1}{y^2} \cos(2\alpha) + 1}
\]

If we then add our two versions of the integral (the left-most and the right-most integrals in the previous line, remembering that \(x\) and \(y\) are just dummy variables of integration) we have

\[
2I = \int_0^\infty \frac{(1 + x^2) \; dx}{x^4 + 2x^2 \cos(2\alpha) + 1}
\]

or,

\[
I = \frac{1}{2} \int_0^\infty \frac{(1 + x^2) \; dx}{x^4 + 2x^2 \cos(2\alpha) + 1}
\]

And since the integrand is even, we can write

\[
I = \frac{1}{4} \int_0^\infty \frac{(1 + x^2) \; dx}{x^4 + 2x^2 \cos(2\alpha) + 1}.
\]
Because \( \cos(2\alpha) = 1 - 2\sin^2(\alpha) \) you can show by direct multiplication that

\[
x^4 + 2x^2 \cos(2\alpha) + 1 = \left[ x^2 - 2x \sin(\alpha) + 1 \right] \left[ x^2 + 2x \sin(\alpha) + 1 \right]
\]

and so

\[
I = \frac{1}{4} \int_{-\infty}^{\infty} \frac{(1 + x^2) \, dx}{\left[ x^2 - 2x \sin(\alpha) + 1 \right]\left[ x^2 + 2x \sin(\alpha) + 1 \right]}.
\]

Now, since the integrand is even, then if we include in the numerator of the integrand an odd function like \(2x \sin(\alpha)\) we do not change the value of the integral, and so

\[
I = \frac{1}{4} \int_{-\infty}^{\infty} \frac{[x^2 - 2x \sin(\alpha) + 1] \, dx}{[x^2 - 2x \sin(\alpha) + 1][x^2 + 2x \sin(\alpha) + 1]}
\]

or

\[
I = \frac{1}{4} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x \sin(\alpha) + 1}.
\]

Or, since \( \sin^2(\alpha) + \cos^2(\alpha) = 1 \) then

\[
I = \frac{1}{4} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x \sin(\alpha) + \sin^2(\alpha) + \cos^2(\alpha)} = \frac{1}{4} \int_{-\infty}^{\infty} \frac{dx}{[x + \sin(\alpha)]^2 + \cos^2(\alpha)}.
\]

Let \( u = x + \sin(\alpha) \) (and so \( du = dx \)), and then

\[
I = \frac{1}{4} \int_{-\infty}^{\infty} \frac{du}{u^2 + \cos^2(\alpha)} = \frac{1}{4} \left( \frac{1}{\cos(\alpha)} \right) \left[ \tan^{-1}\left\{ \frac{u}{\cos(\alpha)} \right\} \right]_{-\infty}^{\infty}
\]

\[
= \frac{1}{4 \cos(\alpha)} \left[ \tan^{-1}\{ \infty \} - \tan^{-1}\{-\infty \} \right]
\]

and so, at last,

\[
(2.3.3) \quad \int_{0}^{\infty} \frac{dx}{x^4 + 2x^2 \cos(2\alpha) + 1} = \frac{\pi}{4 \cos(\alpha)}.
\]

Special, interesting cases occur for some obvious values of \( \alpha \). Specifically, for \( \alpha = \frac{\pi}{4} \) we have

\[
(2.3.4) \quad \int_{0}^{\infty} \frac{dx}{x^4 + 1} = \int_{0}^{\infty} \frac{x^2 \, dx}{x^4 + 1} = \frac{\pi \sqrt{2}}{4}.
\]
These two integrals are therefore equal to 1.11072 ... and MATLAB agrees, as\n\[ \text{quad}(@((x^(4+1)),0,1000)) = 1.11072 \ldots \] \text{and quad}(@((x^4+1),0,10000)) = 1.11062\ldots. \text{For } \alpha = 30^\circ \text{ (that is, } \alpha = \frac{\pi}{6} \text{)} \text{ we have}\n\begin{equation}
\int_0^\infty \frac{dx}{x^4 + x^2 + 1} = \frac{\pi}{2\sqrt{3}}. \tag{2.3.5}
\end{equation}
This is equal to 0.906899 \ldots; and indeed \text{quad}(@((x^4-x^2+1),0,10000)) = 0.9068993 \ldots. \text{For } \alpha = 60^\circ \text{ (that is, } \alpha = \frac{\pi}{3} \text{)} \text{ we have}\n\begin{equation}
\int_0^\infty \frac{dx}{x^4 - x^2 + 1} = \frac{\pi}{2}. \tag{2.3.6}
\end{equation}
This is equal to 1.570796 \ldots; and indeed \text{quad}(@((x^4-x^2+1),0,10000)) = 1.570796 \ldots. \text{And finally, for } \alpha = 0 \text{ we have}\n\begin{equation}
\int_0^\infty \frac{dx}{x^4 + 2x^2 + 1} = \frac{\pi}{4}. \tag{2.3.7}
\end{equation}
This is equal to 0.785398 \ldots; and \text{quad}(@((x^4+2x^2+1),0,10000)) = 0.785398 \ldots.\n
Here’s a new trick, one using a difference equation to evaluate a class of definite integrals indexed on an integer-valued variable. Specifically,
\[ I_n(\alpha) = \int_0^\pi \frac{\cos(n\theta) - \cos(n\alpha)}{\cos(\theta) - \cos(\alpha)} \, d\theta \]
where \( \alpha \) is a constant and \( n \) is a non-negative integer (\( n = 0, 1, 2, 3, \ldots \)). The first two integrals are easy to do by inspection: \( I_0(\alpha) = 0 \) and \( I_1(\alpha) = \pi \). For \( n > 1 \), however, things get more difficult. What I’ll do next is perfectly understandable as we go through the analysis step-by-step, but I have no idea what motivated the person who first did this. The mystery of mathematical genius!

*If* you recall the trigonometric identity
\[ \cos((n+1)\theta) + \cos((n-1)\theta) = 2 \cos(\theta) \cos(n\theta) \]
then perhaps you’d think of taking a look at the quantity \( I_{n+1}(\alpha) + I_{n-1}(\alpha) \) to see if it is related in some ‘nice’ way to \( I_n(\alpha) \). So, imagining that we have been so inspired, let’s take a look at the quantity \( Q = AI_{n+1}(\alpha) + BI_n(\alpha) + CI_{n-1}(\alpha) \), where \( A, B, \) and \( C \) are constants, to see what we get.
Thus,

\[ Q = \int_0^\pi \frac{A[\cos \{(n+1)\theta\} - \cos \{(n+1)\alpha\}] + B[\cos (n\theta) - \cos (n\alpha)]}{\cos (\theta) - \cos (\alpha)} \, d\theta. \]

Suppose we now set \( A = C = 1 \) and \( B = -2 \cos(\alpha) \). Then,

\[ Q = \int_0^\pi \frac{\cos \{(n+1)\theta\} + \cos \{(n-1)\theta\} - 2 \cos (\alpha) \cos (n\theta)}{\cos (\theta) - \cos (\alpha)} \, d\theta. \]

From our trig identity the second term in the numerator vanishes and the first term reduces to

\[ Q = \left. \int_0^\pi \frac{2 \cos (\theta) \cos (n\theta) - 2 \cos (\alpha) \cos (n\theta)}{\cos (\theta) - \cos (\alpha)} \, d\theta = \int_0^\pi \frac{2 \cos (n\theta) \cos (\theta) - \cos (\alpha) \cos (n\theta)}{\cos (\theta) - \cos (\alpha)} \, d\theta \right| = 0, n = 1, 2, \ldots \]

That is, we have the following second-order, linear difference equation:

\[ I_{n+1}(\alpha) - 2 \cos (\alpha) I_n(\alpha) + I_{n-1}(\alpha) = 0, \quad n = 1, 2, 3, \ldots, \]

with the conditions \( I_0(\alpha) = 0 \) and \( I_1(\alpha) = \pi \).

It is well-known that such a so-called recursive equation has solutions of the form \( I_n = Ce^{sn} \) where \( C \) and \( s \) are constants. So,

\[ Ce^{s(n+1)} - 2 \cos (\alpha) Ce^{sn} + Ce^{s(n-1)} = 0 \]

or, cancelling the common factor of \( Ce^{sn} \),

\[ e^s - 2 \cos (\alpha) + e^{-s} = 0 \]

or,

\[ e^{2s} - 2 \cos (\alpha)e^s + 1 = 0. \]

This is a quadratic in \( e^s \), and so

\[ e^s = \frac{2 \cos (\alpha) \pm \sqrt{4 \cos^2(\alpha) - 4}}{2} = \frac{2 \cos (\alpha) \pm 2i \sqrt{1 - \cos^2(\alpha)}}{2} = \cos (\alpha) \pm i \sin (\alpha), \]
where $i = \sqrt{-1}$. Now, from Euler’s fabulous formula we have $e^{s} = e^{\pm is}$ and thus $s = \pm is$. This means that the general solution for $I_n(\alpha)$ is

$$I_n(\alpha) = C_1 e^{in\alpha} + C_2 e^{-in\alpha}.$$ 

Since $I_0(\alpha) = 0$ then $C_1 + C_2 = 0$ or, $C_2 = -C_1$. Also, as $I_1(\alpha) = \pi$ we have

$$C_1 e^{i\alpha} - C_1 e^{-i\alpha} = \pi = C_1 i 2 \sin(\alpha)$$

which says that

$$C_1 = \frac{\pi}{i 2 \sin(\alpha)} \text{ and } C_2 = -\frac{\pi}{i 2 \sin(\alpha)}.$$ 

Thus,

$$I_n(\alpha) = \frac{\pi}{2 \sin(\alpha)} \left[ \frac{e^{in\alpha} - e^{-in\alpha}}{i} \right] = \frac{\pi}{2 \sin(\alpha)} \left[ \frac{i 2 \sin(n\alpha)}{i} \right]$$

or, at last, using $x$ as the dummy variable of integration,

\begin{equation}
\int_0^{\infty} \frac{\pi \cos(nx) - \cos(n\alpha)}{\cos(x) - \cos(\alpha)} \, dx = \frac{\pi \sin(n\alpha)}{\sin(\alpha)}.
\end{equation}

For example, if $n = 6$ and $\alpha = \frac{\pi}{11}$ this result says our integral is equal to $\pi \frac{\sin(6\alpha)}{\sin(\frac{\pi}{11})}$ which is equal to 11.03747399... , and quad agrees because \textit{quad}(@\texttt{(x)}(\texttt{cos(6*xi)}-\texttt{cos(6*pi/11)))/(\texttt{cos(x)}-\texttt{cos(pi/11))},0,\texttt{pi}) = 11.03747399... .

Here’s a final, quick example of recursion used to solve an entire class of integrals:

$$I_n = \int_0^{\infty} x^{2n} e^{-x^2} \, dx, \, n = 0, 1, 2, 3, \ldots$$

We start by observing that

$$\frac{d}{dx} \left( x^{2n-1} e^{-x^2} \right) = (2n - 1) x^{2n-2} e^{-x^2} - 2x^{2n} e^{-x^2}, \, n \geq 1.$$ 

So, integrating both sides,

$$\int_0^{\infty} \frac{d}{dx} \left( x^{2n-1} e^{-x^2} \right) \, dx = (2n - 1) \int_0^{\infty} x^{2n-2} e^{-x^2} \, dx - 2 \int_0^{\infty} x^{2n} e^{-x^2} \, dx.$$ 

The right-most integral is $I_n$, and the middle integral is $I_{n-1}$. So,
\[
\int_0^\infty \frac{d}{dx} \left( x^{2n-1} e^{-x^2} \right) dx = (2n - 1)I_{n-1} - 2I_n.
\]

Now, notice that the remaining integral (on the left-hand-side) is simply the integral of a derivative and so is very easy to do! In fact,

\[
\int_0^\infty \frac{d}{dx} \left( x^{2n-1} e^{-x^2} \right) dx = \left( x^{2n-1} e^{-x^2} \right)|_0^\infty = 0
\]
since \(x^{2n-1} e^{-x^2} = 0\) at \(x = 0\) and as \(x \to \infty\). So, we immediately have the recurrence

\[
I_n = \frac{2n - 1}{2} I_{n-1} = \frac{2n(2n - 1)}{4n} I_{n-1}.
\]

For the first few values \(n\) we have:

\[
I_1 = \frac{2}{(4)(1)} I_0,
\]

\[
I_2 = \frac{(4)(3)}{(4)(2)} I_1 = \frac{(4)(3)(2)}{(4)(2)(4)(1)} I_0,
\]

\[
I_3 = \frac{(6)(5)}{(4)(3)} I_2 = \frac{(6)(5)(4)(3)(2)}{(4)(3)(4)(2)(4)(1)} I_0,
\]

and by now you should see the pattern:

\[
I_n = \frac{(2n)!}{4^n n!} I_0.
\]

This is a nice result, but of course the next question is obvious: what is \(I_0\)? In fact,

\[
I_0 = \int_0^\infty e^{-x^2} dx = \frac{1}{2} \sqrt{\pi},
\]

which we have not shown (yet). In the next chapter, as result (3.1.4), I will show you (using a new trick) that

\[
\int_{-\infty}^\infty e^{-x^2/2} dx = \sqrt{2\pi}.
\]

If you make the change of variable \(y = x\sqrt{2}\) (and remember that \(\int_{-\infty}^\infty f(x) dx = 2 \int_0^\infty f(x)dx\) if \(f(x)\) is even) then the value of \(I_0\) immediately follows. Thus,
If \( n = 5 \) this says the integral is equal to 26.171388, . . ., while \( \text{quad}(@x(x.^10).*exp(-x.^2),0,10) = 26.1713896 . . . \)

### 2.4 Another Old Trick: Euler’s Log-Sine Integral

In 1769 Euler computed (for \( a = 1 \)) the value of (where \( a \geq 0 \))

\[
I = \int_{0}^{\pi/2} \ln\{a \sin (x)\}dx,
\]

which is equal to

\[
\int_{0}^{\pi/2} \ln\{a \cos (x)\}dx.
\]

The two integrals are equal because the integrands take-on the same values over the integration interval (\( \sin(x) \) and \( \cos(x) \) are mirror-images of each other over that interval). For many years it was commonly claimed in textbooks that these are quite difficult integrals to do, best tackled with the powerful techniques of contour integration. As you’ll see with the following analysis, however, that is simply not the case.

So, to start we notice that

\[
I = \frac{1}{2} \int_{0}^{\pi/2} \ln\{a \sin (x)\} + \ln\{a \cos (x)\}dx = \frac{1}{2} \int_{0}^{\pi/2} \ln\{a^2 \sin (x) \cos (x)\}dx.
\]

Since \( \sin(2x) = 2\sin(x)\cos(x) \), we have \( \sin (x) \cos (x) = \frac{1}{2} \sin (2x) \) and therefore

\[
I = \frac{1}{2} \int_{0}^{\pi/2} \ln\left\{a^2 \sin(2x)\right\}dx = \frac{1}{2} \int_{0}^{\pi/2} \left[ \ln(a) + \ln\left(\frac{1}{2}\right) + \ln\{a \sin (2x)\}\right]dx
\]

\[
= \frac{\pi}{4} \ln(a) - \frac{\pi}{4} \ln(2) + \frac{1}{2} \int_{0}^{\pi/2} \ln\{a \sin (2x)\}dx
\]
In the last integral, let \( t = 2x \) (and so \( dx = \frac{1}{2} dt \)). Thus,

\[
\frac{1}{2} \int_0^{\pi/2} \ln \{a \sin (2x)\} \, dx = \frac{1}{2} \int_0^{\pi} \ln \{a \sin (t)\} \frac{1}{2} \, dt = \frac{1}{2} I
\]

where the last equality follows because (think of how \( \sin(t) \) varies over the interval \( 0 \) to \( \pi \))

\[
\int_0^{\pi/2} \ln \{a \sin (t)\} \, dt = \frac{1}{2} \int_0^{\pi} \ln \{a \sin (t)\} \, dt.
\]

So,

\[
I = \frac{\pi}{4} \ln(a) - \frac{\pi}{4} \ln(2) + \frac{1}{2} \ln(\frac{a}{2}) + \frac{1}{2} I
\]

or,

\[
\frac{1}{2} I = \frac{\pi}{4} \ln(\frac{a}{2})
\]

and so, at last, for \( a \geq 0 \),

\[
(2.4.1) \quad \int_0^{\pi/2} \ln \{a \sin (x)\} \, dx = \int_0^{\pi/2} \ln \{a \cos (x)\} \, dx = \frac{\pi}{2} \ln \left( \frac{a}{2} \right).
\]

Special cases of interest are \( a = 1 \) (Euler’s integral) for which both integrals equal \(-\frac{\pi}{2} \ln(2) = -1.088793\ldots\) and \( a = 2 \) for which both integrals are equal to zero. We can check both of these cases with \textit{quad}; \textit{quad}(\textit{@}(x)\ln(\sin(x)),0,\pi/2) = -1.0888035\ldots\) and \textit{quad}(\textit{@}(x)\ln(\cos(x)),0,\pi/2) = -1.0888043\ldots\), while \textit{quad}(\textit{@}(x)\ln(2*\sin(x)),0,\pi/2) = -1.0459 \times 10^{-5}\) and \textit{quad}(\textit{@}(x)\ln(2*\cos(x)),0,\pi/2) = -1.1340 \times 10^{-5}. We’ll see Euler’s log-sine integral again, in Chap. 7.

With the result for \( a = 1 \), we can now calculate the interesting integral

\[
\int_0^{\pi/2} \ln \left( \frac{\sin (x)}{x} \right) \, dx.
\]
That’s because this integral is

\[
\int_0^{\pi/2} \ln \{ \sin(x) \} \, dx - \int_0^\pi \ln \{ x \} \, dx = -\frac{\pi}{2} \ln(2) - [x \ln(x) - x]_0^{\pi/2}
\]

\[
= -\frac{\pi}{2} \ln(2) - \left[ \frac{\pi}{2} \ln \left( \frac{\pi}{2} \right) - \frac{\pi}{2} \right]
\]

\[
= -\frac{\pi}{2} \ln(2) - \left[ \frac{\pi}{2} \ln(\pi) - \frac{\pi}{2} \ln(2) - \frac{\pi}{2} \right]
\]

\[
= -\frac{\pi}{2} \ln(2) - \frac{\pi}{2} \ln(\pi) + \frac{\pi}{2} \ln(2) + \frac{\pi}{2}
\]

\[
= -\frac{\pi}{2} \ln(\pi) + \frac{\pi}{2}
\]

and so

\[
(2.4.2) \quad \int_0^{\pi/2} \ln \left\{ \frac{\sin(x)}{x} \right\} \, dx = \frac{\pi}{2} \left[ 1 - \ln(\pi) \right].
\]

Our result says this integral is equal to \(-0.22734\ldots\) and \texttt{quad} agrees, as \texttt{quad(@(x)log(sin(x)/x),0,pi/2)} \(= -0.22734\ldots\).

With a simple change of variable in Euler’s log-sine integral we can get yet another pretty result. Since \(\sin^2(\theta) = 1 - \cos^2(\theta)\) then

\[
\frac{\sin^2(\theta)}{\cos^2(\theta)} = \tan^2(\theta) = \frac{1}{\cos^2(\theta)} - 1
\]

and so

\[
\tan^2(\theta) + 1 = \frac{1}{\cos^2(\theta)}
\]

which says

\[
\ln \left\{ \frac{1}{\cos^2(\theta)} \right\} = -\ln \{ \cos^2(\theta) \} = -2\ln \{ \cos(\theta) \} = \ln \{ \tan^2(\theta) + 1 \}.
\]

That is,

\[
\ln \{ \cos(\theta) \} = -\frac{1}{2} \ln \{ \tan^2(\theta) + 1 \}.
\]
So, in the integral
\[
\int_0^{\pi/2} \ln \{ \cos(x) \} \, dx
\]
replace the dummy variable of integration \( x \) with \( \theta \) and write
\[
\int_0^{\pi/2} \ln \{ \cos(\theta) \} \, d\theta = \int_0^{\pi/2} \left[ -\frac{1}{2} \ln \{ \tan^2(\theta) + 1 \} \right] \, d\theta = -\frac{\pi}{2} \ln(2)
\]
or,
\[
\int_0^{\pi/2} \ln \{ \tan^2(\theta) + 1 \} \, d\theta = \pi \ln(2).
\]

Now, change variable to \( x = \tan(\theta) \). Then
\[
\frac{dx}{d\theta} = \frac{1}{\cos^2(\theta)}
\]
and thus
\[
d\theta = \cos^2(\theta) \, dx = \frac{1}{\tan^2(\theta) + 1} \, dx = \frac{1}{x^2 + 1} \, dx.
\]

So, since \( x = 0 \) when \( \theta = 0 \) and \( x = \infty \) when \( \theta = \frac{\pi}{2} \), we have

\[
\int_0^{\pi/2} \ln \{ \tan^2(\theta) + 1 \} \, d\theta = \pi \ln(2).
\]

That is, this integral is equal to 2.177586 . . . , and \( \text{quad} \) agrees, as \( \text{quad}(\log(x^2+1),((x^2+1),0,1e6)) = 2.1775581 . . . \). To end this discussion, here’s a little calculation for you to play around with: writing
\[
\int_0^{\infty} \frac{\ln(x^2 + 1)}{x^2 + 1} \, dx
\]
as \( \int_0^1 + \int_1^{\infty} \), make the change of variable \( u = \frac{1}{x} \) in the last integral and show that this leads to

\[
\int_0^{\infty} \frac{\ln(x^2 + 1)}{x^2 + 1} \, dx = \frac{\pi \ln(2)}{2}.
\]

This is equal to 1.088793 . . . , and MATLAB agrees because \( \text{quad}(\log(x+(1/x)),((x^2+1),0,1)) = 1.088799 . . . \).
The substitution $u = \frac{1}{x}$ is a trick well-worth keeping in mind. Here’s another use of it to derive a result that almost surely would be much more difficult to get otherwise. Consider the integral

$$\int_0^{\infty} \frac{\ln(x^a + 1)}{x^2 - bx + 1} \, dx,$$

where $a \neq 0$ and $b$ are constants. If we let $u = \frac{1}{x}$ (and so $dx = -\frac{1}{u^2} \, du$), we have

$$\int_0^{\infty} \frac{\ln(x^a + 1)}{x^2 - bx + 1} \, dx = \int_0^{\infty} \frac{\ln\left(\frac{1}{u^a} + 1\right) - \frac{1}{u^2}}{u^2 - \frac{1}{u} + 1} \, du = \int_0^{\infty} \frac{\ln\left(\frac{1 + u^a}{u^a}\right)}{1 - bu + u^2} \, du$$

That is,

$$\int_0^{\infty} \frac{\ln(x^a + 1)}{x^2 - bx + 1} \, dx = \int_0^{\infty} \frac{\ln(1 + u^a)}{1 - bu + u^2} \, du - a \int_0^{\infty} \frac{\ln(u)}{1 - bu + u^2} \, du.$$

and so we immediately have

$$\int_0^{\infty} \frac{\ln(x^a + 1)}{x^2 - bx + 1} \, dx = \int_0^{\infty} \frac{\ln(1 + x^a)}{1 - bx + x^2} \, dx - a \int_0^{\infty} \frac{\ln(x)}{1 - bx + x^2} \, dx$$

Notice that the case of $b = 0$ in (2.4.5) reduces this result to (1.5.1),

$$\int_0^{\infty} \frac{\ln(x)}{1 - bx + x^2} \, dx = 0,$$

what we also get when we set $b = 1$ in (2.1.3). The value of $b$ can’t be just anything, however, and you’ll be asked more on this point in the challenge problem section.

To end the chapter, let me remind you of a simple technique that you encountered way back in high school algebra—‘completing the square.’ This is a ‘trick’ that is well-worth keeping in mind when faced with an integral with a quadratic polynomial in the denominator of the integrand (but see also Challenge Problem 2 for its use in a cubic denominator, and look back at the final integration of Sect. 1.6, too). As another example, let’s calculate

$$\int_0^{1} \frac{1 - x}{1 + x + x^2} \, dx.$$
We’ll start by rewriting the denominator of the integrand by completing the square:

\[ x^2 + x + 1 = x^2 + x + \frac{1}{4} + \left(1 - \frac{1}{4}\right) = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}. \]

Thus,

\[
\int_0^1 \frac{1 - x}{1 + x + x^2} \, dx = \int_0^1 \frac{dx}{(x + \frac{1}{2})^2 + \frac{3}{4}} - \int_0^1 \frac{x}{(x + \frac{1}{2})^2 + \frac{3}{4}} \, dx.
\]

Now, change variable to

\[ u = x + \frac{1}{2} \]

(and so \(dx = du\)). Then, our integral becomes

\[
\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{du}{u^2 + \frac{3}{4}} - \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{u - \frac{1}{2}}{u^2 + \frac{3}{4}} \, du = \frac{3}{2} \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{du}{u^2 + \frac{3}{4}} - \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{u}{u^2 + \frac{3}{4}} \, du.
\]

The first integral on the right is

\[
\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{du}{u^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}/2}\right) \bigg|_{1/2}^{3/2}
\]

\[
= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) \bigg|_{1/2}^{3/2} = \frac{2}{\sqrt{3}} \left[ \tan^{-1}\left(\frac{3}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \right]
\]

\[
= \frac{2}{\sqrt{3}} \left[ \tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}\left(\frac{3}{\sqrt{3}}\right) \right] = \frac{2}{\sqrt{3}} \left[ \frac{\pi}{3} - \frac{\pi}{6} \right] = \frac{\pi}{3\sqrt{3}}.
\]

So,

\[
\int_0^1 \frac{1 - x}{1 + x + x^2} \, dx = \frac{\pi}{2\sqrt{3}} - \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{u}{u^2 + \frac{3}{4}} \, du.
\]

In the integral on the right, change variable to

\[ t = u^2 + \frac{3}{4}. \]
Then, \( dt = 2u \ du \) and
\[
\int_{1/2}^{3/4} \frac{u}{u^2 + \frac{3}{4}} \, du = \int_{1}^{3} \frac{u \ dt}{\sqrt{2u}} = \frac{1}{2} \int_{1}^{3} \frac{dt}{\sqrt{t}} = \frac{1}{2} \ln(t) \bigg|_{1}^{3} = \frac{1}{2} \ln(3).
\]

So,
\[
\int_{0}^{1} \frac{1 - x}{1 + x + x^2} \, dx = \frac{\pi}{2\sqrt{3}} - \frac{1}{2} \ln(3)
\]
or,

\[
(2.4.6)
\int_{0}^{1} \frac{1 - x}{1 + x + x^2} \, dx = \frac{1}{2} \left[ \frac{\pi}{\sqrt{3}} - \ln(3) \right]
\]

which equals 0.35759 \ldots, and MATLAB agrees because \( \text{quad}(@(x)(1-x)/(1+x+x.\,^2),0,1) = 0.357593 \ldots \)

### 2.5 Challenge Problems

That was all pretty straightforward, but here are some problems that \textbf{will}, I think, give your brain a really good workout. And yet, like nearly everything else in this book, \textit{if you see the trick} they will unfold for you like a butterfly at rest.

**(C2.1):** According to Edwards’ \textit{A Treatise on the Integral Calculus} (see the end of the Preface), the following question appeared on an 1886 exam at the University of Cambridge: show that
\[
\int_{x^4}^{0} \frac{\ln(x)}{\sqrt{4x - x^2}} \, dx = 0.
\]

Edwards included no solution and, given that it took me about 5 h spread over 3 days to do it (in my quiet office, under no pressure), my awe for the expected math level of some of the undergraduate students at nineteenth century Cambridge is unbounded. See if you can do it faster than I did (and if not, my solution is at the end of the book). A quick numerical check by MATLAB should convince you that the given answer is correct:
\[
\text{quad}(@(x)\log(x)/\sqrt{4x-x^2},0,4) = 0.0000066.
\]

**(C2.2):** Calculate the value of \( \int_{0}^{1} \frac{dx}{x^3 + 1} \). Hint: first confirm the validity of the partial fraction expansion
\[
\frac{1}{x^3 + 1} = \frac{1}{3} \left[ \frac{1}{x+1} - \frac{x-2}{x^2-x+1} \right]
\]
and so
\[
\int_{0}^{1} \frac{dx}{x^3 + 1} = \frac{1}{3} \int_{0}^{1} \frac{dx}{x+1} - \frac{1}{3} \int_{0}^{1} \frac{x-2}{x^2-x+1} \, dx.
\]
The first integral on the right will yield to an obvious change
of variable, and the second integral on the right will yield to another change of
variable, one just as obvious if you first complete the square in the denominator of
the integrand. Your theoretical answer should have the numerical value 0.8356488 . . .

(C2.3): Here’s a pretty little recursive problem for you to work through. Suppose
you know the value of

\[ \int_{1}^{0} \frac{dx}{x^4 + 1}. \]

This integral is not particularly difficult to do—we did it in (2.3.4)—with a value of \( \frac{\pi}{2\sqrt{2}} \). The point here is that with this knowledge you then also immediately know
the values of

\[ \int_{1}^{0} \frac{(x^4 + 1)^m}{dx} \]

for all integer m > 1 (and not just for m = 1). Show this is so by deriving the recursion

\[ \int_{0}^{\infty} \frac{dx}{(x^4 + 1)^{m+1}} = \frac{4m - 1}{4m} \int_{0}^{\infty} \frac{dx}{(x^4 + 1)^m}. \]

For example,

\[ \int_{0}^{\infty} \frac{dx}{(x^4 + 1)^3} = \frac{(4)(2) - 1}{(4)(2)} \int_{0}^{\infty} \frac{dx}{(x^4 + 1)^2} = \frac{7}{8} \int_{0}^{\infty} \frac{dx}{(x^4 + 1)^2} \]

and

\[ \int_{0}^{\infty} \frac{dx}{(x^4 + 1)^2} = \frac{(4)(1) - 1}{(4)(1)} \int_{0}^{\infty} \frac{dx}{x^4 + 1} = \frac{3}{4} \int_{0}^{\infty} \frac{dx}{x^4 + 1}. \]

Thus,

\[ \int_{0}^{\infty} \frac{dx}{(x^4 + 1)^3} = \left( \frac{7}{8} \right) \left( \frac{3}{4} \right) \int_{0}^{\infty} \frac{dx}{x^4 + 1} = \left( \frac{21}{32} \right) \frac{\pi}{2\sqrt{2}} = \frac{21\pi}{64\sqrt{2}} = 0.72891 . . . . \]

MATLAB agrees, as \( \text{quad(@(x)1/((x^4+1)^3)),0,1000) = 0.72891 . . . . } \)

Hint: Start with \( \int_{0}^{\infty} \frac{dx}{(x^4 + 1)^m} \) and integrate by parts.

(C2.4): For what values of b does the integral in (2.4.5) make sense? Hint: think
about where any singularities in the integrand are located.

(C2.5): Show that \( \int_{0}^{\infty} \frac{\ln(1 + x)}{x\sqrt{x}} \) \( dx = 2\pi \). Hint: integrate by parts.
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