Chapter 2
Counting

We often want to know the answer to the question, "How many?" In this chapter we shall look at some of the rules that help us to answer this question.

2.1 Some Counting Principles

One obvious question, given a set S, is "how many elements are there in S? We shall denote that number by $|S|$, the *order* of S.

We shall start with an example.

Sample Problem 2.1 *Suppose there are 30 students in Dr. Green's finite mathematics course and 40 in his calculus section. If these are his only classes, and if 20 of the students are taking both subjects, how many students does he have altogether?*

Solution. We shall use the notation S for the set of students in the finite mathematics class and T for calculus. So $|S| = 30$ and $|T| = 40$, and $|S \cap T| = 20$. We want to know the number of elements of $|S \cup T|$.

If we wrote a list of all the students in the two classes, we would write $|S| + |T| = 30 + 40$ names. But we have duplicated the 20 names in $|S \cap T|$. Therefore the total number is

$$|S \cup T| = |S| + |T| - |S \cap T| = 30 + 40 - 20 = 50,$$

so he has 50 students.

This is the simplest case of a rule called the *principle of inclusion and exclusion*, and you can remember it in the following way. To list all members of the union of two sets, list all members of the first set and all members of the second set. This ensures that all members are included. However, some elements will be listed twice, so it is necessary to exclude the duplicates. If S is the set of all the objects that

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have some property A and T is the set of all the objects with property B, then (2.1) expresses the way to count the objects that have *either* property A *or* property B:

(i) count the objects with property A;
(ii) count the objects with property B;
(iii) count the objects with both properties;
(iv) subtract the third answer from the sum of the other two.

Putting it another way, if you list all members of S, then list all the members of T, you will cover all members of $S \cup T$, but those in $S \cap T$ will be listed twice. To count all members of $S \cup T$, you could count all members of both lists, then subtract the number of duplicates. In other words,

$$|S \cup T| = |S| + |T| - |S \cap T|. \tag{2.1}$$

A similar argument can be applied to three or more sets. For example, suppose you want to know the number of elements in $S \cup T \cup W$. If you add $|S|$, $|T|$ and $|W|$, you have added all the elements of $S \cap T$ twice, and similarly those of $S \cap W$ and $T \cap W$. So subtract the orders of those sets. Then every element of one or two of the sets has been counted twice. But you are missing the elements of $S \cap T \cap W$—they were added three times and subtracted three times. So

$$|S \cup T \cup W| = |S| + |T| + |W| - |S \cap T| - |S \cap W| - |T \cap W| + |S \cap T \cap W|.$$

This can be generalized to any number of sets.

A variation of this rule, called the *rule of sum*, can be simply expressed by saying "the number of objects with property A equals the number that have both property A and property B, plus the number that have property A but not property B"; if we again define sets S and T to be the collections of all objects having properties A and B respectively, the rule is

$$|S| = |S \cap T| + |S \backslash T|. \tag{2.2}$$

If we rewrite (2.2) as

$$|S \backslash T| = |S| - |S \cap T|$$

and substitute into (2.1), we obtain

$$|S \cup T| = |T| + |S \backslash T|. \tag{2.3}$$

Sample Problem 2.2 *Suppose the set S has 35 elements, T has 17 elements, and $S \cap T$ has seven elements. Find $|S \cup T|$ and $|S \backslash T|$.*
Solution. From (2.1) we see

$$|S \cup T| = |S| + |T| - |S \cap T| = 35 + 17 - 7 = 45.$$

From (2.2) we get

$$|S\backslash T| = |S| - |S\cap T| = 35 - 7 = 28.$$

Your Turn. Suppose $|S| = 40$, $|T| = 30$ and $|S\cap T| = 20$. Find $|S\cup T|$ and $|S\backslash T|$.

It is sometimes useful to break an event down into several parts, forming what we shall call a *compound event*. For example, suppose you are planning a trip from Los Angeles to Paris, with a stopover in New York. You have two options for the flight to New York: a direct flight with Delta or an American flight that stops in Chicago. For the second leg, you consider the direct flight with Air France, a British Airways flight through London, and Lufthansa stopping in Frankfurt. There are two ways to make the first flight and three to make the second, for a total of six combinations.

Suppose S is the set of available flights for the first leg and T is the set of available flights for the second leg. Then the flight combinations correspond to the members of the Cartesian product $S\times T$, and in this example

$$|S| = 2,\ \ |T| = 3,\ \ |S\times T| = |S|\times|T| = |2\times 3| = 6.$$

The correspondence between compound events and Cartesian products applies in general. If S is the set of cases where A occurs, and T is the set of cases where B occurs, then the possible combinations correspond to the set $S\times T$, which has $|S|\times|T|$ elements. This idea is usually applied without mentioning the sets S and T. Suppose the event A can occur in a ways and the event B can occur in b ways, then the combination of events A and B can occur in ab ways. This very obvious principle is sometimes called the *multiplication principle* or *rule of product*. It can be extended to three or more sets.

Sample Problem 2.3 *To open a bicycle lock you must know a three-number combination. You must first turn to the left until the first number is reached, then back to the right until the second number, then left to the third number. Any number from 1 to 36 can be used. How many combinations are possible?*

Solution. There are 36 ways to choose the first number, 36 ways to choose the second, and 36 ways to choose the third. So there are $36\times 36\times 36$ combinations.

Your Turn. Your debit card has a 4-digit PIN. If you can use any digits, how many PINs are possible?

Sample Problem 2.4 *A true–false test consists of four questions. Assuming you answer all questions, how many ways are there to answer the test ?*

Solution. There are two ways to answer the first question, two ways to answer the second, two ways to answer the third, and two ways to answer the fourth. So the total is $2\times 2\times 2\times 2 = 16$.

The multiplication principle only works when the events are performed independently—if the result of A is somehow used to affect the performance of B, some combined results may be impossible. In the airline example, if the Delta flight leaves too late to connect with the Air France flight, then your choices are not independent, and only five combinations would be available.

Sometimes the order of the elements in a set is unimportant, but sometimes the order is significant. In that case we would like to know how many ways there are to order the elements of the set. For example, if you have three cards, an Ace, King and Queen (abbreviated as A, K, Q), they can be ordered in six ways: AKQ, AQK, KAQ, KQA, QAK, and QKA. The three-element set $\{A, K, Q\}$ has six orderings. So obviously any three-element set has three orderings.

More generally, suppose S is a set with n elements. How many different ways are there to order the elements of S?

We solve this by treating the ordering as a compound event with n parts. There are n ways to choose the first element of the ordered set. Whichever element is chosen, there remain $n-1$ possible choices for the second element. When two elements have been selected, there are $n-2$ choices for the third element.

In this way, we see that there are $n \times (n-1) \times n-2 \times \ldots \times 3 \times 2 \times 1$ ways to order S. This number is called n factorial, and denoted $n!$. So

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

For convenience we define $0!$ to equal 1.

The different ways of ordering the set S are called *permutations of S.*

Sample Problem 2.5 *Evaluate* $10!$.
Solution. $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362880$.
Your Turn. Evaluate $6!$.

Sample Problem 2.6 *A committee of three people—chair, secretary, and treasurer—is to be elected by a club with* 11 *members. If every member is eligible to stand for each position, how many different committees are possible?*
Solution. We can treat the selection of the committee as a compound event with three parts: choose the chair, choose the secretary, and choose the treasurer. These parts can be performed in 11, 10, and 9 ways respectively. So there are $11 \times 10 \times 9$ committees possible.
Your Turn. What is the number of committees if there are 9 members?

This method is often combined with the multiplication principle.

Sample Problem 2.7 *Four boys and four girls are to sit along a bench. The boys must sit together, as must the girls. How many ways can this be done?*
Solution. We treat this as a compound event with three parts. First, it is decided whether the boys are to be on the left or on the right. This can be done in two ways. Then the ordering of the boys is chosen. This can be done in $4! = 24$ ways. Finally, the girls are ordered. This can be accomplished in $4! = 24$ ways. So there are $2 \times 24 \times 24 = 1152$ arrangements.
Your Turn. How many ways could five boys and four girls be seated on two benches, if the boys must sit on the back bench and the girls on the front?

2.2 Arrangements

Suppose S is a set with s elements. We often need to know how many k-element *ordered* subsets or *k-sequences* or *arrangements* of size k can be chosen from S. This number is denoted $P(s,k)$. For example, if S is the three-element set $\{x,y,z\}$, the possible two-element ordered subsets are xy, yx, xz, zx, yz and zx; so $P(3,2) = 6$.

In particular $P(s,s)$ denotes the number of s-sequences that can be chosen from an s-set, or the number of arrangements of the set S. But these are precisely the permutations of S. It follows that $P(s,s) = s!$. For this reason arrangements of size k are often called *permutations of size k*.

Given an s-set $S = \{x_1, x_2, \ldots, x_s\}$, there are s different sequences of length 1 on S, namely (x_1), (x_2), ..., and (x_s). So $P(s,1) = s$. There are $s \times (s-1)$ sequences of length 2, because each sequence of length 1 can be extended to length 2 in $s-1$ different ways, and no two of these $s \times (s-1)$ extensions will ever be equal. So $P(s,2) = s(s-1)$. Similarly we find

$$
\begin{aligned}
P(s,3) &= s \times (s-1) \times (s-2), \\
P(s,4) &= s \times (s-1) \times (s-2) \times (s-3), \\
&\ldots, \\
P(s,k) &= s \times (s-1) \times (s-2) \ldots (s-k+1).
\end{aligned}
$$

So $P(s,k)$ is calculated by multiplying, s, $s-1$, $s-2, \ldots$ until there are k factors.

It follows that

$$P(s,k) = s!/(s-k)!. \tag{2.4}$$

Sample Problem 2.8 *Calculate* $P(10,3)$.
Solution. There are two ways to calculate $P(10,3)$. We could say $P(10,3) = 10 \times 9 \times 8 = 720$. Or we could use the formula:

$$P(10,3) = 10!/7! = 362880/5040 = 720.$$

The first way is easier.
Your Turn. Calculate $P(6,4)$.

Several of the problems in the preceding section, such as those on selecting a committee, asked for the number of sequences of a certain length, and their solutions can sometimes be stated compactly by using arrangements.

Sample Problem 2.9 *A committee of three people—chair, secretary, and treasurer—is to be elected by a club with* 14 *members. If every member is eligible to stand for each position, how many different committees are possible?*
Solution. We can treat the committee as an ordered set of three elements chosen from the 14-element set of members. So the answer is $P(14,3)$, or 2,184.

Your Turn. What is the number of committees if there are 12 members?

Sometimes an added condition makes the solution of a problem easier, not harder. For example, arranging people around a circular table is no more difficult than arranging them in a line, and sometimes easier.

Suppose n people are to sit around a circular table. We start by arbitrarily labeling one seat at the table as "1," the one to its left as "2," and so on. Then there are $n!$ different ways of putting the n people into the n seats. However, we have counted two arrangements as different if one is obtained from the other by shifting every person one place to the left, because these two arrangements put different people in "1"; but they are clearly the same arrangement for the purposes of the question. Each arrangement is one of a set of n, all obtained from the others by shifting in a circular fashion. So the number of truly different arrangements is $n!/n$, which equals $(n-1)!$.

Sample Problem 2.10 *How many ways can you make a necklace by threading together seven different beads?*
Solution. Suppose you put the beads on a table before threading them. There would be $(n-1)! = 6! = 720$ ways to arrange them in a circle. However, after the beads are threaded, the necklace could be flipped over, so every necklace has been counted twice (for example, $abcdefg$ and $agfedcb$ are the same necklace). Therefore, the total number is $6!/2 = 360$.
Your Turn. How many ways could the three boys and four girls be arranged around a circular table if the boys must sit together and the girls as well?

Sample Problem 2.11 *Colette's Copying Company has eight photocopying machines and seven employees who can operate them. There are four copying jobs to be done simultaneously. How many ways are there to allocate these jobs to operators and machines?*
Solution. Call the jobs A, B, C, D. Choose two arrangements: first, which four operators should do the jobs; second, which machines should be used. The operator choice can be made in $P(7,4)$ ways, and the machines in $P(8,4)$ ways. In each case, the first member of the sequence is the one allocated to job A, the second to job B, and so on. There are $P(7,4) \times P(8,4) = 7 \times 6 \times 5 \times 4 \times 8 \times 7 \times 6 \times 5 = 1{,}411{,}200$ ways.

Sample Problem 2.12 *The club in Sample Problem* 2.9 *wishes to elect a by-laws committee with three members—Chair, Secretary, and Legal Officer—and requires that no members of the main club committee be members of the by-laws committee. In how many different ways can the two committees be chosen?*
Solution. Suppose the main committee is chosen first. There are $P(14,3) = 2{,}184$ ways to do this. After the election, there are 11 members eligible for election to the by-laws committee, so it can be chosen in $P(11,3) = 990$ ways. So there are a grand total of $2{,}184 \times 990 = 2{,}162{,}160$.

In the preceding Sample Problem we see that, even in small problems, the numbers get quite large. It might be better to report the answer in its factored form, as $14 \times 13 \times 12 \times 11 \times 10 \times 9$. This form of answer also makes it clear that we could have solved the problem by treating the two committees as one six-member sequence, with $P(14,6)$ possible solutions.

In some cases, we want to talk about collections of objects with repetitions. For example, consider the letters in the word *ASSESS*. In how many ways can you order these letters? The set of letters involved is $\{A,S,E\}$, but there are six letters in the word, and the orderings will have six elements. For clarity, we often talk of *distinguishable* and *indistinguishable* elements. We could say the set of letters in *ASSESS* has six elements: four (indistinguishable) S's, an A, and an E.

One way to tackle these problems is to assume the "indistinguishable" objects can be distinguished, and then take this into account. For example, if the letters in the word *ASSESS* were written on tiles with numbers as subscripts, like scrabble tiles, we could label them so that no two copies of the same letter get the same subscript, for example $A_1S_1S_2E_1S_3S_4$. Then all the six letters are different, and there are $6!$ orderings. Say you have each of these orderings written on slips of paper. Now collect together into one pile all the slips that differ only in their subscripts. For example, $A_1E_1S_1S_2S_3S_4$ and $A_1E_1S_2S_1S_3S_4$ will be in the same pile, as will $A_1E_1S_2S_3S_1S_4$, $A_1E_1S_4S_2S_3S_1$, and several others. In fact, we can work out how many slips there are in a pile. There are four letters S, and one each of the others. Two slips will be in the same pile when they have the letters in the same order, but the subscripts on the S's are in different order. There are $4! = 24$ ways to order the four subscripts, so there are $4!$ slips in each pile. Therefore, there are $6!/4! = 30$ piles. Two orderings can be distinguished if and only if their slips are in different piles, so there are $6!/4! = 30$ distinguishable orderings of *ASSESS*.

The same principle can be applied with several repeated letters. For example, if *SUCCESS* is written as $S_1U_1C_1C_2E_1S_2S_3$, we see that there are $2!$ ways of ordering the C's and $3!$ ways if ordering the S's, so each pile will contain $3! \times 2!$ slips, and the number of distinguishable orderings is $7!/(3! \times 2!) = 420$.

Sample Problem 2.13 *In how many distinguishable ways can you order the letters of the word MISSISSIPPI?*

Solution. There are one M, four I's, four S's and two P's, for a total of 11 letters. So the number of orderings is $11!/(4! \times 4! \times 2!)$ or 34650.

Your Turn. In how many distinguishable ways can you order the letters of the word *BANANA*?

2.3 Selections

Suppose you are giving a party, and you want to order three different pizzas from a list of 12 types that your local store sells. How many ways can you make your choice? If you were to list the three types, starting with your first choice, then the second, and finally the third, there would be 12 possible first choices, 11 second (you want different types, so no repeats are allowed), and 10 third. So the number of ways is $12 \times 11 \times 10 = 1320$. Essentially, you are calculating $P(12,3)$. But there would be six possible lists that give the same set of three pizzas, in different order. So there are really $1320/6 = 220$ possible choices.

Essentially, you are calculating the number of possible sets of three types of pizza you could choose from a set of 12 types. There are a number of situations similar to this: given a set S, we want to know how many different subsets of a given size are contained in S. These are called *selections* or *combinations*. We shall write $C(s,k)$ or $\binom{s}{k}$ for the number of different k-subsets of an s-set; it is usual to read the symbol as "s choose k." $\binom{s}{k}$ is often called the *choice function* (of s and k).

We can use the formula (2.4) to derive expressions for the numbers $C(s,k)$. Suppose S is a set with s elements. It is clear that every k-set that we choose from S gives rise to exactly $k!$ distinct k-sequences on S and that the same k-sequence never arises from different k-sets. So the number of k-sequences on S is $k!$ times the number of k-sets on S, or

$$\binom{s}{k} = \frac{P(s,k)}{k!} = \frac{s!}{(s-k)!k!} \tag{2.5}$$

When calculating $\binom{s}{k}$ in practice, you would usually calculate $P(s,k)$, then divide by $k!$. So

$$\binom{s}{k} = \frac{s \times (s-1) \times (s-2) \times \ldots \times (s-k+1)}{1 \times 2 \times 3 \times \ldots \times k}.$$

There are k factors in the denominator and in the numerator.

We agreed to say $0! = 1$. In combination with (2.5) this yields $\binom{s}{0} = 1$. This makes sense: it is possible to choose *no* elements from a set, but one cannot imagine different ways of doing so. We also define $\binom{s}{k} = 0$ if $k > s$. Again this makes sense—there is no way to choose more than s elements from an s-set.

Sample Problem 2.14 *Calculate* $C(8,5)$ *and* $\binom{6}{6}$.

Solution.

$$C(8,5) = \frac{8 \times 7 \times 6 \times 5 \times 4}{5 \times 4 \times 3 \times 2 \times 1} = 56.$$

$$\binom{6}{6} = \frac{6!}{0! \times 6!} = 1.$$

There is no need for calculation: the terms 6! in the numerator and denominator cancel.

Your Turn. Calculate $C(9,5)$ and $\binom{6}{0}$.

Sample Problem 2.15 *A student must answer five of the eight questions on a test. How many different ways can she answer, assuming there is no restriction on her choice and the order in which she answers them is unimportant?*
Solution. $\binom{8}{5} = 56$ ways.
Your Turn. How many ways can she answer if she must choose five, one of which is Question 1?

Sample Problem 2.16 *Computers read strings consisting of the digits* 0 *and* 1. *Such a string with k entries is called a k-bit string. How many* 8*-bit strings are there that contain exactly five* 1*s?*
Solution. To specify a string, it is sufficient to say which positions have 1s. There are $C(8,5)$ choices, so the answer is $C(8,5) = 56$.
Your Turn. How many 8-bit strings contain exactly four 1s?

Sample Problem 2.17 *How many ways can a committee of three men and two women be chosen from six men and four women?*
Solution. The three men can be chosen in $\binom{6}{3}$ ways; the two women can be chosen in $\binom{4}{2}$ ways. Using the multiplication principle, the total number of committees possible with no restrictions is

$$\begin{aligned}\binom{6}{3} \times \binom{4}{2} &= \frac{6!}{3!3!} \times \frac{4!}{2!2!} \\ &= 120.\end{aligned}$$

Your Turn. You wish to borrow two mystery books and three westerns from your friend. He owns five mysteries and seven westerns. How many different selections can you make?

Sample Problem 2.18 *How many different "words" of five letters can you make from the letters of the word REPUBLICAN, if every word must contain two different vowels and three different consonants?*
Solution. The three consonants can be chosen in $\binom{6}{3} = 20$ ways, and the vowels in $\binom{4}{2} = 6$ ways. After the choice is made, the letters can be arranged in $5! = 120$ ways. So there are $20 \times 6 \times 120 = 14400$ "words."
Your Turn. What is the answer if you use the word *DEMOCRAT*?

Multiple Choice Questions 2

1. If $|S| = 5$, $|T| = 7$ and $|S \cap T| = 4$, what is $S \cup T$?
A. 2 B. 6 C. 4 D. 8

2. If $|S| = 5$, $|T| = 7$ and $|S \cup T| = 10$, what is $S \cap T$?
A. 2 B. 6 C. 4 D. 8

3. Say there are 25 books on your shelf, each either hardback or paperback. If 10 are paperback, how many are hardback?
A. 10 B. 25 C. 15 D. 40

4. Judy wants one cereal, one juice and one coffee for breakfast. If she can choose from five cereals, three juices and two coffees (either caffeinated or decaf), how many different breakfasts are possible?
A. 5 B. 30 C. 10 D. 60

5. You have six books on the shelf at the back of your desk. In how many ways can they be ordered?
A. 6 B. 120 C. 720 D. 5040

6. Five people are to sit at a round table. How many ways can they be seated?
A. 12 B. 60 C. 24 D. 120

7. In how many ways can you arrange the letters of the word *LIFELINE*?
A. 120 B. 5040 C. 6720 D. 40320

8. Eight students participate in a car wash for charity. In how many ways can you choose two of the students to hold the signs advertising the car wash?
A. 8 B. 16 C. 28 D. 56

9. There are nine students from whom four are going to be chosen to represent their school at a conference. If Jack, Anna or Chris, but only one of them, must be chosen, in how many ways can the students be chosen to go to the conference?
A. 60 C. 2160 B. 126 D. 3024

10. Evaluate $C(4,2) \times C(5,3)$.
A. 720 C. 60 B. 480 D. 16

Exercises 2

1. Suppose $|S| = 19$, $|T| = 11$ and $|S \cap T| = 8$. Find $|S \cup T|$ and $|S \backslash T|$.

2. Suppose $|S| = 22$, $|T| = 12$ and $|S \cup T| = 28$. Find $|S \cap T|$ and $|S \backslash T|$.

3. Suppose $|T| = 37$, $|S \cap T| = 7$ and $|S \backslash T| = 14$. Find $|S|$ and $|S \cup T|$.

4. Suppose $|S| = 44$, $|T| = 18$ and $|S \cap T| = 12$. Find $|S \cup T|$ and $|S \backslash T|$.

5. In a survey of 1100 voters it was found that 275 will vote in favor of a $\frac{1}{2}\%$ increase in sales tax for Public Safety funding, 550 will vote in favor of a $\frac{1}{4}\%$ increase in income tax for school funding, and 200 of these will vote for both tax increases.

(i) How many favor the school tax but not the Public Safety tax?
(ii) How many will vote for neither option?

6. Among 1000 telephone subscribers it was found that 475 have answering machines and 250 call waiting. Moreover 150 had both options.
  (i) How many had either an answering machine or call waiting?
  (ii) How many had neither option?

7. Out of 400 people surveyed, 100 said they plan to buy a new house within the next three years, and 200 expected to buy a new car in that period. Of these, 50 planned to make both kinds of expenditure. Use (2.1) and the rule of sum to find out how many planned to buy a new house but not buy a new car, and how many planned to buy a new car but not buy a new house.

8. There are three roads from town X to town Y, four roads from town Y to town Z, and two roads from town X to town Z.
  (i) How many routes are there from town X to town Z with a stopover in town Y?
  (ii) How many routes are there in total from town X to town Z?
(Assume that no road is traveled twice.)

9. List all different permutations of the set $\{A,B,C\}$.

10. The three boys and four girls in the choir are to sit on two benches. The boys must sit on the back bench and the girls on the front. How many different ways can they be seated?

11. A multiple-choice quiz contains eight questions. Each has three possible answers.
  (i) If you must answer every question, how many different answer sheets are possible?
  (ii) If you may either answer a question or leave a blank, how many different answer sheets are possible?

12. In how many different ways can you order the letters of the word "BREAK"?

13. Calculate:
(i) $P(8,3)$ (ii) $P(4,4)$ (iii) $P(5,4)$ (iv) $P(9,2)$

14. Three men and four women sit in a row. How many different ways can they do it if:
  (i) the men must sit together;
  (ii) the women must sit together?

15. John likes to arrange his books. He has four western, seven mystery and six science fiction books, all different.
  (i) In how many ways can he arrange them on a shelf?
  (ii) In how many ways can he arrange them if all the books on the same subject must be grouped together?

16. Your PIN number consists of four digits. No repetitions are allowed, and 0 is not to be used.
  (i) How many PIN numbers are possible?
  (ii) How many PIN numbers are smaller than 4000?
  (iii) How many PIN numbers are even?
  (iv) How many PIN numbers contain no number greater than 6?

17. A football league consists of eight teams. Each team must play each other team twice: once as home team, once as visitors.
  (i) How many games must be played?
  (ii) If each pair plays only once (you don't care which is the home team), how many games must be played?

18. Seven stereo systems are to be arranged in a line against the wall of the appliance department.
  (i) How many ways can this be done?
  (ii) How many ways can they be arranged if the most expensive model must be in the middle?

19. In how many ways can you arrange the letters of the following words?
  (i) *TODDLER* (ii) *OFFERED* (iii) *BORROW*
  (iv) *ARROWROOT* (v) *MOOSEWOOD* (vi) *APPLESEED*

20. There are ten speakers in a debate, five on each side. It is agreed that the first speaker must speak in favor of the proposition, followed by a speaker against it, then one in favor, then one against. The remaining six speakers may speak in any order. In how many different ways can the debate be scheduled?

21. List all the selections of size 3 that can be made from the set $\{A, B, C, D, E\}$.

22. Calculate the following quantities:
  (i) $C(8,3)$ (ii) $C(9,4)$ (iii) $\binom{6}{3}$
  (iv) $C(7,3)$ (v) $C(7,7)$ (vi) $\binom{8}{6}$

23. The Student Council consists of six juniors and 12 seniors. A committee of two juniors and three seniors is to be formed. How many ways can this be done?

24. A test has 12 questions, and you must answer nine of them.
  (i) How many ways can you choose which questions to answer?
  (ii) In how many ways can you make your choice if you must include Question 1 or Question 2 (or maybe both)?

25. A state lottery requires you to choose five different numbers from $\{1, 2, \ldots, 49\}$. The order in which the numbers are chosen does not matter.
  (i) How many possible choices are there?
  (ii) The state then draws six different numbers. You win if all five of your numbers are chosen. How many of the possible choices are winners?

26. A businessman wishes to pack three different ties for a business trip. If he has six ties available, how many different selections could be made?

27. There are 18 undergraduates and 14 graduates in the Math club. A committee of five members is to be selected. Calculate how many ways this can be done, if:
  (i) There is no restriction.
  (ii) There must be exactly two graduates.
  (iii) There must be at least two graduates and two undergraduates.

28. A Euchre deck of cards contains 25 cards: 6 spades, 6 hearts, 6 diamonds, 6 clubs, and a joker. A five-card hand is dealt.
  (i) How many different hands are possible?
  (ii) How many of those hands contain only spades?
  (iii) How many hands contain three spades and two clubs?

(iv) How many hands contain the joker?
(v) How many hands contain only spades and clubs, at least one of each?

29. Suppose the Senate contains 49 Democrats and 51 Republicans. A committee of six must be chosen.
(i) How many different committees can be chosen?
(ii) How many of these possible committees contain exactly three Democrats and three Republicans?

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