On Representation of an Integer by $X^2 + Y^2 + Z^2$ and the Modular Equations of Degree 3 and 5

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There are always flowers for those who want to see them

Abstract I discuss a variety of results involving $s(n)$, the number of representations of n as a sum of three squares. One of my objectives is to reveal numerous interesting connections between the properties of this function and certain modular equations of degree 3 and 5. In particular, I show that

$$s(25n) = (6 - (-n|5))\, s(n) - 5s\left(\frac{n}{25}\right)$$

follows easily from the well known Ramanujan modular equation of degree 5. Moreover, I establish new relations between $s(n)$ and $h(n)$, $g(n)$, the number of representations of n by the ternary quadratic forms

$$2x^2 + 2y^2 + 2z^2 - yz + zx + xy, \quad x^2 + y^2 + 3z^2 + xy,$$

respectively.

Finally, I propose a remarkable new identity for $s(p^2n) - ps(n)$ with p being an odd prime. This identity makes nontrivial use of the ternary quadratic forms with discriminants p^2, $16p^2$.

Key words and Phrases Ternary quadratic forms • Sum of three squares • Modular equations • θ-function identities

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1 Introduction

Let $(a, b, c, d, e, f)(n)$ denote the number of representations of n by the ternary form $ax^2 + by^2 + cz^2 + dyz + ezx + fxy$. I will assume that $(a, b, c, d, e, f)(n) = 0$, whenever $n \notin Z$. Let $s(n)$ denote the number of representations of n by ternary form $x^2+y^2+z^2$. In [14], Hirschhorn and Sellers proved in a completely elementary manner that

$$s(p^2n) = (p + 1 - (-n|p))\, s(n) - ps\left(\frac{n}{p^2}\right), \tag{1.1}$$

when $p = 3$. Here $(a|p)$ denotes the Legendre symbol. It should be pointed out that the authors of [14] proved (1.1) for all odd prime numbers p by an appeal to the theory of modular forms.

In Sect. 2, I will show that (1.1) with $p = 5$ follows easily from the well-known identity for $\phi(q)^2 - \phi(q^5)^2$ with

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{1.2}$$

Here and throughout, q is a complex number with $|q| < 1$. I will also provide an elementary proof of the following

Theorem 1.1. *If* $n \equiv 1, 2$ *mod* 4*, then*

$$s(25n) - 5s(n) = 4(2, 2, 2, -1, 1, 1)(n), \tag{1.3}$$

and

Theorem 1.2. *If* $n \equiv 1, 2$ *mod* 4*, then*

$$s(9n) - 3s(n) = 2(1, 1, 3, 0, 0, 1)(n). \tag{1.4}$$

In Sect. 5, I will show how to remove the parity restrictions in the above theorems by proving Theorems 5.2 and 5.3. Section 6 contains my new Proposition 6.1, which generalizes Theorems 1.1, 1.2, 5.2 and 5.3. A reader with no vested interest in q-series may want to proceed directly to Sect. 6. However, a motivated reader may decide to walk slowly through the initial sections to experience suffering which will later turn into joy.

Let me point out that two ternary forms $2x^2 + 2y^2 + 2z^2 - yz + zx + xy$ and $x^2 + y^2 + 3z^2 + xy$ both have class number one. This implies that these forms are both regular [11, 16, 17]. For a recent discussion of the relation between the Ramanujan modular equations and certain ternary quadratic forms the reader is invited to examine [2]. And it goes without saying that one should not forget the timeless classic [1].

I begin by recalling some standard notations, definitions, and useful formulas.

$$(a;q)_\infty := \prod_{j\geq 0}(1-aq^j), \tag{1.5}$$

and

$$E(q) := \prod_{j\geq 1}(1-q^j). \tag{1.6}$$

Note that

$$E(-q) = \frac{E(q^2)^3}{E(q^4)E(q)}, \tag{1.7}$$

Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{(n-1)n}{2}} b^{\frac{(n+1)n}{2}}, \quad |ab|<1. \tag{1.8}$$

In Ramanujan's notation, the celebrated Jacobi triple product identity takes the shape [5], p. 35

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab|<1. \tag{1.9}$$

Note that $\phi(q)$ can be interpreted as

$$\phi(q) = f(q,q) = \frac{E(q^2)^5}{E(q^4)^2E(q)^2}, \tag{1.10}$$

where the product on the right follows easily from (1.8). We shall also require

$$\phi(-q) = \frac{E(q)^2}{E(q^2)}. \tag{1.11}$$

Next we define

$$\psi(q) = f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{2n^2+n}. \tag{1.12}$$

It is not hard to check that

$$\psi(q) = \frac{1}{2}f(1,q) = \sum_{n\geq 0} q^{\frac{(n+1)n}{2}} = \frac{E(q^2)^2}{E(q)}, \tag{1.13}$$

$$\sum_{n=-\infty}^{\infty} q^{(4n+1)^2} = \sum_{n=-\infty}^{\infty} q^{(4n+3)^2} = q\psi(q^8), \tag{1.14}$$

and that

$$f(q,q^9)f(q^3,q^7) = \frac{E(q^{20})E(q^5)E(q^2)^2}{E(q^4)E(q)}, \tag{1.15}$$

$$f(q,q^4)f(q^2,q^3) = \frac{E(q^5)^3E(q^2)}{E(q^{10})E(q)}. \tag{1.16}$$

The function $f(a,b)$ may be dissected in many different ways. We will use the following trivial dissections [5], pp. 40, 49

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \tag{1.17}$$

$$\phi(q) = \phi(q^9) + 2qf(q^3,q^{15}), \tag{1.18}$$

$$\phi(q) = \phi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4f(q^5,q^{45}). \tag{1.19}$$

We will also require a special case of Schröter's formula [5], p. 45

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af\left(\frac{b}{c},ac^2d\right)f\left(\frac{b}{d},acd^2\right), \tag{1.20}$$

provided $ab = cd$. Setting $a = b = c = d = q$ in (1.20) we obtain

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2. \tag{1.21}$$

Iterating, we find that

$$\phi(q)^2 = \phi(q^4)^2 + 4q\psi(q^4)^2 + 4q^2\psi(q^8)^2. \tag{1.22}$$

Next, we set $a = q, b = q^9, c = q^3, d = q^7$ in (1.20) and square the result. This way we have

$$\begin{aligned} f(q,q^9)^2f(q^3,q^7)^2 &= f(q^4,q^{16})^2f(q^8,q^{12})^2 \\ &+2qf(q^4,q^{16})f(q^8,q^{12})f(q^6,q^{14})f(q^2,q^{18}) + q^2f(q^6,q^{14})^2f(q^2,q^{18})^2. \end{aligned} \tag{1.23}$$

Finally, we multiply both sides in (1.23) by

$$\frac{E(q^4)\phi(q^5)}{E(q^{20})E(q^{10})^2},$$

and use (1.10), (1.13), (1.15) and (1.16) to arrive at

$$\begin{aligned} \phi(q)f(q^2,q^8)f(q^4,q^6) &= \psi(q^4)\phi(q^5)\phi(q^{10}) \\ &+ 2q\psi(q^2)\psi(q^{10})\phi(q^5) + q^2\psi(q^{20})\phi(q^2)\phi(q^5). \end{aligned} \tag{1.24}$$

This result will come in handy in my proof of (1.3) with $n \equiv 2 \bmod 4$. To deal with the case $n \equiv 1 \bmod 4$ in (1.3) I will require another identity

$$\phi(q)\phi(q^5) + \sum_{m,n} q^{2m^2+2nm+3n^2} = 2\Pi_1(q), \tag{1.25}$$

where

$$\Pi_1(q) = \frac{E(q^{10})E(q^5)E(q^4)E(q^2)}{E(q^{20})E(q)}. \tag{1.26}$$

This formula was discovered and proven in [4]. The proof of (1.25), given in [4], used only a special case of the Ramanujan ${}_1\psi_1$ summation formula [6], p. 64. Multiplying both sides in (1.25) by $\psi(q^{10})$ and utilizing (1.13) and (1.15) we can rewrite (1.25) as

$$\psi(q^{10})\phi(q)\phi(q^5)+\psi(q^{10})\sum_{m,n} q^{2m^2+2nm+3n^2} = 2\psi(q^2)f(q,q^9)f(q^3,q^7). \tag{1.27}$$

2 The Ternary Implications of the Fundamental Modular Equation of Degree 5

In this section we will make an extensive use of a well-known modular equation of degree 5

$$\phi(q)^2 - \phi(q^5)^2 = 4qf(q,q^9)f(q^3,q^7) \tag{2.1}$$

to prove (1.1) with $p = 5$. We note that (2.1) has an attractive companion

$$5\phi(q^5)^2 - \phi(q)^2 = 4\Pi_2(q), \tag{2.2}$$

where

$$\Pi_2(q) = \frac{E(q^{10})^2E(q^4)E(q)}{E(q^{20})E(q^5)}. \tag{2.3}$$

Both (2.1) and (2.2) are discussed in [5]. We remark that the right hand side of (2.1) was interpreted in terms of so-called self-conjugate 5-cores in [12]. To proceed further I will need a sifting operator $S_{t,s}$. It is defined by its action on power series as follows

$$S_{t,s}\sum_{n\geq 0} c(n)q^n = \sum_{k\geq 0} c(tk+s)q^k. \tag{2.4}$$

Here t, s are integers such that $0 \leq s < t$. Making use of (1.19), we find that

$$S_{5,0}\phi(q)^2 = \phi(q^5)^2 + 8qf(q,q^9)f(q^3,q^7). \tag{2.5}$$

And so

$$S_{5,0}(\phi(q)^2 - \phi(q^5)^2) = -(\phi(q)^2 - \phi(q^5)^2) + 8qf(q,q^9)f(q^3,q^7). \tag{2.6}$$

Employing (2.1) twice, we see that

$$S_{5,0}(qf(q,q^9)f(q^3,q^7)) = qf(q,q^9)f(q^3,q^7). \tag{2.7}$$

Analogously, we can check that

$$S_{5,0}\phi(q)^3 = \phi(q^5)^3 + 24q\phi(q^5)f(q,q^9)f(q^3,q^7), \tag{2.8}$$

and that

$$S_{5,1}\phi(q)^3 = 6f(q^3,q^7)(\phi(q^5)^2 + 4qf(q,q^9)f(q^3,q^7)) = 6f(q^3,q^7)\phi(q)^2, \tag{2.9}$$

$$S_{5,4}\phi(q)^3 = 6f(q,q^9)(\phi(q^5)^2 + 4qf(q,q^9)f(q^3,q^7)) = 6f(q,q^9)\phi(q)^2. \tag{2.10}$$

We note, in passing, that thanks to (1.9), the right hand side in (2.9) can be rewritten as an infinite product

$$\sum_{n=0}^{\infty} s(5n+1)q^n = 6\prod_{j=1}^{\infty}(1-q^{2j})^2(1-q^{10j})$$
$$(1+q^{-1+2j})^4(1+q^{-3+10j})(1+q^{-7+10j}).$$

Cooper and Hirschhorn studied the generating functions of subsequences of $s(n)$ that could be represented by a single, simple infinite product. For example, (2.9), (2.10) and (4.17) are the formulas (3.1), (3.2) and (1.1) in [10].

With the aid of (1.19) we can combine (2.9) and (2.10) into a single elegant statement

$$S_{5,r}(\phi(q)^3 - 3\phi(q)\phi(q^5)^2) = 0, \tag{2.11}$$

where $r = 1, 4$. Next, we apply $S_{5,0}$ to both sides of (2.8) to obtain, with a little help from (2.7)

$$S_{25,0}\phi(q)^3 = \phi(q)^3 + 24q\phi(q)f(q,q^9)f(q^3,q^7). \tag{2.12}$$

Subtracting $5\phi(q)^3$ and making use of (2.1) again, we deduce that

$$\begin{aligned} S_{25,0}\phi(q)^3 - 5\phi(q)^3 &= -4\phi(q)^3 + 6\phi(q)(\phi(q)^2 - \phi(q^5)^2) \\ &= 2(\phi(q)^3 - 3\phi(q)\phi(q^5)^2). \end{aligned} \tag{2.13}$$

Finally, we apply $S_{5,r}$ with $r = 1, 4$ to both sides of (2.13) to find that

$$S_{125,25r}\phi(q)^3 - 5S_{5,r}\phi(q)^3 = 0. \tag{2.14}$$

But it is plain that

$$\phi(q)^3 = \sum_{n=0}^{\infty} s(n)q^n. \tag{2.15}$$

And so the equation (2.14) can be interpreted as

$$s(25n) - 5s(n) = 0, \tag{2.16}$$

when $n \equiv 1, 4 \bmod 5$. Thus, the proof of (1.1) with $p = 5$ and $n \equiv 1, 4 \bmod 5$ is complete.

We now turn our attention to the $n \equiv 2, 3 \bmod 5$ case. Subtracting $2\phi(q)^3$ from the extremes of (2.13), we end up with the formula

$$S_{25,0}\phi(q)^3 - 7\phi(q)^3 = -6\phi(q)\phi(q^5)^2. \tag{2.17}$$

It is now clear that for $r = 2, 3$

$$S_{5,r}(S_{25,0}\phi(q)^3 - 7\phi(q)^3) = -6\phi(q)^2 S_{5,r}\phi(q) = 0, \tag{2.18}$$

where in the last step we took advantage of the dissection formula (1.19). Obviously, (2.18) is equivalent to

$$s(25n) - 7s(n) = 0, \tag{2.19}$$

when $n \equiv 2, 3 \bmod 5$. And so we completed the proof of (1.1) with $p = 5$ and $n \equiv 2, 3 \bmod 5$. All that remains to do is to take care of the $n \equiv 0 \bmod 5$ case. Adding $\phi(q)^3$ to both sides of (2.17) and applying $S_{5,0}$ to the result, we get

$$S_{5,0}(S_{25,0}\phi(q)^3 - 6\phi(q)^3) = S_{5,0}(\phi(q)^3 - 6\phi(q)\phi(q^5)^2). \tag{2.20}$$

Next, we utilize (1.19), (2.1) and (2.8) to process the right hand side of (2.20) as follows

$$\begin{aligned} S_{5,0}(\phi(q)^3 - 6\phi(q)\phi(q^5)^2) &= \phi(q^5)^3 + 6\phi(q^5)(\phi(q)^2 - \phi(q^5)^2) - 6\phi(q^5)\phi(q)^2 \\ &= -5\phi(q^5)^3. \end{aligned}$$

Hence, we have shown that

$$S_{125,0}\phi(q)^3 - 6S_{5,0}\phi(q)^3 = -5\phi(q^5)^3. \tag{2.21}$$

Consequently,

$$s(25n) - 6s(n) = -5s\left(\frac{n}{25}\right), \tag{2.22}$$

when $5|n$. This concludes our proof of (1.1) with $p = 5$.

3 Proof of Theorem 1.1

I begin by observing that Theorem 1.1 is equivalent to the following statement

$$S_{100,25r}\phi(q)^3 - 5S_{4,r}\phi(q)^3 = 4S_{4,r}T(q), \tag{3.1}$$

where

$$T(q) := \sum_{x,y,z} q^{2x^2+2y^2+2z^2-yz+zx+xy} \tag{3.2}$$

and $r = 1, 2$. It is not hard to verify that

$$S_{4,1}T(q) = 6S_{4,1}X(1,q), \tag{3.3}$$

and that

$$S_{4,2}T(q) = 3S_{4,2}(X(0,q) + X(2,q)). \tag{3.4}$$

Here

$$X(r,q) := \sum_{\substack{x,\\ y\equiv -z\equiv r \bmod 4}} q^{2x^2+2y^2+2z^2-yz+zx+xy}. \tag{3.5}$$

It takes very little effort to check that

$$2x^2+2y^2+2z^2-zy+zx+xy = 2\left(x+\frac{y+z}{4}\right)^2+\frac{5}{8}(y+z)^2+\frac{5}{4}(y-z)^2. \tag{3.6}$$

Hence

$$\begin{aligned} X(r,q) &= \sum_{\substack{x,\\ y\equiv -z\equiv r \bmod 4}} q^{2\left(x+\frac{y+z}{4}\right)^2+10\left(\frac{y+z}{4}\right)^2+20\left(\frac{y-z}{4}\right)^2} \\ &= \sum_{\substack{u,\\ w\equiv v+\frac{r}{2} \bmod 2}} q^{2u^2+10v^2+20w^2}, \end{aligned} \tag{3.7}$$

for $r = 0, 2$. It is now evident that

$$X(0,q) + X(2,q) = \sum_{u,v,w} q^{2u^2+10v^2+20w^2} = \phi(q^2)\phi(q^{10})\phi(q^{20}). \tag{3.8}$$

Using this last result in (3.4), we find that

$$S_{4,2}T(q) = 3\phi(q^5)S_{4,2}(\phi(q^2)\phi(q^{10})). \tag{3.9}$$

Recalling (1.17), we obtain at once that

$$4S_{4,2}T(q) = 24\phi(q^5)(\psi(q^4)\phi(q^{10}) + 6q^2\phi(q^2)\psi(q^{20})). \tag{3.10}$$

We now consider $X(r,q)$ with $r=1,3$.

$$X(r,q)=\sum_{\substack{u,\\ v\equiv w \bmod 2}} q^{2u^2+10v^2+5(2w+r)^2}.$$

Recalling (1.14), we get

$$X(1,q)=X(3,q)=\sum_{u,v,\tilde{w}} q^{2n^2+10v^2+5(4\tilde{w}+1)^2}=q^5\phi(q^2)\phi(q^{10})\psi(q^{40}). \quad (3.11)$$

Using (1.17), (3.3) and (3.11), we deduce that

$$S_{4,1}T(q)=6q\psi(q^{10})S_{4,0}(\phi(q^2)\phi(q^{10}))=6q\psi(q^{10})(\phi(q^2)\phi(q^{10})+4q^3\psi(q^4)\psi(q^{20})).$$

Also, it is not hard to check that

$$\begin{aligned}\sum_{m,n} q^{2m^2+2nm+3n^2} &= \sum_{m,n} q^{2(m+n)^2+10n^2}+q^3\sum_{m,n} q^{2(m+n+1)(m+n)+10(n+1)n}\\ &= \phi(q^2)\phi(q^{10})+4q^3\psi(q^4)\psi(q^{20}). \end{aligned} \quad (3.12)$$

This implies that

$$4S_{4,1}T(q)=24q\psi(q^{10})\sum_{m,n} q^{2m^2+2nm+3n^2}. \quad (3.13)$$

Next, we employ (2.13) to get

$$S_{100,25r}\phi(q)^3-5S_{4,r}\phi(q)^3=2S_{4,r}(\phi(q)^3-3\phi(q)\phi(q^5)^2). \quad (3.14)$$

With the aid of (1.17), (1.22), (2.1) and (2.2) we verify that

$$S_{4,1}(\phi(q)^3-3\phi(q)\phi(q^5)^2)=24q\psi(q^2)f(q,q^9)f(q^3,q^7)-12q\phi(q)\phi(q^5)\psi(q^{10}), \quad (3.15)$$

$$S_{4,2}(\phi(q)^3-3\phi(q)\phi(q^5)^2)=-24q\psi(q^2)\psi(q^5)^2+12\phi(q)f(q^2,q^8)f(q^4,q^6). \quad (3.16)$$

Utilizing these results in (3.14) we obtain

$$S_{100,25}\phi(q)^3-5S_{4,1}\phi(q)^3=48q\psi(q^2)f(q,q^9)f(q^3,q^7)-24q\phi(q)\phi(q^5)\psi(q^{10}), \quad (3.17)$$

$$S_{100,50}\phi(q)^3-5S_{4,2}\phi(q)^3=-48q\psi(q^2)\psi(q^5)^2+24\phi(q)f(q^2,q^8)f(q^4,q^6). \quad (3.18)$$

Recalling (3.13), we see that (3.1) with $r=1$ is equivalent to

$$2\psi(q^2)f(q,q^9)f(q^3,q^7)-\phi(q)\phi(q^5)\psi(q^{10})=\psi(q^{10})\sum_{m,n} q^{2m^2+2nm+3n^2},$$

which is, essentially, (1.27). Analogously, employing (3.10), we find that (3.1) with $r = 2$ is equivalent to

$$-2q\psi(q^2)\psi(q^5)^2 + \phi(q)f(q^2,q^8)f(q^4,q^6) = \phi(q^5)\psi(q^4)\phi(q^{10}) + 6q^2\phi(q^2)\phi(q^5)\psi(q^{20}),$$

which is, essentially, (1.24). The proof of Theorem 1.1 is now complete.

In Sect. 5 we will generalize Theorem 1.1. To this end we need to define

$$Y(r,q) := \sum_{\substack{x,\\ y+z\equiv r \bmod 4}} q^{2x^2+2y^2+2z^2-yz+zx+xy}, \tag{3.19}$$

where $r = 0,1,2,3$. Observe that the condition $y + z \equiv r \bmod 4$ allows us to introduce new summation variables u, v, w, defined as $x = w - v$, $y = 2u + v + r$, $z = 2u - v$. Using (3.6), it is easy to see that

$$2x^2 + 2y^2 + 2z^2 - zy + zx + xy = 2r^2 + w(2w + r) + 5v(v + r) + 5u(2u + r).$$

Hence

$$Y(0,q) = \phi(q^2)\phi(q^5)\phi(q^{10}), \tag{3.20}$$

$$Y(2,q) = 4q^3\phi(q^5)\psi(q^4)\psi(q^{20}), \tag{3.21}$$

$$Y(1,q) = Y(3,q) = 2q^2\psi(q)\psi(q^5)\psi(q^{10}). \tag{3.22}$$

Employing (3.12), (3.20)–(3.22), we derive

$$T(q) = \sum_{r=0}^{3} Y(r,q) = \phi(q^5)\sum_{m,n} q^{2m^2+2nm+3n^2} + 4q^2\psi(q)\psi(q^5)\psi(q^{10}). \tag{3.23}$$

It is easy to see that

$$\sum_{\substack{x,\\ y+z\equiv 1 \bmod 2}} q^{2x^2+2y^2+2z^2-yz+zx+xy} = Y(1,q) + Y(3,q) = 4q^2\psi(q)\psi(q^5)\psi(q^{10}), \tag{3.24}$$

and that

$$\sum_{\substack{x,\\ y+z\equiv 1 \bmod 2}} q^{2x^2+2y^2+2z^2-yz+zx+xy} = 2Z(q), \tag{3.25}$$

where

$$Z(q) := \sum_{\substack{x,\\ y\equiv 0 \bmod 2,\\ z\equiv 1 \bmod 2}} q^{2x^2+2y^2+2z^2-yz+zx+xy}. \tag{3.26}$$

It is worthwhile to point out that $Z(q)$ has six equivalent representations. For example, one has

$$Z(q) := \sum_{\substack{x\equiv 0 \bmod 2,\\ y\equiv 1 \bmod 2,\\ z}} q^{2x^2+2y^2+2z^2-yz+zx+xy}.$$

From (3.24), (3.25) we deduce that

$$Z(q) = 2q^2\psi(q)\psi(q^5)\psi(q^{10}). \tag{3.27}$$

We conclude this Section that by proving that

$$\sum_{\substack{x+y\equiv 1 \bmod 2,\\ y\equiv z \bmod 2}} q^{2x^2+2y^2+2z^2-yz+zx+xy} = Z(q). \tag{3.28}$$

Indeed, the left hand side of (3.28) can be rewritten as

$$\sum_{\substack{x\equiv 0 \bmod 2,\\ y\equiv 1 \bmod 2,\\ z\equiv 1 \bmod 2}} q^{2x^2+2y^2+2z^2-yz+zx+xy} + \sum_{\substack{x\equiv 1 \bmod 2,\\ y\equiv 0 \bmod 2,\\ z\equiv 0 \bmod 2}} q^{2x^2+2y^2+2z^2-yz+zx+xy}.$$

Now observe that

$$\sum_{\substack{x\equiv 1 \bmod 2,\\ y\equiv 0 \bmod 2,\\ z\equiv 0 \bmod 2}} q^{2x^2+2y^2+2z^2-yz+zx+xy} = \sum_{\substack{x\equiv 0 \bmod 2,\\ y\equiv 1 \bmod 2,\\ z\equiv 0 \bmod 2}} q^{2x^2+2y^2+2z^2-yz+zx+xy}.$$

And so the left hand side of (3.28) becomes

$$\sum_{\substack{x\equiv 0 \bmod 2,\\ y\equiv 1 \bmod 2,\\ z}} q^{2x^2+2y^2+2z^2-yz+zx+xy} = \sum_{\substack{x,\\ y\equiv 0 \bmod 2,\\ z\equiv 1 \bmod 2}} q^{2x^2+2y^2+2z^2-yz+zx+xy} = Z(q),$$

as desired.

4 Cubic Modular Identities Revisited

As in the last section, I begin by observing that Theorem 1.2 is equivalent to the following statement

$$S_{36,9r}\phi(q)^3 - 3S_{4,r}\phi(q)^3 = 4S_{4,r}\phi(q^3)a(q), \tag{4.1}$$

where

$$a(q) := \sum_{x,y} q^{x^2+xy+y^2},$$

and $r = 1, 2$. The function $a(q)$ was extensively studied in the literature [7–9, 13]. It appeared in Borwein's cubic analogue of Jacobi's celebrated theta function identity [8]. I will record below some useful formulas

$$4a(q^2)\phi(q^3) = \phi(q)^3 + 3\tfrac{\phi(q^3)^4}{\phi(q)}, \tag{4.2}$$

$$a(q) = a(q^3) + 6q\tfrac{E(q^9)^3}{E(q^3)}, \tag{4.3}$$

$$a(q) = \phi(q)\phi(q^3) + 4q\psi(q^2)\psi(q^6), \tag{4.4}$$

$$a(q) = 2\phi(q)\phi(q^3) - \phi(-q)\phi(-q^3), \tag{4.5}$$

$$2a(q^2) - a(q) = \tfrac{\phi(-q)^3}{\phi(-q^3)} \tag{4.6}$$

$$a(q) = a(q^4) + 6q\psi(q^2)\psi(q^6). \tag{4.7}$$

Formula (4.2) appears as equation (6.4) in [7]. Identities (4.3)–(4.6) are discussed in [9]. In order to prove (4.7), the authors of [13] have shown that

$$2q\psi(q^2)\psi(q^6) = \sum_{u\not\equiv v \bmod 2} q^{u^2+3v^2}. \tag{4.8}$$

We have at once that

$$\begin{aligned} 2q\psi(q^2)\psi(q^6) &= \sum_{\substack{u\equiv 1 \bmod 2,\\ v\equiv 0 \bmod 2}} q^{u^2+3v^2} + \sum_{\substack{u\equiv 0 \bmod 2,\\ v\equiv 1 \bmod 2}} q^{u^2+3v^2} \\ &= 2q\psi(q^8)\phi(q^{12}) + 2q^3\phi(q^4)\psi(q^{24}). \end{aligned} \tag{4.9}$$

Combining (4.7) and (4.9), we have a pretty neat dissection of $a(q)$ mod 4

$$a(q) = a(q^4) + 6q\psi(q^8)\phi(q^{12}) + 6q^3\phi(q^4)\psi(q^{24}). \tag{4.10}$$

In [19], L.C. Shen discussed two well-known modular identities of degree 3

$$\phi(q)^2 - \phi(q^3)^2 = 4q\frac{\psi(q)\psi(q^3)\psi(q^6)}{\psi(q^2)}, \tag{4.11}$$

and

$$\phi(q)^2 + \phi(q^3)^2 = 2\frac{\psi(q)f(q,q^2)f(q^2,q^4)}{\psi(q^2)}. \tag{4.12}$$

Multiplying (4.11) and (4.12), and using

$$f(q, q^2) = \frac{E(q^3)^2 E(q^2)}{E(q^6) E(q)}, \tag{4.13}$$

$$f(q, q^5) = \frac{E(q^{12}) E(q^3) E(q^2)^2}{E(q^6) E(q^4) E(q)} \tag{4.14}$$

together with (1.13) we have

$$\phi(q)^4 - \phi(q^3)^4 = 8q\phi(q^3) f(q, q^5)^3. \tag{4.15}$$

Next, we rewrite (4.15) as

$$\frac{\phi(q)^4}{\phi(q^3)} = \phi(q^3)^3 + 8q f(q, q^5)^3. \tag{4.16}$$

Recalling (1.18), we can recognize the expression on the right as

$$\phi(q^3)^3 + 8q f(q, q^5)^3 = S_{3,0}(\phi(q^9) + 2q f(q^3, q^{15}))^3 = S_{3,0}\phi(q)^3.$$

And so

$$S_{3,0}\phi(q)^3 = \frac{\phi(q)^4}{\phi(q^3)}. \tag{4.17}$$

Next, we want to show that

$$S_{9,0}\phi(q)^3 = \frac{4\phi(q)^4 - 3\phi(q^3)^4}{\phi(q)}. \tag{4.18}$$

To this end, we apply $S_{3,0}$ to both sides of (4.17). Utilizing (1.18), we find that

$$S_{9,0}\phi(q)^3 = \frac{\phi(q^3)^4 + 4(8q\phi(q^3) f(q, q^5)^3)}{\phi(q)}. \tag{4.19}$$

The statement in (4.18) follows immediately from (4.15) and (4.19). Moreover, we have

$$S_{9,0}\phi(q)^3 - 5\phi(q)^3 = -\phi(q)^3 - 3\frac{\phi(q^3)^4}{\phi(q)} = -4a(q^2)\phi(q^3), \tag{4.20}$$

where we used (4.2) in the last step. Adding $2\phi(q)^3$ to the extremes in (4.20) we derive

$$S_{9,0}\phi(q)^3 - 3\phi(q)^3 = 2\phi(q)^3 - 4a(q^2)\phi(q^3). \tag{4.21}$$

This result will come in handy in my proof of Theorem 5.2 in the next section.

5 Proof of Theorems 1.2, 5.2 and 5.3

I begin this section by providing an easy proof of two formulas in (4.1). All I need is the following

Lemma 5.1. *If* $r = 1, 2$, *then*

$$S_{4,r}(\phi(q)^3 - 2a(q^2)\phi(q^3)) = S_{4,r}(a(q)\phi(q^3)). \tag{5.1}$$

Proof: This lemma is a straightforward corollary of (1.17), (4.7) and (4.10). Next, we apply $S_{4,r}$ with $r = 1, 2$ to (4.21) and use (5.1) to obtain

$$S_{36,9r}\phi(q)^3 - 3S_{4,r}\phi(q)^3 = 2S_{4,r}(\phi(q)^3 - 2\phi(q^3)a(q^2)) = 2S_{4,r}(a(q)\phi(q^3)), \tag{5.2}$$

which is (4.1), as desired. The proof of Theorem 1.2 is now complete. We can do much better, if we realize that (5.1) is an immediate consequence of the following elegant result

$$\phi(q)^3 = \phi(q^3)(a(q) + 2a(q^2) - 2a(q^4)). \tag{5.3}$$

To prove it, we divide both sides by $\phi(q^3)$ and obtain

$$\frac{\phi(q)^3}{\phi(q^3)} = 2a(q^2) - a(q) + 2(a(q) - a(q^4)). \tag{5.4}$$

Using (4.6) and (4.7) in (5.4), we see that (5.3) is equivalent to

$$\frac{\phi(q)^3}{\phi(q^3)} - \frac{\phi(-q)^3}{\phi(-q^3)} = 12q\psi(q^2)\psi(q^6). \tag{5.5}$$

To verify (5.5), I replace q by $-q$ in (4.6) and subtract (4.6) to find with the aid of (4.5) the following

$$\frac{\phi(q)^3}{\phi(q^3)} - \frac{\phi(-q)^3}{\phi(-q^3)} = a(q) - a(-q) = 3(\phi(q)\phi(q^3) - \phi(-q)\phi(-q^3)). \tag{5.6}$$

Subtracting (4.4) from (4.5) we obtain

$$\phi(q)\phi(q^3) - \phi(-q)\phi(-q^3) = 4q\psi(q^2)\psi(q^6). \tag{5.7}$$

Hence,

$$\frac{\phi(q)^3}{\phi(q^3)} - \frac{\phi(-q)^3}{\phi(-q^3)} = 12q\psi(q^2)\psi(q^6), \tag{5.8}$$

as desired. This completes the proof of (5.3). We are now in a position to improve on (5.2). Indeed, it follows from (4.21) and (5.3) that

$$S_{9,0}\phi(q)^3 - 3\phi(q)^3 = 2\phi(q^3)a(q) - 4\phi(q^3)a(q^4). \tag{5.9}$$

Consequently, we can extend Theorem 1.2 as

Theorem 5.2.

$$s(9n) - 3s(n) = 2(1,1,3,0,0,1)(n) - 4(4,3,4,0,4,0)(n). \tag{5.10}$$

It is worthwhile to point out that Theorem 1.1 can be extended in a similar manner as

Theorem 5.3.

$$s(25n) - 5s(n) = 4(2,2,2,-1,1,1)(n) - 8(7,8,8,-4,8,8)(n). \tag{5.11}$$

It is easy to check that $(7,8,8,-4,8,8)(n) = 0$ when $n \equiv 1,2 \bmod 4$. And so (5.11) reduces to (1.3) when $n \equiv 1,2 \bmod 4$. Recalling (2.13), we see that all that is required to prove Theorem 5.3 is

$$\phi(q)^3 - 3\phi(q)\phi(q^5)^2 = 2T(q) - 4\tilde{T}(q), \tag{5.12}$$

where $T(q)$ was defined in (3.2), and

$$\tilde{T}(q) := \sum_{x,y,z} q^{7x^2+8y^2+8z^2-4yz+8zx+8xy}. \tag{5.13}$$

Making easy changes of summation variables $y \to x+y$ and $z \to x+z$ in (3.2) we find that

$$T(q) = \sum_{x,y,z} q^{7x^2+2y^2+2z^2-yz+4zx+4xy}. \tag{5.14}$$

In a similar fashion one can prove that

$$\tilde{T}(q) = \sum_{x \equiv y \equiv z \bmod 2} q^{2x^2+2y^2+2z^2-yz+zx+xy}. \tag{5.15}$$

Combining (3.2), (3.25), (3.27), (3.28) and (5.15), we can easily derive that

$$T(q) - \tilde{T}(q) = 2Z(q) + Z(q) = 6q^2\psi(q)\psi(q^5)\psi(q^{10}). \tag{5.16}$$

Hence we can rewite the right hand side of (5.12) as

$$2T(q) - 4\tilde{T}(q) = 24q^2\psi(q)\psi(q^5)\psi(q^{10}) - 2T(q).$$

Recalling (3.23), we see that (5.12) is equivalent to

$$\phi(q)^3 - 3\phi(q)\phi(q^5)^2 = 16q^2\psi(q)\psi(q^5)\psi(q^{10}) - 2\phi(q^5)\sum_{m,n} q^{2m^2+2nm+3n^2}. \tag{5.17}$$

To prove the above identity we subtract $2\phi(q)\phi(q^5)^2$ from both sides and use (1.25), (2.2) to find that

$$\phi(q)\Pi_2(q) = \phi(q^5)\Pi_1(q) - 4q^2\psi(q)\psi(q^5)\psi(q^{10}). \tag{5.18}$$

Next, we multiply both sides of (5.18) by

$$\frac{E(q^{20})E(q^5)E(q)}{E(q^{10})^2E(q^4)E(q^2)},$$

and use (1.11) to end up with

$$\phi(-q^2)^2 - \phi(-q^{10})^2 = -4q^2\frac{E(q^{20})^3E(q^2)}{E(q^{10})E(q^4)}.$$

Finally, replacing q^2 by q in the above, we deduce that (5.12) is equivalent to

$$\phi(-q)^2 - \phi(-q^5)^2 = -4q\frac{E(q^{10})^3E(q)}{E(q^5)E(q^2)}.$$

Employing (1.7) and (1.15), we see that the last identity is nothing else but (2.1) with q replaced by $-q$. Hence (5.12) is true. This completes my proof of the Theorem 5.3.

6 Bold Proposition

I now proceed to describe the generalization of Theorem 1.2 for any odd prime p. Observe that the ternary quadratic form $x^2 + y^2 + 3z^2 + xy$ in this theorem has the discriminants 3^2. We remind the reader that a discriminant of a ternary form $ax^2 + by^2 + cz^2 + dyz + ezx + fxy$ is defined as

$$\frac{1}{2}\det\begin{bmatrix} 2a & f & e \\ f & 2b & d \\ e & d & 2c \end{bmatrix}.$$

Using [18] it is easy to check that all ternary forms with the discriminant p^2 belong to the same genus, say $TG_{1,p}$. Let $|\text{Aut}(f)|$ denote the number of integral automorphs of a ternary quadratic form f, and let $R_f(n)$ denote the number of representations of n by f. Let p be an odd prime and $n \not\equiv 3 \bmod 4$. I propose that

$$s(p^2n) - ps(n) = 48\sum_{f\in TG_{1,p}}\frac{R_f(n)}{|\text{Aut}(f)|} - 96\sum_{f\in TG_{1,p}}\frac{R_f\left(\frac{n}{4}\right)}{|\text{Aut}(f)|}. \tag{6.1}$$

Clearly, one wants to know if the parity restriction on n in (6.1) can be removed. In other words, the question is whether a straightforward generalizion of Theorem 5.2 exists. Fortunately, the answer is "yes". However, the answer involves the second genus of ternary forms $TG_{2,p}$ with discriminant $16p^2$. Note that, in general, there are 12 genera of the ternary forms with the discriminant $16p^2$ [18]. However, when $p \equiv 3 \bmod 4$ one can create $TG_{2,p}$ from some binary quadratic form of discriminant $-p$. It is a well known fact that all binary forms with the discriminant $-p$ belong to the same genus, say BG_p. Let $ax^2 + bxz + cz^2$ be some binary form $\in BG_p$. We can convert it into ternary form

$$f(x, y, z) := 4ax^2 + py^2 + 4cz^2 + 4|b|xz.$$

Next, we extend f to a genus that contains f. This genus is, in fact, $TG_{2,p}$ when $p \equiv 3 \bmod 4$. It can be shown that the map

$$BG_p \to TG_{2,p}$$

does not depend on which specific binary form from BG_p we have choosen as our starting point. I would like to comment that somewhat similar construction was employed in [2] to define the so-called S-genus. Let me illustrate this map for $p = 23$. In this case,

$$BG_{23} = \{x^2 + xz + 6z^2, 2x^2 + xz + 3z^2, 2x^2 - xz + 3z^2\}.$$

Choosing a binary form $x^2 + xz + 6z^2$ as a starting point one gets

$$\begin{aligned}&\{x^2 + xz + 6z^2\} \to \{4x^2 + 23y^2 + 24z^2 + 4xz\} \to\\ &\quad \{4x^2 + 23y^2 + 24z^2 + 4xz, 8x^2 + 23y^2 + 12z^2 + 4xz, 3x^2\\ &\quad + 31y^2 + 31z^2 - 30yz + 2zx + 2xy\}.\end{aligned}$$

We note that

$$\begin{aligned}TG_{2,23} := \{&4x^2 + 23y^2 + 24z^2 + 4xz, 8x^2 + 23y^2 + 12z^2 + 4xz, 3x^2\\ &+ 31y^2 + 31z^2 - 30yz + 2zx + 2xy\}\end{aligned}$$

is just one out of 12 possible genera of the ternary form with the discriminant $8{,}464$. It is instructive to compare $TG_{2,23}$ and

$$TG_{1,23} := \{x^2+6y^2+23z^2+xy, 2x^2+3y^2+23z^2+xy, 3x^2+8y^2+8z^2-7yz+2zx+2xy\}.$$

Clearly,

$$|TG_{1,23}| = |TG_{2,23}|.$$

Moreover,

$$\begin{aligned}&|\mathrm{Aut}(3x^2 + 8y^2 + 8z^2 - 7yz + 2zx + 2xy)|\\ &\quad = |\mathrm{Aut}(3x^2 + 31y^2 + 31z^2 - 30yz + 2zx + 2xy)| = 12,\end{aligned}$$

$$|\text{Aut}(x^2+6y^2+23z^2+xy)| = |\text{Aut}(4x^2+23y^2+24z^2+4xz)| = 8,$$
$$|\text{Aut}(2x^2+3y^2+23z^2+xy)| = |\text{Aut}(8x^2+23y^2+12z^2+4xz)| = 4.$$

It is a bit less obvious that

$$(3,31,31,-30,2,2)(4n) = (3,8,8,-7,2,2)(n),$$

$$(4,23,24,0,4,0)(4n) = (1,6,23,0,0,1)(n),$$
$$(8,23,12,0,4,0)(4n) = (2,3,23,0,0,1)(n),$$

and that

$$(3,31,31,-30,2,2)(m) = (4,23,24,0,4,0)(m) = (8,12,23,0,0,4)(m) = 0,$$

whenever $m \equiv 1,2 \bmod 4$. I propose that the above properties are, in fact, the signature properties of $TG_{2,p}$. In other words, for any odd prime p there exists an automorphism preserving bijection

$$H : TG_{2,p} \to TG_{1,p},$$

such that , for any $f \in TG_{2,p}$,

$$|\text{Aut}(f)| = |\text{Aut}H(f)|,$$

$$R_f(4n) = R_{H(f)}(n), \tag{6.2}$$

and

$$R_f(m) = 0, \quad \text{when} \quad m \equiv 1,2 \bmod 4. \tag{6.3}$$

Jagy [15] suggested that $TG_{1,p} \cup TG_{2,p}$ does not represent any integer that is quadratic residue mod p when $p \equiv 1 \bmod 4$, and when $p \equiv 3 \bmod 4$ this union does not represent any integer that is a quadratic nonresidue mod p. That is for any $f \in TG_{1,p} \cup TG_{2,p}$

$$R_f(n) = 0,$$

when $(-n|p) = 1$. In addition, he pointed out that $TG_{2,p}$ represents a proper subset of those numbers represented by $TG_{1,p}$. Lastly, he observed that both $TG_{1,p}$ and $TG_{2,p}$ are anisotropic at p. I discuss one more example. This time I choose $p = 17$. Here one has

$$TG_{1,17} := \{3x^2+5y^2+6z^2+yz+2zx+3xy, 3x^2+6y^2+6z^2-5yz+2zx+2xy\},$$

and

$$TG_{2,17} := \{7x^2+11y^2+20z^2-8yz+4zx+6xy, 3x^2+23y^2+23z^2-22yz+2zx+2xy\}.$$

Note that

$$|\mathrm{Aut}(3x^2+5y^2+6z^2+yz+2zx+3xy)| = |\mathrm{Aut}(7x^2+11y^2+20z^2-8yz+4zx+6xy)| = 4,$$

$$|\mathrm{Aut}(3x^2+6y^2+6z^2-5yz+2zx+2xy)| = |\mathrm{Aut}(3x^2+23y^2+23z^2-22yz+2zx+2xy)| = 12,$$

$$(3,23,23,-22,2,2)(4n) = (3,6,6,-5,2,2)(n),$$
$$(7,11,20,-8,4,6)(4n) = (3,5,6,1,2,3)(n),$$
$$(7,11,20,-8,4,6)(m) = (3,23,23,-22,2,2)(m) = 0,$$

whenever $m \equiv 1, 2 \bmod 4$. It is worthwhile to point out that there are exactly 12 genera with the discriminant 4,624. Only three of those have the correct cardinality

$$|TG_{2,17}| = 2,$$

$$|\{3x^2 + 6y^2 + 68z^2 + 2xy, \quad 10x^2 + 11y^2 + 14z^2 + 2yz + 4zx + 10xy\}| = 2,$$
$$|\{5x^2 + 7y^2 + 34z^2 + 2xy, \quad 6x^2 + 12y^2 + 17z^2 + 4xy\}| = 2.$$

Note, however, that

$$|\mathrm{Aut}(3x^2+6y^2+68z^2+2xy)| = |\mathrm{Aut}(10x^2+11y^2+14z^2+2yz+4zx+10xy)| = 4,$$

and

$$|\mathrm{Aut}(5x^2 + 7y^2 + 34z^2 + 2xy)| = |\mathrm{Aut}(6x^2 + 12y^2 + 17z^2 + 4xy)| = 4.$$

And so, $TG_{2,17}$ is a unique genus with the desired properties.

I would like to conclude this discussion of $TG_{2,p}$ by providing a more explicit description valid in three special cases. If $p \equiv 3 \bmod 4$, then $TG_{2,p}$ is the genus that contains

$$4x^2 + py^2 + (p+1)z^2 + 4zx.$$

I remark that the above form was obtained from the principal binary form $x^2 + xz + \frac{p+1}{4}z^2$. If $p \equiv 2 \bmod 3$, then $TG_{2,p}$ is the genus that contains

$$x^2 + \frac{4p+1}{3}y^2 + \frac{4p+1}{3}z^2 + \frac{2-4p}{3}yz + 2zx + 2xy.$$

If $p \equiv 5 \bmod 8$, then $TG_{2,p}$ is the genus that contains

$$8x^2 + \frac{p+1}{2}y^2 + (p+2)z^2 + 2yz + 8zx + 4xy.$$

Observe that the smallest prime to escape the above net of three special cases is $p = 73$. I am now ready to unveil the promised extension of (6.1).

Proposition 6.1. *Let p be an odd prime, then*

$$s(p^2n) - ps(n) = 48 \sum_{f \in TG_{1,p}} \frac{R_f(n)}{|Aut(f)|} - 96 \sum_{f \in TG_{2,p}} \frac{R_f(n)}{|Aut(f)|}. \qquad (6.4)$$

The proof of this neat result with $p \geq 7$ is beyond the scope of this paper and will be given in [3]. Note, that (6.1) follows easily from (6.2) to (6.4).
Below I illustrate Proposition 6.1 with some initial examples

$$s(7^2n) - 7s(n) = 6(1,2,7,0,0,1)(n) - 12(4,7,8,0,4,0)(n), \qquad (6.5)$$

$$\begin{aligned} s(11^2n) - 11s(n) &= 4(3,4,4,-3,2,2)(n) + 6(1,3,11,0,0,1)(n) \\ &\quad - 8(3,15,15-14,2,2)(n) - 12(4,11,12,0,4,0)(n), \end{aligned} \qquad (6.6)$$

$$s(13^2n) - 13s(n) = 12(2,5,5,-3,1,1)(n) - 24(8,7,15,2,8,4)(n), \qquad (6.7)$$

$$\begin{aligned} s(17^2n) - 17s(n) &= 12(3,5,6,1,2,3)(n) + 4(3,6,6,-5,2,2)(n) \\ &\quad - 24(7,11,20,-8,4,6)(n) - 8(3,23,23,-22,2,2)(n), \end{aligned} \qquad (6.8)$$

$$\begin{aligned} s(19^2n) - 19s(n) &= 6(1,5,19,0,0,1)(n) + 12(4,5,6,5,1,2)(n) \\ &\quad - 12(4,19,20,0,4,0)(n) - 24(7,11,23,-10,6,2)(n), \end{aligned} \qquad (6.9)$$

$$\begin{aligned} s(23^2n) - 23s(n) &= 4(3,8,8,-7,2,2)(n) + 6(1,6,23,0,0,1)(n) \\ &\quad + 12(2,3,23,0,0,1)(n) - 8(3,31,31,-30,2,2)(n) \\ &\quad - 12(4,23,24,0,4,0)(n) - 24(8,23,12,0,4,0)(n), \end{aligned} \qquad (6.10)$$

Finally, I note that (6.5) implies the following impressive identity

$$\begin{aligned} &8q\psi(-q)E(q^2)^2 S_{7,5}(-q;q^2)_\infty \\ &\quad = \phi(q)^3 + \phi(q^7)\sum_{m,n}(q^{m^2+mn+2n^2} - 2q^{4m^2+4mn+8n^2}). \end{aligned}$$

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