Transmission-Line Super Theory as Antenna Theory for Linear Structures

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Abstract A new generalized transmission-line theory is presented to treat multiconductor as well as antenna systems. Maxwell’s equations are cast into the form of classical telegrapher’s equations. Two quite different examples are calculated to illustrate the wide use of this theory.

Keywords Transmission-line theory • Antenna theory • Electromagnetic compatibility

1 Introduction

Antennas and transmission lines play an important role in electromagnetic compatibility (EMC). They are necessary for connecting electronic components and devices, but, at the same time, they are ideally suited for gathering unwanted electromagnetic energy from their environment. This coupled energy is often the reason for the failure of electronic devices. Since nowadays, the packing density of electronic components is increasing with a simultaneous increase of the used frequencies, it also seems necessary to extend and adapt the existing theories for estimating energy coupling at high frequencies to non-uniform linear structures. In this context, it can be definitely stated that classical transmission-line theory is no longer sufficient. This was the reason for the development of the presented new, generalized transmission-line theory for linear structures presented here which enables one to deal with cables and antennas alike.
2 Fundamentals of Transmission-Line Super Theory and Its Application

2.1 Geometrical Description of a System of Non-uniform Conductors

Consider a total number of \( N \) non-uniform conductors which are geometrically described by three-dimensional curves \( C_i(l_i) \) \((i = 1, 2, \ldots, N)\) and are parameterized by their natural parameter, the arc length \( l_i \). All conductors have circular cross sections of the same small radius \( r_0 \) and an individual total length of \( L_i \). The curves represent the centre lines of the conductors. Then one can represent the surface of the transmission lines using the so-called Frenet frame which is composed of three vectors: the tangential unit vector \( T_i(l_i) \), the normal unit vector \( N_i(l_i) \), and the binormal unit vector \( B_i(l_i) = T_i(l_i) \times N_i(l_i) \). These vectors are derived from the centre curves as

\[
T_i(l_i) = \frac{\partial C_i(l_i)}{\partial l_i}, \quad (1a)
\]

\[
N_i(l_i) = \frac{1}{\kappa_i(l_i)} \frac{\partial T_i(l_i)}{\partial l_i}, \quad (1b)
\]

with the quantity \( \kappa_i(l_i) \) being the curvature of the \( i \)th conductor. Then the surface points of the conductors are expressed by

\[
C_i^S(l_i, \alpha) = C_i(l_i) + N_i(l_i)r_0 \cos \alpha + B_i(l_i)r_0 \sin \alpha, \quad \alpha \in [0, 2\pi]. \quad (2)
\]

Since all of the conductors are circular tubes, the tangential vectors along the centre lines are also the tangential vectors to the surfaces, i.e.

\[
\frac{\partial C_i^S(l_i)}{\partial l_i} = \frac{\partial C_i(l_i)}{\partial l_i} = T_i(l_i). \quad (3)
\]

The last two summands on the right-hand side of Eq. (2) vanish by differentiation with respect to \( l_i \).

There is, however, one situation where Eq. (1b) cannot be applied to the normal unit vector: In the case of a straight wire or wires with linear curve parts, the curvature is zero and Eq. (1b) is not defined. Then one has to obtain the necessary normal vector in a different way. A simple construction of \( N_i(l_i) \) is obtained by a 90° rotation of the tangent vector \( T_i(l_i) \).

In the following examples, the circular wires will be conducted above a perfectly conducting plane. Therefore, one also needs the mirrored curves \( \tilde{C}_i(l_i) \) and the mirrored tangential vectors \( \tilde{T}_i(l_i) \), which are easily obtained from the
original quantities. Furthermore it is assumed that the transmission lines are lossless
and can be treated in the thin-wire approximation [1, 2], i.e. only axial currents are
considered, and they are concentrated on the axes (centre lines) of the wires. Only
the boundary conditions [see Eq. (12)] have to be measured on the boundary of the
conductors for one direction \( \alpha \), e.g. \( \alpha = 0^\circ \). Under these restrictions, all physical
equations are presented below.

In order to express the upcoming equations in a more compact matrix form,
another formal step is necessary: the introduction of bijective mappings which
transform all arc lengths \( l_i \) to only one parameter \( l \) as follows:

\[
l_i = l_i(l) \quad \text{for } \forall l_i(0 \leq l \leq L) \quad \text{with } l_i(0) = 0 \quad \text{and } l_i(L) = L_i. \tag{4}
\]

A simple example for such mappings is

\[
l_i = \frac{L_i}{L} l \quad \text{with } L = 1 \text{ m and } l \in [0, 1], \quad \text{or } L = \frac{1}{N} \sum_{i=1}^{N} L_i. \tag{5}
\]

Of course, if one uses this new single parameter, all equations have to be
represented with respect to \( l \) like the derivative of the potential

\[
\frac{\partial \varphi_i(l_i)}{\partial l_i} = \frac{\partial \varphi_i(l)}{\partial l} \left( \frac{dl_i(l)}{dl} \right)^{-1}.
\tag{6}
\]

Note that the mappings in Eqs. (4) or (5) are not unique. They are only
introduced to simplify the mathematical representations and operations. Therefore,
all physical quantities that are represented with respect to this common parameter
have no direct physical meaning. After having obtained the final desired results,
they have to be transformed back into the physical space where they represent
observable quantities.

### 2.2 Derivation of the New Telegrapher’s Equation (Antenna Equation) from the Mixed-Potential Integral Equation

The scattered electrical potential \( \varphi(r(l_i)) \) can be written with the aid of the scalar
Green’s function of free space

\[
G(r) = \frac{\exp(-jkr)}{r} (j \text{ is the imaginary unit}) \tag{7}
\]

as
\[ \varphi(r) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \int_{l_0}^{L_i} q_i'(l_i) \left( G\left(\left| r - \mathbf{C}_i(l_i) \right| \right) - G\left(\left| r - \tilde{\mathbf{C}}_i(l_i) \right| \right) \right) dl_i. \quad (8) \]

In Eq. (8) the charge per unit length is related to the mirrored charge per unit length by \( q_i'(l_i) = -q_i(l_i) \).

Using the continuity equation

\[ q_i'(l_i) = -\frac{1}{j\omega} \frac{\partial i_i(l_i)}{\partial l_i} \quad (i_i \text{ denotes the axial current along wire } i), \]

Eq. (8) can be rewritten as

\[ \varphi(r) = -\frac{1}{j4\pi\varepsilon_0\omega} \sum_{i=1}^{N} \int_{l_0}^{L_i} \frac{\partial i_i(l_i)}{\partial l_i} \left( G\left(\left| r - \mathbf{C}_i(l_i) \right| \right) - G\left(\left| r - \tilde{\mathbf{C}}_i(l_i) \right| \right) \right) dl_i. \quad (10) \]

Similar to the scalar potential, the magnetic vector potential can be expressed as

\[ \mathbf{A}(r) = \frac{\mu_0}{4\pi} \sum_{i=1}^{N} \int_{l_0}^{L_i} i_i(l_i) \left( \mathbf{T}_i(l_i) G\left(\left| r - \mathbf{C}_i(l_i) \right| \right) - \tilde{\mathbf{T}}_i(l_i) G\left(\left| r - \tilde{\mathbf{C}}_i(l_i) \right| \right) \right) dl_i. \quad (11) \]

Now, the potential equations (10) and (11) allow for expressing the total electrical field (exciting and scattered) on the surface of the conductors in curvilinear coordinates as (boundary condition with \( \alpha = 0^\circ \))

\[ \left( \mathbf{E}^{\text{total}}(r = \mathbf{C}_i(l_i) + \mathbf{N}_i(l_i)r_0) \right)_{l_i} = 0 = \mathbf{T}_i(l_i)\mathbf{E}^{\text{total}}(l_i). \quad (12) \]

In explicit form and after some rearrangement, Eq. (12) reads

\[ \frac{\partial \varphi_i(l_i)}{\partial l_i} + j\omega \frac{\mu_0}{4\pi} \sum_{j=1}^{N} \int_{l_0}^{L_j} i_j(l_j) \left[ \mathbf{T}_i(l_i) \cdot \mathbf{T}_j(l_j) G_{ij}(l_i, l_j; k) \right. \\
- \mathbf{T}_i(l_i) \cdot \tilde{\mathbf{T}}_j(l_j) G_{ij}(l_i, l_j; k) \left. \right] dl_j = \mathbf{E}^{\text{exc}}(l_i) \cdot \mathbf{T}_i(l_i). \quad (13) \]

Here the abbreviations

\[ G_{ij}(l_i, l_j; k) := G\left(\left| \mathbf{C}_i(l_i) + \mathbf{N}_i(l_i)r_0 - \mathbf{C}_j(l_j) \right| \right) \]
\[ G_{ij}(l_i, l_j; k) := G \left( C_i(l_i) + N_i(l_i) r_0 - C_j(l_j) \right) \]  

(14)

are used.

Similarly, Eq. (10) is rewritten on the boundary of the conductor \( i \) as

\[ \varphi_i(l_i) + \frac{1}{j 4 \pi \varepsilon_0 \omega} \sum_{j=1}^{N} \int_{0}^{L} \frac{\partial i_j(l_j)}{\partial l_j} \left[ G_{ij}(l_i, l_j; k) - \tilde{G}_{ij}(l_i, l_j; k) \right] dl_j = 0 \]  

(15)

Using at this stage the general parameter \( l \) as introduced in Eqs. (4), (5), and (6), Eqs. (13) and (15) can be represented in a compact matrix form

\[ \frac{\partial \Phi}{\partial l} + j \omega \frac{\mu_0}{4 \pi} \int_{0}^{L} G_L(l, l'; k) \cdot i(l') dl' = \mathbf{v}_{\text{exc}}(l) \]  

(16)

\[ \int_{0}^{L} G_C(l, l'; k) \cdot \frac{\partial i(l)}{\partial l} dl' + j \omega 4 \pi \varepsilon_0 \varphi(l) = 0. \]  

(17)

The above matrices and vectors are defined as follows:

\[ \Phi(l) := (\varphi_1(l), \varphi_2(l), \ldots, \varphi_N(l))^T, \quad i(l) := (i_1(l), i_2(l), \ldots, i_N(l))^T \]  

(18)

and

\[ \mathbf{v}_{\text{exc}}(l) := \left( \mathbf{E}^{\text{exc}}(C_1(l)) \cdot \mathbf{T}_1(l), \mathbf{E}^{\text{exc}}(C_2(l)) \cdot \mathbf{T}_2(l), \ldots, \mathbf{E}^{\text{exc}}(C_N(l)) \cdot \mathbf{T}_N(l) \right)^T, \]  

(19)

\[ \left( G_L(l, l'; k) \right)_{ij} := T_i(l) \cdot \mathbf{T}_j(l') G \left( \left| C_i(l) + N_i(l) r_0 - C_j(l') \right| \right) \]  

(20a)

\[ - T_i(l) \cdot \tilde{T}_j(l') G \left( \left| C_i(l) + N_i(l) r_0 - \tilde{C}_j(l') \right| \right) \]  

(20b)

and

\[ \left( G_C(l, l'; k) \right)_{ij} := G \left( \left| C_i(l) + N_i(l) r_0 - C_j(l') \right| \right) - G \left( \left| C_i(l) + N_i(l) r_0 - \tilde{C}_j(l') \right| \right). \]  

(21a)

Remember, the transformation of all equations into the representation with respect to the general parameter \( l \) has to be correctly applied to all operations, coordinate transformation and operator transformation; see, e.g. Eq. (6). Equations (16) and (17) are the so-called mixed-potential integral equation (MPIE), which represent Maxwell’s equations in the Lorenz gauge for a non-uniform multiconductor system. Observe, however, that all quantities depend on the
(“artificial”) parameter \( l \) and therefore do not have a direct physical meaning. In addition, the potential \( \phi \) is not uniquely defined due to its general dependency on the integration contour.

If one replaces the derivative of the potential in Eq. (16) using Eqs. (17) and (9), then Eq. (16) can be written in the form which will be used later:

\[
\frac{1}{4\pi\varepsilon_0} \int_0^L G_C(l, l'; k) q'(l') dl' + j\omega \frac{\mu_0}{4\pi} \int_0^L G_L(l, l'; k) \cdot i(l') dl' = \mathbf{v}_{\text{exc}}(l). \tag{21b}
\]

In the low-frequency limit (\( k \to 0 \)), when radiation is still absent, the currents and their derivatives can be pulled out of the integrals in Eq. (21b), because the phase-independent Green’s functions have very strong peaks at the values \( l' = l \) and therefore act almost like delta functions, resulting in

\[
\left( \int_0^L G_L(l, l'; 0) dl' \right) \cdot i(l) = : G^0_L(l) \cdot i(l), \tag{22a}
\]

\[
\left( \int_0^L G_C(l, l'; 0) dl' \right) \cdot \frac{\partial}{\partial l} i(l) = : G^0_C(l) \cdot \frac{\partial}{\partial l} i(l). \tag{22b}
\]

Then the MPIE can be simplified and written in a more compact and familiar form

\[
\begin{bmatrix}
\frac{\partial}{\partial l} \phi(l) \\
\frac{\partial}{\partial l} i(l)
\end{bmatrix} + j\omega \begin{bmatrix}
0 & L'(l) \\
C'(l) & 0
\end{bmatrix} \cdot \begin{bmatrix}
\phi(l) \\
i(l)
\end{bmatrix} = \begin{bmatrix}
\mathbf{v}_{\text{exc}}(l) \\
0
\end{bmatrix}. \tag{23}
\]

This equation is the classical transmission-line equation for multiconductors with the exception that the elements of the parameter matrix (inductances and capacitances) depend on the location. In Eq. (23) the usual definitions have been made:

\[
L'(l) = \frac{\mu_0}{4\pi} G^0_L(l) \tag{24a}
\]

and

\[
C'(l) = 4\pi\varepsilon_0 \left( G^0_C(l) \right)^{-1}. \tag{24b}
\]

Also the solution of Eq. (23) is well known [3], even if the block matrix is completely occupied and its elements are all complex valued:
\[
\begin{align*}
\left(\Phi(l), i(l)\right) &= M^l_{l_0} \left\{ -j\omega P^{*}(0) \right\} \left(\Phi(l_0), i(l_0)\right) + \int_{l_0}^{l} M^l_{l'} \left\{ -j\omega P^{*}(0) \right\} \cdot \vec{v}_{exc}(l') \, dl'.
\end{align*}
\] (25)

The block matrix \( P^{*}(0) \) represents the block matrix in Eq. (23). The quantity \( L_{l_0} \) is the so-called Matrizant or product integral [3]. In order to find an explicit solution for Eq. (25), one needs to know \( P^{*}(0) \) and the exciting sources. The general procedure for how to solve Eq. (25) will be given below.

Note that the first-order differential equation system Eq. (23) can be written as a second-order wave equation for the current.

\[
\left(\frac{\partial^2}{\partial l^2} + C' \cdot \frac{\partial C'^{-1}}{\partial l} + \omega^2 C' \cdot L'\right) \cdot i(l) = -j\omega C' v_{exc}(l).
\] (26)

Remember, this equation is valid in the low-frequency approximation. New in this otherwise classical equation is the occurrence of the first derivative of the current vector due to losses caused by reflections along the lines. As expected, the solutions of Eq. (26) describe forward and backward running current waves. This result gives rise to the assumption that even in the exact case (all frequencies are allowed), the current fulfills a second-order wave equation of the general form

\[
\left(\frac{\partial^2}{\partial l^2} + j\omega P_{11}(l) \cdot \frac{\partial}{\partial l} + \omega^2 P_{12}(l)\right) \cdot i(l) = -j\omega q''_{exc}(l).
\] (27)

With the aid of the continuity equation (9), this equation can be converted into a coupled first-order differential equation system:

\[
\frac{\partial}{\partial l} \left[ \begin{array}{c}
q'(l) \\
i(l)
\end{array} \right] + j\omega \left[ \begin{array}{cc}
P_{11}(l) & P_{12}(l) \\
1 & 0
\end{array} \right] \left[ \begin{array}{c}
q'(l) \\
i(l)
\end{array} \right] = \left[ \begin{array}{c}
q''_{exc}(l) \\
0
\end{array} \right].
\] (28)

In Eq. (28) the charge per unit length, \( q' \), now appears as the physical quantity instead of the potential. But again, the solution of Eq. (28) is known:

\[
\left(\begin{array}{c}
q'(l) \\
i(l)
\end{array}\right) = M^l_{l_0} \left\{ -j\omega P \right\} \left(\begin{array}{c}
q'(l_0) \\
i(l_0)
\end{array}\right) + \int_{l_0}^{l} M^l_{l'} \left\{ -j\omega P \right\} \cdot q''_{exc}(l') \, dl'.
\] (29)

The next subsection will show how the Matrizant and the parameter matrix are calculated by an iterative procedure.
3 Iterative Methods for the Solution of the Matrizant and Solution of the TLST Equations

In the next step towards the generalized telegrapher’s equation, it will be proven that Eq. (28) indeed follows from the MPIE Eqs. (21b) and (9). For this purpose, Eq. (21b) is rearranged as

\[
\begin{bmatrix}
 j \omega & 1 \frac{\partial}{\partial l}
\end{bmatrix}_{(N,2N)} \int_0^L \mathbf{G}(l,l';k) \begin{bmatrix}
 q'(l') \\
 i(l')
\end{bmatrix} dl' = v'_{\text{exc}}(l).
\]  

(30)

Here the block matrix is given by

\[
\mathbf{G} := \begin{bmatrix}
 0 & \frac{\mu_0}{4\pi} \mathbf{G}_L \\
 4\pi\varepsilon_0 \mathbf{G}_C & 0
\end{bmatrix}.
\]  

(31)

Then Eq. (29) is inserted into Eq. (30) and after some minor conversions becomes

\[
\begin{bmatrix}
 I_{21} & I_{22}
\end{bmatrix} \frac{\partial q'}{\partial l} + \begin{bmatrix}
 I_{11} & I_{12}
\end{bmatrix} \frac{\partial i}{\partial l} + \left( j \omega I_{11} + \frac{\partial I_{21}}{\partial l} \right) q' + \left( j \omega I_{12} + \frac{\partial I_{22}}{\partial l} \right) i + j \omega I_{01} + \frac{\partial I_{02}}{\partial l} = v'_{\text{exc}}.
\]  

(32)

Note that Eq. (30) [and therefore also Eq. (32)] implicitly contains the continuity equation. Here the components \(I_{ij}\) are elements of the block matrices

\[
\mathbf{I}(l) = \begin{bmatrix}
 I_{10}(l) \\
 I_{20}(l)
\end{bmatrix} = \int_0^L \mathbf{G}(l,l') \begin{bmatrix}
 I_{10}'(l') \\
 I_{20}'(l')
\end{bmatrix} \begin{bmatrix}
 -j \omega \mathbf{P} \\
 0
\end{bmatrix} dl'.
\]  

(33a)

(33b)

Collecting now the summands in Eq. (32) in terms of block vectors and block matrices, one indeed obtains Eq. (28) and thereby confirming the assumption that the current obeys a wave equation with the following parameters and renormalized source term:

\[
\mathbf{P} = \begin{bmatrix}
 I_{21} & I_{22} & 0 \\
 0 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
 I_{11} + \frac{1}{j \omega} \frac{\partial I_{21}}{\partial l} & I_{12} + \frac{1}{j \omega} \frac{\partial I_{22}}{\partial l} \\
 \frac{1}{j \omega} \frac{\partial I_{21}}{\partial l} & \frac{1}{j \omega} \frac{\partial I_{22}}{\partial l}
\end{bmatrix}.
\]  

(34a)
\[ q'' _{\text{exc}} = I^{-1} _{21} \left( v' _{\text{exc}} - j\omega I_{10} - \frac{\partial I_{20}}{\partial l} \right). \]  

(34b)

In other words, Maxwell’s equations applied to linear thin conducting structures have been cast into the form of the transmission-line equations which are known from classical transmission-line theory (TLT). No restrictions are made on frequencies or heights of the lines above ground. Thus, in particular, radiation is included. There is, however, an essential difference compared to classical TLT: The parameter matrix is fully occupied, and its elements are complex valued. This can be better seen if the Eq. (28) is transformed into the potential-current representation, which is done with the aid of the Eqs. (9) and (17) as well as with the relations

\[ \varphi = I_{21} q' + I_{22} i + I_{20} \]  

(35a)

or

\[ \begin{bmatrix} \varphi \\ i \end{bmatrix} = \begin{bmatrix} I_{21} & I_{22} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q' \\ i \end{bmatrix} + \begin{bmatrix} I_{20} \\ 0 \end{bmatrix} \]  

(35b)

and results in

\[ \frac{\partial}{\partial l} \begin{bmatrix} \varphi(l) \\ i(l) \end{bmatrix} + j\omega \mathbf{P}^*(l) \begin{bmatrix} \varphi(l) \\ i(l) \end{bmatrix} = \begin{bmatrix} \varphi'(l) \\ i'_s(l) \end{bmatrix} \]  

(36a)

with

\[ \mathbf{P}^* = \begin{bmatrix} I_{11} I_{21}^{-1} & I_{12} - I_{11} I_{21}^{-1} \\ I_{21} I_{21}^{-1} & -I_{21}^{-1} I_{22} \end{bmatrix} \]  

(36b)

and

\[ \begin{bmatrix} \varphi'_s \\ i'_s \end{bmatrix} = \begin{bmatrix} v' _{\text{exc}} + j\omega (I_{11} I_{21}^{-1} I_{20} - I_{10}) \\ j\omega I_{21}^{-1} I_{20} \end{bmatrix}. \]  

(37)

With the knowledge of the block matrix \( \mathbf{I} \), one can determine the parameter matrices \( \mathbf{P} \) and \( \mathbf{P}^* \) in the charge-current representation (28) and in the potential-current representation Eq. (36a), respectively. To calculate the parameter matrix, it is preferable to use the charge-current representation, and in a last step, the current is obtained from Eq. (36a) with the chosen boundary conditions (see below).

To solve Eqs. (28), (34a), and (34b), one needs to know the parameter matrix \( \mathbf{P} \) and the Matrizant \( \mathbf{M}^f / \{ -j\omega \mathbf{P} \} \), both operators enter the key Eqs. (33a) and (33b) for \( \mathbf{I} \) and the block vector \( [I_{10} \quad I_{20}]^T \). However, both depend on each other in an implicit equation. Thus, one may start an iteration procedure to calculate \( \mathbf{I} \),
beginning with $M_j\{ -j\omega \mathbf{P} \} = \mathbf{T}$ and $\begin{bmatrix} \mathbf{I}_0(0) & \mathbf{I}_0(0) \end{bmatrix}^T = [0 \ 0]^T$ in a general series expansion for $\mathbf{T}$

$$\mathbf{T} = \mathbf{T}_0 + j\omega \mathbf{T}_1 + \ldots$$

In the low-frequency limit ($\omega \to 0$) together with the above assumptions, this results in

$$\mathbf{I}_0(l) \equiv \mathbf{I}_0(l) = \int_{k-\Delta l}^{K} \mathbf{G}(l, l'; k)|_{k-\Delta l} dl.$$

Now, the calculations are performed following the chain [4]:

$$\begin{align*}
\mathbf{I}_0(l) &\rightarrow \mathbf{P}_0(l) \rightarrow M_j\{ -j\omega \mathbf{P}_0(l) \} \rightarrow \mathbf{I}_1(l) \rightarrow \mathbf{P}_1(l) \rightarrow M_j\{ -j\omega \mathbf{P}_1(l) \} \rightarrow \ldots
\end{align*}$$

This iterative procedure is terminated when convergence is achieved. In practical applications [4, 5], convergence was already reached after the first iteration with $\mathbf{I}_1$. Then from $\mathbf{I}_1$ one finds $\mathbf{P}_1^*(\mathbf{I}_1)$ (Eq. (36b)) and can solve Eq. (36a) to get the final result for current and potential.

### 4 Radiated Power

The average power radiated by a lossless multiconductor system above conducting ground, which is excited by lumped sources, was derived in [6]. This power is obtained by the difference of the power that is fed into the system at the beginning of the conductors and the power that arrives at the end of the conductors:

$$P_{\text{rad}} = -\frac{1}{2} \text{Re} \sum_{i=1}^{N} \int_{l_0-\Delta}^{L_{i-\Delta}} \frac{d}{dl_i} (\mathbf{P}_i(\mathbf{P}_i^* - \mathbf{P}_i^*) dl_i$$

with $\Delta \to 0$.  

When transforming Eq. (41) into the matrix notation, it becomes via Eq. (36a)

$$P_{\text{rad}} = j\omega \int_{0}^{L} \left[ \mathbf{\Phi}^*(\mathbf{P}_{22}^* - \mathbf{P}_{11}^*) + i^* \left( \mathbf{P}_{11}^* - \mathbf{P}_{22}^* \right) \mathbf{\Phi} + i^* \left( \mathbf{P}_{12}^* - \mathbf{P}_{21}^* \right) i + \mathbf{\Phi}^* \left( \mathbf{P}_{21}^* - \mathbf{P}_{21}^* \right) \mathbf{\Phi} \right] d\mathbf{r}.$$
The quantities which carry the upper index * denote the transposed complex conjugate vectors and matrices. The formula Eq. (42) simplified to one conductor reads

\[ P_{\text{rad}} = -\frac{\omega}{2} \int_0^L \left[ \text{Im}(P_{12}^*) |i|^2 + \text{Im}(P_{21}^*) |\varphi|^2 + \text{Im} \left( \varphi \left( P_{11}^* - P_{22}^{*(\uparrow)} \right) i^* \right) \right]. \] (43)

The element \( P_{22}^{*(\uparrow)} \) is the complex conjugate of \( P_{22}^* \).

It can be recognized from Eq. (42) that all block matrices \( P_{ij}^* \) contribute to the radiated power with both their real and their imaginary parts.

## 5 Application Examples

Two completely different application examples are investigated: a simple transmission line above ground with vertical risers and a linear antenna with different angles relative to the ground plane. In the two examples, the line and the antenna are both fed by a voltage source of 1 V at their near ends and have a load impedance of 50 \( \Omega \). Figures 1 and 2 show their configuration and their geometry. The transmission line is terminated with its (classical) characteristic impedance of 359 \( \Omega \). The antenna is open ended.

Figures 3 and 4 display the real parts of the inductances per unit length along the wire/antenna. In Fig. 3 it can be seen that this parameter shows a symmetrical behaviour relative to the central point of the line. For low frequencies \( \lambda \gg h \) around this central point, the inductance per unit length equals the value of classical transmission-line theory \( L_0 = \left( \frac{\mu_0}{2 \pi} \right) \ln \left( \frac{2h}{\lambda} \right) \).

Comparing a point from the central

![Fig. 1 Geometry, feeding and termination of the TL](image1)

![Fig. 2 Geometry and feeding of the open-ended antenna](image2)
part of the wire where the wire is present at both sides, with the bend points, it can be seen that in the latter the line is mainly present at only one side. Consequently, the inductance decreases to its minimum values. In Fig. 4 the real part of the inductance per unit length shows a very similar behaviour for $\alpha = 0^\circ$ as in Fig. 3.

Fig. 3 The real part of the inductance per unit length along the TL at 1 GHz

Fig. 4 The real part of the inductance per unit length along the antenna

part of the wire where the wire is present at both sides, with the bend points, it can be seen that in the latter the line is mainly present at only one side. Consequently, the inductance decreases to its minimum values. In Fig. 4 the real part of the inductance per unit length shows a very similar behaviour for $\alpha = 0^\circ$ as in Fig. 3,
except at the right end, due to the missing riser. However, with growing angle size, the inductance increases and starts to oscillate with the eigenvalue $\lambda/2$ of the current, caused by increasing radiation losses. The biggest inductance occurs at $90^\circ$.

Unlike in the previous case, Figs. 5 and 6 now represent the graphs of those quantities which do not have an analogue in classical transmission-line theory:

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**Fig. 5** Imaginary part of the capacitance per unit length along the TL at 1 GHz

**Fig. 6** Imaginary part of the capacitance per unit length along the antenna
namely, the imaginary parts of the complex capacitance. Although their values are relatively small, they nevertheless contribute to the radiation losses of the otherwise lossless line. Note that at the bend points of the wire and antenna, the imaginary parts of the capacitances take their maxima, indicating that at these points, an essential part of radiation is created. Also oscillations along the lines with $\lambda/2$ eigenvalue can be observed. For the antenna, these oscillations increase with the angle $\alpha$.

As soon as the parameter block matrix is known, the super transmission-line Eq. (36a) can be solved by applying known methods [5] to obtain the current for the given boundary conditions. In the case of the transmission-line configuration, the wire is terminated at the far end by its classical (constant) characteristic TL impedance. Note, however, that in the present case, the expression $(P^{*}_{12}/P^{*}_{21})^{0.5}$ is a function of the local coordinate and frequency. Therefore, the question arises, “At which location shall this value be chosen?” In [7] it was shown that the maximum of the above square root along the line (which equals the constant classical characteristic impedance value) led to the least oscillations of the current along the wire. A constant value of the current like in classical TL theory cannot be achieved due to reflections at all non-uniformities. These reflections, of course, cause standing waves of the wire and antenna. In Figs. 7 and 8, the currents ring with frequencies of 70 MHz and 150 MHz, respectively, according to their $\lambda/2$ eigenvalues which correspond to their total length. As can be seen in Fig. 8, the angle $\alpha$ has only a minor influence on the very pronounced current amplitude oscillations along the antenna.

Figures 9 and 10 display the radiated power from the TL and from the antenna at different angles. It can be observed that the radiated power grows with frequency.
Fig. 8  Current along the antenna

Fig. 9  Radiated power of the TL
Due to the resonance phenomena, the power oscillates with the eigenfrequency of the wire (70 MHz) and the antenna (150 MHz), respectively. Since the transmission line is matched with its “characteristic impedance”, the power oscillations are smaller than in the open-ended antenna. The influence of the angle of the antenna on the radiated power becomes more noticeable at lower frequencies \( f < 900 \text{ MHz} \) than at higher frequencies. But above 45°, the maximum power only changes marginally for frequencies \( f \geq 1 \text{ GHz} \).

### 6 Conclusion

It was shown that the TLST can also be used as an antenna theory for linear structures. The derived Eqs. (36a), (36b), and (37) are Maxwell’s equations for a multi-wire system. When the potential and current for such a system are known, one can use these quantities to calculate the total electric field and the magnetic field [5], and, therefore, the Poynting vector as well.

The radiated power results from the parameter block matrices, the currents and the potentials. In its structure, this new theory resembles the classical transmission-line theory very much. There are, however, essential differences: The transmission-line parameters are now complex and are frequency and location dependent. They depend on the chosen gauge and coordinate system (see [4]) and, therefore, lose their physical meaning, unlike in the classical theory. The above examples show the wide range of possible applications of the new theory.
References
