CHAPTER 2

The Simplex Method

In this chapter we present the simplex method as it applies to linear programming problems in standard form.

1. An Example

We first illustrate how the simplex method works on a specific example:

$$
\begin{array}{lrcl}
\text{maximize} & 5x_1 + 4x_2 + 3x_3 & & \\
\text{subject to} & 2x_1 + 3x_2 + x_3 & \leq & 5 \\
& 4x_1 + x_2 + 2x_3 & \leq & 11 \\
& 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
& x_1,\ x_2,\ x_3 & \geq & 0.
\end{array}
\tag{2.1}
$$

We start by adding so-called *slack variables*. For each of the less-than inequalities in (2.1) we introduce a new variable that represents the difference between the right-hand side and the left-hand side. For example, for the first inequality,

$$2x_1 + 3x_2 + x_3 \leq 5,$$

we introduce the slack variable w_1 defined by

$$w_1 = 5 - 2x_1 - 3x_2 - x_3.$$

It is clear then that this definition of w_1, together with the stipulation that w_1 be nonnegative, is equivalent to the original constraint. We carry out this procedure for each of the less-than constraints to get an equivalent representation of the problem:

$$
\begin{array}{lrcl}
\text{maximize} & \zeta = & & 5x_1 + 4x_2 + 3x_3 \\
\text{subject to} & w_1 = & & 5 - 2x_1 - 3x_2 - x_3 \\
& w_2 = & & 11 - 4x_1 - x_2 - 2x_3 \\
& w_3 = & & 8 - 3x_1 - 4x_2 - 2x_3 \\
& x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 & \geq & 0.
\end{array}
\tag{2.2}
$$

Note that we have included a notation, ζ, for the value of the objective function, $5x_1 + 4x_2 + 3x_3$.

The simplex method is an iterative process in which we start with a solution $x_1, x_2, \ldots, w_3$ that satisfies the equations and nonnegativities in (2.2) and then look for a new solution $\bar{x}_1, \bar{x}_2, \ldots, \bar{w}_3$, which is better in the sense that it has a larger objective function value:

$$5\bar{x}_1 + 4\bar{x}_2 + 3\bar{x}_3 > 5x_1 + 4x_2 + 3x_3.$$

R.J. Vanderbei, *Linear Programming*, International Series in Operations Research & Management Science 196, DOI 10.1007/978-1-4614-7630-6_2,

We continue this process until we arrive at a solution that can't be improved. This final solution is then an optimal solution.

To start the iterative process, we need an initial feasible solution $x_1, x_2, \ldots, w_3$. For our example, this is easy. We simply set all the original variables to zero and use the defining equations to determine the slack variables:

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad w_1 = 5, \quad w_2 = 11, \quad w_3 = 8.$$

The objective function value associated with this solution is $\zeta = 0$.

We now ask whether this solution can be improved. Since the coefficient of x_1 in the objective function is positive, if we increase the value of x_1 from zero to some positive value, we will increase ζ. But as we change its value, the values of the slack variables will also change. We must make sure that we don't let any of them go negative. Since x_2 and x_3 are currently set to 0, we see that $w_1 = 5 - 2x_1$, and so keeping w_1 nonnegative imposes the restriction that x_1 must not exceed $5/2$. Similarly, the nonnegativity of w_2 imposes the bound that $x_1 \leq 11/4$, and the nonnegativity of w_3 introduces the bound that $x_1 \leq 8/3$. Since all of these conditions must be met, we see that x_1 cannot be made larger than the smallest of these bounds: $x_1 \leq 5/2$. Our new, improved solution then is

$$x_1 = \frac{5}{2}, \quad x_2 = 0, \quad x_3 = 0, \quad w_1 = 0, \quad w_2 = 1, \quad w_3 = \frac{1}{2}.$$

This first step was straightforward. It is less obvious how to proceed. What made the first step easy was the fact that we had one group of variables that were initially zero and we had the rest explicitly expressed in terms of these. This property can be arranged even for our new solution. Indeed, we simply must rewrite the equations in (2.2) in such a way that x_1, w_2, w_3, and ζ are expressed as functions of w_1, x_2, and x_3. That is, the roles of x_1 and w_1 must be swapped. To this end, we use the equation for w_1 in (2.2) to solve for x_1:

$$x_1 = \frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3.$$

The equations for w_2, w_3, and ζ must also be doctored so that x_1 does not appear on the right. The easiest way to accomplish this is to do so-called *row operations* on the equations in (2.2). For example, if we take the equation for w_2 and subtract two times the equation for w_1 and then bring the w_1 term to the right-hand side, we get

$$w_2 = 1 + 2w_1 + 5x_2.$$

Performing analogous row operations for w_3 and ζ, we can rewrite the equations in (2.2) as

$$\begin{array}{rl} \zeta = & 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\ \hline x_1 = & 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\ w_2 = & 1 + 2w_1 + 5x_2 \\ w_3 = & 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3. \end{array} \tag{2.3}$$

Note that we can recover our current solution by setting the "independent" variables to zero and using the equations to read off the values for the "dependent" variables.

Now we see that increasing w_1 or x_2 will bring about a *decrease* in the objective function value, and so x_3, being the only variable with a positive coefficient, is the only independent variable that we can increase to obtain a further increase in the objective function. Again, we need to determine how much this variable can be increased without violating the requirement that all the dependent variables remain nonnegative. This time we see that the equation for w_2 is not affected by changes in x_3, but the equations for x_1 and w_3 do impose bounds, namely $x_3 \leq 5$ and $x_3 \leq 1$, respectively. The latter is the tighter bound, and so the new solution is

$$x_1 = 2, \quad x_2 = 0, \quad x_3 = 1, \quad w_1 = 0, \quad w_2 = 1, \quad w_3 = 0.$$

The corresponding objective function value is $\zeta = 13$.

Once again, we must determine whether it is possible to increase the objective function further and, if so, how. Therefore, we need to write our equations with ζ, x_1, w_2, and x_3 written as functions of w_1, x_2, and w_3. Solving the last equation in (2.3) for x_3, we get

$$x_3 = 1 + 3w_1 + x_2 - 2w_3.$$

Also, performing the appropriate row operations, we can eliminate x_3 from the other equations. The result of these operations is

$$\begin{array}{rrrrr} \zeta = & 13 - & w_1 - & 3x_2 - & w_3 \\ \hline x_1 = & 2 - & 2w_1 - & 2x_2 + & w_3 \\ w_2 = & 1 + & 2w_1 + & 5x_2 & \\ x_3 = & 1 + & 3w_1 + & x_2 - & 2w_3. \end{array} \tag{2.4}$$

We are now ready to begin the third iteration. The first step is to identify an independent variable for which an increase in its value would produce a corresponding increase in ζ. But this time there is no such variable, since all the variables have negative coefficients in the expression for ζ. This fact not only brings the simplex method to a standstill but also proves that the current solution is optimal. The reason is quite simple. Since the equations in (2.4) are completely equivalent to those in (2.2) and, since all the variables must be nonnegative, it follows that $\zeta \leq 13$ for every feasible solution. Since our current solution attains the value of 13, we see that it is indeed optimal.

1.1. Dictionaries, Bases, Etc. The systems of equations (2.2), (2.3), and (2.4) that we have encountered along the way are called *dictionaries*. With the exception of ζ, the variables that appear on the left (i.e., the variables that we have been referring to as the dependent variables) are called *basic variables*. Those on the right (i.e., the independent variables) are called *nonbasic variables*. The solutions we have obtained by setting the nonbasic variables to zero are called *basic feasible solutions*.

2. The Simplex Method

Consider the general linear programming problem presented in standard form:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, 2, \ldots, m \\ & x_j \geq 0 & j = 1, 2, \ldots, n. \end{array}$$

Our first task is to introduce slack variables and a name for the objective function value:

$$\begin{aligned} \zeta &= \sum_{j=1}^{n} c_j x_j \\ w_i &= b_i - \sum_{j=1}^{n} a_{ij} x_j \qquad i = 1, 2, \ldots, m. \end{aligned} \tag{2.5}$$

As we saw in our example, as the simplex method proceeds, the slack variables become intertwined with the original variables, and the whole collection is treated the same. Therefore, it is at times convenient to have a notation in which the slack variables are more or less indistinguishable from the original variables. So we simply add them to the end of the list of x-variables:

$$(x_1, \ldots, x_n, w_1, \ldots, w_m) = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}).$$

That is, we let $x_{n+i} = w_i$. With this notation, we can rewrite (2.5) as

$$\begin{aligned} \zeta &= \sum_{j=1}^{n} c_j x_j \\ x_{n+i} &= b_i - \sum_{j=1}^{n} a_{ij} x_j \qquad i = 1, 2, \ldots, m. \end{aligned}$$

This is the starting dictionary. As the simplex method progresses, it moves from one dictionary to another in its search for an optimal solution. Each dictionary has m basic variables and n nonbasic variables. Let $\mathcal{B}$ denote the collection of indices from $\{1, 2, \ldots, n+m\}$ corresponding to the basic variables, and let $\mathcal{N}$ denote the indices corresponding to the nonbasic variables. Initially, we have $\mathcal{N} = \{1, 2, \ldots, n\}$ and $\mathcal{B} = \{n+1, n+2, \ldots, n+m\}$, but this of course changes after the first iteration. Down the road, the current dictionary will look like this:

$$\begin{aligned} \zeta &= \bar{\zeta} + \sum_{j \in \mathcal{N}} \bar{c}_j x_j \\ x_i &= \bar{b}_i - \sum_{j \in \mathcal{N}} \bar{a}_{ij} x_j \qquad i \in \mathcal{B}. \end{aligned} \tag{2.6}$$

Note that we have put bars over the coefficients to indicate that they change as the algorithm progresses.

Within each iteration of the simplex method, exactly one variable goes from nonbasic to basic and exactly one variable goes from basic to nonbasic. We saw this process in our example, but let us now describe it in general.

The variable that goes from nonbasic to basic is called the *entering variable*. It is chosen with the aim of increasing ζ; that is, one whose coefficient is positive: *pick* k *from* $\{j \in \mathcal{N} : \bar{c}_j > 0\}$. Note that if this set is empty, then the current solution is optimal. If the set consists of more than one element (as is normally the case), then we have a choice of which element to pick. There are several possible selection criteria, some of which will be discussed in the next chapter. For now, suffice it to say that we usually pick an index k having the largest coefficient (which again could leave us with a choice).

The variable that goes from basic to nonbasic is called the *leaving variable*. It is chosen to preserve nonnegativity of the current basic variables. Once we have decided that x_k will be the entering variable, its value will be increased from zero to a positive value. This increase will change the values of the basic variables:

$$x_i = \bar{b}_i - \bar{a}_{ik} x_k, \qquad i \in \mathcal{B}.$$

We must ensure that each of these variables remains nonnegative. Hence, we require that

$$\bar{b}_i - \bar{a}_{ik} x_k \geq 0, \qquad i \in \mathcal{B}. \tag{2.7}$$

Of these expressions, the only ones that can go negative as x_k increases are those for which $\bar{a}_{ik}$ is positive; the rest remain fixed or increase. Hence, we can restrict our attention to those i's for which $\bar{a}_{ik}$ is positive. And for such an i, the value of x_k at which the expression becomes zero is

$$x_k = \bar{b}_i / \bar{a}_{ik}.$$

Since we don't want any of these to go negative, we must raise x_k only to the smallest of all of these values:

$$x_k = \min_{i \in \mathcal{B}: \bar{a}_{ik} > 0} \bar{b}_i / \bar{a}_{ik}.$$

Therefore, with a certain amount of latitude remaining, the rule for selecting the leaving variable is *pick* l *from* $\{i \in \mathcal{B} : \bar{a}_{ik} > 0$ *and* $\bar{b}_i / \bar{a}_{ik}$ *is minimal*$\}$.

The rule just given for selecting a leaving variable describes exactly the process by which we use the rule in practice. That is, we look only at those variables for which $\bar{a}_{ik}$ is positive and among those we select one with the smallest value of the ratio $\bar{b}_i / \bar{a}_{ik}$. There is, however, another, entirely equivalent, way to write this rule which we will often use. To derive this alternate expression we use the convention that $0/0 = 0$ and rewrite inequalities (2.7) as

$$\frac{1}{x_k} \geq \frac{\bar{a}_{ik}}{\bar{b}_i}, \qquad i \in \mathcal{B}$$

(we shall discuss shortly what happens when one of these ratios is an indeterminate form $0/0$ as well as what it means if none of the ratios are positive). Since we wish to take the largest possible increase in x_k, we see that

$$x_k = \left(\max_{i \in \mathcal{B}} \frac{\bar{a}_{ik}}{\bar{b}_i} \right)^{-1}.$$

Hence, the rule for selecting the leaving variable is as follows: *pick l from* $\{i \in \mathcal{B} : \bar{a}_{ik}/\bar{b}_i$ *is maximal*$\}$.

The main difference between these two ways of writing the rule is that in one we minimize the ratio of $\bar{a}_{ik}$ to $\bar{b}_i$ whereas in the other we maximize the reciprocal ratio. Of course, in the minimize formulation one must take care about the sign of the $\bar{a}_{ik}$'s. In the remainder of this book we will encounter these types of ratios often. We will always write them in the maximize form since that is shorter to write, acknowledging full well the fact that it is often more convenient, in practice, to do it the other way.

Once the leaving-basic and entering-nonbasic variables have been selected, the move from the current dictionary to the new dictionary involves appropriate row operations to achieve the interchange. This step from one dictionary to the next is called a *pivot*.

As mentioned above, there is often more than one choice for the entering and the leaving variables. Particular rules that make the choice unambiguous are called *pivot rules*.

3. Initialization

In the previous section, we presented the simplex method. However, we only considered problems for which the right-hand sides were all nonnegative. This ensured that the initial dictionary was feasible. In this section, we discuss what to do when this is not the case.

Given a linear programming problem

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{j=1}^n c_j x_j \\ \text{subject to} & \displaystyle\sum_{j=1}^n a_{ij} x_j \leq b_i \qquad i = 1, 2, \ldots, m \\ & \qquad x_j \geq 0 \qquad j = 1, 2, \ldots, n, \end{array}$$

the initial dictionary that we introduced in the preceding section was

$$\begin{aligned} \zeta &= \qquad \sum_{j=1}^n c_j x_j \\ w_i &= b_i - \sum_{j=1}^n a_{ij} x_j \qquad i = 1, 2, \ldots, m. \end{aligned}$$

The solution associated with this dictionary is obtained by setting each x_j to zero and setting each w_i equal to the corresponding b_i. This solution is feasible if and only if all the right-hand sides are nonnegative. But what if they are not? We handle this difficulty by introducing an *auxiliary problem* for which

(1) A feasible dictionary is easy to find and
(2) The optimal dictionary provides a feasible dictionary for the original problem.

The auxiliary problem is

$$\begin{array}{lll} \text{maximize} & -x_0 & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij}x_j - x_0 \leq b_i & i = 1, 2, \ldots, m \\ & x_j \geq 0 & j = 0, 1, \ldots, n. \end{array}$$

It is easy to give a feasible solution to this auxiliary problem. Indeed, we simply set $x_j = 0$, for $j = 1, \ldots, n$, and then pick x_0 sufficiently large. It is also easy to see that the original problem has a feasible solution if and only if the auxiliary problem has a feasible solution with $x_0 = 0$. In other words, the original problem has a feasible solution if and only if the optimal solution to the auxiliary problem has objective value zero.

Even though the auxiliary problem clearly has feasible solutions, we have not yet shown that it has an easily obtained feasible dictionary. It is best to illustrate how to obtain a feasible dictionary with an example:

$$\begin{array}{lrcr} \text{maximize} & -2x_1 - x_2 & & \\ \text{subject to} & -x_1 + x_2 & \leq & -1 \\ & -x_1 - 2x_2 & \leq & -2 \\ & x_2 & \leq & 1 \\ & x_1,\ x_2 & \geq & 0. \end{array}$$

The auxiliary problem is

$$\begin{array}{lrcr} \text{maximize} & -x_0 & & \\ \text{subject to} & -x_1 + x_2 - x_0 & \leq & -1 \\ & -x_1 - 2x_2 - x_0 & \leq & -2 \\ & x_2 - x_0 & \leq & 1 \\ & x_0,\ x_1,\ x_2 & \geq & 0. \end{array}$$

Next we introduce slack variables and write down an initial *infeasible dictionary*:

$$\begin{array}{rcl} \xi & = & - x_0 \\ \hline w_1 & = & -1 + x_1 - x_2 + x_0 \\ w_2 & = & -2 + x_1 + 2x_2 + x_0 \\ w_3 & = & 1 - x_2 + x_0. \end{array}$$

This dictionary is infeasible, but it is easy to convert it into a feasible dictionary. In fact, all we need to do is one pivot with variable x_0 entering and the "most infeasible variable," w_2, leaving the basis:

$$\begin{array}{rcl} \xi & = & -2 + x_1 + 2x_2 - w_2 \\ \hline w_1 & = & 1 - 3x_2 + w_2 \\ x_0 & = & 2 - x_1 - 2x_2 + w_2 \\ w_3 & = & 3 - x_1 - 3x_2 + w_2. \end{array}$$

Note that we now have a feasible dictionary, so we can apply the simplex method as defined earlier in this chapter. For the first step, we pick x_2 to enter and w_1 to leave the basis:

$$\begin{array}{rcl}
\xi &=& -1.33 + x_1 - 0.67w_1 - 0.33w_2 \\ \hline
x_2 &=& 0.33 \qquad - 0.33w_1 + 0.33w_2 \\
x_0 &=& 1.33 - x_1 + 0.67w_1 + 0.33w_2 \\
w_3 &=& 2 - x_1 + \quad w_1 .
\end{array}$$

Now, for the second step, we pick x_1 to enter and x_0 to leave the basis:

$$\begin{array}{rcl}
\xi &=& 0 - x_0 \\ \hline
x_2 &=& 0.33 \qquad - 0.33w_1 + 0.33w_2 \\
x_1 &=& 1.33 - x_0 + 0.67w_1 + 0.33w_2 \\
w_3 &=& 0.67 + x_0 + 0.33w_1 - 0.33w_2 .
\end{array}$$

This dictionary is optimal for the auxiliary problem. We now drop x_0 from the equations and reintroduce the original objective function:

$$\zeta = -2x_1 - x_2 = -3 - w_1 - w_2.$$

Hence, the starting feasible dictionary for the original problem is

$$\begin{array}{rcl}
\zeta &=& -3 - \quad w_1 - \quad w_2 \\ \hline
x_2 &=& 0.33 - 0.33w_1 + 0.33w_2 \\
x_1 &=& 1.33 + 0.67w_1 + 0.33w_2 \\
w_3 &=& 0.67 + 0.33w_1 - 0.33w_2 .
\end{array}$$

As it turns out, this dictionary is optimal for the original problem (since the coefficients of all the variables in the equation for ζ are negative), but we can't expect to be this lucky in general. All we normally can expect is that the dictionary so obtained will be feasible for the original problem, at which point we continue to apply the simplex method until an optimal solution is reached.

The process of solving the auxiliary problem to find an initial feasible solution is often referred to as *Phase I*, whereas the process of going from a feasible solution to an optimal solution is called *Phase II*.

4. Unboundedness

In this section, we discuss how to detect when the objective function value is unbounded.

Let us now take a closer look at the "leaving variable" computation: *pick l from $\{i \in \mathcal{B} : \bar{a}_{ik}/\bar{b}_i$ is maximal$\}$*. We avoided the issue before, but now we must face what to do if a denominator in one of these ratios vanishes. If the numerator is nonzero, then it is easy to see that the ratio should be interpreted as plus or minus infinity depending on the sign of the numerator. For the case of $0/0$, the correct convention (as we'll see momentarily) is to take this as a zero.

What if all of the ratios, $\bar{a}_{ik}/\bar{b}_i$, are nonpositive? In that case, none of the basic variables will become zero as the entering variable increases. Hence, the entering variable can be increased indefinitely to produce an arbitrarily large objective value.

In such situations, we say that the problem is *unbounded*. For example, consider the following dictionary:

$$\begin{array}{rcl} \zeta & = & 5 + \ \ x_3 - \ \ x_1 \\ \hline x_2 & = & 5 + 2x_3 - 3x_1 \\ x_4 & = & 7 \qquad\quad - 4x_1 \\ x_5 & = & \qquad\qquad\quad x_1. \end{array}$$

The entering variable is x_3 and the ratios are

$$-2/5, \quad -0/7, \quad 0/0.$$

Since none of these ratios is positive, the problem is unbounded.

In the next chapter, we will investigate what happens when some of these ratios take the value $+\infty$.

5. Geometry

When the number of variables in a linear programming problem is three or less, we can graph the set of feasible solutions together with the level sets of the objective function. From this picture, it is usually a trivial matter to write down the optimal solution. To illustrate, consider the following problem:

$$\begin{array}{lrcl} \text{maximize} & 3x_1 + 2x_2 & & \\ \text{subject to} & -x_1 + 3x_2 & \leq & 12 \\ & x_1 + \ \ x_2 & \leq & 8 \\ & 2x_1 - \ \ x_2 & \leq & 10 \\ & x_1, \ x_2 & \geq & 0. \end{array}$$

Each constraint (including the nonnegativity constraints on the variables) is a half-plane. These half-planes can be determined by first graphing the equation one obtains by replacing the inequality with an equality and then asking whether or not

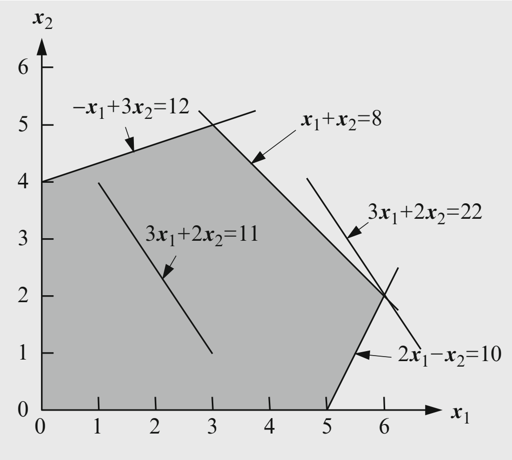

FIGURE 2.1. The set of feasible solutions together with level sets of the objective function.

some specific point that doesn't satisfy the equality (often $(0,0)$ can be used) satisfies the inequality constraint. The set of feasible solutions is just the intersection of these half-planes. For the problem given above, this set is shown in Figure 2.1. Also shown are two level sets of the objective function. One of them indicates points at which the objective function value is 11. This level set passes through the middle of the set of feasible solutions. As the objective function value increases, the corresponding level set moves to the right. The level set corresponding to the case where the objective function equals 22 is the last level set that touches the set of feasible solutions. Clearly, this is the maximum value of the objective function. The optimal solution is the intersection of this level set with the set of feasible solutions. Hence, we see from Figure 2.1 that the optimal solution is $(x_1, x_2) = (6, 2)$.

Exercises

Solve the following linear programming problems. If you wish, you may check your arithmetic by using the simple online pivot tool:

www.princeton.edu/~rvdb/JAVA/pivot/simple.html

2.1

$$\begin{array}{ll} \text{maximize} & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ \text{subject to} & 2x_1 + x_2 + x_3 + 3x_4 \leq 5 \\ & x_1 + 3x_2 + x_3 + 2x_4 \leq 3 \\ & x_1,\ x_2,\ x_3,\ x_4 \geq 0\,. \end{array}$$

2.2

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 4 \\ & 2x_1 + 3x_2 \leq 3 \\ & 4x_1 + x_2 \leq 5 \\ & x_1 + 5x_2 \leq 1 \\ & x_1,\ x_2 \geq 0\,. \end{array}$$

2.3

$$\begin{array}{ll} \text{maximize} & 2x_1 - 6x_2 \\ \text{subject to} & -x_1 - x_2 - x_3 \leq -2 \\ & 2x_1 - x_2 + x_3 \leq 1 \\ & x_1,\ x_2,\ x_3 \geq 0\,. \end{array}$$

2.4

$$\begin{array}{ll} \text{maximize} & -x_1 - 3x_2 - x_3 \\ \text{subject to} & 2x_1 - 5x_2 + x_3 \leq -5 \\ & 2x_1 - x_2 + 2x_3 \leq 4 \\ & x_1,\ x_2,\ x_3 \geq 0\,. \end{array}$$

2.5

$$\begin{array}{ll} \text{maximize} & x_1 + 3x_2 \\ \text{subject to} & -x_1 - x_2 \leq -3 \\ & -x_1 + x_2 \leq -1 \\ & x_1 + 2x_2 \leq 4 \\ & x_1,\ x_2 \geq 0\,. \end{array}$$

2.6

$$\begin{array}{lrcr} \text{maximize} & x_1 + 3x_2 & & \\ \text{subject to} & -x_1 - x_2 & \leq & -3 \\ & -x_1 + x_2 & \leq & -1 \\ & x_1 + 2x_2 & \leq & 2 \\ & x_1,\ x_2 & \geq & 0\,. \end{array}$$

2.7

$$\begin{array}{lrcr} \text{maximize} & x_1 + 3x_2 & & \\ \text{subject to} & -x_1 - x_2 & \leq & -3 \\ & -x_1 + x_2 & \leq & -1 \\ & -x_1 + 2x_2 & \leq & 2 \\ & x_1,\ x_2 & \geq & 0\,. \end{array}$$

2.8

$$\begin{array}{lrcr} \text{maximize} & 3x_1 + 2x_2 & & \\ \text{subject to} & x_1 - 2x_2 & \leq & 1 \\ & x_1 - x_2 & \leq & 2 \\ & 2x_1 - x_2 & \leq & 6 \\ & x_1 & \leq & 5 \\ & 2x_1 + x_2 & \leq & 16 \\ & x_1 + x_2 & \leq & 12 \\ & x_1 + 2x_2 & \leq & 21 \\ & x_2 & \leq & 10 \\ & x_1,\ x_2 & \geq & 0\,. \end{array}$$

2.9

$$\begin{array}{lrcr} \text{maximize} & 2x_1 + 3x_2 + 4x_3 & & \\ \text{subject to} & - 2x_2 - 3x_3 & \geq & -5 \\ & x_1 + x_2 + 2x_3 & \leq & 4 \\ & x_1 + 2x_2 + 3x_3 & \leq & 7 \\ & x_1,\ x_2,\ x_3 & \geq & 0\,. \end{array}$$

2.10

$$\begin{array}{lrcr} \text{maximize} & 6x_1 + 8x_2 + 5x_3 + 9x_4 & & \\ \text{subject to} & x_1 + x_2 + x_3 + x_4 & = & 1 \\ & x_1,\ x_2,\ x_3,\ x_4 & \geq & 0\,. \end{array}$$

2.11

$$\begin{array}{lrcr} \text{minimize} & x_{12} + 8x_{13} + 9x_{14} + 2x_{23} + 7x_{24} + 3x_{34} & & \\ \text{subject to} & x_{12} + x_{13} + x_{14} & \geq & 1 \\ & -x_{12} + x_{23} + x_{24} & = & 0 \\ & -x_{13} - x_{23} + x_{34} & = & 0 \\ & x_{14} + x_{24} + x_{34} & \leq & 1 \\ & x_{12}, x_{13}, \ldots, x_{34} & \geq & 0\,. \end{array}$$

2.12 Using today's date (MMYY) for the seed value, solve 10 initially feasible problems using the online pivot tool:

www.princeton.edu/∼rvdb/JAVA/pivot/primal.html

2.13 Using today's date (MMYY) for the seed value, solve 10 not necessarily feasible problems using the online pivot tool:

www.princeton.edu/∼rvdb/JAVA/pivot/primal_x0.html

2.14 Consider the following dictionary:

$$\begin{array}{rcl} \zeta &=& 3 + x_1 + 6x_2 \\ \hline w_1 &=& 1 + x_1 - x_2 \\ w_2 &=& 5 - 2x_1 - 3x_2 \,. \end{array}$$

Using the largest coefficient rule to pick the entering variable, compute the dictionary that results after *one pivot*.

2.15 Show that the following dictionary cannot be the optimal dictionary for any linear programming problem in which w_1 and w_2 are the initial slack variables:

$$\begin{array}{rcl} \zeta &=& 4 - w_1 - 2x_2 \\ \hline x_1 &=& 3 \phantom{{}- w_1} - 2x_2 \\ w_2 &=& 1 + w_1 - x_2 \,. \end{array}$$

Hint: if it could, what was the original problem from whence it came?

2.16 Graph the feasible region for Exercise 2.8. Indicate on the graph the sequence of basic solutions produced by the simplex method.

2.17 Give an example showing that the variable that becomes basic in one iteration of the simplex method can become nonbasic in the next iteration.

2.18 Show that the variable that becomes nonbasic in one iteration of the simplex method cannot become basic in the next iteration.

2.19 Solve the following linear programming problem:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{j=1}^{n} p_j x_j \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} q_j x_j \le \beta \\ & \qquad x_j \le 1 \qquad j = 1, 2, \ldots, n \\ & \qquad x_j \ge 0 \qquad j = 1, 2, \ldots, n. \end{array}$$

Here, the numbers p_j, $j = 1, 2, \ldots, n$, are positive and sum to one. The same is true of the q_j's:

$$\sum_{j=1}^{n} q_j = 1$$
$$q_j > 0.$$

Furthermore (with only minor loss of generality), you may assume that

$$\frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_n}{q_n}.$$

Finally, the parameter β is a small positive number. See Exercise 1.3 for the motivation for this problem.

Notes

The simplex method was invented by G.B. Dantzig in 1949. His monograph (Dantzig 1963) is the classical reference. Most texts describe the simplex method as a sequence of pivots on a table of numbers called the *simplex tableau*. Following Chvátal (1983), we have developed the algorithm using the more memorable dictionary notation.

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