Explicit Bounds Concerning Non-trivial Zeros of the Riemann Zeta Function

Mehdi Hassani

Dedicated to Professor Hari M. Srivastava

Abstract In this paper, we get explicit upper and lower bounds for γ_n, where $0 < \gamma_1 < \gamma_2 < \gamma_3 < \cdots$ are consecutive ordinates of non-trivial zeros $\rho = \beta + \mathrm{i}\gamma$ of the Riemann zeta function. Meanwhile, we obtain the asymptotic relation $\gamma_n \log^2 n - 2\pi n \log n \sim 2\pi n \log\log n$ as $n \to \infty$.

1 Introduction

The Riemann zeta function is defined for $\mathrm{Re}(s) > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and extended by analytic continuation to the complex plan with a simple pole at $s = 1$ with residues 1. It is known [3, 7] that

$$N(T) := \sum_{\substack{0<\gamma\leqslant T \\ \zeta(\beta+\mathrm{i}\gamma)=0}} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi \mathrm{e}} + O(\log T). \tag{1}$$

As a consequence of (1) we get

$$\sum_{\substack{0<\gamma\leqslant T \\ \zeta(\beta+\mathrm{i}\gamma)=0}} \frac{1}{\gamma} = \frac{1}{4\pi} \log^2 T - \frac{\log(2\pi)}{2\pi} \log T + E(T),$$

with $E(T) = O(1)$. Recently, we obtained an explicit form of this approximate formula by proving that $\frac{3}{50} < E(T) < \frac{109}{250}$ for any $T \geqslant \gamma_1$ (see [1, Theorem 1]), where

M. Hassani (✉)
Department of Mathematics, University of Zanjan, University Blvd., 45371-38791, Zanjan, Iran
e-mail: mehdi.hassani@znu.ac.ir

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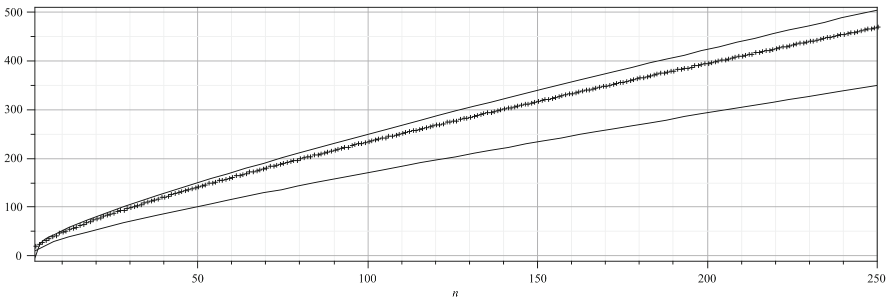

Fig. 1 Graph of the point set (n, γ_n) for $2 \leqslant n \leqslant 250$ and functions $\frac{2\pi n}{\log n}(1 + a\frac{\log\log n}{\log n})$ with $a = 3/4$ and $a = 5/2$

$$\gamma_1 = \min\{\gamma > 0 : \zeta(\beta + \mathrm{i}\gamma) = 0\} \approxeq 14.134725141734693790457251983562.$$

More generally, we set $0 < \gamma_1 < \gamma_2 < \gamma_3 < \cdots$ to be consecutive ordinates of the imaginary parts of non-trivial zeros $\rho = \beta + \mathrm{i}\gamma$ of $\zeta(s)$. Another consequence of (1) is

$$\gamma_n \sim \frac{2\pi n}{\log n}, \quad \text{as } n \to \infty.$$

Our intention in writing this note is to obtain explicit forms of this approximate formula. More precisely, we show the following.

Theorem 1.1. *For any integer $n \geqslant 5$ we have*

$$\frac{2\pi n}{\log n}\left(1 + \frac{3}{4}\frac{\log\log n}{\log n}\right) \leqslant \gamma_n \leqslant \frac{2\pi n}{\log n}\left(1 + \frac{5}{2}\frac{\log\log n}{\log n}\right). \tag{2}$$

Figure 1 shows graph of the point set (n, γ_n) for $2 \leqslant n \leqslant 250$, and lower and upper bounds appeared in (2). We note that the left-hand side of (2) is valid for $2 \leqslant n \leqslant 4$, too.

One may obtain better bounds for γ_n by using numerical information, which we obtain during proof of Theorem 1.1. More precisely, by considering Tables 1 and 2, we have the following.

Theorem 1.2. *Assume that we choose pairs λ and n_λ from Table 1, and also we choose pairs η and n_η from Table 2. Then, we have*

$$\frac{2\pi n}{\log n}\left(1 + \frac{\lambda}{2\pi}\frac{\log\log n}{\log n}\right) \leqslant \gamma_n \quad \textit{and} \quad \gamma_n \leqslant \frac{2\pi n}{\log n}\left(1 + \frac{\eta}{2\pi}\frac{\log\log n}{\log n}\right),$$

respectively, for $n \geqslant n_\lambda$ and for $n \geqslant n_\eta$.

Table 1 Some values of λ and n_λ for which the inequality (11) is valid for $n \geqslant n_\lambda$

λ	$n_\lambda \approxeq$	λ	$n_\lambda \approxeq$
-2π	3.9	$3\pi/2$	4984.5
$-\pi$	5.1	$5\pi/3$	392062.1
-1	7.8	$7\pi/4$	138610176.5
0	10.7	$9\pi/5$	2499273431483.9
1	16.7	$11\pi/6$	109511051064367600190250.3
π	97.1	5.795	876581819433015771165641491644046075.5

Table 2 Some values of η and n_η for which the inequality (12) is valid for $n \geqslant n_\eta$

η	$n_\eta \approxeq$	η	$n_\eta \approxeq$
20π	8.8	6π	1197.1
10π	11.7	5π	26245.8
8π	64.3	4π	80727920.5
7π	217.7	3π	742199235320620698359223515347877.7

On the other hand, we mention that the constants $\frac{3}{4}$ and $\frac{5}{2}$ in Theorem 1.1, as more as, the constants $\frac{\lambda}{2\pi}$ and $\frac{\eta}{2\pi}$ in Theorem 1.2, are not optimal. More precisely, if we let

$$R_n := \frac{\frac{\gamma_n}{\frac{2\pi n}{\log n}} - 1}{\frac{\log\log n}{\log n}}, \tag{3}$$

then Theorem 1.1 yields that $\frac{3}{4} \leqslant R_n \leqslant \frac{5}{2}$ for any integer $n \geqslant 5$. But, the proof of above theorems includes an argument in its heart, which implies that $\lim_{n\to\infty} R_n = 1$. Indeed, we show the following.

Theorem 1.3. *Let*

$$\Lambda_n = \frac{\gamma_n \log^2 n - 2\pi n \log n}{n \log\log n}. \tag{4}$$

Then, we have $\lim_{n\to\infty} \Lambda_n = 2\pi$.

Corollary 1.1. *For any real* $\varepsilon \in (0, 1)$, *there exists positive integer* n_ε *such that for* $n \geqslant n_\varepsilon$ *we have*

$$\frac{2\pi n}{\log n}\left(1 + (1-\varepsilon)\frac{\log\log n}{\log n}\right) \leqslant \gamma_n \leqslant \frac{2\pi n}{\log n}\left(1 + (1+\varepsilon)\frac{\log\log n}{\log n}\right).$$

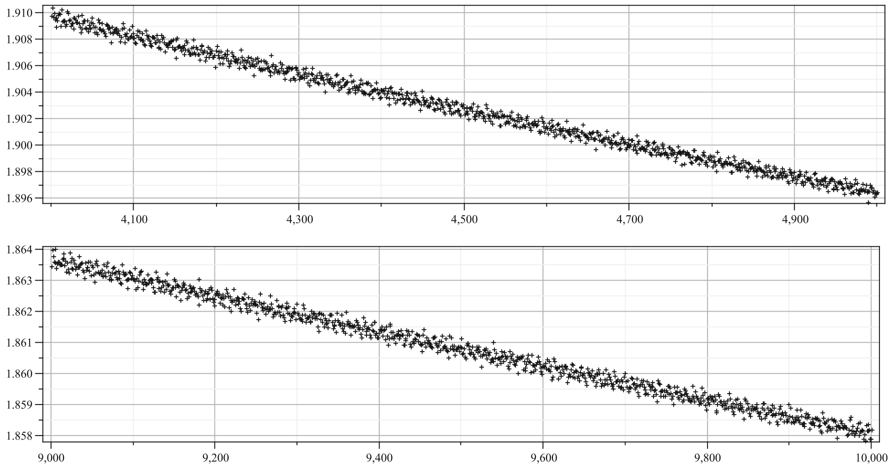

Fig. 2 Graph of the pointset (n, R_n) for $4000 \leqslant n \leqslant 5000$ and $9000 \leqslant n \leqslant 10000$, where R_n is defined by (3)

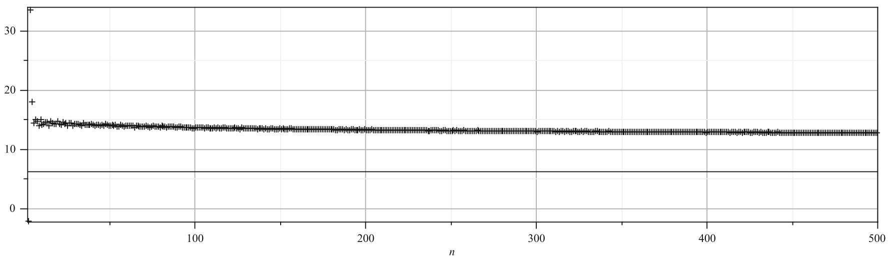

Fig. 3 Graph of the point set (n, Λ_n) for $2 \leqslant n \leqslant 500$, where Λ_n is defined by (4), and horizontal line at height 2π

Remark 1.1. Figure 2 pictures some values of R_n for several values of n. As our computations show, one may have the inequality $R_n > 1$ for $n \geqslant 3$. This means that one may have the validity of the left-hand side of (2) with 1 instead of $\frac{3}{4}$, for any integer $n \geqslant 3$. This conjecture is pictured in Fig. 3 in another point of view, where we plot values of Λ_n for $2 \leqslant n \leqslant 500$ and horizontal line at height 2π. Also, it seems that there exists a positive integer $m \approxeq 250$ such that $R_{n+m} \geqslant R_n$ for any integer $n \geqslant 3$.

Remark 1.2. The truth of Corollary 1.1 asserts that as $n \to \infty$ we have

$$\gamma_n = \frac{2\pi n}{\log n}\left(1 + (1 + o(1))\frac{\log\log n}{\log n}\right).$$

One may ask for such asymptotic expansions with more precise terms.

In the next two sections, we prove our results. To generate figures which appeared on present paper, as more as, during proofs, we will do several computations running over the numbers γ_n, all of which have been done by using Maple software and are based on the tables of zeros of the Riemann zeta function due to Odlyzko [4].

2 Lambert *W* Function, the Key of Proof

The main idea to get explicit results similar to (2) is applying an explicit version of the Riemann–von Mangoldt formula (1). This can be found in the following result due to Rosser, which is Theorem 19 of [6].

Proposition 2.1. *For any* $T \geqslant 2$ *we have* $|N(T) - F(T)| \leqslant R(T)$, *with*

$$F(T) = \frac{T}{2\pi} \log \frac{T}{2\pi \mathrm{e}} + \frac{7}{8} \quad \textit{and} \quad R(T) = \frac{137}{1000} \log T + \frac{443}{1000} \log\log T + \frac{397}{250}.$$

For our purpose, we need to modify the truth of above proposition as follows. For the whole text, we set

$$\ell = \frac{14}{25} \quad \text{and} \quad u = \frac{11}{50}.$$

Lemma 2.1. *Let*

$$L(T) = \frac{1}{2\pi} T \log T - \ell T \quad \textit{and} \quad U(T) = \frac{1}{2\pi} T \log T - uT. \tag{5}$$

Then, for $T \geqslant \gamma_1 - 10^{-5}$ *we have*

$$L(T) \leqslant N(T) \leqslant U(T). \tag{6}$$

Moreover, $U(T)$ *and* $L(T)$ *are strictly increasing for* $T \geqslant \mathrm{e}^{2\pi u - 1} \approxeq 1.465653$ *and* $T \geqslant \mathrm{e}^{2\pi \ell - 1} \approxeq 12.411008$, *respectively.*

Proof. We consider Proposition 2.1 to write $F(T) - R(T) \leqslant N(T) \leqslant F(T) + R(T)$ for $T \geqslant 2$. On the other hand, for $T \geqslant \gamma_1 - 10^{-5}$ we have $F(T) + R(T) \leqslant U(T)$ and $L(T) \leqslant F(T) - R(T)$. This proves both sides of (6). Monotonicity of the functions $U(T)$ and $L(T)$ is straightforward. □

The following lemma brings lower and upper bounds for $N(T)$ to bounds for γ_n in terms of inverses of mentioned bounds for $N(T)$.

Lemma 2.2. *Assume that* $L(T)$ *and* $U(T)$ *are defined as in* (5), *and denote by* $L^{-1}(T)$ *and* $U^{-1}(T)$ *their inverses, respectively. Then, for any integer* $n \geqslant 1$ *we have*

$$U^{-1}(n) \leqslant \gamma_n \leqslant L^{-1}(n). \tag{7}$$

Proof. Assume that $n \geqslant 1$ is any arbitrary integer and $\delta \in (0, 1)$ is any arbitrary real. We have $N(\gamma_n) = n$. Thus, we get $N(\gamma_n + \delta) \geqslant n$ and $N(\gamma_n - \delta) \leqslant n - 1$. Therefore, we obtain

$$1 + N(\gamma_n - \delta) \leqslant n \leqslant N(\gamma_n + \delta). \tag{8}$$

Right-hand sides of (6) and (8) give $\gamma_n + \delta \geqslant U^{-1}(n)$. Thus, we get $\gamma_n \geqslant U^{-1}(n)$. Similarly, left-hand sides of (6) and (8) give $L(\gamma_n - \delta) \leqslant N(\gamma_n - \delta) \leqslant n - 1 < n$. So, we get $\gamma_n - \delta \leqslant L^{-1}(n)$, and this implies validity of $\gamma_n \leqslant L^{-1}(n)$. □

In order to use inequalities (7), we need formulas for the inverses of the functions $L(T)$ and $U(T)$. This may be done in terms of the Lambert W function $W(x)$, which is defined by the relation $W(x)\mathrm{e}^{W(x)} = x$ for $x \in [-\mathrm{e}^{-1}, +\infty)$. The Lambert W function has the asymptotic expansion $W(x) = \log x + O(\log\log x)$ as $x \to \infty$, (see [5, p. 111]). The following lemma summarizes what we need about the inverses of the functions $L(T)$ and $U(T)$.

Lemma 2.3. *Assume that a and b are some positive real numbers, and let*

$$f(T) = \frac{1}{a}T\log T - bT.$$

We denote the inverse function of f by f^{-1}. Then, for $T \geqslant \mathrm{e}^{ab-1}$ the function f is strictly increasing and we have

$$f^{-1}(T) = \frac{aT}{W(a\mathrm{e}^{-ab}T)}. \tag{9}$$

In particular, as $T \to +\infty$, we obtain $f^{-1}(T) \sim aT/\log T$.

Proof. Assume that $T > 0$. Then, by definition of the Lambert W function, we imply that $f(\mathrm{e}^{W(a\mathrm{e}^{-ab}T)+ab}) = T$ or equivalently $f^{-1}(T) = \mathrm{e}^{W(a\mathrm{e}^{-ab}T)+ab}$. Definition of the Lambert W function also gives that $aT = W(a\mathrm{e}^{-ab}T)\mathrm{e}^{W(a\mathrm{e}^{-ab}T)+ab}$. Thus, we obtain (9). The asymptotic relation comes from $W(a\mathrm{e}^{-ab}T) \sim \log T$, which is valid as $T \to +\infty$. □

Finally, to get our desired explicit results, we need some explicit bounds for the Lambert W function. The following proposition, which is Theorem 2.8 of [2], offers such sharp bounds.

Proposition 2.2. *Assume that $\alpha > 0$ is real, and let*

$$\omega_\alpha(x) := \log x - \log\log x + \alpha\frac{\log\log x}{\log x}.$$

Then, for every $x \geqslant \mathrm{e}$ we have

$$\omega_{\frac{1}{2}}(x) \leqslant W(x) \leqslant \omega_{\frac{\mathrm{e}}{\mathrm{e}-1}}(x), \tag{10}$$

with equality only for $x = \mathrm{e}$.

3 Proof of Results

3.1 *Proof of the Left-Hand Side of* (2)

We let $c_u = 2\pi \mathrm{e}^{-2\pi u}$. By applying the validity of Lemma 2.3, considering the left-hand side of (7), and considering the right-hand side of (10), we obtain

$$\gamma_n \geqslant U^{-1}(n) = \frac{2\pi n}{W(c_u n)} \geqslant \frac{2\pi n}{\omega_{\frac{\mathrm{e}}{\mathrm{e}-1}}(c_u n)},$$

for $c_u n \geqslant \mathrm{e}$ or equivalently for $n \geqslant \frac{\mathrm{e}}{c_u} \approxeq 1.7$. Moreover, by computation, for any integer $n \geqslant 1$ we get

$$\gamma_n \geqslant \frac{2\pi n}{\omega_{\frac{\mathrm{e}}{\mathrm{e}-1}}(c_u n)} := g(n),$$

say. We let

$$h(n) := \frac{g(n) - \frac{2\pi n}{\log n}}{\frac{n \log\log n}{\log^2 n}}.$$

Now, we note that the function $h : (\mathrm{e}, +\infty) \longrightarrow (-\infty, 2\pi)$ defined by $h(n)$ is continuous and strictly increasing. Moreover, we have

$$\lim_{n \to \mathrm{e}^+} h(n) = -\infty \quad \text{and} \quad \lim_{n \to +\infty} h(n) = 2\pi.$$

Therefore, for any real $\lambda \in (-\infty, 2\pi)$, there exists unique $n_\lambda \in (\mathrm{e}, +\infty)$ such that $h(n) \geqslant \lambda$ for $n \geqslant n_\lambda$ with equality only for $n = n_\lambda$. Hence, for $n \geqslant n_\lambda$ we obtain

$$\gamma_n \geqslant \frac{2\pi n}{\log n} + \lambda \frac{n \log\log n}{\log^2 n}. \tag{11}$$

In Table 1 we list some values of λ and related values of n_λ. We use information of this table choosing $\lambda = \frac{3\pi}{2}$, from which we obtain the inequality

$$\gamma_n \geqslant \frac{2\pi n}{\log n} + \frac{3\pi n \log\log n}{2\log^2 n},$$

for $n \geqslant 4985$. By computation, we confirm validity of it for $2 \leqslant n \leqslant 4984$, too. This completes the proof of left-hand side of (2).

3.2 *Proof of the Right-Hand Side of* (2)

Let $c_\ell = 2\pi \mathrm{e}^{-2\pi\ell}$. We use the validity of Lemma 2.3, the right-hand side of (7), and the left-hand side of (10) to get

$$\gamma_n \leqslant L^{-1}(n) = \frac{2\pi n}{W(c_\ell n)} \leqslant \frac{2\pi n}{\omega_{1/2}(c_\ell n)},$$

for $c_\ell n \geqslant \mathrm{e}$ or equivalently for $n \geqslant \frac{\mathrm{e}}{c_\ell} \approxeq 14.6$. As more as, by computation, for any integer $n \geqslant 8$, we obtain

$$\gamma_n \leqslant \frac{2\pi n}{\omega_{1/2}(c_\ell n)} := v(n),$$

say. We set

$$z(n) := \frac{v(n) - \frac{2\pi n}{\log n}}{\frac{n \log\log n}{\log^2 n}}.$$

Also, we let

$$y_1 := \lim_{n \to 1/c_\ell +} \frac{1}{z(n)} = \frac{\log(-\log c_\ell)}{2\pi \log c_\ell} \approxeq -0.049167.$$

We note that the function $y : (1/c_\ell, +\infty) \to (y_1, 1/(2\pi))$ defined by $y(n) = 1/z(n)$ is continuous and strictly increasing. Thus, there exists unique $n_0 > 1/c_\ell$ such that $y(n_0) = 0$. By computation, we observe that $n_0 \approxeq 7.745051$. Now, we note that the function $z : (n_0, +\infty) \to (2\pi, +\infty)$ defined by $z(n)$ is continuous and strictly decreasing. Moreover, we have

$$\lim_{n \to n_0^+} z(n) = +\infty \quad \text{and} \quad \lim_{n \to +\infty} z(n) = 2\pi.$$

Therefore, for any $\eta \in (2\pi, +\infty)$, there exists unique $n_\eta \in (n_0, +\infty)$ such that $z(n) \leqslant \eta$ for $n \geqslant n_\eta$ with equality only for $n = n_\eta$, and consequently, for $n \geqslant n_\eta$ we get

$$\gamma_n \leqslant \frac{2\pi n}{\log n} + \eta \frac{n \log \log n}{\log^2 n}. \tag{12}$$

Table 2 includes some values of η and related values of n_η. Considering our computational tools, we choose $\eta = 5\pi$ from this table, from which for $n \geqslant 26246$ we obtain the inequality

$$\gamma_n \leqslant \frac{2\pi n}{\log n} + \frac{5\pi n \log \log n}{\log^2 n}.$$

By computation, we confirm validity of it for $5 \leqslant n \leqslant 26245$, too. This completes the proof of right-hand side of (2).

3.3 Proof of Theorem 1.3

We note that inequalities (11) and (12) imply

$$\liminf_{n\to\infty} \Lambda_n \geqslant 2\pi \quad \text{and} \quad \limsup_{n\to\infty} \Lambda_n \leqslant 2\pi,$$

respectively. This gives assertion of Theorem 1.3.

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