Chapter 2

Hermite's Theorem

We will begin with the proof that e is transcendental, a result first proved by Charles Hermite in 1873.

Theorem 2.1 *e is transcendental.*

Proof. We make the observation that for a polynomial f and a complex number t,

$$\int_0^t e^{-u} f(u) du = [-e^{-u} f(u)]_0^t + \int_0^t e^{-u} f'(u) du$$

which is easily seen on integrating by parts. Here the integral is taken over the line joining 0 and t. If we let

$$I(t, f) := \int_0^t e^{t-u} f(u) du,$$

then we see that

$$I(t, f) = e^t f(0) - f(t) + I(t, f').$$

If f is a polynomial of degree m, then iterating this relation gives

$$I(t, f) = e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t). \tag{2.1}$$

If F is the polynomial obtained from f by replacing each coefficient of f by its absolute value, then it is easy to see from the definition of $I(t, f)$ that

$$|I(t, f)| \leq |t| e^{|t|} F(|t|). \tag{2.2}$$

M.R. Murty and P. Rath, *Transcendental Numbers*, DOI 10.1007/978-1-4939-0832-5_2,

With these observations, we are now ready to prove the theorem. Suppose e is algebraic of degree n. Then

$$a_n e^n + a_{n-1} e^{n-1} + \cdots + a_1 e + a_0 = 0 \tag{2.3}$$

for some integers a_i and $a_0 a_n \neq 0$. We will consider the combination

$$J := \sum_{k=0}^{n} a_k I(k, f)$$

with

$$f(x) = x^{p-1}(x-1)^p \cdots (x-n)^p$$

where $p > |a_0|$ is a large prime. Using (2.3), we see that

$$J = -\sum_{j=0}^{m} \sum_{k=0}^{n} a_k f^{(j)}(k)$$

where $m = (n+1)p - 1$. Since f has a zero of order p at $1, 2, \ldots, n$ and a zero of order $p-1$ at 0, we have that the summation actually starts from $j = p-1$. For $j = p-1$, the contribution from f is

$$f^{(p-1)}(0) = (p-1)!(-1)^{np} n!^p.$$

Thus if $n < p$, then $f^{(p-1)}(0)$ is divisible by $(p-1)!$ but not by p. If $j \geq p$, we see that $f^{(j)}(0)$ and $f^{(j)}(k)$ are divisible by $p!$ for $1 \leq k \leq n$. Hence J is a non-zero integer divisible by $(p-1)!$ and consequently

$$(p-1)! \leq |J|.$$

On the other hand, our estimate (2.2) shows that

$$|J| \leq \sum_{k=0}^{n} |a_k| e^k F(k) k \leq A n e^n (2n)!^p$$

where A is the maximum of the absolute values of the a_k's. The elementary observation

$$e^p \geq \frac{p^{p-1}}{(p-1)!}$$

gives

$$p^{p-1} e^{-p} \leq (p-1)! \leq |J| \leq A n e^n (2n)!^p.$$

For p sufficiently large, this is a contradiction. □

Exercises

1. Show that for any polynomial f, we have
$$\int_0^\pi f(x)\sin x dx = f(\pi) + f(0) - \int_0^\pi f''(x)\sin x dx.$$

2. Utilise the identity in the previous exercise to show π is irrational as follows. Suppose $\pi = a/b$ with a, b coprime integers. Let
$$f(x) = \frac{x^n(a-bx)^n}{n!}.$$
Prove that
$$\int_0^\pi f(x)\sin x\ dx$$
is a non-zero integer and derive a contradiction from this.

3. Use Euler's identity
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
to prove that there are infinitely many primes.

4. Use the series $\sum_{n=0}^{\infty} 1/n!$ to show that e is irrational.

5. Show that e is not algebraic of degree 2 by considering the relation
$$Ae + Be^{-1} + C = 0, \quad A, B, C \in \mathbb{Z},$$
and using the infinite series for e and e^{-1} and arguing as in the previous exercise.

6. Prove that $e^{\sqrt{2}}$ is irrational (Hint: Consider the series expansion for $\alpha = e^{\sqrt{2}} + e^{-\sqrt{2}}$).

http://www.springer.com/978-1-4939-0831-8

Transcendental Numbers
Murty, M.R.; Rath, P.
2014, XIV, 217 p., Softcover
ISBN: 978-1-4939-0831-8