Chapter 1
Introduction

> . . . the aim of mathematical physics is not only to facilitate for the physicist the numerical calculation of certain constants or the integration of certain differential equations. It is besides, it is above all, to reveal to him the hidden harmony of things in making him see them in a new way.
>
> (H. Poincaré, *The Value of Science*, Dover, 1958, p. 79)

1.1 Aim of the Book

The purpose of this book is to demonstrate the existence of a mathematical structure that is common to all physical theories of the macrocosm and to explain the origin of this common structure. The starting point of this investigation is the analysis of physical variables under a new profile: we take into consideration all those geometric features that are usually overlooked in physics books.

A detailed analysis of physical variables and of equations of the theories of the macrocosm makes it possible to put forward as evidence a natural association of physical variables with elementary geometric elements, such as points, lines, surface and volumes. This association makes it possible to build a classification diagram of physical variables and equations *that is the same for all theories, both classic and relativistic*.

The association of 'global' variables with space elements is a new perspective in the description of physics and is the *raison d'être* of this book.

The revelation of a common mathematical structure arose from the ardent desire to explain why analogies exist between physical theories that are very different in their physical content. The book gives an answer to this question and explains *why* all these theories have this common structure.

(a) The role of global variables. As is obvious, the mathematical description of physics relies on the very existence of quantitative attributes of physical systems, a fact that makes possible the introduction of physical quantities. Since

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these quantities are linked by equations, we are accustomed to detecting analogies between different physical theories through the similarity of the equations that describe their laws. Contrary to this practice, we have realized that to explain the origin of the analogies, we must not start from the similarities of differential equations, but from 'global' physical variables, i.e. those variables that are neither a density nor a rate of other variables (Chap. 5). We must investigate their origins, their operative definitions and the role they play in the corresponding theory. The presentation of a physical theory by starting with global variables, instead of field variables, brings out a simple *topological structure* that the differential formulation is not able to show.

What is remarkable is that, in general, global variables arise *directly* from physical measurements: this fact is in contrast to our practice of deducing such global variables from space and time integration of field variables, a fact that leads us to call them *integral variables*. The use of global variables from the very beginning leads to the *global formulation* and then to the *algebraic formulation* of physical theories, while the use of field variables leads to the *differential formulation*. The global formulation must precede the differential formulation because it does not impose those purely mathematical restrictions on the field functions that are indispensable to perform the derivatives but that are not required on physical grounds.

Global variables are linked with domains such as *volumes* **V**, *surfaces* **S** and *lines* **L**: it is here that topology enters the scene. But we will see that there are also global variables associated with *points* **P**, e.g. temperature and electric potential. We will call **P**, **L**, **S** and **V** space elements, and we will say that a variable is *global in space* when it is not the volume density, surface density, or line density of another variable. Global variables can also be associated with *instants* **I** and *intervals* **T**. We will call **I** and **T** time elements, and we will say that a variable is *global in time* when it is not the rate of another variable.

We will show that this association of global variables with space and time elements is a general property of all physical theories of the macrocosm. This association is intrinsic to the very definition of each physical variable and, when the variable is measurable, is reflected in its measurement process. The field functions, which arise as *densities* of space global variables and as *rates* of time global variables, inherit the association with space and time elements of the corresponding space or time global variable (Chap. 5).

This natural association makes it possible to describe physical theories without using the differential formulation from the outset. A formulation based on algebraic topology emphasizes the topological, geometric and mathematical structures that are common to all physical theories and that the differential formulation leaves in the shadow.

Remark. Regarding the microcosm, which is described by quantum mechanics and which replaces observables with operators, the author does not have sufficient knowledge to identify the source variables, nor to identify space and time elements with which these operators are associated. Despite this, the variables and the equations of relativistic quantum mechanics for particles with integer spin, such as those of Klein–Gordon and Proca, find their place in a diagram similar to that of the relativistic formulation of electromagnetism. Moreover, we remark that both Bohm and Schönberg used algebraic topology for quantum mechanics, as do we in this book.[1] We have included in Chap. 13 six tables dealing with quantum mechanics that are formally similar to the classification diagram presented here, with the purpose of inciting theoretical physicists to find their justification.

(b) The two kinds of orientations of space and time elements. In the association of global variables with space and time elements, the notion of *orientation* plays a key role. In fact, space and time elements can be equipped with two types of orientation: *inner* or *outer* orientation. It is shown the every global physical variable is associated with a space and a time element, each of which has an inner or an outer orientation (Chap. 3, p. 39).

We will denote the space and time elements endowed with inner orientation by placing a bar over the boldface, uppercase letters, i.e. $\bar{\mathbf{P}}, \bar{\mathbf{L}}, \bar{\mathbf{S}}, \bar{\mathbf{V}}, \bar{\mathbf{I}}, \bar{\mathbf{T}}$, and the space and time elements endowed with outer orientation by placing a tilde over the boldface, uppercase letters, i.e. $\tilde{\mathbf{P}}, \tilde{\mathbf{L}}, \tilde{\mathbf{S}}, \tilde{\mathbf{V}}, \tilde{\mathbf{I}}, \tilde{\mathbf{T}}$.

This association requires consideration of all the possible combinations between the global variables and the oriented space and time elements. As shown in Table 1.1, 32 couples are formed by an oriented space element and an oriented time element.

Table 1.1 The 32 combinations of space and time elements

	$\bar{\mathbf{I}}$	$\bar{\mathbf{T}}$		$\tilde{\mathbf{I}}$	$\tilde{\mathbf{T}}$
$\bar{\mathbf{P}}$	$[\bar{\mathbf{I}}, \bar{\mathbf{P}}]$	$[\bar{\mathbf{T}}, \bar{\mathbf{P}}]$	$\tilde{\mathbf{P}}$	$[\tilde{\mathbf{I}}, \tilde{\mathbf{P}}]$	$[\tilde{\mathbf{T}}, \tilde{\mathbf{P}}]$
$\bar{\mathbf{L}}$	$[\bar{\mathbf{I}}, \bar{\mathbf{L}}]$	$[\bar{\mathbf{T}}, \bar{\mathbf{L}}]$	$\tilde{\mathbf{L}}$	$[\tilde{\mathbf{I}}, \tilde{\mathbf{L}}]$	$[\tilde{\mathbf{T}}, \tilde{\mathbf{L}}]$
$\bar{\mathbf{S}}$	$[\bar{\mathbf{I}}, \bar{\mathbf{S}}]$	$[\bar{\mathbf{T}}, \bar{\mathbf{S}}]$	$\tilde{\mathbf{S}}$	$[\tilde{\mathbf{I}}, \tilde{\mathbf{S}}]$	$[\tilde{\mathbf{T}}, \tilde{\mathbf{S}}]$
$\bar{\mathbf{V}}$	$[\bar{\mathbf{I}}, \bar{\mathbf{V}}]$	$[\bar{\mathbf{T}}, \bar{\mathbf{V}}]$	$\tilde{\mathbf{V}}$	$[\tilde{\mathbf{I}}, \tilde{\mathbf{V}}]$	$[\tilde{\mathbf{T}}, \tilde{\mathbf{V}}]$

First group

	$\tilde{\mathbf{I}}$	$\tilde{\mathbf{T}}$		$\bar{\mathbf{I}}$	$\bar{\mathbf{T}}$
$\bar{\mathbf{P}}$	$[\tilde{\mathbf{I}}, \bar{\mathbf{P}}]$	$[\tilde{\mathbf{T}}, \bar{\mathbf{P}}]$	$\tilde{\mathbf{P}}$	$[\bar{\mathbf{I}}, \tilde{\mathbf{P}}]$	$[\bar{\mathbf{T}}, \tilde{\mathbf{P}}]$
$\bar{\mathbf{L}}$	$[\tilde{\mathbf{I}}, \bar{\mathbf{L}}]$	$[\tilde{\mathbf{T}}, \bar{\mathbf{L}}]$	$\tilde{\mathbf{L}}$	$[\bar{\mathbf{I}}, \tilde{\mathbf{L}}]$	$[\bar{\mathbf{T}}, \tilde{\mathbf{L}}]$
$\bar{\mathbf{S}}$	$[\tilde{\mathbf{I}}, \bar{\mathbf{S}}]$	$[\tilde{\mathbf{T}}, \bar{\mathbf{S}}]$	$\tilde{\mathbf{S}}$	$[\bar{\mathbf{I}}, \tilde{\mathbf{S}}]$	$[\bar{\mathbf{T}}, \tilde{\mathbf{S}}]$
$\bar{\mathbf{V}}$	$[\tilde{\mathbf{I}}, \bar{\mathbf{V}}]$	$[\tilde{\mathbf{T}}, \bar{\mathbf{V}}]$	$\tilde{\mathbf{V}}$	$[\bar{\mathbf{I}}, \tilde{\mathbf{V}}]$	$[\bar{\mathbf{T}}, \tilde{\mathbf{V}}]$

Second group

More precisely, these 32 couples can be divided into two groups, each consisting of 16 elements. The 16 couples of the first group are those combinations of time and space elements both of which are endowed with the same kind of orientation, either inner or outer. The 16 couples of the second group are those combinations of time and space elements that are endowed with opposite orientations, one inner the other outer. These two groups are organized in Table 1.2 (p. 7).

[1] Bohm et al. [17]; Schönberg [202].

What is surprising is that *each physical variable of every physical theory (of the macrocosm) can be matched with one of these 32 couples*. This is somewhat similar to the fact that each crystal can be classified in one of the 32 classes of symmetry[2] or that each chemical element can be placed in one of the boxes of Mendeleev's table.

It has been found that the variables of physical theories that can be associated with the elements of the first group, i.e. the one on the left side of Table 1.1, belong to *mechanical theories*, whereas the variables of the physical theories that can be associated with the elements of the second group, on the right side of Table 1.1, belong to *field theories*. In this way we have obtained a classification of global variables that is valid for both the classic and relativistic versions of every theory. This fact reveals that the marriage between physics and mathematics is possible through the intermediation of topology and geometry. This is a natural consequence of the fact that physical phenomena arise in space.

(c) Role of cell complexes. The differential formulation, which is based on field variables (i.e. point variables), makes use of coordinate systems. The algebraic formulation, based on global variables, requires a proper reference structure. The need to consider oriented space elements to create the algebraic formulation suggests the need to introduce into the working region of a suitable reference structure whose elements are endowed with spatial extension. A *cell complex*, as defined in algebraic topology (usually in the restricted form of *simplicial complex*), provides the appropriate tool (Chap. 4).

The four kinds of space elements that make up a cell complex, i.e. *vertices*, *edges*, *faces* and *volumes*, can be considered as cells of different dimensions: they have zero, one, two, or three dimensions, respectively. Following the terminology of algebraic topology, we will denote these elements by the terms 0-cells, 1-cells, 2-cells and 3-cells, respectively. In general, we will speak about p-dimensional cells or p-cells for short. In particular, to recover the traditional differential formulation, one can use a cell complex formed by a coordinate system (p. 65).

Once all the p-cells of a complex are endowed with an inner orientation, we obtain a reference structure for those global variables that are associated with space elements with an *inner* orientation. Still lacking is a structure whose p-cells correspond to an *outer* orientation. In fact, we need to locate variables that are associated with space elements endowed with an *outer* orientation. The idea now is to use a second complex that is staggered with respect to the first complex (Fig. 1.1). This is the *dual* cell complex whose outer orientation is automatically induced by the inner orientation of the first complex denoted as *primal*.

Note that every vertex of the primal complex, a 0-cell, is contained in a 3-cell of the dual complex, and vice versa. Moreover, every edge of the primal complex, a 1-cell, is crossed by a face of the dual complex, a 2-cell, and vice versa; and so on. In general, every p-cell of the primal is contained into (or intersects or contains) a

[2] The coincidence of the number of symmetry classes in crystal classification and of the distinct space-time elements is purely a matter of chance.

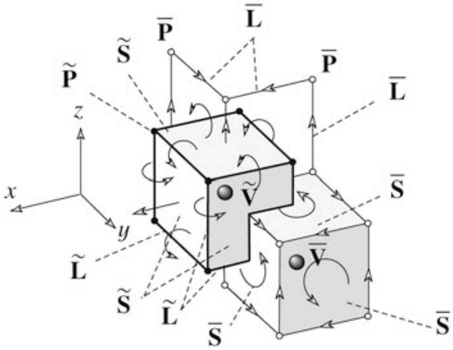


Fig. 1.1 Cell complex in space (*light lines*) and its dual (*heavy lines*)

$(3-p)$-cell of the dual. Conversely, every p-cell of the dual, contains (or intersects or is contained into) a $(3-p)$-cell of the primal. A pair formed of a p-cell of one complex and of the corresponding $(3-p)$-cell of the other complex may be called a dual pair. One can view this correspondence by a pair of boxes (Fig. 1.2a). The boxes corresponding to each dual pair are on the same level. This figure represents the four kinds of cells (pieces of oriented space elements) with four elliptic boxes arranged vertically: four boxes for the primal cell complex and four boxes for the dual cell complex.

A great merit of a pair of cell complexes, a primal and a dual, is to enable a classification of the eight oriented space elements, four equipped with an inner orientation, $\overline{\mathbf{P}}, \overline{\mathbf{L}}, \overline{\mathbf{S}}, \overline{\mathbf{V}}$, and four with an outer orientation, $\widetilde{\mathbf{P}}, \widetilde{\mathbf{L}}, \widetilde{\mathbf{S}}, \widetilde{\mathbf{V}}$.

The differential formulation, while making implicit use of a cell complex, that formed by a coordinate system, lacks a support structure for variables associated with an outer orientation. In contrast, in the topological formulation, this role is played brilliantly by the dual complex.

(d) Classification diagram for physical variables. If we take into account the association of physical variables with the oriented space elements, then we can use the same classification diagram of space elements as a classification diagram for the physical variables associated with them. In fact, we can store the global variables in the appropriate elliptic boxes (Fig. 1.2b). In this way, the two cell complexes, endowed with the two kinds of orientation, become a *topological frame* for the classification of global variables in every physical theory of the macrocosm.

Since the field variables inherit the same association of the corresponding global variables, it is natural to put them in the same boxes as the relative global variables. This leads to the building of a classification diagram also for field variables and, hence, for the differential formulation of physical theories.

The physical variables of every physical theory can be divided into *configuration*, *source* and *energy* variables. Analysing the configuration variables we will see that they are associated with space elements endowed with an inner orienta-

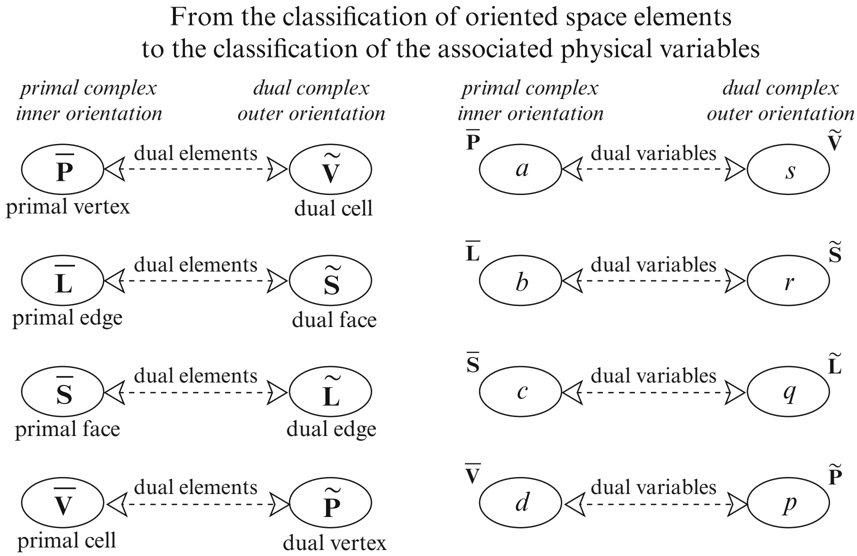


Fig. 1.2 *Left* Classification of eight oriented space elements. *Right* Corresponding classification of physical variables of a physical theory

tion, whereas the source variables are associated with space elements endowed with an outer orientation.[3] It follows that the *configuration* variables can be associated with the cells of the *primal* complex, while the *source* variables can be associated with the cells of the *dual* complex.

Since global physical variables are associated both with an oriented space element and an oriented time element, it is useful to have a classification diagram that takes into account space and time elements. Such a diagram can be obtained by doubling the diagram of Fig. 1.2b and shifting the diagram to the rear, as shown in Fig. 1.3.[4]

The four different combinations of the oriented space and time elements shown in Table 1.1 can be organized in the two diagrams of Table 1.2.

(e) Classification of equations. What about the equations? The equations of every physical theory can be obtained by composing elementary equations of different types. These include *defining* equations, *topological* equations, *phenomenological* equations, and equations of *behaviour*. Since equations are links between the physical variables, they connect the rounded boxes. In the classification

[3] The reason for this astonishing correspondence is not clear, but we take it as an assumption supported by the evidence.

[4] See also Fig. 8.9 on p. 236.

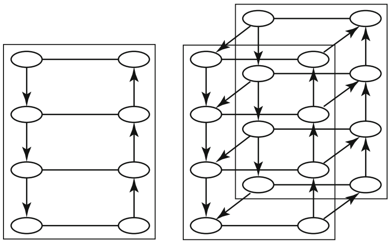
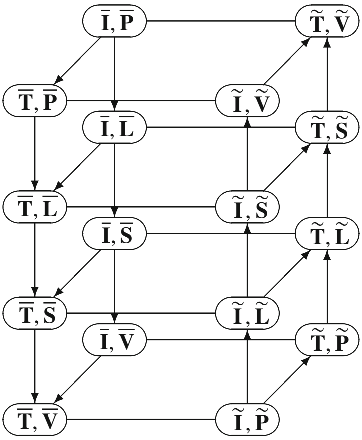


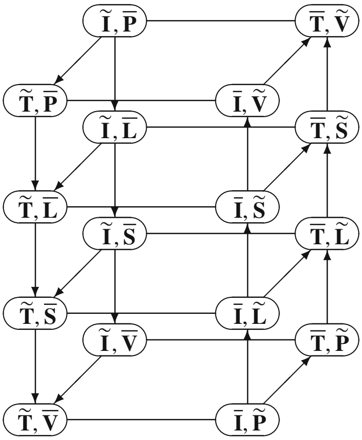
Fig. 1.3 Generation of space-time diagram from the space diagram

Table 1.2 Two classification diagrams for variables of physical theories



Mechanical theories

Particle mechanics, analytical mechanics,
mechanics of deformable solids,
fluid dynamics



Field theories

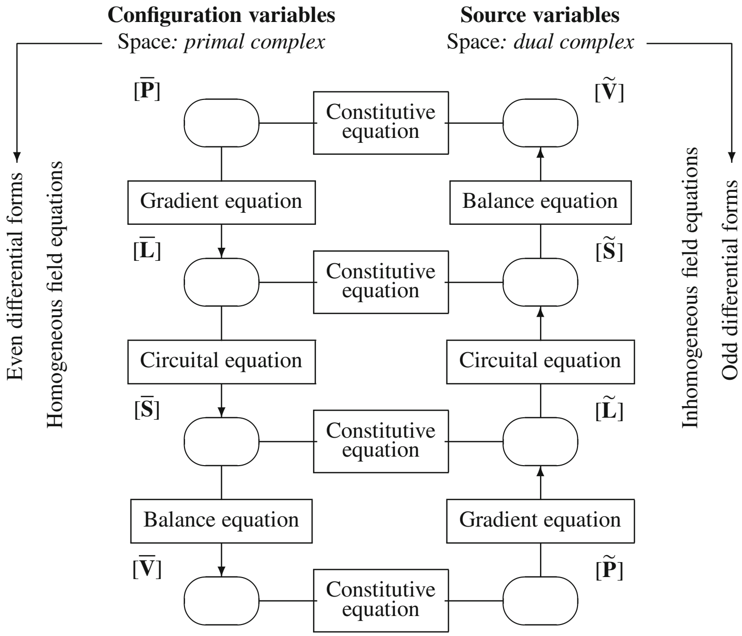
Electromagnetism, gravitation,
irreversible thermodynamics,
thermal conduction, diffusion

diagrams, we will represent equations inside rectangular boxes (Table 1.3). We will show that:

- Topological equations connect the physical variables associated with cells of different dimensions of the same complex (primal or dual); they are indicated by arrows;
- Constitutive equations link the physical variables associated with cells in the primal complex to those associated with cells in the dual complex.

The association of global variables of a physical theory with the cells of different dimensions of a cell complex gives rise to a space distribution of global variables known in algebraic topology by the (unfortunate) name of *cochain* but

Table 1.3 Two kinds of equations of a physical field: field equations and constitutive equations



conveniently called today *discrete form*. All topological equations can be obtained from a very simple and elegant process known as the *coboundary process* in algebraic topology. When we deduce field variables from the corresponding global variables, the discrete forms become exterior differential forms and the coboundary process transforms into the *exterior differential* of a form. More precisely, while the configuration variables, associated with space elements endowed with an inner orientation, can be described by exterior differential forms of an even kind, the source variables, associated with space elements endowed with an outer orientation, can be described by differential forms of an odd kind (≡ *twisted* differential forms).

The unifying power of the coboundary process on the discrete forms, that is, the algebraic ancestor of the exterior differential on the exterior differential forms, is manifested in the fact that it generates the three typical differential operators, which give rise to the gradient, curl and divergence.

Concepts like *global variable, oriented space element, cell complex, chain, cochain* and *coboundary operator* are usually ignored in the traditional description of physics, which is based almost exclusively on differential equations.

The differential formulation hides the geometric and topological structures of physical variables and of physical laws because it reduces every physical variable to a field variable, i.e. it deprives physical variables of their geometric content. The differential machinery disregards the geometric description from the outset and in so doing obscures the structure that is common to different physical theories based on the notion of homologous variables. This is why our investigation should not start from field variables, as the differential formulation does, but from *global variables*.

The classification diagram presented in the book reveals a substantial unity of all physical theories of the macrocosm. To stress this unity, we have included in all diagrams, inside a small icon, the frame in which all present theories find their home. This frame was called by Bossavit the Maxwell house because the variables and the equations of electromagnetism fit nicely inside the frame. The present book shows that the frame is not only a house for Maxwell's equations but for all equations of physical theories of the macrocosm.

The classification of variables and equations obtained in this way exhibits a general structure that displays many known properties of physical equations including, for example, the possibility of a variational formulation of the fundamental equation, the existence of reciprocity theorems, the uniqueness of the solution of the fundamental problem of a theory, compatibility conditions, gauge invariance and the possibility of knowing if a constitutive equation is reversible or not.

Another interesting aspect of the common structure is that the classification is applicable both to the algebraic formulation, which is performed on global variables, and to the differential formulation, which utilizes field variables.

1.2 Analogies in Physics

As was stated previously, the general framework that we want to set up in this book arises from the question of why analogies exist in physics. It is therefore necessary to understand what an analogy is and exemplify the importance that analogies have had and continue to have today in physics.

In the development of scientific thinking, an essential role has been played by the discovery that physical theories exhibit structural similarities, commonly called *analogies*. Duhem wrote:

> The history of physics shows us that the search for analogies between two distinct categories of phenomena has perhaps been the surest and most fruitful method of all the procedures put into play in the construction of physical theories.[5]

An analogy can be defined as the 'invariance of a relation or statement under changes of the elements involved in it'.[6] The corresponding elements of two

[5] Duhem [60, p. 95].

[6] Rosen [193, Chap. 4].

fields are called *homologous*. Analogy is perhaps the weakest form of invariance: invariance in form.

Analogies are not simply a formal fact; they are a structural fact. Analogies reveal the existence of an underlying structure, though they do not explain why that structure exists.

Scientific activity is strongly based on the use of analogies. We use them, more or less consciously, because they are like roads already traced on the land that we are investigating. Analogies are a fundamental tool for knowledge because they allow us to explore a new field using the established knowledge of another field; we need only find the homologous entities of the two fields.

Analogies are a powerful tool in discovery, learning and teaching, i.e. in the creation and transmission of knowledge.

Analogies have another important merit: when dealing with a set of abstract notions, one may find an analogy with a set of concrete notions; in this way, the abstract notions can be more easily understood by referring to the concrete ones. The understanding of abstract notions is greatly facilitated by a well-chosen analogy! In fact, analogies have played an important role in the development of physics. Often a theory is built using a one-to-one correspondence between its physical variables and those of another theory (homologous variables). For example, the analogy between heat conduction in solids, which is transmitted by contact, and electrostatics, which is deduced from the laws of action at a distance, supported the idea that electromagnetic action was also transmitted by contact.[7]

Similarly, Poisson introduced the idea of electric potential by analogy to the notion of temperature in a thermal field, a subject previously treated by Fourier in his book on heat conduction. Another analogy arises from a comparison of the propagation of waves in a material continuum (solid or fluid) and the propagation of electromagnetic waves in free space. In both cases, we have the typical phenomena of reflection, refraction, interference, diffraction, polarization and others.

In an analogy between two physical phenomena, homologous variables differ in many senses: they have different physical meanings, different physical dimensions, and, in general, different mathematical natures. For example, the homolog of a scalar variable may be a vector variable. The existence of similarities despite these differences may be the reason for the fascination that similarities have on many people.

Since a relation between physical variables is expressed by an equation, it follows that analogies in physics are easily captured by the similarity of equations.

In addion, similarity of the equations in various theories allows us to use the same mathematical formalism. For example, the existence of analogies between field theories is shown by the ubiquitous presence of operators such as ‘grad’, ‘curl’ and ‘div’ and by the equations of Laplace, Poisson and d’Alembert arising from them.

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[7] Whittaker [254, Vol. I, p. 241].

Going one step further, we arrive at the formalism of mathematical field theory, which can be used to investigate the possible forms of, for example, field equations (elliptic, parabolic, hyperbolic, linear, or nonlinear), variational principles, invariance properties and conservation laws.

One of the impressive facts underlining the power of analogies in physics and engineering is that they allow for the construction of many mathematical formalisms, such as:

- The formalism of *dynamical systems*;
- The formalism of *generalized network theory*;
- The formalism of *irreversible thermodynamics*;
- The formalism of *mathematical field theory*;
- The formalism of *variational principles*;
- The formalism of the *first quantization*;
- The formalism of the *second quantization*.

We live among formalisms! Mathematics is universally applied because it is the king of formalisms: differential and integral calculus, matrix calculus, vector calculus, operator theory and group theory are all mathematical theories whose application to different fields of science enables great economy of thought, labour and time.

When faced with analogies in physics, two approaches are possible: one is to accept them as a matter of fact and to use them to construct a formalism, the other is to question the reasons for their existence.

This book aims to provide an answer to the latter, i.e.

What is the reason for analogies between physical theories?

The answer to this question forms the core of this book. The same question was raised by Richard Feynman[8]: *Why are the equations from different phenomena so similar?* His answer is as follows:

> We might say: 'It is the underlying unity of nature.' But what does that mean? What could such a statement mean? It could mean simply that the equations are similar for different phenomena; but then, of course, we have given no explanation. The 'underlying unity' might mean that everything is made out of the same stuff, and therefore obeys the same equations. That sounds like a good explanation, but let us think. The electrostatic potential, the diffusion of neutrons, heat flow – are we really dealing with the same stuff? Can we really imagine that the electrostatic potential is *physically* identical to the temperature, or to the density of particles? Certainly ϕ is not *exactly* the same as the thermal energy of particles. The displacement of a membrane is certainly not like a temperature. Why, then, is there 'an underlying unity'? A closer look at the physics of the various subjects shows, in fact, that the equations are not really identical. The equation we found for neutron diffusion is only an approximation that is good when the distance over which we are looking is large compared with the mean free path. If we look more closely, we would see the individual neutrons running around. Certainly the motion of an individual neutron

[8] Feynman et al. [69, Vol. II; pp. 12–12].

> is a completely different thing from the smooth variation we get from solving the differential equation. The differential equation is an approximation, because we assume that the neutrons are smoothly distributed in space. Is it possible that *this* is the clue? That the thing which is common to all the phenomena is the *space* the framework into which the physics is put?

As we can see, Feynman did not give an answer to the question, even though his claim that *space* is responsible has hit the nail on the head.

Analogies show that alongside the traditional criteria for classifying physical quantities, there is one arising from the fact that some physical quantities are related to lines and others to surfaces. In 1871, Maxwell published an article entitled 'Remarks on the Mathematical Classification of Physical Quantities' [152, p.227], in which he wrote:

> Of the factors which compose it [energy], one is referred to unit of length, and the other to unit of area. This gives what I regard as a very important distinction among vector quantities.

Considering that there are vectors relative to lengths and others relative to areas, he suggested that physical variables could be classified according to their reference to a geometric element, like lines and surfaces. When this is done, analogies make a fourfold distinction of polar–axial and line–surface vectors, a distinction absent in books on vector calculus where with the same vector one performs circulation along a line and flux across a surface.[9] The key point of the present analysis is that global physical variables have a natural association with the four space elements and the two time elements. Once this association has been realized, we can easily see that the *homologous variables of two physical theories are those associated with the same space element*. This fact is at the base of the existence of analogies in physics. Hence, every physical variable can be classified according to the space and time element with which it is associated.

In the same article Maxwell said

> It is only through the progress of science in recent times that we have become acquainted with so large a number of physical quantities that a classification of them is desirable. [. . .] But the classification which I now refer to is founded on the mathematical or formal analogy of the different quantities, and not on the matter to which they belong.

He called this

> [. . .] a mathematical classification of quantities. A knowledge of the mathematical classification of quantities is of great use both to the original investigator and to the ordinary student of the science.

Speaking about analogies Maxwell said

> But it is evident that all analogies of this kind depend on principles of a more fundamental nature; and that, if we had a true mathematical classification of quantities, we should be able at once to detect the analogy between any system of quantities presented to us

[9] Post [184, p. 630].

> and other systems of quantities in known sciences, so that we should lose no time in availing ourselves of the mathematical labours of those who had already solved problems essentially the same. [...] At the same time, I think that the progress of science, both in the way of discovery and in the way of diffusion, would be greatly aided if more attention were paid in a direct way to the classification of quantities.

In the classification diagram that will be presented in this book, the relations between variables, i.e. the equations, are displayed in an ordered way such that they immediately reveal analogies. We want to show that the classification of physical variables and equations is the key to explaining analogies in physics, and it is the root of many common mathematical properties of physical theories. The structure revealed in this way allows us to obtain a classification diagram of variables and equations that is the same for all physical theories.

In this book we will consider the following six physical theories:

1. Particle mechanics (and part of analytical mechanics),
2. Electromagnetism,
3. Mechanics of deformable solids,
4. Fluid dynamics,
5. Thermal conduction,
6. Gravitation,

and we will unfold the mathematical structure of these physical theories. Although the classification needs the global formulation, the diagrams are written using the differential formulation because, to date, this has been the most used language in physics.

1.3 Role of a Classification

What is the role of a classification? Many physicists believe that the value of a classification lies essentially in its ability to predict a *new* property, a *new* fact, a *new* phenomenon: ‘if you do not hope to discover something new, what is the interest?’

Let us take as an example the periodic table of chemical elements. It is true that Mendeleev’s classification has made possible the prediction of the existence of new chemical elements on the basis of their physical and chemical properties, and it was a big help to discover them. But the value of this classification has not been exhausted at all in the discovery of new chemical elements. If the predictive aspect was the predominant one, how are we to explain the fact that Mendeleev’s table is still hanging on the walls in the classrooms of colleges and universities around the world? Perhaps this is done to encourage students to discover some new elements? How do we justify the teaching of the classification of crystals into 32 classes of symmetry? Why do we teach the Linnaean classification of botany? Perhaps to stimulate the discovery of a new type of animal or vegetal organism?

One of the advantages of Mendeleev's table is in the spatial arrangement of the elements within a table, not simply in a list. In fact, an order can be obtained simply by making a list of the chemical elements in increasing atomic weight next to the indication of the physical and chemical characteristics of each element. Such a list would contain all the information necessary to make the combinations to create chemical compounds. But it was soon realized that in such a list, there would be a periodic behaviour of the chemical properties of the elements and the period would be composed of eight elements. This fact suggested organizing the elements in rows of eight elements. In this way, we pass from a one-dimensional list to a two-dimensional table. Much later it was realized that eight is the maximum number of electrons that can stay in the outer orbital of an atom.

Doing a classification means *dividing into classes* according to certain criteria. The primary goal of a classification is to impose order on a set of elements, but the value of a classification depends strongly on the criteria used to classify.

The classification of physical quantities that we introduce in this book is based on the *role* that the variables have in their theory. This leads to a first criterion of dividing the variables into three classes – configuration, source and energy variables. A second criterion concerns the space element with which the variables are associated. Since each physical variable can be classified according to both criteria, this allows the formation of a two-dimensional diagram. A third criterion arises considering the association of a variable with a time element. This leads to a three-dimensional diagram represented in an assonometric view (Table 1.2).

One of the features of the classification diagram is the clear-cut distinction between the *field equations*, such as balance equations, circuital equations and equations for the formation of gradients (vertical links), and the *constitutive equations* which describe both reversible and irreversible processes (horizontal links).

One thing that catches the eye looking at the diagram, as shown in Fig. 8.7 (p. 232), are the irreversible links: these connect a variable located in the left column of the front part of the diagram with another variable located in the right column of the rear part of the diagram.

The diagram gives a systematic procedure to obtain the fundamental equation of a theory: simply combine the equations encountered in the path that goes from the potential to the source.

1.4 Role of Geometry in Physics

The mathematical description of physics requires the intermediary role of geometry because all physical phenomena arise in space and physical variables are introduced with reference to lines or surfaces or volumes, and only a few of them refer directly to points. For example, the light flux detected by a photocell depends on the area of the surface and on its position in space, its orientation; a strain-gauge

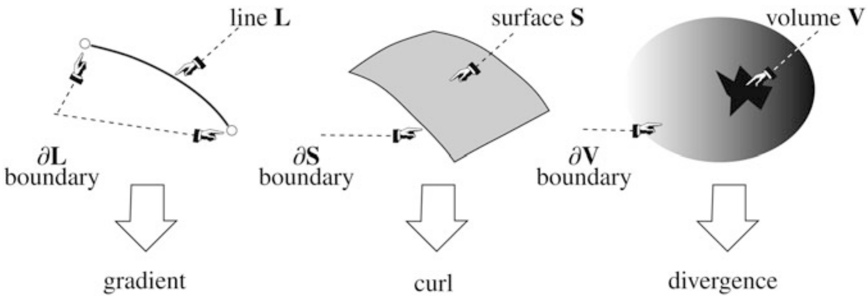

Fig. 1.4 Topological notions at origin of three main differential operators

measures the variation of the distance between two points by the increase of the electric resistance of a wire.

Geometry enters into physics at two levels: by means of *topological* concepts and *metric* concepts. So Coulomb's law requires the distance between two point charges and is therefore a *metric* law. In contrast, the first law of the electrostatic field says that the sum of the voltages along any closed line is zero. This is a *topological* law because the shape of the closed line and its extension are not involved.

1.4.1 Topological Concepts

Recall that topological properties are those properties of geometric figures which are invariant under continuous deformations without introducing tears and overlaps. Stated in more mathematical language, they are invariant under *homeomorphisms*. These concepts differ from metric concepts because no measures of lengths, areas, volumes or angles are involved.

In physical theories, there are three kinds of topological equations: *balance equations, circuital equations* and *equations forming gradients.*

Figure 1.4 shows the topological ingredients which we use, implicitly or explicitly, in physical theories, i.e. line surfaces, volumes and their boundaries: they are involved in the construction of the three differential operators which are ubiquitous in physics – the gradient, the curl and the divergence.

Balance equations. These are the most important physical equations. They play a pivotal role in all physical theories and include the balance of mass, momentum, angular momentum, energy, entropy, electric charge and the number of particles. A balance law states that, given a space region and a time interval, the amount of a given physical variable produced inside the space region in the time interval can be divided into two parts: one part is stored inside the region in the time interval, the other flows across the boundary of the region in the same interval.

A particular case arises when the production vanishes: in this case the balance law is reduced to a *conservation law*.

A distinctive feature of the balance laws is that they can be applied to regions of whatever shape and extension and for whatever interval of time. The size of the space region, a metric concept, and the duration of the interval, a chronometric concept, are not involved. Hence balance laws are described by topological equations.

Circuital equations. These equations assert that the amount of a given physical variable associated with a closed curve is equal to the amount of another physical quantity associated with a surface enclosed by this curve. Such equations may express a law or simply the definition of a physical variable. These include André-Marie Ampère circuital law, Faraday's induction law and Kelvin's circulation theorem.

In fluid dynamics, the vortex flux through a surface is *defined* as the velocity's circulation along the boundary of the surface. In electromagnetism, the magnetic flux across a surface is *defined* as the impulse of the electromotive force[10] across the boundary of the surface. A circuital equation is valid for any shape of the surface and for any area; hence, circuital equations are topological equations.

Equations forming gradients. These are the third kind of topological equation. When we form the difference in temperature between two points, we define a new variable by means of a simple relation which does not involve metric concepts. Hence the equations forming differences are *topological equations*. In general this is a preliminary step towards defining the average temperature gradient, i.e. dividing this temperature difference by the distance of the two points. The division by a distance introduces a metric attribute and leads to the temperature gradient along the direction of the line connecting the two points.

Besides these three classes of equations there are other topological concepts.

Connectivity of a region. The existence, in a region, of closed lines which are not contractible to a point and of closed surfaces which are not contractible to a closed line leads to the concept of multiply connected regions with respect to lines and surfaces, respectively. Hence connectivity is a topological concept. We remark that the projection of a figure, as in axonometry and in perspective, does not respect connectivity because distinct lines in space may have projections which intersect, and it does not respect either the topological properties or the metric properties, as seen in the prospective of a cube.

Orientation. The concept of orientation plays a crucial role in the mathematical description of physics. Both the inner orientation and the outer orientation, which we will present in detail in what follows, are used in the description of physical laws. The concept of orientation, which has a combinatorial nature, is a topological concept.

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[10] See pp. 290 and 291.

1.5 Algebraic Formulation of Fields

The use of global variables leads to an *algebraic* (or *discrete* or *direct* or *finite* formulation) of physical theories, which is an alternative to the *differential formulation*. Starting from the algebraic formulation one can easily deduce the differential formulation. This way of presenting physical theories is very useful in teaching because it makes a strong appeal to physical measurements and avoids premature recourse to differential operators whose symbolism is exceedingly abstract for the average university student.

For example, the Faraday electrostatics law, which states that the electric charge Ψ collected on the boundary of a volume is equal to the charge contained inside the volume, can be grasped more easily than the statement $\nabla \cdot \mathbf{D} = \rho$, which states the same thing but in differential formulation.

What is more important is that *the algebraic formulation can be immediately used for computational physics* using a cell complex and its dual in the computational domain. This formulation avoids the discretization of differential equations, which is necessary in the existing numerical methods, because it uses global variables and balance equations in a global form. Hence, from a computational point of view (which is not considered in this book) the direct algebraic formulation avoids all the typical difficulties linked to differentiability such as the use of generalized functions (e.g. Dirac delta function), the splitting of physical laws into differential equations in the regions of regularity and jump conditions across the surfaces of discontinuity.

The numerical method, which is based on a direct algebraic formulation, is called the *cell method*.[11]

1.6 Summary

In short, a description of physical theories using global variables as a starting point

1. Shows the link between global variables and the oriented space and time elements;
2. Maintains a close link with physical measurements because global variables are, in general, the variables we measure;
3. Permits a numerical formulation of physical theories from the very outset, i.e. without discretizing the differential equations.

The global formulation (algebraic–topological formulation) which we want to reveal gives an answer to the question of why analogies exist. Moreover, it allows us to deduce

[11] The Web site: http://discretephysics.dicar.units.it/ has collected a large number of papers dealing with this method.

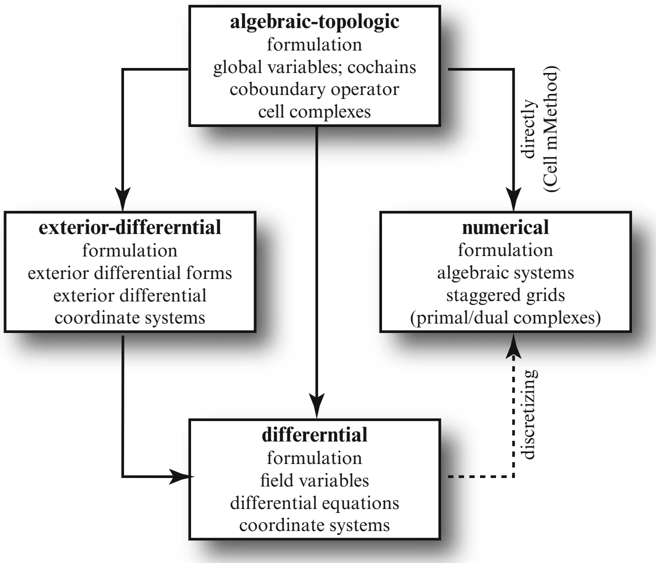

Fig. 1.5 The four mathematical formulations of physics: three of them are known

- The traditional differential formulation,
- The exterior differential formulation,
- The numerical formulation,

as shown in Fig. 1.5. A further consequence of this association is that balance, circuital and gradient forming equations can be described through a single algebraic process which has inherited an unusual name derived from algebraic topology: the coboundary process. Despite its name, the process is so simple and so intuitive that it can be presented to undergraduate students. Algebraic topology also enables the formulation of physical laws in global form, *in the large*, i.e. considering also multiply connected domains.

1.6.1 Notation Used in Book

For symbols of physical quantities we have followed the indication of the International Union of Pure and Applied Physics (IUPAP). Page 485 contains a list of the physical variables treated in this book. The units used are those of the SI system.

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