Preface

This book originated from the notes I kept while teaching the graduate course *Analysis* on Lebesgue’s theory of integration and differentiation at San Francisco State University (SFSU). This course was rightly considered by many students as a difficult one, mainly because some ideas and proofs were presented in their textbooks in unnatural and counterintuitive ways, albeit rigorous ones. These students also had problems connecting the material they learned in undergraduate real analysis classes with this course. The Mathematics Department of SFSU is a master’s mathematics department. Most students who received a MS degree from our department do not pursue a higher degree. When teaching this course I wanted my students to get a feel for the theory, appreciate its importance, and be ready to learn more about it, should the need arise. Consequently, my goal in writing this book was to present Lebesgue’s theory in the most elementary way possible by sacrificing the generality of the theory. For this, the theory is built constructively for measures and integrals over bounded sets only. However, the reader will find all main theorems of the theory here, of course not in their ultimate generality.

The first chapter presents selected topics from the real analysis that I felt are needed to review in order to fill the gaps between what the reader probably learned some time ago or missed completely and what is required to master the material presented in the rest of the book. For instance, one can hardly find properties of summable families (Sect. 1.4) in textbooks on real analysis. Several conventions that are used throughout the book are also found in Chap. 1.

The Lebesgue measure of a bounded set and measurable functions are the subject of the second chapter. Because bounded open and closed sets have relatively simple structures, their measures are introduced first. Then the outer and inner measures of a bounded set are introduced by approximating the set by open and closed sets, respectively. A measurable set is defined as a bounded set for which its inner and outer measures are equal; its Lebesgue measure is the common value of these two measures. We proceed then by
establishing standard properties of the Lebesgue measure and measurable sets. Lebesgue measurable functions and their convergence properties are covered in the last two sections of Chap. 2. Undoubtedly, the highest point of this chapter is Egorov’s Theorem, which is important in establishing convergence properties of integrals in Chap. 3.

I follow most expositions in Chap. 3 where main elements of the theory of Lebesgue integral are presented. Again the theory is developed for functions over bounded sets only. However, the main convergence theorems—the Bounded Convergence Theorem, the Monotone Convergence Theorem, and Dominated Convergence Theorem—are proved in this chapter, establishing the “passage of the limit under the integral sign.”

The main topics of Chap. 4 are Lebesgue’s theorem about differentiability of monotone functions and his versions of the fundamental theorems of calculus. I chose to present functions of bounded variations (BV-functions) and their properties first and then prove the Lebesgue theorem for BV-functions. The proof is elementary albeit a nontrivial one. To make it more accessible, I dissect the proof into a number of lemmas and two theorems. The last two sections of Chap. 4 cover absolutely continuous functions and the fundamental theorems of calculus due to Lebesgue.

A distinguished feature of this book is that it limits attention in Chaps. 2 and 3 to bounded subsets of the real line. In the Appendix, I present a way to remedy this limitation.

There are 187 exercises in the book (there is an exercise section at the end of each chapter). Most exercises are “proof” problems, that is, the reader is invited to prove a statement in the exercise.

I have received help from many people in the process of working on the drafts of this book. First and foremost, I am greatly indebted to my students for correcting several errors in the lecture notes from which this text was derived and providing other valuable feedback. I wish to thank my colleague Eric Hayashi and an anonymous referee for reading parts of the manuscript carefully and suggesting many mathematical and stylistic corrections. My special thanks go to Sheldon Axler for his endorsement of this project and many comments which materially improved the original draft of the book. Last but not least, I wish to thank my Springer editor Kaitlin Leach for her support throughout the preparation of this book.

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