Chapter 2
Governing Equations

Abstract In this chapter fundamental governing equations for propagation of a harmonic disturbance on the surface of an elastic half-space is presented. The elastic media is assumed to be isotropic, continuous and infinite. Although this subject has been treated in various approaches in a number of references, it may be appropriate to develop a general procedure for solution of a number of essential soil dynamic problems based on the first principles of elasticity. The main scope of this chapter is to present a general systematic solution for addressing the problems considered in the next two chapters. This general solution is presented in two-dimensional Fourier domain in terms of elastic dilatation, and two elastic rotation components, which are used for decoupling of the involved partial differential equations. This chapter is also presenting the required fundamental of elasto-dynamics to achieve the set goal for the chapter.

Keywords Stress–strain relation • Elastic rotations • Boundary stresses • Boundary displacements

In this chapter fundamental governing equations for propagation of a harmonic disturbance on the surface of an elastic half-space is presented. The elastic media is assumed to be isotropic, continuous and infinite. Although this subject has been treated in various approaches in a number of references such as Lamb (1904), Timoshenko and Goodier (1951), Sneddon (1951), Ewing and Jardetzky (1957), and many others, it may be appropriate to develop a general procedure for solution of a number of essential soil dynamic problems based on the first principles of elasticity. The main scope of this chapter is to present a general systematic solution for addressing the problems considered in the next two chapters. This general solution is presented in two-dimensional Fourier domain in terms of elastic dilatation, and two elastic rotation components, which are used for decoupling of the involved partial differential equations. This chapter is also presenting the required fundamental of elasto-dynamics to achieve the set goal for the chapter.
2.1 Derivation of Equations of Motion

The equations of motion for an elastic half-space medium can be obtained by deriving the equilibrium equations for an infinitesimal element in terms of the applied stresses. This element is assumed to be an elastic body with an applied orthogonal stresses as shown in Fig. 2.1. The equations of motion for the element can be expressed by writing Newton’s second law in three directions (see Shames and Cozzarelli 1991). These equations are:

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = \rho \frac{\partial^2 u}{\partial t^2} \tag{2.1}
\]

\[
\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y = \rho \frac{\partial^2 v}{\partial t^2} \tag{2.2}
\]

\[
\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + Z = \rho \frac{\partial^2 w}{\partial t^2} \tag{2.3}
\]

where \(\rho\) is the mass density of the medium; \(\sigma_{ij}\) and \(\tau_{ij}\) are direct and shear stresses on the plane perpendicular to \(j\) direction and along the direction of \(i\); \(X, Y, \) and \(Z\) are internal forces per unit volume in the \(x, y,\) and \(z\) directions; \(u, v,\) and \(w\) are displacements of a point in the medium in the \(x, y,\) and \(z\) directions.

2.2 Stress–Strain Relation

For a homogeneous isotropic material, i.e. a material having the same properties in all directions, Hooke’s Law can be written by the following equations:
\[ \varepsilon_{xx} = \frac{1}{E} \left[ \sigma_{xx} - \nu \left( \sigma_{yy} + \sigma_{zz} \right) \right] \] (2.4)

\[ \varepsilon_{yy} = \frac{1}{E} \left[ \sigma_{yy} - \nu \left( \sigma_{zz} + \sigma_{xx} \right) \right] \] (2.5)

\[ \varepsilon_{zz} = \frac{1}{E} \left[ \sigma_{zz} - \nu \left( \sigma_{xx} + \sigma_{yy} \right) \right] \] (2.6)

where \( E \) and \( \nu \) are elastic modulus and Poisson ratio of the medium. Adding Eqs. (2.4)–(2.6) yields:

\[ E \left( \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \right) = (1 - 2\nu) \left( \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \right) \] (2.7)

After substituting \( \sigma_{yy} + \sigma_{zz} \) from Eq. (2.4) into Eq. (2.7) and simplifying, this yields:

\[ \sigma_{xx} = \frac{E \nu}{(1 + \nu)(1 - 2\nu)} \varepsilon + \frac{E}{(1 + \nu)} \varepsilon_{xx} \] (2.8)

\[ \varepsilon = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \] (2.9)

Introducing Lame constants (\( \lambda \) and \( G \)) the above equation can be written as:

\[ \sigma_{xx} = \lambda \varepsilon + 2G \varepsilon_{xx} \] (2.10)

Following the same procedure, \( \sigma_{yy} \) and \( \sigma_{zz} \) become

\[ \sigma_{yy} = \lambda \varepsilon + 2G \varepsilon_{yy} \] (2.11)

\[ \sigma_{zz} = \lambda \varepsilon + 2G \varepsilon_{zz} \] (2.12)

The familiar equations relating shear stresses to strains are:

\[ \tau_{xy} = \tau_{yx} = G \gamma_{xy} \] (2.13)

\[ \tau_{yz} = \tau_{zy} = G \gamma_{yz} \] (2.14)

\[ \tau_{zx} = \tau_{xz} = G \gamma_{zx} \] (2.15)

### 2.3 Strains in Terms of Displacements

Since strains generally vary from point to point, the mathematical definitions of strain must relate to an infinitesimal element. Consider an element in the \((x, y)\) plane (Fig. 2.2). During straining, point \( A \) experiences displacements \( u \) and \( v \) in the \( x \) and \( y \) directions. Displacements of other points are also shown in Fig. 2.2.
On this basis, the mathematical expressions for the linear strains as described by Ford and Alexander (1963) and many others are

\[ \varepsilon_{xx} = \frac{\partial u}{\partial x} \]  
\[ \varepsilon_{yy} = \frac{\partial v}{\partial y} \]  
\[ \varepsilon_{zz} = \frac{\partial w}{\partial z} \]  

These three strains are direct strains, but shear strains due to small deformation can be presented in terms of rotation. On the other hand, angles of rotation of any side of the element are very small, therefore, by definition:

\[ \gamma_{xy} = \theta + \varphi = \frac{(v + \frac{\partial v}{\partial x} dx) - v}{\frac{\partial u}{\partial x} + dx} + \frac{(u + \frac{\partial u}{\partial y} dy) - u}{\frac{\partial v}{\partial y} + dy} \]  

Because \( dx >> \frac{\partial u}{\partial x} \) and \( dy >> \frac{\partial v}{\partial y} \), Eq. (2.19) becomes

\[ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \]  

Consequently, other shear strains are given by

\[ \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \]  
\[ \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \]
2.4 Elastic Rotations in Terms of Displacements

Based on Fig. 2.2 the elastic rotation about the $z$-axis is defined to be:

$$\omega_z = \frac{1}{2} (\theta - \varphi)$$ \hspace{1cm} (2.23)

Considering that both angles of $\theta$ and $\varphi$ are very small then this component of elastic rotation can be expressed as

$$\omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$ \hspace{1cm} (2.24)

Similarly, other rotations are

$$\omega_x = \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)$$ \hspace{1cm} (2.25)

$$\omega_y = \frac{1}{2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)$$ \hspace{1cm} (2.26)

2.5 Equations of Motion

Now, having substituted Eqs. (2.10)–(2.12) and (2.16)–(2.18) into Eqs. (2.1)–(2.3), the equations of motion for the element become:

$$\frac{\partial}{\partial x} \left( \lambda \varepsilon + 2G \varepsilon_{xx} \right) + \frac{\partial}{\partial y} G \gamma_{yx} + \frac{\partial}{\partial z} G \gamma_{zx} + X = \rho \frac{\partial^2 u}{\partial t^2}$$ \hspace{1cm} (2.27)

$$\frac{\partial}{\partial y} \left( \lambda \varepsilon + 2G \varepsilon_{yy} \right) + \frac{\partial}{\partial z} G \gamma_{yz} + \frac{\partial}{\partial x} G \gamma_{xy} + Y = \rho \frac{\partial^2 v}{\partial t^2}$$ \hspace{1cm} (2.28)

$$\frac{\partial}{\partial z} \left( \lambda \varepsilon + 2G \varepsilon_{zz} \right) + \frac{\partial}{\partial x} G \gamma_{xz} + \frac{\partial}{\partial y} G \gamma_{yz} + Z = \rho \frac{\partial^2 w}{\partial t^2}$$ \hspace{1cm} (2.29)

Substituting for strains in Eqs. (2.27)–(2.29), in terms of volumetric strain and displacements, the equations of motion become:

$$(\lambda + G) \frac{\partial \varepsilon}{\partial x} + G \nabla^2 u + X = \rho \frac{\partial^2 u}{\partial t^2}$$ \hspace{1cm} (2.30)

$$(\lambda + G) \frac{\partial \varepsilon}{\partial y} + G \nabla^2 v + Y = \rho \frac{\partial^2 v}{\partial t^2}$$ \hspace{1cm} (2.31)

$$(\lambda + G) \frac{\partial \varepsilon}{\partial z} + G \nabla^2 w + Z = \rho \frac{\partial^2 w}{\partial t^2}$$ \hspace{1cm} (2.32)
where \( \nabla^2 \) is the Laplacian operator.

If the body force is neglected and variations of the displacements are harmonic with circular frequency of \( \Omega \), then Eqs. (2.30)–(2.32) can be written in the following form:

\[
(\lambda + G) \frac{\partial \varepsilon}{\partial x} + G \nabla^2 u = -\rho \Omega^2 u \\
(\lambda + G) \frac{\partial \varepsilon}{\partial y} + G \nabla^2 v = -\rho \Omega^2 v \\
(\lambda + G) \frac{\partial \varepsilon}{\partial z} + G \nabla^2 w = -\rho \Omega^2 w
\]  

(2.33)  
(2.34)  
(2.35)

The above equations can be represented in terms of dilatation and rotations by differentiating Eqs. (2.33)–(2.35) with respect to \( x \), \( y \), and \( z \), then adding them together:

\[
(\lambda + G) \nabla^2 \varepsilon + (G \nabla^2 + \rho \Omega^2) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0
\]

(2.36)

Taking into account the definition of volumetric strain, the above equation becomes

\[
(\nabla^2 + 2G) \nabla^2 \varepsilon + \rho \Omega^2 \varepsilon = 0
\]

or

\[
(\nabla^2 + \Omega^2_1) \varepsilon = 0
\]

(2.37)  
(2.38)

where

\[
\Omega^2_1 = \frac{\Omega^2}{C^2_1}
\]

(2.39)

\[
C^2_1 = \frac{\lambda + 2G}{\rho}
\]

(2.40)

\( C_1 \) is the velocity of waves of dilatation in the media. Subtracting the derivative of (2.34) with respect to \( x \) from the derivative of (2.33) with respect to \( y \) gives

\[
(\lambda + G) \frac{\partial^2 \varepsilon}{\partial x \partial y} + G \nabla^2 \frac{\partial u}{\partial y} - (\lambda + G) \frac{\partial^2 \varepsilon}{\partial x \partial y} - G \nabla^2 \frac{\partial u}{\partial x} = -\rho \Omega^2 \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right)
\]

(2.41)

After simplifying the above equation yields:

\[
(\nabla^2 + \Omega^2_1) \omega_z = 0
\]

(2.42)
2.6 Displacements in Terms of Dilatation and Rotation Components

where

\[ \Omega_2^2 = \frac{\Omega^2}{C_2^3} \]  (2.43)

\[ C_2^2 = \frac{G}{\rho} \]  (2.44)

\( C_2 \) is velocity of waves of distortion.

Finally, subtracting the derivative of Eq. (2.34) with respect to \( z \) from the derivative of Eq. (2.35) with respect to \( y \) results:

\[ (\nabla^2 + \Omega_2^2) \omega_x = 0 \]  (2.45)

2.6 Displacements in Terms of Dilatation and Rotation Components

Since the general solutions of equations of motion are given in terms of dilatation and rotation components, in order to determine displacements, they must be given in terms of dilatation and rotation components. Using Eqs. (2.24)–(2.26), the Laplacian operator of the displacement in the \( x \)-direction in terms of dilatation and rotation components is

\[
\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \\
= \frac{\partial \varepsilon_{xx}}{\partial x} + \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} - 2\omega_z \right) + \frac{\partial}{\partial z} \left( \frac{\partial w}{\partial x} + 2\omega_y \right) \\
= \frac{\partial \varepsilon_{xx}}{\partial x} + \frac{\partial \varepsilon_{yy}}{\partial y} + \frac{\partial \varepsilon_{zz}}{\partial z} + 2 \frac{\partial \omega_y}{\partial x} - 2 \frac{\partial \omega_z}{\partial y} \\
= \frac{\partial \varepsilon}{\partial x} + 2 \frac{\partial \omega_y}{\partial z} - 2 \frac{\partial \omega_z}{\partial y} 
\]  (2.46)

Similarly, the corresponding expression for the Laplacian operator of \( v \) and \( w \) are

\[
\nabla^2 v = \frac{\partial \varepsilon}{\partial y} + 2 \frac{\partial \omega_z}{\partial x} - 2 \frac{\partial \omega_x}{\partial z} 
\]  (2.47)

\[
\nabla^2 w = \frac{\partial \varepsilon}{\partial z} + 2 \frac{\partial \omega_x}{\partial y} - 2 \frac{\partial \omega_y}{\partial x} 
\]  (2.48)

By substituting these equations into Eqs. (2.33)–(2.35), the displacements will be found in terms of dilatation and components of rotations:
\[(\lambda + G) \frac{\partial \varepsilon}{\partial x} + G \left( \frac{\partial \varepsilon}{\partial x} + 2 \frac{\partial \omega_y}{\partial z} - 2 \frac{\partial \omega_z}{\partial y} \right) = -\rho \Omega^2 u \] (2.49)

\[(\lambda + G) \frac{\partial \varepsilon}{\partial y} + G \left( \frac{\partial \varepsilon}{\partial y} + 2 \frac{\partial \omega_z}{\partial x} - 2 \frac{\partial \omega_x}{\partial z} \right) = -\rho \Omega^2 v \] (2.50)

\[(\lambda + G) \frac{\partial \varepsilon}{\partial z} + G \left( \frac{\partial \varepsilon}{\partial z} + 2 \frac{\partial \omega_x}{\partial y} - 2 \frac{\partial \omega_y}{\partial x} \right) = -\rho \Omega^2 w \] (2.51)

After some rearrangement and reduction, the above equations become:

\[
u = - \frac{1}{\Omega_1^2} \frac{\partial \varepsilon}{\partial x} + \frac{2}{\Omega_2^2} \frac{\partial \omega_z}{\partial y} - \frac{2}{\Omega_2^2} \frac{\partial \omega_y}{\partial z} \] (2.52)

\[
v = - \frac{1}{\Omega_1^2} \frac{\partial \varepsilon}{\partial y} + \frac{2}{\Omega_2^2} \frac{\partial \omega_x}{\partial z} - \frac{2}{\Omega_2^2} \frac{\partial \omega_z}{\partial x} \] (2.53)

\[
w = - \frac{1}{\Omega_1^2} \frac{\partial \varepsilon}{\partial z} + \frac{2}{\Omega_2^2} \frac{\partial \omega_y}{\partial x} - \frac{2}{\Omega_2^2} \frac{\partial \omega_x}{\partial y} \] (2.54)

### 2.7 Stresses in Terms of Dilatation and Rotation Components

In order to get the stresses in terms of the dilatation and rotation components, Eq. (2.11) can be written as follows:

\[
\sigma_{yy} = \lambda \varepsilon + 2G \varepsilon_{yy} = (\lambda + 2G) \varepsilon_{yy} + \lambda \varepsilon_{xx} + \lambda \varepsilon_{zz} \] (2.55)

and, in terms of displacement components, it is

\[
\sigma_{yy} = (\lambda + 2G) \frac{\partial v}{\partial y} + \lambda \frac{\partial w}{\partial z} + \lambda \frac{\partial u}{\partial x} \] (2.56)

Having substituted derivatives of Eqs. (2.52)–(2.54) in the above relation, it yields:

\[
\sigma_{yy} = - \left( \frac{\lambda + 2G}{\Omega_1^2} \frac{\partial^2 \varepsilon}{\partial y^2} \right) + 2 \left( \frac{\lambda + 2G}{\Omega_2^2} \right) \left( \frac{\partial^2 \omega_z}{\partial y \partial z} - \frac{\partial^2 \omega_y}{\partial y \partial x} \right)
- \left( \frac{\lambda}{\Omega_1^2} \frac{\partial^2 \varepsilon}{\partial x^2} + 2 \frac{\lambda}{\Omega_2^2} \left( \frac{\partial^2 \omega_z}{\partial x \partial y} - \frac{\partial^2 \omega_y}{\partial x \partial z} \right) \right)
- \left( \frac{\lambda}{\Omega_1^2} \frac{\partial^2 \varepsilon}{\partial z^2} + 2 \frac{\lambda}{\Omega_2^2} \left( \frac{\partial^2 \omega_y}{\partial y \partial z} - \frac{\partial^2 \omega_x}{\partial y \partial z} \right) \right) \] (2.57)
2.7 Stresses in Terms of Dilatation and Rotation Components

simplification, the above equation yields

\[
\sigma_{yy} = -\left[ \frac{\lambda + 2G}{\Omega_1^2} \frac{\partial^2 \varepsilon}{\partial y^2} - \frac{4G}{\Omega_2^2} \left( \frac{\partial^2 \omega_x}{\partial y \partial z} - \frac{\partial^2 \omega_z}{\partial x \partial y} \right) + \frac{\lambda}{\Omega_1^2} \left( \frac{\partial^2 \varepsilon}{\partial x^2} + \frac{\partial^2 \varepsilon}{\partial z^2} \right) \right]
\]

(2.58)

Similarly, the other direct stresses are

\[
\sigma_{xx} = -\left[ \frac{\lambda + 2G}{\Omega_1^2} \frac{\partial^2 \varepsilon}{\partial x^2} - \frac{4G}{\Omega_2^2} \left( \frac{\partial^2 \omega_z}{\partial x \partial z} - \frac{\partial^2 \omega_x}{\partial y \partial z} \right) + \frac{\lambda}{\Omega_1^2} \left( \frac{\partial^2 \varepsilon}{\partial y^2} + \frac{\partial^2 \varepsilon}{\partial z^2} \right) \right]
\]

(2.59)

and

\[
\sigma_{zz} = -\left[ \frac{\lambda + 2G}{\Omega_1^2} \frac{\partial^2 \varepsilon}{\partial z^2} - \frac{4G}{\Omega_2^2} \left( \frac{\partial^2 \omega_x}{\partial x \partial z} - \frac{\partial^2 \omega_y}{\partial y \partial z} \right) + \frac{\lambda}{\Omega_1^2} \left( \frac{\partial^2 \varepsilon}{\partial y^2} + \frac{\partial^2 \varepsilon}{\partial x^2} \right) \right]
\]

(2.60)

Shear stresses in the dilatation strain and the elastic rotation components can be obtained as presented in the following. For instance, the shear stress, \( \tau_{xy} \), in terms of displacements, is:

\[
\tau_{xy} = G \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)
\]

(2.61)

Substituting for \( u \) and \( v \) from Eqs. (2.52)–(2.54), this shear stress be presented by the following equation

\[
\tau_{xy} = G \left[ -2 \frac{\partial^2 \varepsilon}{\partial x \partial y} + \frac{2}{\Omega_2^2} \left( \frac{\partial^2 \omega_z}{\partial y^2} + \frac{\partial^2 \omega_x}{\partial x^2} - \frac{\partial^2 \omega_x}{\partial x \partial y} - \frac{\partial^2 \omega_y}{\partial y \partial z} \right) \right]
\]

(2.62)

Differentiating Eqs. (2.24)–(2.26) with respect to \( x, y, \) and \( z, \)

\[
\frac{\partial \omega_x}{\partial x} = \frac{1}{2} \left( \frac{\partial^2 w}{\partial y \partial z} - \frac{\partial^2 v}{\partial z \partial x} \right)
\]

(2.63)

\[
\frac{\partial \omega_y}{\partial y} = \frac{1}{2} \left( \frac{\partial^2 u}{\partial z \partial y} - \frac{\partial^2 w}{\partial x \partial y} \right)
\]

(2.64)

\[
\frac{\partial \omega_z}{\partial z} = \frac{1}{2} \left( \frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 u}{\partial y \partial z} \right)
\]

(2.65)

And adding them all it yields:

\[
\frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_y}{\partial y} + \frac{\partial \omega_z}{\partial z} = 0
\]

(2.66)
The above equations demonstrate that the $\omega_y$ depends on the other two elastic rotations $\omega_x$ and $\omega_z$. After substituting $\partial \omega_y / \partial y$ from the above equation into Eq. (2.62) and simplifying, the shear stress $\tau_{xy}$ is obtained:

$$
\tau_{xy} = 2G \left[ -\frac{1}{\Omega_1^2} \frac{\partial^2 \varepsilon}{\partial x \partial y} + \frac{1}{\Omega_2^2} \left( \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} \right) \omega_z + \frac{2}{\Omega_2^2} \frac{\partial^2 \omega_x}{\partial x \partial z} \right] \tag{2.67}
$$

Following the same procedure, the shear stress $\tau_{yz}$ will be presented by the following equation:

$$
\tau_{yz} = 2G \left[ -\frac{1}{\Omega_1^2} \frac{\partial^2 \varepsilon}{\partial y \partial z} + \frac{1}{\Omega_2^2} \left( \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) \omega_x + \frac{2}{\Omega_2^2} \frac{\partial^2 \omega_z}{\partial y \partial z} \right] \tag{2.68}
$$

### 2.8 Fourier Transformation of Equations of Motion, Boundary Stresses, and Displacements

In order to reduce the three partial differential equations of (2.38), (2.42), and (2.45) into three ordinary differential equations in terms of $y$, complex Fourier transformation was used. The boundary conditions must then be treated in the same way, so that, instead of having relations in partial derivatives with respect to $x$, $y$, and $z$, relations are obtained in terms of the derivatives with respect to $y$ only.

By applying the two-dimensional Fourier transforms (see Appendix A) on the equations of motion, the reduced equations become

$$
\frac{\partial^2}{\partial y^2} - \left( p^2 + q^2 - \Omega_1^2 \right) \ddot{\varepsilon} = 0 \quad \tag{2.69}
$$

$$
\frac{\partial^2}{\partial y^2} - \left( p^2 + q^2 - \Omega_2^2 \right) \ddot{\omega}_x = 0 \quad \tag{2.70}
$$

$$
\frac{\partial^2}{\partial y^2} - \left( p^2 + q^2 - \Omega_2^2 \right) \ddot{\omega}_z = 0 \quad \tag{2.71}
$$

where $\ddot{\varepsilon}$, $\ddot{\omega}_x$, and $\ddot{\omega}_z$ are the corresponding double complex Fourier transformations of $\varepsilon$, $\omega_x$, and $\omega_z$.

The transformed stresses at the surface of the half-space ($(x, z)$ plane) in terms of the transform of the dilatation and elastic rotation components are

$$
\ddot{\sigma}_{yy} = -\left[ \frac{\lambda + 2G}{\Omega_1^2} \frac{\partial^2 \ddot{\varepsilon}}{\partial y^2} + \frac{4iGq}{\Omega_2^2} \frac{\partial \ddot{\omega}_x}{\partial y} - \frac{4iGq}{\Omega_2^2} \frac{\partial \ddot{\omega}_z}{\partial y} - \frac{\lambda \ddot{\varepsilon}}{\Omega_1^2} \right] \left( p^2 + q^2 \right) \tag{2.72}
$$

For the shear stresses,

$$
\ddot{\tau}_{xy} = 2G \left[ \frac{ip}{\Omega_1^2} \frac{d \ddot{\varepsilon}}{dy} + \frac{1}{\Omega_2^2} \left( \frac{d^2}{d^2} + p^2 - q^2 \right) \ddot{\omega}_x - \frac{2pq}{\Omega_2^2} \ddot{\omega}_z \right] \tag{2.73}
$$
and
\[ \bar{\tau}_{yz} = 2G \left[ i \frac{q}{\Omega_1} \frac{d \bar{\varepsilon}}{dy} - \frac{1}{\Omega_2^2} \left( \frac{d^2}{d^2} - p^2 + q^2 \right) \bar{\omega}_x + \frac{2pq}{\Omega_2^2} \bar{\omega}_z \right] \] (2.74)

The double complex Fourier transform of the displacements will be transforms of Eqs. (2.52)–(2.54)

\[ \bar{u} = \frac{i}{\Omega_1} p \bar{\varepsilon} + \frac{2}{\Omega_2} \frac{\partial \bar{\omega}_z}{\partial y} + \frac{2i}{\Omega_2} q \bar{\omega}_y \] (2.75)

\[ \bar{v} = \frac{2}{\Omega_2} (i p \bar{\omega}_z - i q \bar{\omega}_x) - \frac{1}{\Omega_2} \frac{\partial \bar{\varepsilon}}{\partial y} \] (2.76)

\[ \bar{w} = \frac{i}{\Omega_1} q \bar{\varepsilon} - \frac{2}{\Omega_2} \frac{\partial \bar{\omega}_x}{\partial y} - \frac{2i}{\Omega_2} p \bar{\omega}_y \] (2.77)

\( \bar{\omega}_y \) can be obtained by transforming equation (2.26).

\[ \bar{\omega}_y = -\frac{1}{2} i q \bar{u} + \frac{1}{2} i p \bar{w} \] (2.78)

Substituting \( \bar{\omega}_y \) from the above equation into Eqs. (2.75) and (2.77), and solving for \( \bar{u} \) and \( \bar{w} \) results in:

\[ \bar{u} = \frac{i}{\Omega_1} p \bar{\varepsilon} + \frac{2pq}{\Omega_2} \frac{\partial \bar{\omega}_x}{\partial y} + \frac{2}{\Omega_2} \frac{(\Omega_2^2 - p^2)}{\Omega_2^2 - q^2 - p^2} \frac{\partial \bar{\omega}_z}{\partial y} \] (2.79)

\[ \bar{v} = -\frac{1}{\Omega_1} \frac{d \bar{\varepsilon}}{dy} - \frac{2i}{\Omega_2} q \bar{\omega}_x + \frac{2ip}{\Omega_2^2} \bar{\omega}_z \] (2.80)

\[ \bar{w} = \frac{i}{\Omega_1} q \bar{\varepsilon} - \frac{2pq}{\Omega_2} \frac{\partial \bar{\omega}_x}{\partial y} - \frac{2}{\Omega_2} \frac{(\Omega_2^2 - q^2)}{\Omega_2^2 - q^2 - p^2} \frac{\partial \bar{\omega}_x}{\partial y} \] (2.81)

### 2.9 General Solution of Transformed Equations of Motion

The general solution of Eqs. (2.69)–(2.71) as a function of \( y \) is

\[ \bar{\varepsilon} = A_x \exp(-\gamma_1 y) \] (2.82)

\[ \bar{\omega}_x = A_x \exp(-\gamma_2 y) \] (2.83)

\[ \bar{\omega}_z = A_z \exp(-\gamma_3 y) \] (2.84)
where values of $\gamma_1$, $\gamma_2$, and $\gamma_3$ must be negative to satisfy the boundary conditions requirements for $\bar{\varepsilon}$, $\hat{\omega}_x$, and $\hat{\omega}_z$ as $y$ approaches to infinity. Substituting the above in Eqs. (2.69)–(2.71) results in:

\[
\gamma_1 = p^2 + q^2 - \Omega_1^2 \\
\gamma_2 = p^2 + q^2 - \Omega_2^2 \\
\gamma_3 = p^2 + q^2 - \Omega_2^2
\] (2.85) (2.86) (2.87)

From the last two equations, it is obvious that

\[
\gamma_2 = \gamma_3
\] (2.88)

Values of $A_\varepsilon$, $A_x$, and $A_z$ are functions of $p$ and $q$ and depend on the boundary conditions of the problem. This means that they are dependent on the complex double Fourier transform of the stresses, which act as external excitations on the surface of the half-space medium. In order to evaluate these arbitrary functions, Eqs. (2.72)–(2.74) must be satisfied by the boundary conditions that are expressed by three stress components $\sigma_{xy}$, $\tau_{xy}$, and $\tau_{yz}$, which are applied on the surface of elastic half-space ($x; z$) where $y$ is zero. Substituting $y = 0$ in Eqs. (2.82)–(2.84) and placing them into Eqs. (2.72)–(2.74) the double complex Fourier transform of the stresses on the surface of half-space will be presented by the following equations:

\[
\frac{-2q^2 - 2p^2 + \Omega_1^2}{\Omega_1^2} A_\varepsilon + \frac{4i \gamma_2 p}{\Omega_2^2} A_x - \frac{4i \gamma_2 p}{\Omega_2^2} A_z = \frac{\bar{\sigma}_{yy} (p,q)}{G} \\
\frac{-2i \gamma_1 p}{\Omega_1^2} A_\varepsilon - \frac{4pq}{\Omega_2^2} A_x + \frac{2(2p^2 - \Omega_2^2)}{\Omega_2^2} A_z = \frac{\bar{\tau}_{xy} (p,q)}{G} \\
\frac{-2i \gamma_1 q}{\Omega_1^2} A_\varepsilon + \frac{2(-2q^2 + \Omega_2^2)}{\Omega_2^2} A_x + \frac{4pq}{\Omega_2^2} A_z = \frac{\bar{\tau}_{yz} (p,q)}{G}
\] (2.89) (2.90) (2.91)

The solutions to the above set of equations are given by the values of $A_\varepsilon$, $A_x$, and $A_z$. These values can be expressed as

\[
A_\varepsilon = \frac{D_\varepsilon}{D} \\
A_x = \frac{D_x}{D} \\
A_z = \frac{D_z}{D}
\] (2.92) (2.93) (2.94)
where

\[
D = \begin{bmatrix}
-2q^2 - 2p^2 + \Omega_2^2 & 4i\gamma_2q & -4i\gamma_2p \\
\Omega_1^2 & \Omega_2^2 & \Omega_2^2 \\
-2i\gamma_1p & -4pq & 2\left(2p^2 - \Omega_2^2\right)
\end{bmatrix}
\]  

(2.95)

or

\[
D = \frac{4}{\Omega_1^2\Omega_2^2} \Phi (p, q)
\]  

(2.96)

and

\[
\Phi (p, q) = \left[2 \left(p^2 + q^2\right) - \Omega_2^2\right]^2 - 4\gamma_1\gamma_2 \left(p^2 + q^2\right)
\]  

(2.97)

This is the well-known function associated with Rayleigh surface waves, also

\[
D = \begin{bmatrix}
\tilde{\sigma}_{yy}/G & 4i\gamma_2q & -4i\gamma_2p \\
\tilde{\tau}_{xy}/G & -4pq & 2\left(2p^2 - \Omega_2^2\right) \\
\tilde{\tau}_{yz}/G & 2\left(\Omega_2^2 - 2q^2\right) & 4pq
\end{bmatrix}
\]  

\[
\frac{1}{\Omega_2^4}
\]  

(2.98)

\[
D_x = \begin{bmatrix}
-2q^2 - 2p^2 + \Omega_2^2 & \tilde{\sigma}_{yy}/G & -4i\gamma_2p \\
\tilde{\tau}_{xy}/G & 2\left(2p^2 - \Omega_2^2\right) & 4pq \\
-2i\gamma_1q & \tilde{\tau}_{yz}/G & \Omega_2^2
\end{bmatrix}
\]  

\[
\frac{1}{\Omega_1^2\Omega_2^2}
\]  

(2.99)

\[
D_z = \begin{bmatrix}
-2q^2 - 2p^2 + \Omega_2^2 & 4i\gamma_2q & \tilde{\sigma}_{yy}/G \\
-2i\gamma_1p & -4pq & \tilde{\tau}_{xy}/G \\
-2i\gamma_1q & 2\left(\Omega_2^2 - 2q^2\right) & \tilde{\tau}_{yz}/G
\end{bmatrix}
\]  

\[
\frac{1}{\Omega_1^2\Omega_2^2}
\]  

(2.100)

Equations (2.82) through (2.84) along with Eqs. (2.89)–(2.91) present the general solution for the elastic dilatation $\tilde{e}$ and elastic rotations $\tilde{\omega}_x$ and $\tilde{\omega}_z$ for the surface of the medium due to applied stresses on the surface in the Fourier domain. These general equations will be utilized to determine the displacements on the surface of an elastic half-space for the two specific surface stress distributions caused by a vertical/horizontal harmonic point force that will be addressed in Chaps. 3 and 4, respectively.
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