Chapter 2
Properties of Total Dominating Sets and General Bounds

2.1 Introduction

In order to obtain results on the total domination number, we need to first establish properties of TD-sets in graphs. In this chapter, we list properties of minimal TD-sets in a graph. Further we present general bounds relating the total domination number to other parameters.

2.2 Properties of Total Dominating Sets

Recall that if $G = (V,E)$ is a graph, $S \subseteq V$, and $v \in S$, then $\mathrm{pn}(v,S) = \{w \in V \mid N(w) \cap S = \{v\}\}$, $\mathrm{ipn}(v,S) = \mathrm{pn}(v,S) \cap S$ and $\mathrm{epn}(v,S) = \mathrm{pn}(v,S) \setminus S$. The following property of a minimal TD-set in a graph is established by Cockayne, Dawes, and Hedetniemi [39].

Proposition 2.1 ([39]). *Let S be a TD-set in a graph G. Then, S is a minimal TD-set in G if and only if* $|\mathrm{epn}(v,S)| \geq 1$ *or* $|\mathrm{ipn}(v,S)| \geq 1$ *for each* $v \in S$.

Proof. Let S be a minimal TD-set in G and let $v \in S$. If $|\mathrm{epn}(v,S)| = 0$ and $|\mathrm{ipn}(v,S)| = 0$, then every vertex $x \in V(G)$ must be adjacent to a vertex in $S \setminus \{v\}$ as $N(x) \cap S \neq \{v\}$. Hence, $S \setminus \{v\}$ is a TD-set of G, contradicting the minimality of S. Therefore, $|\mathrm{epn}(v,S)| \geq 1$ or $|\mathrm{ipn}(v,S)| \geq 1$ for each $v \in S$. Conversely, if $|\mathrm{epn}(v,S)| \geq 1$ or $|\mathrm{ipn}(v,S)| \geq 1$ for each $v \in S$, then $S \setminus \{v\}$ is not a TD-set, implying that S is a minimal TD-set in G. □

The following stronger property of a minimum TD-set in a graph is established in [102].

Theorem 2.2 ([102]). *If G is a connected graph of order $n \geq 3$ and $G \ncong K_n$, then G has a minimum TD-set S such that every vertex $v \in S$ satisfies* $|\mathrm{epn}(v,S)| \geq 1$ *or is adjacent to a vertex v' of degree* 1 *in $G[S]$ satisfying* $|\mathrm{epn}(v',S)| \geq 1$.

M.A. Henning and A. Yeo, *Total Domination in Graphs*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-6525-6_2,

For a subset S of vertices in a graph G, the *open boundary* of S is defined as $OB(S) = \{v\colon |N(v) \cap S| = 1\}$; that is, $OB(S)$ is the set of vertices totally dominated by exactly one vertex in S. Hedetniemi, Jacobs, Laskar, and Pillone characterized a minimal TD-set by its open boundary as follows (see Theorem 6.10 in [85]).

Theorem 2.3 ([85]). *A TD-set S in a graph G is a minimal TD-set if and only if $OB(S)$ totally dominates S.*

Proof. Suppose first that $OB(S)$ totally dominates S. Let $v \in S$ and let u be a vertex in $OB(S)$ that is adjacent to v. Then, $N(u) \cap S = \{v\}$. If $u \notin S$, then $u \in \mathrm{epn}(v,S)$. If $u \in S$, then $u \in \mathrm{ipn}(v,S)$. Hence, $\mathrm{epn}(v,S) \neq \emptyset$ or $\mathrm{ipn}(v,S) \neq \emptyset$ for every vertex $v \in S$. Thus, by Proposition 2.1, S is a minimal TD-set. To prove the necessity, suppose that S is a minimal TD-set. Let $v \in S$. By Proposition 2.1, $\mathrm{epn}(v,S) \neq \emptyset$ or $\mathrm{ipn}(v,S) \neq \emptyset$. If $\mathrm{epn}(v,S) \neq \emptyset$, then there exists a vertex $u \in V \setminus S$ such that $N(u) \cap S = \{v\}$, and so $u \in OB(S)$. On the other hand, if $\mathrm{ipn}(v,S) \neq \emptyset$, then there exists a vertex $u \in S$ such that $N(u) \cap S = \{v\}$, and so $u \in OB(S)$. In both cases, $OB(S)$ totally dominates the vertex v. □

A graph class is *hereditary* if it is closed under taking induced subgraphs and *additive* if it is closed under a disjoint union of graphs. An additive hereditary graph class is said to be *nontrivial* if it is nonempty and contains K_2. We note that triangle-free graphs or bipartite graphs are examples of a nontrivial additive hereditary graph class. Further we note that the minimal forbidden subgraphs of an additive hereditary graph class are connected. For example the 3-cycle is the only minimal forbidden subgraph of triangle-free graphs, while all odd cycles are the minimal forbidden subgraphs of bipartite graphs.

Using results of Bacsó [9] and Tuza [201] who independently gave a full characterization of the graphs for which every connected induced subgraph has a connected dominating subgraph satisfying an arbitrary prescribed hereditary property, Schaudt [180] derived a similar characterization of the graphs for which any isolate-free induced subgraph has a TD-set that satisfies a prescribed additive hereditary property. If $\mathscr{G}$ is a graph class, then following the notation of Schaudt, we denote by $Total(\mathscr{G})$ the set of isolate-free graphs for which every isolate-free subgraph H has a TD-set that is isomorphic to some member of $\mathscr{G}$. Restricting his attention to graph classes that are hereditary and additive, Schaudt [180] characterized $Total(\mathscr{G})$ in terms of minimal forbidden subgraphs, for arbitrary nontrivial additive hereditary properties $\mathscr{G}$. The following result shows that the corona graphs are the only minimal forbidden subgraphs of $Total(\mathscr{G})$, where recall that a corona graph $H \circ K_1$ is a graph that can be obtained from a graph H by adding a pendant edge to each vertex of H.

Theorem 2.4 ([180]). *Let $\mathscr{G}$ be a nontrivial additive hereditary graph class containing all paths. Then the minimal forbidden subgraphs of $Total(\mathscr{G})$ are the corona graphs of the minimal forbidden subgraphs of $\mathscr{G}$.*

A characterization of additive hereditary graph classes $\mathscr{G}$ which do not contain all paths remains an open problem. The following partial characterizations and sufficient conditions are given by Schaudt [180]. For any $k \geq 3$ and $2 \leq i \leq k-1$, let T_k^i be the graph obtained from the path P_k by attaching a pendant vertex to the i-th vertex of P_k and let $\mathscr{T}_k = \{T_k^i \mid 2 \leq i \leq k-1\}$ be the collection of these graphs.

Theorem 2.5 ([180]). *Let $\mathscr{G}$ be a nontrivial additive hereditary graph class that does not contain all paths and let k be minimal such that $P_k \notin \mathscr{G}$. Then the following hold:*

(a) *If $k = 3$, then the minimal forbidden subgraphs of $Total(\mathscr{G})$ are C_5 and the coronas of the minimal forbidden subgraphs of $\mathscr{G}$.*
(b) *If $k \geq 4$ and $\mathscr{G} \cap \mathscr{T}_{k-1} \neq \emptyset$, then the minimal forbidden subgraphs of $Total(\mathscr{G})$ are the coronas of the minimal forbidden subgraphs of $\mathscr{G}$.*
(c) *If $k \geq 4$, then $Total(\mathscr{G})$ contains all graphs that do not contain a corona of the minimal forbidden subgraphs of $\mathscr{G}$ as subgraph and do not contain any graph of $\{C_i \mid 5 \leq i \leq k+2\} \cup \{P_{k-1} \circ K_1\}$ as subgraph.*

2.3 General Bounds

In this section we present general bounds relating the total domination number to other parameters.

2.3.1 Bounds in Terms of the Order

We begin with the following bound on the total domination number of a connected graph in terms of the order of the graph due to Cockayne et al. [39].

Theorem 2.6 ([39]). *If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq 2n/3$.*

Brigham, Carrington, and Vitray [20] characterized the connected graphs of order at least 3 with total domination number exactly two-thirds their order.

Theorem 2.7 ([20]). *Let G be a connected graph of order $n \geq 3$. Then $\gamma_t(G) = 2n/3$ if and only if G is C_3, C_6, or $F \circ P_2$ for some connected graph F.*

Since Theorems 2.6 and 2.7 are fundamental results on total domination, we present a proof of these two results. The first proof we present is a graph theory proof from [130] and uses the property of minimal TD-sets in Sect. 2.2. We will afterwards give a hypergraph proof of Theorem 2.6.

Proof. Let $G = (V, E)$ be a connected graph of order $n \geq 3$. If $G = K_n$, then $\gamma_t(G) = 2 \leq 2n/3$. Further, if $\gamma_t(G) = 2n/3$, then $G = K_3$. Hence we may assume that $G \neq K_n$. Let S be a $\gamma_t(G)$-set satisfying the statement of Theorem 2.2.

Let $A = \{v \in S \mid \text{epn}(v,S) = \emptyset\}$ and let $B = S \setminus A$. By Theorem 2.2, each vertex $v \in A$ is adjacent to at least one vertex of B which is adjacent to v but to no other vertex of S. Hence, $|S| = |A| + |B| \leq 2|B|$. Let C be the set of all external S-private neighbors. Then, $C \subseteq V \setminus S$. Since each vertex of B has at least one external S-private neighbor, $|C| \geq |B|$. Hence,

$$n - |S| = |V \setminus S| \geq |C| \geq |B| \geq |S|/2, \tag{2.1}$$

and so $\gamma_t(G) = |S| \leq 2n/3$. This proves Theorem 2.6.

To prove Theorem 2.7, suppose that $\gamma_t(G) = 2n/3$ (and still $G \neq K_n$). Then we must have equality throughout the inequality chain (2.1). In particular, $|A| = |B| = |C|$ and $V \setminus S = C$. We deduce that each vertex of B therefore has degree 2 in G and is adjacent to a unique vertex of A and a unique vertex of C. Hence, G contains a spanning subgraph that consists of r disjoint copies of a path P_3 on three vertices, where $r = n/3$

On the one hand, if both sets A and C contain a vertex of degree 2 or more in G, then the connected graph G contains a spanning subgraph H where $H = C_6$ or $H = P_9 \cup (r-3)P_3$. If $H = P_9 \cup (r-3)P_3$, then $\gamma_t(H) = 5 + 2(r-3) = 2r - 1 < 2n/3$. Since adding edges to a graph does not increase its total domination number, $\gamma_t(G) \leq \gamma_t(H) < 2n/3$, contrary to our assumption. Hence, $H = C_6$. But then G can contain no additional edges not in H, and so $G = H = C_6$. Hence in this case, $G = C_6$.

On the other hand, if every vertex of A has degree 1 in G, then by the connectivity of G, the subgraph $G[C]$ is connected and $G = F \circ P_2$ where $F = G[C]$, while if every vertex of C has degree 1 in G, then the subgraph $G[A]$ is connected and $G = F \circ P_2$ where $F = G[A]$. This establishes Theorem 2.7. $\square$

The second proof of Theorem 2.6 that we present is the analogous hypergraph proof of this result. This second proof serves to gently introduce the reader who may not be familiar with hypergraphs to the transition from total domination in graphs to transversals in hypergraphs. We shall first need the following hypergraph result which was proven independently by several authors (see, e.g., [35] for a more general result).

Theorem 2.8. *If $H = (V,E)$ is a connected hypergraph with all edges of size at least two, then $3\tau(H) \leq |V| + |E|$.*

Proof. We proceed by induction on the number $n = |V| \geq 1$ of vertices. Let H have size $m = |E|$. If $m = 0$, then $\tau(H) = 0$ and the theorem holds. Therefore the theorem holds when $n = 1$. Assume, then, that $n \geq 2$ and that the statement holds for all connected hypergraphs $H = (V',E')$ with all edges of size at least two where $|V'| < n$. Let $H = (V,E)$ be a connected hypergraph with all edges of size at least two where $|V| = n$ and $|E| = m$.

If $\Delta(H) = 1$, then $m = 1$, $n \geq 2$ and $\tau(H) = 1$, implying that the theorem holds. Hence, we may assume that $\Delta(H) \geq 2$. Let v be a vertex of maximum degree in H, and so $d_H(v) = \Delta(H)$. Let $H' = H - v$. Then, $H' = (V',E')$ is a hypergraph of order $|V'| = n - 1$ and size $|E'| \leq m - 2$ with all edges of size at least two. Let T' be a $\tau(H')$-transversal in H'. Applying the induction hypothesis to each component of

H', we have that $3\tau(H') = 3|T'| \le |V'| + |E'| \le |V| + |E| - 3$. Since $T = T' \cup \{v\}$ is a transversal of H, we have that $3\tau(H) \le 3|T| = 3(|T'| + 1) \le |V| + |E|$, establishing the desired upper bound. □

Recall that a 2-uniform hypergraph is a graph and that a complete graph is a graph in which every two vertices is adjacent. We remark that with a bit more work one can show that equality holds in Theorem 2.8 if and only if H is a complete graph on two or three vertices. We omit the details. We next present our second proof of Theorem 2.6.

Proof (Analogous hypergraph proof of Theorem 2.6). Let $G = (V, E)$ be a connected graph of order $n \ge 3$. We first consider the case when $\delta(G) \ge 2$. Let H_G be the *ONH* of G, and so $n(H_G) = m(H_G) = n(G) = n$. By Theorems 1.2 and 2.8, $\gamma_t(G) = \tau(H_G) \le (n(H_G) + m(H_G))/3 = 2n/3$, which establishes the desired upper bound. Therefore we may assume that $\delta(G) = 1$.

Let X be the set of all vertices of degree one in G, and so $X = \{v \in V(G) \mid d_G(v) = 1\}$. By assumption $\delta(G) = 1$, and so $|X| \ge 1$. Since G is a connected graph on at least three vertices, the set X is an independent set in G. Let $Y = N(X)$ and let $Z = N(Y) \setminus (X \cup Y)$. Then, $|X| \ge |Y|$. We now consider the following two cases:

Case 1. $|Z| < |Y|$*:* If $V(G) = X \cup Y \cup Z$, then $Y \cup Z$ is a TD-set of G. Further since $|X| \ge |Y|$ and $|Y| > |Z|$, we have that $2n = 2|X| + 2|Y| + 2|Z| > 3|Y| + 3|Z|$, and so $\gamma_t(G) \le |Y| + |Z| < 2n/3$. Hence we may assume that $V(G) \ne X \cup Y \cup Z$, for otherwise the desired result follows. We now consider the hypergraph $H = H_G - (X \cup Y \cup Z)$, and so H is obtained from H_G by removing all vertices in $X \cup Y \cup Z$ and all edges that intersect $X \cup Y \cup Z$. Applying Theorem 2.8 to every component of H, we get $\tau(H) \le (n(H) + m(H))/3$. Every $\tau(H)$-set can be extended to a transversal in H_G by adding to it the set $Y \cup Z$. Hence by Theorem 1.2 and the fact that $|Z| < |Y|$ and $|Y| \le |X|$, we get the following, which completes the proof of Case 1:

$$\begin{aligned}
\gamma_t(G) &= \tau(H_G) \\
&\le |Y| + |Z| + \tau(H) \\
&\le |Y| + |Z| + \left(\tfrac{n(H)+m(H)}{3}\right) \\
&\le |Y| + |Z| + \left(\tfrac{n(H_G)-|X|-|Y|-|Z|}{3}\right) + \left(\tfrac{m(H_G)-|X|-|Y|-|Z|}{3}\right) \\
&= \left(\tfrac{n(H_G)+m(H_G)}{3}\right) + \left(\tfrac{|Y|+|Z|-2|X|}{3}\right) \\
&< \tfrac{2n}{3}.
\end{aligned}$$

Case 2. $|Z| \ge |Y|$*:* Let $H = H_G - Y$, and so H is obtained from H_G by removing all vertices in Y and all edges that intersect Y. We note that there are at least $|X| + |Z|$ edges in H_G that intersect Y, and so $m(H) \le m(H_G) - |X| - |Z| \le m(H_G) - 2|Y| = n - 2|Y|$. Further, $n(H) = n(H_G) - |Y| = n - |Y|$, and so $n(H) + m(H) \le 2n - 3|Y|$. Every $\tau(H)$-set can be extended to a transversal in H_G by adding to it the set Y. Hence by Theorems 1.2 and 2.8, we have that $\gamma_t(G) = \tau(H_G) \le |Y| + \tau(H) \le |Y| + (n(H) + m(H))/3 \le |Y| + (2n - 3|Y|)/3 = 2n/3$, which establishes the desired result. □

We remark that with a bit more work the above hypergraph proof of Theorem 2.6 can be expanded to give a hypergraph proof of Theorem 2.7. We omit the details. A more detailed discussion of bounds on the total domination number of a graph in terms of its order is given in Chap. 5.

2.3.2 Bounds in Terms of the Order and Size

The total domination number of a cycle C_n or path P_n on $n \geq 3$ vertices is easy to compute.

Observation 2.9. *For $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. In other words,*

$$\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 1,3 \pmod 4 \\ \frac{n}{2}+1 & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

In this section, we therefore restrict our attention to graphs with maximum degree at least 3. The following upper bound on the total domination number of a graph in terms of both its order and size is given in [106, 184].

Theorem 2.10 ([106,184]). *If G is a connected graph with $\Delta(G) \leq 3$ and of order n and size m, then $\gamma_t(G) \leq n - m/3$.*

Note that if G is a cubic graph of order n and size m, then $2m = \sum_{v \in V(G)} d(v) = 3n$, which by Theorem 2.10 implies that $\gamma_t(G) \leq n/2$. In Sect. 5.5 we will generalize this result on cubic graphs to minimum degree three graphs. A more detailed discussion of bounds on the total domination number of a graph in terms of both its order and size is given in Chap. 8.

2.3.3 Bounds in Terms of Maximum Degree

The following result provides a trivial lower bound on the total domination number of a graph in terms of the maximum degree of the graph.

Theorem 2.11. *If G is a graph of order n with no isolated vertex, then $\gamma_t(G) \geq n/\Delta(G)$.*

Proof. Let S be a $\gamma_t(G)$-set. Since every vertex is totally dominated by the set S, every vertex belongs to the open neighborhood of at least one vertex in S. Hence, $V(G) = \cup_{v \in S} N_G(v)$, implying that

$$n = |\bigcup_{v \in S} N_G(v)| \leq \sum_{v \in S} |N_G(v)| \leq |S| \cdot \Delta(G),$$

or, equivalently, $\gamma_t(G) = |S| \geq n/\Delta(G)$. $\square$

If G is a connected graph of order $n \geq 2$ with $\Delta(G) = n-1$, then a vertex of maximum degree and an arbitrary neighbor of such a vertex form a TD-set in G, and so $\gamma_t(G) = 2 = n - \Delta(G) + 1$ in this trivial case. In the more interesting case when $\Delta(G) \leq n-2$, Cockayne, Dawes, and Hedetniemi [39] established the following upper bound of the total domination number of a graph in terms of the order and maximum degree of the graph.

Theorem 2.12 ([39]). *If G is a connected graph of order $n \geq 3$ and $\Delta(G) \leq n-2$, then $\gamma_t(G) \leq n - \Delta(G)$.*

Haynes and Markus [100] established the following property of graphs that achieve equality in the upper bound of Theorem 2.12. For this purpose, for $1 \leq k \leq n$ they define the *generalized maximum degree* of a graph G, denoted by $\Delta_k(G)$, to be $\max\{|N(S)| \colon S \subseteq V$ and $|S| = k\}$ and noted that $\Delta_1(G) = \Delta(G)$.

Theorem 2.13 ([100]). *Let G be a connected graph of order $n \geq 3$ with $\Delta = \Delta(G) \leq n-2$. Then, $\gamma_t(G) = n - \Delta$ if and only if $\Delta_k(G) = \Delta + k$ for all $k \in \{2, \ldots, \gamma_t(G)\}$.*

We remark that one direction of Theorem 2.13 can be slightly strengthened as follows.

Theorem 2.14. *Let G be a connected graph of order $n \geq 3$ with $\Delta = \Delta(G) \leq n-2$. If $\Delta_{n-\Delta-1}(G) = \Delta + (n - \Delta - 1) = n-1$, then $\gamma_t(G) = n - \Delta$.*

Proof. If $\Delta_{n-\Delta-1}(G) = n-1$, then $|N(S)| < n$ for all $S \subseteq V(G)$, with $|S| = n - \Delta - 1$, implying that $\gamma_t(G) \geq n - \Delta$. By Theorem 2.12, $\gamma_t(G) \leq n - \Delta$. Consequently, $\gamma_t(G) = n - \Delta$. □

A constructive characterization of connected triangle-free graphs G that achieve equality in Theorem 2.12 can be found in [42].

2.3.4 Bounds in Terms of Radius and Diameter

In this section, we present some bounds relating the total domination number in a connected graph with its radius or diameter. DeLaViña et al. [44] showed that the radius provides a lower bound for the total domination number.

Theorem 2.15 ([44]). *If G is a connected graph of order at least two, then $\gamma_t(G) \geq \mathrm{rad}(G)$.*

The following characterization of the case of equality for Theorem 2.15 is given in [44].

Theorem 2.16 ([44]). *Let G be a connected graph of order at least two and let S be a $\gamma_t(G)$-set. Then, $\gamma_t(G) = \mathrm{rad}(G)$ if and only if $G[S]$ has size $\mathrm{rad}(G)/2$.*

Note that if $n \equiv 0 \pmod 4$ and $G = P_n$ (a path of order n), then $\mathrm{rad}(G) = \gamma_t(G) = n/2$, implying that the bound in Theorem 2.16 is sharp for paths of order congruent to zero modulo four. Since $\mathrm{diam}(G) \le 2\mathrm{rad}(G)$ for all connected graphs G, Theorem 2.15 implies that $\gamma_t(G) \ge \mathrm{diam}(G)/2$. One can in fact do slightly better.

Theorem 2.17 ([44]). *If G is a connected graph of order at least two, then $\gamma_t(G) \ge (\mathrm{diam}(G)+1)/2$.*

The following stronger result is established in [138].

Theorem 2.18 ([138]). *If $G = (V,E)$ is a connected graph and $x_1, x_2, x_3 \in V$, then*

$$\gamma_t(G) \ge \frac{1}{4}\left(d(x_1,x_2) + d(x_1,x_3) + d(x_2,x_3)\right).$$

Furthermore if $\gamma_t(G) = \frac{1}{4}(d(x_1,x_2) + d(x_1,x_3) + d(x_2,x_3))$, then the multiset $\{d(x_1,x_2),\, d(x_1,x_3), d(x_2,x_3)\}$ is equal to $\{2,3,3\}$ modulo four.

To illustrate the sharpness of the bound in Theorem 2.18, take, for example, the graph G to be the double star (a tree with exactly two vertices that are not leaves) on five vertices. Let x_1 and x_2 be two leaves with a common neighbor and let x_3 be a leaf at distance 3 from x_1 in G. Then, $\gamma_t(G) = 2$ and $d(x_1,x_2) + d(x_1,x_3) + d(x_2,x_3) = 8$, implying that $\gamma_t(G) = \frac{1}{4}(d(x_1,x_2) + d(x_1,x_3) + d(x_2,x_3))$.

We remark that Theorem 2.17 is an immediate consequence of Theorem 2.18, due to the following argument. Let G be a connected graph and let x_1 and x_3 be two vertices in G with $d(x_1,x_3) = \mathrm{diam}(G)$. Applying Theorem 2.18 with $x_1 = x_2$ and noting that $d(x_1,x_2) = 0 \notin \{2,3\}$ modulo four, we have that $\gamma_t(G) > (d(x_1,x_2) + d(x_1,x_3) + d(x_2,x_3))/4 = \mathrm{diam}(G)/2$, implying that $\gamma_t(G) \ge (\mathrm{diam}(G)+1)/2$.

The result of Theorem 2.18 is extended to four vertices in [138].

Theorem 2.19 ([138]). *If $G = (V,E)$ is a connected graph and $x_1, x_2, x_3, x_4 \in V$, then $\gamma_t(G) \ge \frac{1}{8}\big(d(x_1,x_2) + d(x_1,x_3) + d(x_1,x_4) + d(x_2,x_3) + d(x_2,x_4) + d(x_3,x_4)\big)$, and this result is best possible.*

We pose a more general problem in Sect. 18.13 where five or more vertices are considered.

The *center* of a graph G, denoted by $C(G)$, is the set of all vertices of minimum eccentricity. Since every vertex in $C(G)$ is at distance at most $\mathrm{rad}(G)$ from every other vertex, we note that $\mathrm{ecc}(C(G)) \le \mathrm{rad}(G)$. When $\mathrm{ecc}(C(G)) = \mathrm{rad}(G)$, the following theorem provides a slight improvement on Theorem 2.15.

Theorem 2.20 ([44]). *If G is a connected graph of order at least two, then $\gamma_t(G) \ge \mathrm{ecc}(C(G)) + 1$.*

The *periphery* of a graph G, denoted by $B(G)$, is the set of all vertices of maximum eccentricity. A lower bound on the total domination number of a graph in terms of the eccentricity of its periphery, $\mathrm{ecc}(B(G))$, is given in [138].

Theorem 2.21 ([138]). *If G is a connected graph of order at least two, then $\gamma_t(G) \ge (3\mathrm{ecc}(B)+2)/4$.*

2.3.5 Bounds in Terms of Girth

The girth of a graph can be used to provide both lower and upper bounds for the total domination number as the following two results illustrate.

Theorem 2.22 ([44]). *If G is a graph of girth g, then $\gamma_t(G) \ge g/2$.*

Theorem 2.23 ([137]). *If G is a connected graph of order n, girth $g \ge 3$, and with $\delta(G) \ge 2$, then*

$$\gamma_t(G) \le \frac{n}{2} + \max\left(1, \frac{n}{2(g+1)}\right),$$

and this bound is sharp.

As $n \ge g$, we have that $(\frac{1}{2} + \frac{1}{g})n \ge \frac{n}{2} + 1$, and so as an immediate consequence of Theorem 2.23, we have the following weaker result.

Theorem 2.24 ([132]). *If G is a graph of order n, minimum degree at least two, and girth $g \ge 3$, then*

$$\gamma_t(G) \le \left(\frac{1}{2} + \frac{1}{g}\right)n.$$

Note that if $n \equiv 2 \pmod 4$ and $G = C_n$, then G has order n, girth $g = n$, and $\gamma_t(G) = (n+2)/2 = \left(\frac{1}{2} + \frac{1}{g}\right)n$, implying that the bound in Theorem 2.24, and therefore also of Theorem 2.23, is sharp for cycles of length congruent to two modulo four. A more detailed discussion of bounds on the total domination number of a graph in terms of its girth is given in Chap. 7. In particular, the sharpness of the bound in Theorem 2.23 is discussed in Sect. 7.3.

2.4 Bounds in Terms of the Domination Number

Every TD-set in a graph is also a dominating set in the graph, implying that $\gamma(G) \le \gamma_t(G)$ for all graphs G with no isolated vertex. Furthermore if S is a $\gamma(G)$-set in a graph G and X is obtained by picking an arbitrary neighbor of each vertex in S, then $|X| \le |S|$ and the set $X \cup S$ is a TD-set in G. Hence, $\gamma_t(G) \le |X| + |S| \le 2|S| = 2\gamma(G)$. This implies the following relationship between the domination and total domination numbers of a graph with no isolated vertex first observed by Bollobás and Cockayne [16].

Theorem 2.25 ([16]). *For every graph G with no isolated vertex, $\gamma(G) \le \gamma_t(G) \le 2\gamma(G)$.*

A more detailed discussion of bounds on the total domination number of a graph in terms of its domination number is given in Sects. 4.6 and 4.7.

http://www.springer.com/978-1-4614-6524-9

Total Domination in Graphs
Henning, M.; Yeo, A.
2013, XIV, 178 p., Hardcover
ISBN: 978-1-4614-6524-9