Preface

Elementary arithmetic studies divisibility and factorization of ordinary integers. Algebraic number theory considers the same questions for algebraic numbers, solutions to polynomial equations with integer coefficients. In this setting, (un)fortunately, the uniqueness of prime factorization no longer holds. Remedying and measuring its failure is the starting point of algebraic number theory.

There are many excellent texts both on elementary and on algebraic number theory. However, I needed an intermediate-level book for an undergraduate course with the aim of imparting the flavor and beauty of algebraic number theory with minimal algebraic prerequisites. The notes for that course grew into this book. Restricting to quadratic numbers was a natural choice: hands-on examples abound, while the Galois theory is trivial to the point of invisibility. Indeed, concreteness and computation underpin the approach of the book. Computers are a great research tool; for learning, however, there is no substitute for getting one's hands dirty with calculations. A parallel emphasis is on the interaction between different branches of mathematics, all too often introduced to undergraduates as separate worlds. In this book, the student can see them joining forces to prove beautiful theorems. Linear and abstract algebra together rescue unique factorization; basic plane geometry is essential for proving that unique factorization never fails by much.

The prerequisites for this book are a knowledge of elementary number theory and a passing familiarity with ring theory. Its goal is to give the undergraduate a taste of algebraic number theory right after (or during) the first course in abstract algebra, and before learning Galois theory. Indeed, the book can serve as a rich source of examples for a ring theory course. A bright and brave freshman can read it, with some effort, even before the first course on rings: the necessary theory is covered, albeit briskly, in the Chap. 2. Finally, the book can be a concrete supplement to a beginning graduate course in algebraic number theory.

Here's a brief outline of the contents.

Chapter 1 generalizes, to the extent possible, the arithmetic of $\mathbb{Z}$ to several specific quadratic fields. The examples are chosen to present the range of new phenomena in algebraic number theory. The only abstract algebra required here is the definitions of a ring and a field.

Chapter 2 is a self-contained but quick review of ring theory, with examples geared toward number theory.

Chapter 3 amplifies the standard theory of row- and column-reduction by restricting to matrices with entries in $\mathbb{Z}$.

Chapter 4 is the heart of the book. It develops algebraic number theory in a general quadratic field, and culminates in the proof of unique factorization of ideals.

Chapter 5 proves the finiteness of the ideal class group, with numerous computational examples.

Chapter 6 completes the picture of the arithmetic of quadratic fields by describing the group of units of a real quadratic field. To do this, we develop the theory of continued fractions. This chapter and the next are arguably the most interesting ones, since their techniques and results don't generalize to higher-degree fields.

Chapter 7 goes back to the roots of algebraic number theory—binary quadratic forms. Forgoing the usual elementary presentation, we emphasize the connections with ideals. The main result of the chapter is a precise description of the identification of strict equivalence classes of binary quadratic forms and narrow ideal classes. This chapter has no extra prerequisites, but does require more mathematical maturity, as we introduce the language of group actions and commutative diagrams. Concreteness is not sacrificed, though: we cover in detail the algorithms for reducing both definite and indefinite forms. The section concludes with a presentation of Barbara cubes, a recent development. The appendix assembles the results on the orders in quadratic fields, proved in the exercises throughout the book.

The terms being defined, either in formal definitions or in the running text, are **highlighted** for ease of finding. All propositions, definitions, displayed equations etc., are numbered in the same sequence. Starred exercises come with a hint at the back of the book.

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