Chapter 2

The Mathematics of Finance

2.1 Ancient Mathematics of Finance

Among the difficulties faced by the earliest emerging civilizations was the need for record keeping. Because we have a written language, record keeping is easy enough for us, but the earliest civilizations did not have that tool. Archaeological evidence indicates that the invention of written language was contemporaneous with the development of civilization. In fact, it is hypothesized that the creation of numerical notation for record keeping was the first step in the process of developing written language (see [Sch 94]).

Middle Eastern artifacts in the form of clay tokens that are believed to have represented units of grain have been dated to as early as 8000 BCE. These tokens are believed to have been used to record amounts of stored grain, and they are believed to be the first mechanism used for that record-keeping task. Figure 2.1 illustrates ancient clay accounting tokens—not the earliest but still dating from before 3100 BCE.

After an amount of grain has been recorded, the next step is to make a record of the ownership of the grain represented by the tokens. Here the archaeological evidence shows that the ownership record was maintained by enclosing the tokens in a clay envelope marked by the owner's seal. Figure 2.2 illustrates a clay envelope and accounting tokens that also date from before 3100 BCE.

A clay envelope is not transparent, so once tokens have been sealed inside the envelope, the record, while safe, is inaccessible. The impossibility of seeing what's inside a clay envelope was overcome by the innovation of making

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Figure 2.1 Clay accounting tokens. Susa, Uruk period (4000–3100 BCE).

impressions on the outside of the envelope using the tokens that were to go inside the envelope. In this way, the contents of the envelope could be known without breaking the envelope.

It is apparent to us that, once the contents and ownership are represented on the surface of the envelope, the contents themselves are redundant. The surface of the enclosure itself and the impressed markings thereon provide all the needed information. The next innovation is also apparent to us: Don't bother with the tokens inside the envelope and don't even bother making a spherically shaped clay envelope; simply make the needed marks on a flat surface.

Finally, about 3100 BCE that ultimate step was taken: People began to use a pointed stylus to incise pictures of the tokens in clay tablets instead of impressing the tokens themselves. It is at this point that we can say that a system of writing had been invented. Figure 2.3 illustrates a clay accounting tablet that dates from 3100–2850 BCE.[1]

Figure 2.2 Clay envelope and accounting tokens. Susa, Uruk period (4000–3100 BCE).

[1]Two things that had puzzled archaeologists in the mid-twentieth century were: (1) what was the significance of various bits of clay like those in Fig. 2.1 that were found at nearly every Middle Eastern archaeological site, and (2) why did many early pictographs not look like the things they represented. It was Denise Schmandt-Besserat who recognized that the bits of clay were accounting tokens and that the pictographs represented the tokens.

Figure 2.3 Clay accounting tablet. Susa, period III (3100–2850 BCE).

2.2 Loans and Charging Interest

Civilization requires the division of labor and the transfer of goods among various specialized groups of workers. In particular, the producers of food need to be taxed so that some of their production is available to an organizing authority. Not surprisingly, even in ancient times the payment of some taxes was delayed. The records of those delayed payments are the earliest recorded debts. So it is within a few hundred years of the emergence of written language that written records of debt appeared.

There are also early records (circa 2400 BCE) of debts owed by one individual to another. By 1800 BCE there are records of loans requiring the payment of interest. An example of such an early record (see [Sim 78]) reads as follows:

> One and one-sixth shekels silver, to which the standard interest is to be added, Ilshu-bani, the son of Nabi-ilishu, received from Shamash and from Sin-tajjar. At harvest time he will repay the silver and the interest. Before five witnesses. In month seven of the year that Apil-Sin built the temple of Inanna of Elip.

The preceding loan contract is between the borrower Ilshu-bani and the lender Sin-tajjar. Shamash was the name of a god. The shekel is a unit of measurement equal to approximately 8 grams (although it later became a unit of currency). That there can be a reference to a “standard interest” tells us that the practice of charging interest was commonplace. In fact, the standard interest rate is known to have been 20% for silver and $33\frac{1}{3}\%$ for grain. Many loans were actually for short time periods and would still be paid with the 20% interest, independent of the time period. Thus it may not be fully appropriate to project our notion of annual interest onto the thinking of those ancient Babylonians making a contract. Nonetheless, it was common to pay 1 shekel interest per lunar month on each 1 mina (which equals 60 shekels) that was borrowed. Since a year usually contains 12 lunar months (but can have 13) we arrive at an interest rate of approximately 20% per annum.

Even though charging interest on loans had been a well-established practice among the Babylonians and had spread over most of the Near East, still the practice of charging interest later came to be seen as, at best, disreputable. For instance, Aristotle said (see [Bar 84]):

> The most hated sort [of wealth getting], and with the greatest reason, is usury, which makes a gain out of money itself, and not from the natural object of it. For money was intended to be used in exchange, but not to increase at interest. And this term interest, which means the birth of money from money, is applied to the breeding of money because the offspring resembles the parent. That is why of all modes of getting wealth this is the most unnatural.

Later the Catholic Church took a dim view of charging interest. Some of the Church’s actions against charging interest are as follows: The Council of Nicaea in 325 banned usury among clerics, and the First Council of Carthage in 345 and the Council of Aix in 789 declared it to be reprehensible for laymen to make money by lending with interest. The canonical laws of the Middle Ages absolutely forbade the practice of lending with interest. The Third Council of the Lateran in 1179 and the Second Council of Lyons in 1274 condemn usurers. In the Council of Vienne in the year 1311, it was declared that anyone maintaining that there was no sin in the practice of demanding interest should be punished as a heretic.

The Church’s censure notwithstanding, since loans are essential for commerce, people came up with stratagems that could be used to evade the Church’s prohibition on charging interest. One common dodge was for the interest to be hidden within the premium (*agio*) charged by bankers for currency conversion. A more blatant evasion was to characterize payments to lenders as discretionary gifts.

2.3 Compound Interest

Compound interest is based on the idea that, after a specific time interval, the outstanding interest also becomes part of the loan and thus from that time on interest must be paid on the interest. In ancient Babylonia most loans were extended for time periods of a year or less, so compound interest was not relevant. Even so, the idea behind compound interest was understood. Our evidence for this is a royal inscription describing a conflict dating to 2400 BCE (see [Coo 86]). The inscription tells us that a portion of the grain harvested from a certain disputed property—1 guru of grain we are told—was to be paid by one city–state to another. But the payment was not made, and after several decades the 1 guru of grain together with its interest had increased to 8.64 million gurus (in modern units about 4.5 trillion liters).

If we apply the principle of compound interest to the above "loan" of 1 guru of grain using the interest rate of $33\frac{1}{3}\%$ that the Babylonians applied to loans of grain, then we see that, after 1 year, $\frac{4}{3} \approx 1.33$ guru of grain is owed. After that the debt grows each year by multiplying by $\frac{4}{3}$. After 2 years, the debt becomes $\frac{4}{3} \times \frac{4}{3} = \frac{16}{9} \approx 1.78$ guru of grain. Continuing in this way, we obtain the following table that shows not only is the amount owed increasing, but the rate of growth of the amount owed is also increasing.

Years	Grain owed (gurus)
1	1.33
2	1.78
3	2.37
4	3.16
5	4.21
10	17.76

Years	Grain owed (gurus)
20	315.34
30	5,599
40	99,437
50	1,765,780
55	7,440,986
56	9,921,315

Fibonacci: Leonardo of Pisa

From an engraving by Pelle

Fibonacci

Since the 1970s there has been easy availability of hand-held calculators capable of doing many financial calculations which, if done by hand, would be quite laborious. We now take for granted the calculation of compound interest and the calculation of the payments required on a car loan or a mortgage. Centuries before the computer revolution, there was another revolution in computation that allowed the pencil and paper financial calculations needed for commerce. One of the important pioneers in introducing this revolutionary mathematical notation and technique was Leonardo of Pisa (1170–1250), most often referred to as Fibonacci.

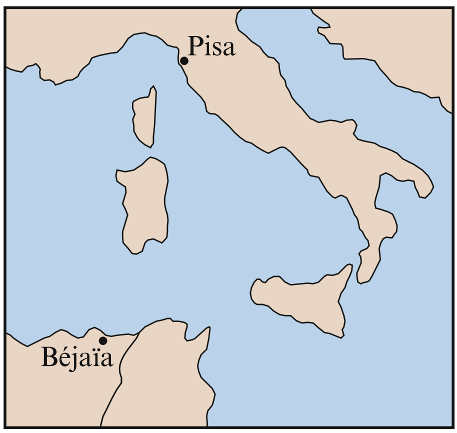

Figure 2.4 Location of Béjaïa.

Fibonacci and his family were part of the growing commercial community that arose after the Dark Ages. The Italian city of Pisa had established the colony of Bugia in North Africa (now Béjaïa in Algeria—see Fig. 2.4). Fibonacci's father was an administrative official in that colony and Fibonacci was brought to the colony at his father's request and received training in Arabic mathematical methods.[2] Fibonacci traveled extensively in the Mediterranean. It is believed that he was earning his living as a merchant, but he

[2]At that time the Arabs were in many ways on the cutting edge of mathematical development. A number of basic ideas of algebra, and also the arabic system of numerals that we use today, were generated by medieval Arabs. The Arabs learned some of their

was also—and more importantly for us—pursuing mathematical knowledge wherever he went. When he returned from his travels, he wrote the book, *Liber Abaci*.[3] That book is the reason Fibonacci is remembered, while the other merchants of those days have been forgotten.

Liber Abaci was published in 1202 and revised in 1228. Because of its antiquity, copies of *Liber Abaci* were necessarily made by hand. The oldest surviving version dates to the 1290s, omits the preliminary material explaining Arabic numerals and mathematical operations, and is written in Italian rather than Latin. *Liber Abaci* is believed to contain the earliest Arabic numeral multiplication table in Western mathematics.

The specific part of *Liber Abaci* with which modern mathematicians are most familiar is the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

in which each number, after the first two, is the sum of the preceding two numbers. Now known as the Fibonacci sequence, this sequence provides the solution to a problem in *Liber Abaci* concerning the breeding of rabbits. In fact, much of *Liber Abaci* is devoted to practical problems of finance and commerce. In 1241, the Republic of Pisa granted Fibonacci a pension for "educating its citizens and for his painstaking, dedicated service."

2.4 Continuously Compounded Interest

In our modern world, a financial institution would never wait a full year before compounding the interest on a loan (or on a savings account). In fact, computers allow financial institutions to use compounding periods as short as they wish. Figure 2.5 shows the effect of more frequent compounding. The simple interest rate considered is 64% (per year), a large value chosen so that the effects of compounding will be visible in the graph. The principal is \$100, and after 1 year, \$100 at 64% simple interest grows to \$164, as shown by the bottom graph in each part of Fig. 2.5. Figure 2.5a shows that, by compounding in the middle of the year, the total interest on \$100 after 1 year increases from \$64 to \$74.25; Fig. 2.5b shows that, if the interest is

ideas from the East Indians. The intercourse between the Arabs and the Indians came about because of medical needs—with the best doctors traveled the best ideas.

[3] This famous text is about arithmetic, pure and simple. That was a cutting-edge topic at the time. In fact *Liber Abaci* was one of the very first Western books to explain the Hindu-Arabic numerals that we use today.

compounded after each quarter year, then the total interest becomes $81.06; and Fig. 2.5c shows that, if the interest is compounded after each eighth of a year, then the total interest becomes $85.09.

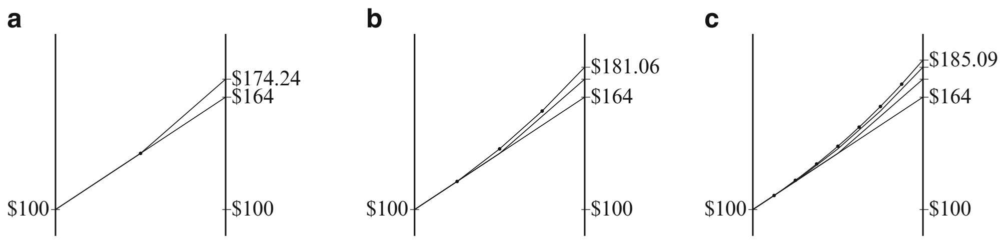

Figure 2.5 Compounding interest. (**a**) Compounding twice per year. (**b**) Compounding four times per year. (**c**) Compounding eight times per year.

Just to be clear about this: When we compound interest in the middle of the year, we calculate half the interest (or 32% applied to the principal) on June 30. Then we calculate the other half of the interest (or 32% applied to the principal plus the first quantity of interest) on December 31. When we compound interest four times a year, we calculate one-fourth of the interest (or 16% applied to the principal) on March 31. Then we calculate another one-fourth of the interest (or 16% applied to the principal plus the first quantity of interest) on June 30. Then we calculate another one-fourth of the interest (or 16% applied to the principal plus the first two quantities of interest) on September 30. Finally, we calculate the last one-fourth of the interest (or 16% applied to the principal plus the first three quantities of interest) on December 31. Other forms of compound interest are calculated in the same way.

So easy have the computations become that it is now commonplace for interest to be compounded continuously. By continuous compounding is meant the limiting value as the number of compounding periods approaches infinity. Figure 2.6 illustrates the effect of continuous compounding. Each graph in Fig. 2.6 shows twice as many compounding periods as the graph immediately below it. As more and more compounding periods are used, the graphs approach the limiting graph that shows continuous compounding and of course is the highest of all the graphs in Fig. 2.6. Even though each increase in the number of compounding periods leads to greater interest, the total interest does not increase to infinity, but instead approaches $89.65.

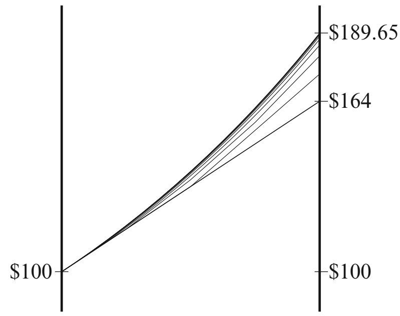

Figure 2.6 The transition from simple interest to continuously compounded interest.

Figure 2.7 compares compound interest to simple interest on a principal amount of $100 at a nominal rate of 10% per annum. The compound interest is computed on the basis of continuous compounding. After 10 years, the compound interest is noticeably more than simple interest, but after 20 years the difference is large and after 30 years it is huge.

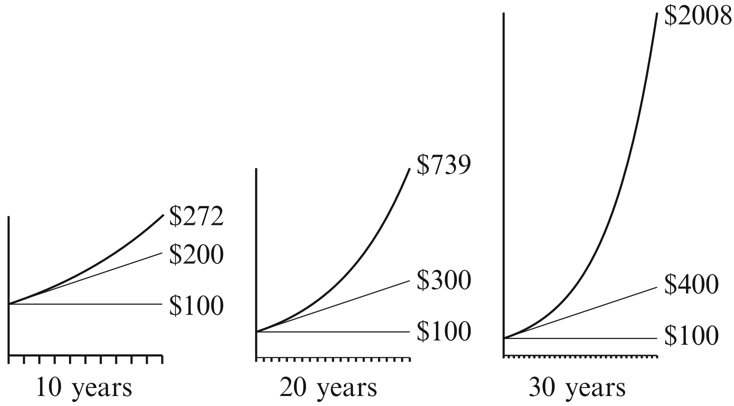

Figure 2.7 Comparison of compound interest and simple interest.

2.5 Raising Capital: Stocks and Bonds

Government endeavors and commercial projects typically require capital far beyond the means of one individual. One way in which a large amount of capital can be raised is by borrowing from many individuals, a process carried out by issuing bonds. The Italian city–states of Venice, Florence, and Genoa were early leaders in this practice, utilizing both voluntary and forced loans as early as the eleventh century. Some such bonds were non-transferable, but in other cases the bonds were transferable and a secondary market for them consequently emerged.

For a commercial enterprise to raise a large amount of capital, an appropriate and effective legal structure must exist. A "joint contract" among individuals is not well suited to this purpose, because then each individual is fully responsible for the enterprise's liabilities. Even if a partnership has been arranged in such a way that individual liability is limited, there remains the serious problem of resolving how a partner can recover his capital and withdraw from the partnership. One quite successful solution to the problem of raising capital for commercial enterprise is the joint-stock company in which transferable shares representing ownership of a limited liability company are issued. To withdraw and recover his or her capital, an owner can simply sell the shares owned.

The seventeenth century Dutch company Verenigde Nederlandsche Geoctroyeerde Oostindische Compagnie is the prototypical example of the joint-stock company for which a secondary market developed. It is the development of a secondary market in financial instruments that led to, and necessitated, the development of stock exchanges on which shares of companies can be bought and sold. The name "Verenigde Nederlandsche Geoctroyeerde Oostindische Compagnie" translates literally to "United Netherlands Chartered East India Company," but it is commonly referred to as the Dutch East India Company. This company played a crucial role in the history of finance.

The Dutch East India Company was chartered in 1602. The original plan had been to liquidate the company in 1612, but the time required for round trips to Asia makes it obvious to us that a 10 year time frame was far too short; in any event, the Dutch East India Company continued in existence until 1795. The equity capitalization of the company remained essentially fixed throughout the life of the company, but the company also issued bonds. One salubrious side effect of the existence of a secondary market in shares in the Dutch East India Company was that shares owned could then be used as collateral for loans. Since shares could be liquidated easily, the lender was much more secure with shares as collateral as opposed to other less liquid collateral. The effect of using more liquid collateral was a lowering of interest rates, further facilitating commercial enterprise.

In England in 1688 a union of Parliamentarians and an invading army led by the Dutch *stadtholder* William III of Orange-Nassau (William of Orange) overthrew King James II. This is the event called the "Glorious Revolution" in English history. After this revolution England adopted the Dutch model for the joint-stock company, as exemplified by the Dutch East India Company. By using the Dutch model, England gained the same benefits from active secondary markets and lowered interest rates.

The Oldest Live Securities in Modern Capital Markets

Dikes are, and have been, crucial for the existence and survival of the Netherlands. The maintenance of dikes is managed by water boards, of which there may have been as many as 3,500 in the nineteenth century. Mergers have brought the number of extant water boards in the Netherlands down to fewer than 60 at present. Water boards had the power to draft citizens into a "dike army" when needed and were granted taxing authority. It still sometimes happened that expenses would exceed what taxes could raise and bonds would be issued. Some such bonds were issued as perpetuities, i.e., bonds that would not be redeemed, but instead would pay interest forever, or synonymously, in perpetuity.

The management areas of water boards in the Netherlands are dictated by nature and so the water boards are separate from government entities and consequently are often insulated from government upheavals. This independence has led to the perpetuities issued by some water boards continuing to function for centuries. In recent years, at least four seventeenth century bonds issued by the Hoogheemraadschap Lekdijk Bovendams have been presented to the successor organization (namely, the water board of Stichtse Rijnlanden) for payment of interest. The oldest of these bonds dates from 1624. On July 1, 2003, Geert Rouwenhorst, Professor of Finance at Yale University and coeditor of [GR 05], personally collected 26 years back interest on a 1,000 Carolus guilder bond issued at 5% interest on May 15, 1648. The currency in use in 1648, the Carolus guilder worth 20 stuivers ("stuiver" remains a nickname for the 5 euro cent coin in the Netherlands) has been succeeded by the Flemish pound, then the guilder, and finally (so far) the euro. The bond under discussion now pays €11.34 annually, which is the modern equivalent of 25 guilders, the interest rate having been reduced to 2.5% during the eighteenth century.

John Law

From an engraving by Langlois

John Law

John Law (1671–1729) was born into a family of Scottish bankers and goldsmiths. Law initially joined the family business and studied banking especially. After his father died in 1688, Law changed direction and moved to London where he lived extravagantly. Despite being reputed to be brilliant at calculating odds, he lost large sums of money gambling. After killing another man in a duel he was forced to escape to the continent.

Law's contribution to economic theory was the pamphlet "Money and Trade Considered with a Proposal for Supplying the Nation with Money." In modern terms, Law proposed that economic activity could be spurred by increasing the money supply, and that the increasing material production would then be sufficient to prevent inflation. Having left the continent for Scotland, Law unsuccessfully attempted to get the Scots to adopt his proposed monetary policies. But after the union of Scotland and England in 1707, he again had to flee to the continent.

In France, after the death of Louis XIV (1638–1715), Law found fertile ground for his financial innovations. At that time, France was bankrupt—having been drained by continuous warfare. Money was in short supply. The opportunity that presented itself to Law was the conversion of the huge government debt into equity. In 1716, Law established the General Bank (Banque Générale Privé) which developed the use of paper money. But the main tool for the conversion of government debt into equity was to be a large trading company along the lines of the Dutch East India Company and the similar British East India Company. In 1717, Law took over the Company of the West (Compagnie d'Occident), which owned the trading rights to Louisiana (at that time a territory far larger than the present day state of Louisiana);[4] this in exchange for also taking over France's short-term debt. That short-term debt was then converted into long-term debt

[4]In fact one of the important events in the development of the United States was the so-called "Louisiana Purchase" by President Thomas Jefferson in 1803. This added considerably to the land mass of the country. The acquisition was celebrated by the 1903 World's Fair held in St. Louis. In fact many of the Washington University campus buildings were originally erected as administration buildings for the fair. The fair introduced to the world, among other things, cotton candy and hot dogs.

at a lower interest rate. The effect was to ease France's debt problem while simultaneously establishing an income stream for the Company of the West.

The Company of the West issued shares and embarked on a series of acquisitions. The resulting company is generally known as the Mississippi Company. Another step in building Law's financial empire was the conversion of the General Bank mentioned above into the Royal Bank. As part of the conversion from General to Royal Bank, the king's Council was entrusted with the power to determine the bank's note issue. The result was the printing of an excess of money, and that served to drive up the price of shares in the Mississippi Company. In 1720, the Royal Bank and the Mississippi Company were united and Law was appointed Controller General of Finances for France.

By 1720, France was awash in liquidity, but investors began to lose confidence in the Mississippi Company. The King of France, who was himself a large shareholder, was one of the first to lose confidence. He sold his entire holdings, and the market value of shares in the Mississippi Company collapsed. Law was dismissed from his post, and he fled France. He died in Venice, a poor man.

2.6 The Standard Model for Stock Prices

As mentioned above, the advent of computers has made possible many computations that, say 100 years ago, a person could only dream of doing. Not surprisingly, it was military needs (such as the calculation of artillery trajectories) that spurred the development of the earliest electromechanical and electronic computers during the 1940s. But it was not too long afterwards that electronic computers entered the civilian realm and major calculations were done to satisfy intellectual curiosity. One such computation was performed by the British statistician Maurice Kendall. Kendall's goal was to find the underlying price cycles that were presumed to exist in stock prices. Instead of finding such underlying price cycles, Kendall concluded that stock prices move randomly (see [Ken 53]). This work was published in 1953, and at that time Kendall's conclusion that stock prices move randomly was difficult to accept. Part of the difficulty was the absence of any theoretical explanation.

About a decade later a theoretical explanation for the random movement of stock prices emerged. The nascent justification was the *efficient market theory* first described in Eugene Fama's 1964 University of Chicago Ph.D. thesis.[5] This efficient market theory postulates that prices on any widely traded asset already reflect all known information. A prototypical example of a widely traded asset would be shares of stock trading on one of the world's major exchanges. Since insider trading and market manipulation via rumor are (now) explicitly illegal, there is reason to believe that the efficient market theory may hold a measure of truth.

If all available information is already reflected in stock prices, then we may wonder what it is that stock prices are doing, since clearly stock prices are neither constant nor steadily increasing or decreasing. One possible explanation for the randomness in stock prices is that, since all known information is already included in the existing price of a stock, the only thing that can contribute to changing the prices is the emergence of previously unknown (and hence unpredictable) information. Tautologically, the unpredictable is unpredictable, so the effect is that stock prices change randomly.

Let's look at randomness as a model of stock price movements. A common example of a random phenomenon is the outcome of a coin toss. Figure 2.8 illustrates the outcome of 500 computer simulated coin tosses. Reading from left to right, each outcome is indicated by a line segment that is above the midline (for a head) or a line segment that is below the midline (for a tail).

Figure 2.8 500 coin tosses.

A random walk is the motion in one dimension generated by taking one step up or down in response to the outcome of a coin toss. Figure 2.9 illustrates the random walk that is based on the coin tosses shown in Fig. 2.8.

The standard model for a stock price is that it is a geometric random walk. In a geometric random walk the outcome of the coin toss determines the ratio of the stock price after the time step to the stock price before the

[5] Fama was one of the receipients of the 2013 Nobel Prize in Economics (the others were Richard Shiller and Lars Peter Hansen).

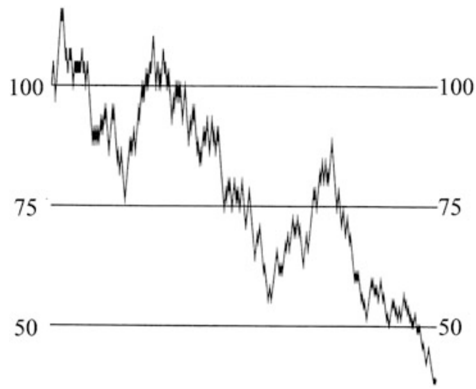

Figure 2.9 Random walked based on 500 coin tosses.

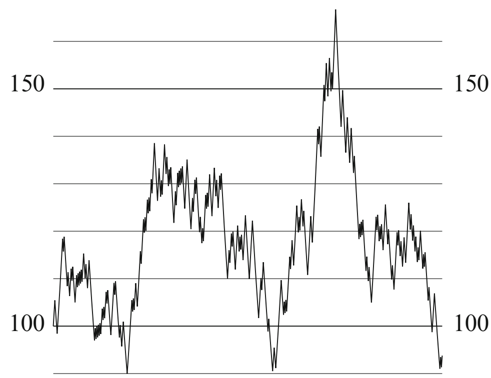

Figure 2.10 Geometric random walk based on 500 coin tosses.

time step. Since it seems to be an observable fact that stock prices rise gradually over time, the standard model also includes a growth factor. To simulate a geometric random walk using the coin-tosses from Fig. 2.8, we multiply or divide the share price by a number greater than 1, the choice of multiplication or division depending on the outcome of the coin toss. Figure 2.10 illustrates the result of this process with a beginning share price of 100. The vertical scales are different in Figs. 2.9 and 2.10, but the ups and downs occur at the same places. The random walk in Fig. 2.9 could become negative if continued, but the geometric random walk can never become negative (because we multiply and divide instead of adding and subtracting).

The geometric random walk model can be converted into a continuous model by letting the length of the time interval decrease to 0 while the step size also decreases appropriately. The resulting continuous process is called geometric Brownian motion. The mathematics involved in describing a geometric Brownian motion is significantly more technical than that required to describe a geometric random walk, so we will be content to model stock prices using a geometric random walk.

Louis Jean-Baptiste Alphonse Bachelier

The French mathematician Louis Bachelier (1870–1946) spent the majority of his career on the faculty at the university in Besançon, the small, isolated capital of the French province Franche-Comté. His relatively obscure academic position contrasts starkly with the fact that his 1900 Ph.D. thesis developed and applied the mathematical theory of Brownian motion 5 years in advance of Einstein's renowned work on the subject.

Various events in Bachelier's life seem to have contributed to the obscurity of his academic career. Bachelier's parents died early, so he was forced to take over the family business. That event hampered his educational plans, so he did not get on the proper track to ultimately obtain a prestigious academic position. The direction in which his own interests led him may have been a problem as well. Bachelier's thesis was concerned with applying probability to the stock market. In 1900, the subject of probability was not considered to be a proper part of mathematics. Further, there is a distinct likelihood that studying something as "sordid" as the stock market would have been considered déclassé. A final problem is that Bachelier may have had a "difficult" personality.

With the advantage of hindsight, Bachelier's work is now recognized as pioneering. It was Bachelier who first used Brownian motion to investigate stock options. His work was not appreciated and not advanced by economists until the 1960s, well after Bachelier's death.

2.7 Parameters in the Standard Model

The general upward drift of the price of a stock is called the *average return.* The average return is an important parameter in the geometric random walk model of a stock price. Likewise, the average return must figure into the continuous model of stock prices based on geometric Brownian motion. The second important parameter describes the magnitude of the fluctuations in the stock price. This parameter is called *volatility.* In fact the average return and the volatility are the only parameters in the standard model for stock prices.

Given their importance, one would most certainly want to know and use actual values for the average return and the volatility. Unfortunately, and disappointingly, statistical theory tells us that the average return cannot be accurately computed from data. The difficulty is that additional data does not narrow the margin of error. Consequently, the average return of a stock is effectively unknowable.

In contrast to the average return, the volatility can be estimated from data, and the accuracy and reliability of the estimate can be quantified in much the way the accuracy and reliability of an opinion poll is quantified.

It is appropriate to note here that, over the long term, actual companies and operating conditions for them do change. As a consequence, the parameters associated with the price of the stock of any specific company will be subject to gradual change. Over a period of decades the change can be enormous. For example, early in the twentieth century the industrialist, investor, and art collector Henry Clay Frick (1849–1919) declared that railroad stocks were the "Rembrandts of investments." Contrary to Frick's pronouncement, in 1970 the Penn Central Railroad[6] was forced into bankruptcy. In 1970, Penn Central had assets of $6.5 billion, making its bankruptcy the largest up until that time. Over the same period Rembrandt's work appreciated in value—and continues to appreciate.

2.8 Derivatives

Financial markets deal in two types of instruments. The first is called an *underlying asset*; these are company shares, bonds, foreign currencies, and commodities. The second type of instrument is a *derivative.* A derivative involves a future payment for, or future delivery of, an underlying asset. If there were no underlying assets, then derivatives would not exist.

One might wonder why derivatives *do* exist. Wouldn't it make sense to simply buy or sell the underlying asset rather than fool around with some future contract?

[6]Penn Central was formed by the merger of the Pennsylvania Railroad and the New York Central Railroad, once the two largest and most renowned of U.S. railroads.

One reason for the existence of future contracts is that, in some instances, it is literally impossible to transfer the underlying asset at a particular time: the underlying asset may still be growing in the field. A second reason for the existence of future contracts is to decrease (or increase) risk. For instance, if you know you will need a particular commodity at a particular time, then you can use a future contract to guarantee the availability of the commodity at a known price.

Derivatives emerge when the underlying assets can be readily traded, are available in sufficient quantity, and are subject to price changes. Since the prices of the various types of underlying assets are variable, the prices of the derivatives based on them also vary. One interesting and important problem is to determine an appropriate price for a derivative.

The simplest derivative is a *forward contract*, or more simply a *forward*, in which two parties agree on a price for the underlying asset at a specific future time. The forward is a very old type of derivative; there exists evidence of forward contracts in early records written in cuneiform script.

A more sophisticated type of derivative is an *option*. An option gives the right, but not the obligation, to buy, or sell, the underlying asset at some future time. Trade in derivatives is facilitated by certain clearing institutions, one of the main functions of which is canceling offsetting claims, a process called *netting*. Clearing institutions may also provide coverage against counterparty risk, that is, the risk of one of the parties to a contract defaulting on the contract.

Tulipmania

The classic example of a speculative bubble is provided by the trade in tulip bulbs in Holland in the 1630s. Tulips are very slow to grow from seeds, so they are usually propagated from bulbs. According to Charles Mackay's *Extraordinary Popular Delusions and the Madness of Crowds*, originally published in the mid 1800s, the price of tulip bulbs reached a phenomenal peak in January 1637. This phenomenal peak was followed by a disastrous collapse in February 1637. In fact, this market frenzy and speculative bubble—called "tulipmania"— would not have involved physically transferring actual tulip bulbs, because bulbs are not uprooted and moved during the winter. What was being traded were derivatives for which the underlying asset was tulips. Nonetheless, when the price of an asset seems to be inordinately high, it is often wondered if people are "buying tulips."

2.9 Pricing a Forward

The simplest derivative is a forward contract in which two parties agree on a price for the underlying asset at a specific future time; that is, two parties, say A and B, agree that at a particular time in the future A will deliver the underlying asset to B and B will pay the agreed upon price to A. Note that the forward contract described in this example is *not* a case of paying now for future delivery of the asset; both the asset and the payment will be exchanged at the specified future time.

To determine a price for the forward, we need to assume that the underlying asset is widely traded, can be bought and sold without transaction costs, does not degrade with time, and can be borrowed without cost: shares of stock in a company come close enough to meeting these requirements.

The possibility of borrowing the underlying asset without cost is a non-obvious assumption, but it is very powerful. This assumption makes it possible to "buy negative shares" through the mechanism of selling shares that have been borrowed; this is called *selling short*.[7] The possibility of selling short makes the circumstances of the buyer and the seller equivalent but with plus and minus signs interchanged.

Let's consider the pricing for a forward contract in which A will deliver 1 share of a stock to B 1 year in the future, at which time B will pay the price of $\$P$ to A. Suppose for simplicity that the current price per share of stock is \$100. Our task is to determine $\$P$.

You might reasonably think that the appropriate price for the forward is closely related to the probability distribution for the price of the underlying stock 1 year in the future. For instance, suppose the stock that is currently priced at \$100 per share will, 1 year in the future, have either the price \$200 or \$50, each with equal likelihood. You might think that the appropriate price to set for the forward contract is $\frac{1}{2} \times \$200 + \frac{1}{2} \times \$50 = \$125$.

[7] Those who invest in the stock market today speak of the "bull market" and the "bear market." The bull market is when stock prices go up. You obviously make money in a bull market by buying stock today and selling it later at a higher price. The bear market is different. There you anticipate that the price of a stock will go down. So you sell shares of the stock (which you do not own) today, but do not actually buy the shares until, say, three days later, at which time you deliver them to the buyer. Since the stock market went down, you paid less for the shares than the price that the buyer gave you. So you made money. The just-described process of taking advantage of a falling stock market is called "selling short."

Even with the specific price information in the preceding paragraph, the price of \$125 for 1 share, 1 year in the future, will almost surely be the wrong price. In fact, the correct price for the future has almost nothing to do with what is going to happen to the stock price in the future. It turns out that the correct price in the forward contract is determined by a crucial additional ingredient that we have not yet mentioned; that is, the risk-free interest rate.

The *risk-free interest rate*, which we will denote by r, tells us that if X dollars are invested in a money market account (money market accounts are considered risk-free) at the present time, then the account will grow to $(1+r)X$ dollars in 1 year. Similarly, X dollars can be borrowed, but $(1+r)X$ dollars must be repaid in 1 year. We are assuming that the same interest rate applies to saving and borrowing. We are also simplifying the interest rate model by assuming that the saving and borrowing will be for exactly 1 year. Thus we are using r for the *effective annual interest rate*.

To illustrate the correct way to set the price in a forward contract, let's make the specific assumption that the risk-free interest rate is 10%. In that case, the correct price for the forward contract is the current price per share, \$100, plus 1 year's interest on that price, \$10 per share; that is, the correct price in the forward contract is \$110 per share.

The price of \$110 per share for the forward contract is correct even if we are given the information that, 1 year in the future, the share price will be \$200 or \$50 with equal likelihood. Here's why we are confident that \$110 is the correct price:

• If the price of the forward contract were more than \$110, then we assert that everyone would want to be a seller of the forward, because a risk-free profit could be made on each share sold forward. To make this risk-free profit, at the start of the contract, say $T = 0$, the seller of the forward, A, would borrow \$100 at 10% interest and buy 1 share of stock. Then, 1 year later, at $T = 1$, A would have the 1 share of stock available to fulfill the contract, A could pay back the loan, including the interest, using \$110 of the money received under the forward contract, and there would still be money left over as profit. To make the example concrete, observe that if the forward price is \$125 per share, then the seller could make \$15 profit per share—risk-free.

• If the price of the forward were less than \$110, then we assert that everyone would want to be a buyer of the forward, because a risk-free profit could be made on every share bought forward. To make this risk-free profit, at $T = 0$, the buyer of the forward, B, would borrow 1 share of stock, sell it for \$100, and deposit the \$100 at 10% interest. Then, 1 year later, at $T = 1$, B would use the forward contract to buy 1 share of stock for less than \$110, that 1 share would be used to pay back the borrowed share of stock, and there would still be money left over as profit. To make the example concrete, observe that if the forward price is \$105 per share, then the buyer of the forward makes \$5 profit per share—risk-free.

2.10 Arbitrage

It was possible for us to determine the correct price for a forward contract because we made the fundamental assumption that it is impossible to make a risk-free profit, at least not without tying up some of your own money in the proposed investment. That is, we assumed that there is no arbitrage. *Arbitrage* is trading simultaneously, or nearly so, in different markets to take advantage of price differences. The idea is that the low and high prices may exist simultaneously, but in different markets. Provided the markets are large enough, the arbitrageur can buy at the low price and immediately sell at the high price, making a profit with no risk.

You seldom see opportunities to make risk-free profit, and any such opportunities that do happen to arise in the financial markets tend to disappear quickly, because prices change in response. When we determine the correct price for a derivative, we make the assumption that an arbitrage opportunity is not merely a rare thing, but that it is non-existent. The price for the forward contract that we constructed in the preceding section is thus called the *no-arbitrage price* or *arbitrage-free price*.

An Arbitrage Opportunity: Hoover's Free Travel Offer

In the summer of 1992, Maytag-UK had built up an excess inventory of washing machines and Hoover vacuum cleaners. To clear the backlog, the company started a promotion in which the purchase of £100 or more of merchandise entitled the purchaser to 2 roundtrip tickets to key cities on the continent. The promotion was successful, so the company improved it by offering 2 roundtrip tickets to the US.

Two roundtrip tickets from the UK to the continent were already worth more than the £100 purchase required, but two roundtrip flights to the US were worth a lot more. The public recognized an arbitrage opportunity and bought vacuum cleaners simply to get the airline tickets.

In any contract, there is always counterparty risk, that is, the risk that the other party to the contract will default. Maytag–UK attempted to default on the contract with consumers by not honoring the airline vouchers that consumers had obtained. Litigation resulted, and the company lost tens of millions of pounds sterling and its reputation.

2.11 Call Options

The forward contract, such as the example discussed above (1 share of stock 1 year in the future), is a particularly simple derivative and the appropriate arbitrage-free price for our example was relatively easy to determine. In finance a *call option*, or simply a *call*, is a contract that gives the buyer of the option the right, but not the obligation, to buy the underlying asset, at a specified time, called the *expiry date* or simply the *expiry*, and at a specified price, called the *strike price*. Again, a typical example of an underlying asset would be shares of a stock.

The second type of option is a *put option* which gives the buyer of the option the right, but not the obligation, to sell the underlying asset, at a specified time, the *expiry date*, and at a specified price, the *strike price*. For simplicity, we will consider only call options.

In reality, stock options trade in units that apply to 100 shares of the underlying stock, but instead of exactly modeling reality, we will consider options for 1 share of stock. By considering options on 1 share of stock, we will eliminate a factor of 100 that we would otherwise need to include.

As with derivatives in general, you might think the whole idea of options is silly. If a person thinks the stock will go up, why not simply buy the stock at the current low price and sell it later for more? The point is that options cost far less than the underlying stock, so a person may be able to capture a large profit from an upward movement in price without investing as much money.

A *European call option* is an option to buy the underlying asset that can only be exercised at the expiry date, while an *American call option* allows the purchase of the underlying asset at the strike price at any time up to and including the expiry date. Our goal is to compute the no-arbitrage price for an option. Because a European option can be exercised at only one time, the explanation of the no-arbitrage price will be simpler if we limit our attention to European options, and we shall do so. It is reasonable to assert that the no-arbitrage price is the correct price for an option.

Alfred Winslow Jones

The first modern hedge fund, A.W. Jones & Co., was the 1949 creation of Alfred Winslow Jones (1900–1989). Jones was born in Australia, grew up in the United States, and obtained his undergraduate education at Harvard. After working various jobs unrelated to finance, he enrolled in Columbia University where he earned a doctorate in sociology in 1941.

In the 1940s, Jones worked for *Fortune* magazine, but he wrote mainly on non-financial topics. However, for the March 1949 issue of *Fortune*, Jones wrote an article entitled "Forecasting fashions" about stock market forecasting and forecasters. Apparently Jones learned quite a bit while doing research for this article, because 2 months before it appeared, he formed an investment partnership. Jones directed the investment strategy for that partnership with the goals of decreasing risk and increasing returns by employing short selling, options, and leverage.[8]

"Hedge" is a gambling term describing betting both for and against an outcome. A hedged investment is one in which a profit will be made whether the asset goes up or down. For example, if the market is valuing a call option incorrectly, say the price is too low, then you could short the stock and simultaneously buy the option. Of course, there is the difficulty of knowing what the price of the call option should be. That question will be the main topic of the remainder of this chapter.

A.W. Jones & Co. was a spectacularly successful hedge fund. The company made large profits, apparently with little or no risk. On the other hand, not every company that has called itself a hedge fund has been successful. For example, the ironically named hedge fund Long-Term Capital Management opened in 1994, performed very well for a couple of years, but lost heavily in 1998. Because it was highly leveraged, the company owed a vast amount of money to many banks, so much money that the failure of the company would likely trigger major bank failures. The New York Federal Reserve Bank was forced by those circumstances to orchestrate a bailout—though government money was not used. Long-Term Capital Management finally dissolved in 2000 [Low 00].

[8] "Leverage" is obtained by borrowing money to invest more than could be done otherwise.

2.12 Value of a Call Option at Expiry

At the expiry date of a European option, the decision to be made by the holder of that option is whether to use the option to buy the stock at the strike price. The correct decision at that time may be clear, but we state it explicitly here:

• If at expiry the price of the underlying stock is higher than the strike price of the option, then the option should be used to buy the stock at the strike price. Note that, in this situation, an immediate profit can be made by buying the stock at the strike price, then selling it at the market price. We also conclude that when the market price at expiry exceeds the strike price, then the value of the option at expiry is the market price minus the strike price.

• If at expiry the price of the underlying stock is lower than the strike price, then the option should not be used, because the stock can be purchased more cheaply at the market price. We also conclude that when the market price at expiry is less than the strike price, then the value of the option at expiry is \$0.

If at expiry the price of the underlying stock is exactly equal to the market price, then using the option has no effect on the purchase price of the stock. The value of the option at expiry is again \$0.

2.13 Pricing a Call Option Using a Replicating Portfolio: A Single Time Step

In the preceding section we determined the value of a European option at expiry. A more difficult problem is determining the value of an option before expiry. The full theory of options pricing is technical, so we will address the problem in a simplified setting.

We will find the initial no-arbitrage price for the option under the following simplifying assumptions:

• The underlying stock only trades at the time when the option is purchased and at expiry.

• The stock price at expiry can only be one of two values, one larger than the strike price and the other lower than the strike price. Which of these two values will happen, we don't know.

• The stock price at expiry is determined by some random experiment such as a coin toss—possibly with an unfair coin.

The assumption that the stock price either goes up one "step" or down one "step" is motivated by the geometric random walk model for stock prices. We don't know what the odds are for the random experiment that determines the stock price at expiry, but it will turn out that the no-arbitrage price for the option does not depend on those odds. This surprising independence of the option price and the odds of the stock price going up or down is crucial to the usefulness of the theory.

To find the arbitrage-free initial price of a European option we will use the *replication method*. In the replication method, we determine a portfolio consisting of a mixture of the underlying stock and cash that produces exactly the same result at expiry as the option produces. Then the initial value of that portfolio must be equal to the initial value of the option that it models.

Rather than using formulas and equations, we will simplify further by making specific choices for the prices of the underlying stock and the strike price. After working through that specific example, we will show how to use graphical methods to construct the replicating portfolio.

Our specific example uses the following (arbitrarily chosen) values:

(1) the underlying stock trades only at time $T = 0$ and $T = 1$,

(2) the price of the underlying stock at time $T = 0$ is \$300,

(3) at time $T = 1$, the underlying stock can only take one of two values, a high value of \$700 or a low value of \$400,

(4) the strike price in the call option is \$550.

As was true for finding the no-arbitrage price of a forward in Sect. 2.9, the risk-free interest rate, r, is an essential additional ingredient. So that the effect of the risk-free interest rate will show clearly, we will assume the very high interest rate of 100%. Thus we have one more assumption

(5) the effective risk-free interest rate over one time period is 100%.

Figure 2.11 Possible stock price movements vs. money market.

Figure 2.11 illustrates two things that you could do with \$300 at $T = 0$ and the possible consequences of your choice. You could simply put the \$300 in the risk-free money market account and have \$600 at $T = 1$. This possibility is shown by the green line in Fig. 2.11. A second possibility is to buy 1 share of stock with the \$300. If you buy the stock at $T = 0$ and the stock price at $T = 1$ is the low value of \$400, then you end up with \$200 less than if you had put your \$300 in the money market account. On the other hand, if you buy the stock at $T = 0$ and the stock price at $T = 1$ is the high value of \$700, then you end up with \$100 more than if you had put your \$300 in the money market account.

Notice that if we were to change assumption (3) above by making the high value of the stock at $T = 1$ equal to \$600 or less, then buying the stock would be a bad choice no matter whether the stock went to its high or low value. Similarly, if we were to change assumption (3) above by making the low value of the stock at $T = 1$ equal to \$600 or more, then putting money in the money market account would be a bad choice no matter whether the stock went to its high or low value.

It is convenient to define the value spread of 1 share of the stock to be the difference between the high and low values that the stock could assume at $T = 1$; that is, the value spread equals \$700 − \$400 = \$300. The black arrow pointing up in Fig. 2.12 illustrates that the value spread of the underlying stock is +\$300.

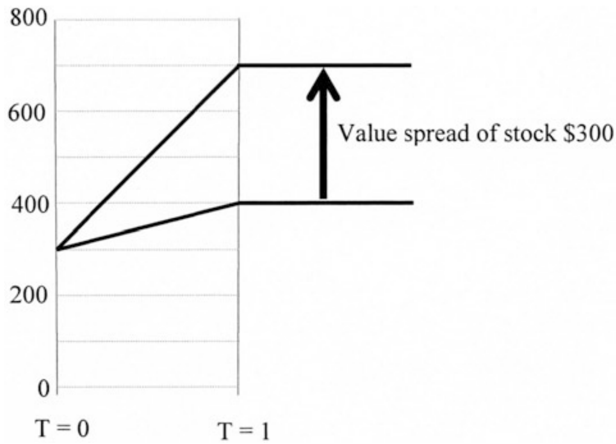

Figure 2.12 The value spread of one share of the underlying stock.

Any derivative based on the underlying stock also will have a value spread at $T = 1$. The value spread for the underlying stock is by definition positive (high minus low). We can make no such general assertion about whether the value spread of a derivative is positive or negative. We also have not yet shown how to go about finding the value spread of a derivative.

We next illustrate how to find the value spread for the European call option in our example. For the particular case of a European call option at expiry, we know how to find the value of the option from the value of the underlying stock and the strike price.

• If the underlying stock price at $T = 1$ assumes the high value \$700, then the option is worth the difference between the stock price and the strike price; that is the value of the option equals \$700 − \$550 = \$150.

• On the other hand, if the stock price at $T = 1$ assumes the low value \$400, then the option is worthless, because if you want to buy a share of stock at $T = 1$, it can be bought for \$400 which is less than the \$550 that the option entitles you to pay.

We see that the value of the call option could be as high as \$150 and as low as \$0 and that gives the call option a value spread of \$150 − \$0 = \$150. The red arrow pointing up in Fig. 2.13 illustrates the fact that the value spread of the call option is +\$150.

Again we note that in general it is possible for the value spread of something to be negative. The value spread of anything is calculated by first

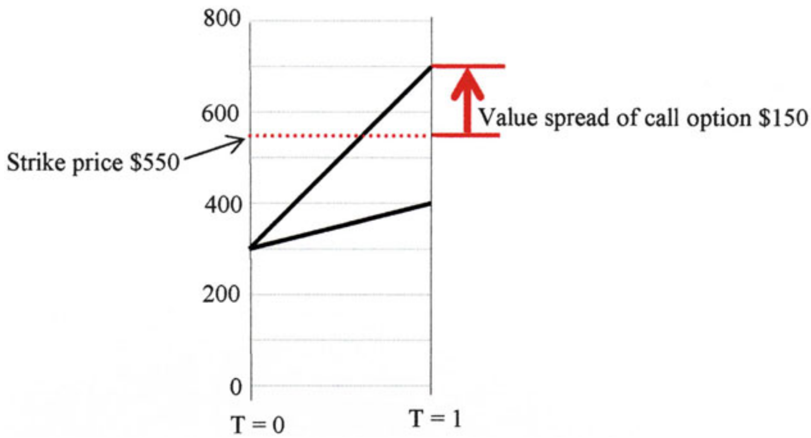

Figure 2.13 The value spread of the derivative: European call option.

computing its value when the stock is assigned its high value and then subtracting its value when the stock is assign its low value. In particular, the value spread for a negative number of shares of the underlying stock must be negative. Recall that a negative number of shares is obtained by selling short, i.e., borrowing the shares and selling them.

In order to determine the no-arbitrage price of the call option, we construct a portfolio consisting of the underlying stock and cash that exactly mimics the behavior of the option at $T = 1$, that is, we construct a replicating portfolio. Since there is no arbitrage, the value of the option at $T = 0$ must equal the value of the replicating portfolio at $T = 0$; were that not the case you would sell whichever is overpriced to buy the other and make a risk-free profit at $T = 1$.

We will construct the replicating portfolio in two steps. First, we construct a preliminary portfolio that matches the value spread of the call option. Because there is no value spread for cash, all of the value spread in the preliminary portfolio must come from the underlying stock. Since the value spread of the option is \$150 and the value spread of 1 share of stock is \$300, we see that $\frac{1}{2}$ share of the stock will produce the desired value spread of \$150. We begin by putting $\frac{1}{2}$ share of stock in the preliminary portfolio. The behavior of the preliminary portfolio is illustrated by the blue arrow in Fig. 2.13.

Figure 2.14 also includes a red arrow that corresponds to the value spread of the call option at $T = 1$. The top and bottom of the red arrow are located

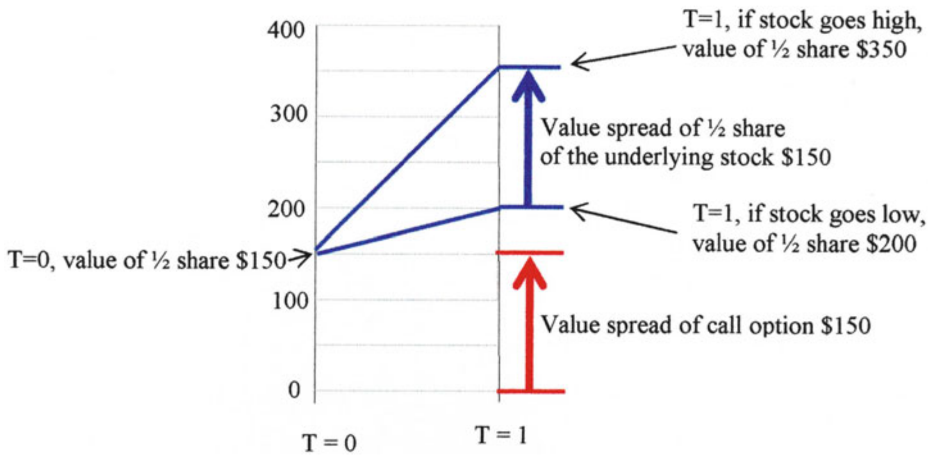

Figure 2.14 The value spread of $\frac{1}{2}$ share of the underlying stock.

at positions that correspond to the possible values of the call option. Likewise, the top and bottom of the blue arrow are located at the positions that correspond to the possible values of the $\frac{1}{2}$ share of stock in the preliminary portfolio.

By construction, the blue arrow in Fig. 2.14 is the same length and points in the same direction as the red arrow in Fig. 2.14. But the blue arrow is positioned higher than the red arrow. In fact, the blue arrow is higher than the red arrow by a distance that corresponds to $200.

The replicating portfolio must exactly match the behavior of the call option when $T = 1$, so we must add cash to the preliminary portfolio so as to leave the value spread unchanged while lowering the ending values by $200. To make the replicating portfolio worth $200 less at $T = 1$, we add to the preliminary portfolio a loan on which we owe $200 at $T = 1$. Since the risk-free interest rate is assumed to be 100%, the loan must be for $100 at $T = 0$. This loan is illustrated by the green line in Fig. 2.15.

Figure 2.15 also illustrates the total value of the replicating portfolio with the blue lines. Notice that the value of the replicating portfolio is $50 at $T = 0$. From this total value at $T = 0$, we conclude that the no-arbitrage price for the call option at $T = 0$ equals $50. The behavior of the option is illustrated in Fig. 2.16.

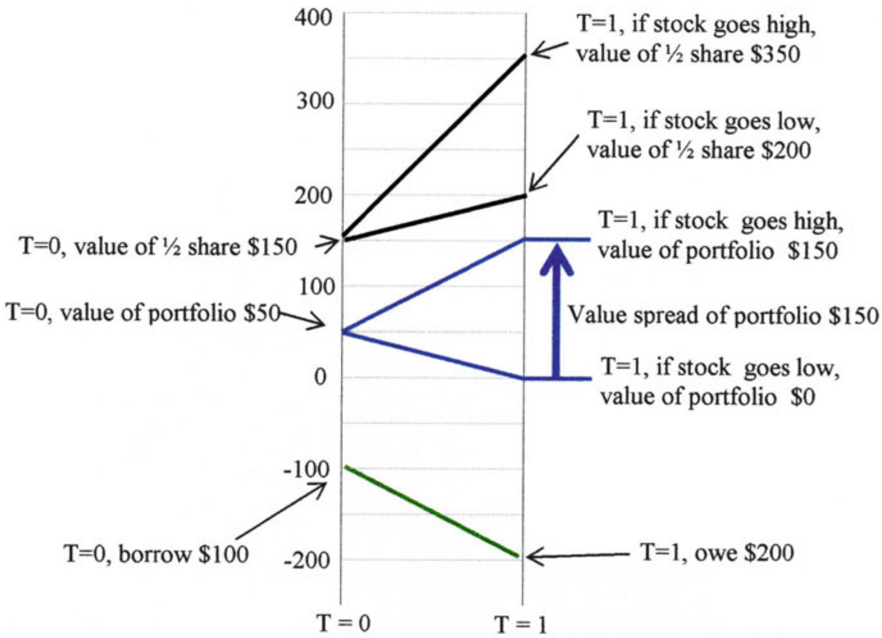

Figure 2.15 Behavior of the replicating portfolio.

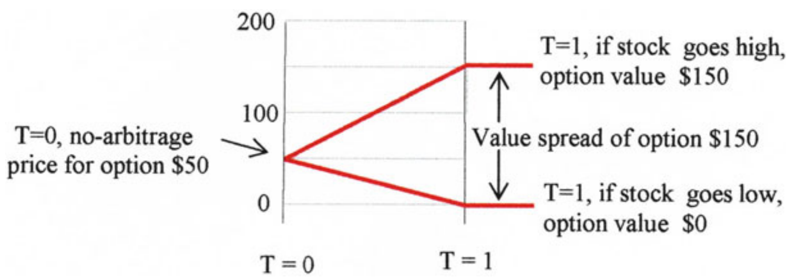

Figure 2.16 Behavior of the value of the European call option.

In review, we found the correct amount of stock for the replicating portfolio by matching the value spread of the option with the appropriate amount of the underlying stock, then we matched the $T = 1$ values themselves by including the appropriate amount of cash at $T = 1$. Once we know how much stock and cash is in the portfolio at $T = 1$, we can then use the stock price at $T = 0$ (which is known) to calculate the value at $T = 0$ of the stock in the portfolio, and we can use the risk-free interest rate (which is known) to calculate the amount of cash in the portfolio at $T = 0$. The total value of the portfolio at $T = 0$ is the no-arbitrage price for the call option at $T = 0$, namely \$50. This value of \$50 is unaffected by the probability that the stock price goes up or down; it does depend on the value spread of the stock and the risk-free interest rate.

The same process,

(1) match the value spread with stock,

(2) shift the final values with cash,

(3) calculate the value at $T = 0$ from the stock price and the interest rate,

can be applied to find the no-arbitrage value of any derivative based on the underlying stock. It is not necessary to know the probability of an upward or downward movement of the stock price. Even if the probability is known, it has no effect on the no-arbitrage valuation for the derivative.

2.14 Pricing a Call Option Using a Replicating Portfolio: Multiple Time Steps

The geometric random walk model for stock prices suggests that we might be able to calculate the no-arbitrage value of a derivative by repeated applications of the method illustrated in the preceding subsection. Figure 2.17 illustrates the tree of stock price movements implied by the geometric random walk model. Each dot corresponds to a specific price for the stock at a specific time. The left-hand side of the figure corresponds to the present, at which time the stock price is known (so there is only one dot). With each time step the stock price can go to a higher value or a lower value. As we move through more time-steps the number of possible stock prices increases. Also, notice that some pairs of dots are connected by more than one path, because there can be more than one way to get from one price to another.

To determine the value of a European call option, we look at a tree like that in Fig. 2.17. The dots on the right-hand side correspond to the stock price at the expiry date, $T = 4$ in the figure. At the expiry date, $T = 4$, the value of an option depends on the strike price, but the strike price is a known quantity. In Fig. 2.18, a strike price has been superimposed over the stock prices. For the two expiry date stock prices that are above the strike price, the option has the values represented by the red arrows. The values

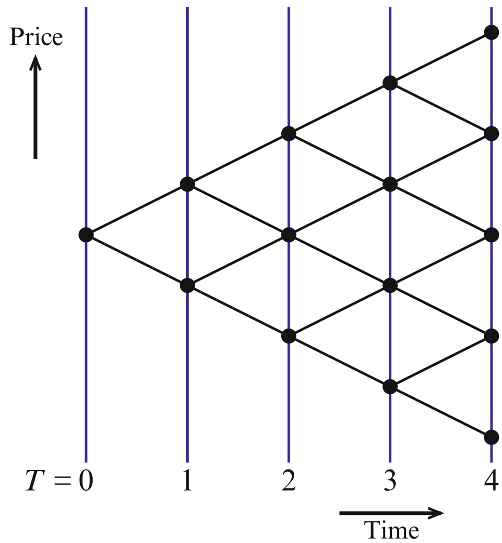

Figure 2.17 The tree structure corresponding to the geometric random walk model for stock prices.

are positive, so the arrows point up. For the three expiry date stock prices that are below the strike price, the value of the option is 0.

For each dot corresponding to a stock price at one time step before the expiry date, that is, at $T = 3$, there are 2 paths leading to 2 of the dots representing stock prices at expiry. For each such set of 3 dots, that is, one dot at $T = 3$ connected to 2 dots at $T = 4$, we can calculate the no-arbitrage value of the option at $T = 3$ using the method described in the preceding section. There are four such no-arbitrage option values at $T = 3$ that must be computed.

Once the no-arbitrage values at $T = 3$ have been computed, we work back one time step to $T = 2$. After the 3 no-arbitrage values of the option at $T = 2$ have been calculated, we work back to $T = 1$, and then finally we compute the no-arbitrage value for the option at $T = 0$.

2.15 Black–Scholes Option Pricing

The multiple time step method for option pricing that we described in the preceding subsection is based on the geometric random walk model for stock prices. The geometric random walk can be extended to its continuous limit,

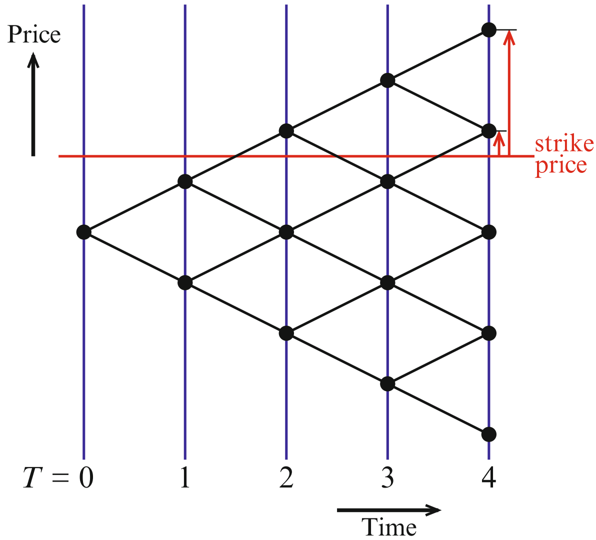

Figure 2.18 Calculate the value of the call option starting at the expiry date. Work backward in time until reaching the present.

that is, to a geometric Brownian motion. The no-arbitrage option pricing obtained in this continuous limit was developed by Fischer Black and Myron Scholes in the 1970s.

The risk-free interest rate clearly plays a crucial role in determining the no-arbitrage price of an option. Because we have been building our option prices without using equations, it is not clear how the option price is affected by the two stock-price parameters, average return and volatility. When the full analysis is carried out, one finds that of the two stock-price parameters, only the volatility enters into the no-arbitrage price for a European call option. Because the average return cannot be precisely estimated, but the volatility can be, the fact that the no-arbitrage price for an option depends only on the volatility is a tremendously important discovery. This insight is one of the main reasons that the work of Black and Scholes is considered to have revolutionized financial mathematics.

Fischer Sheffey Black, Myron Samuel Scholes, and Robert Cox Merton

Photograph courtesy of Alethea Black

Fischer S. Black

2008, Nobel laureates photographer

Myron Scholes

The names of Fischer Black (1938–1995) and Myron Scholes (b. 1941) are often linked in reference to the Black–Scholes equation. The Black–Scholes equation is a stochastic partial differential equation[9] which is used to model the behavior of the price of an option. Black and Scholes's discovery of a rational basis for option pricing began a revolution in finance.

Scholes and Robert C. Merton (b. 1944) extended the initial work of Black and Scholes. It was Scholes and Merton who received the 1997 Nobel Prize in Economics for their part in starting the financial revolution. Black was not so recognized, because he succumbed to throat cancer in 1995 and Nobel Prizes are not awarded posthumously.[10]

Had he lived long enough, Fischer Black would have been one of the few mathematicians to win a Nobel Prize. Of course, we associate Black's name with economics and finance, and his most famous work was indeed in economics, but his interests and background were broader than that. His undergraduate degree from Harvard was in physics and he started his doctoral work in that field. He then switched to mathematics. Ultimately his thesis, *A Deductive Question Answering System*, was on artificial intelligence.[11]

[9] A partial differential equation involves an unknown function and that function's rate of change with respect to two or more variables. If a random variable is also involved, then it is a stochastic partial differential equation.

[10] The matter of posthumous Nobel prizes is complicated. In fact, the 1931 Nobel Prize in Literature was awarded posthumously to Erik Axel Karlfeldt. Since 1974 the Nobel Foundation statutes have prohibited posthumous prizes, with the proviso that if a prize winner is announced but dies before the presentation ceremony, then the award will remain valid. In 2011, Dr. Ralph Steinman died hours before the Nobel Committee selected him for the prize in Medicine (along with two other men). In this case, it was decided that since the committee made the award on the good faith assumption that Dr. Steinman was alive, the award was valid.

[11] A slightly edited version of Black's thesis appears as a chapter in *Semantic Information Processing*, edited by Marvin Minsky.

Robert C. Merton

One of Fischer Black's children is Alethea Black—author of *I Knew You'd Be Lovely.* The name Alethea is from the Greek $\alpha\lambda\eta\theta\epsilon\iota\alpha$ meaning *truth.* We would imagine, as she long did, that the name was chosen because of her mathematician father's love of and quest for truth. In fact, her parents got her name from the character Alethea Staunton played by Karen Black in a 1968 episode of *Judd for the Defense.*

A Look Back

Fisher Black was a Professor of Finance at Harvard University and Myron Scholes a Professor at Stanford. Scholes was awarded the Nobel Prize in Economics in 1997. (Black was deceased at the time, so he received no tribute.) The Black/Scholes option pricing model had really turned the world of finance on its ear, and this recognition seemed ever so appropriate.

Like most universities, Stanford has a monthly, in-house newsletter whose purpose is to inform the members of the university community about recent news and accomplishments of their colleagues. You can imagine that quite a big deal was made of Scholes winning the Nobel Prize, and a substantial article was written about the event.

Unfortunately, the editor assigned to handle the Scholes article carelessly applied a spell checker to the piece. He ended up replacing every occurrence of "Myron Scholes" with "moron schools." And that is how the article went into print. More's the pity for all of us.

The Black/Scholes model for option pricing has had a considerable impact on Wall Street. There are now a good many mathematics Ph.D.s who get jobs in the finance world. At Washington University we had a recent Ph.D. whose first position was in a Wall Street investment firm for a starting salary of $200,000.

But sometimes there is trouble in paradise. The concept of stochastic volatility—fundamental for the Black/Scholes model—has now become a mainstay of the investment world. But not everyone subscribes to that model. The stock market crash of 1987 lent weight to the nay-sayers. There have been other crashes since then.

Some say that the Black/Scholes model has too many simplifying assumptions. Others have asserted that volatility must be modeled in a different way in order to make the Black/Scholes model work effectively.

Any good idea continues to grow and develop. Mathematician James H. Simons founded and ran the historically most successful hedge fund (named Renaissance Technologies). Edward O. Thorp, after writing the famous book *Beat the Dealer* about how to win at blackjack in Las Vegas, wrote another book called *Beat the Market* in which he applied similar strategies to investments. He went on to become a multi-billionaire and is now happily retired (see [Pou 05]). Both men have proprietary mathematical investment schemes, and both certainly took into account the stochastic analysis that goes into the Black/Scholes model.

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