Chapter 2
Discontinuous Galerkin Approximations
and Main Results

The second chapter of the book is twofold. First, we briefly present the approximation schemes under consideration and their main properties. In particular, we introduce the discontinuous Galerkin (DG) semi-discretization of the wave and Klein–Gordon equations using the so-called symmetric interior penalty DG method in its simplest version, in which piecewise linear polynomials are used on uniform meshes. We then emphasize the main difficulties encountered when analyzing the observability inequality for the DG schemes, related to the high-frequency spurious solutions propagating at small group velocities on both physical and spurious modes. In the last part of this chapter, we briefly present our main results concerning the filtering strategies we develop in this book, based on the Fourier truncation method and on the bi-grid filtering technique.

2.1 Discontinuous Galerkin Approximations

As we have seen in the previous chapter, observability inequalities for the $1 - d$ wave equation can be easily derived out of the d’Alembert formula (1.21) and the propagation properties of solutions along characteristics. But observability and controllability issues are more subtle when the continuous wave equation is replaced by a numerical scheme. For instance, it is by now well known that the discrete analogue of the observability property (1.20) is not uniform with respect to the mesh size parameter $h$ for the classical FD and $P_1$-FEM schemes. Indeed, due to the pathological behavior of the spurious high-frequency numerical solutions, the discrete version of the observability constant $C(T)$ in (1.20) blows up as $h \to 0$ for any $T > 0$. The interested reader is referred to the survey articles [27, 70] for a presentation of the state of art on this topic.

The aim of this book is to extend this analysis to a class of more sophisticated schemes, the so-called discontinuous Galerkin (DG) methods.
In several space dimensions, the DG methods easily handle elements of various types and shapes, irregular nonmatching grids, and even varying polynomial orders. They were proposed in the 1970s for the numerical approximation of solutions of hyperbolic equations and, independently, for elliptic and parabolic problems (cf. [6]). Usually, the DG approximations for elliptic and parabolic problems are called interior penalty (IP) methods since the continuity across the element interfaces is weakly enforced by penalizing the classical variational formulations by suitable bilinear forms, the so-called numerical fluxes. In recent years, due to their efficiency in parallel computing, intensive research has been developed on DG methods. We refer to [6, 11] for a unified analysis and comparison of the existing DG methods for elliptic problems.

In this book, we deal with the simplest setting of the DG space semi-discretization of the $1-d$ wave and Klein–Gordon equations with first-order polynomials ($\mathcal{P}_1$) on a uniform grid $\mathcal{G}^h := \{x_j = jh, j \in \mathbb{Z}\}$ of the whole real line generating the partition (triangulation) $\mathcal{T}^h := \{I_j = (x_j, x_{j+1}), j \in \mathbb{Z}\}$ of the whole real line.

When using the $\mathcal{P}_1$-DG method under consideration, numerical solutions can be decomposed into two essential components: the averages and the jumps of the numerical solution along the interfaces. In the $1-d$ case, the jump $[\cdot]$ and the average $\{\cdot\}$ of a function $f$ at the point $x$ are defined as $[f](x) := f(x^-) - f(x^+)$ and $\{f\}(x) := (f(x^+) + f(x^-))/2$, respectively, where $f(x \pm)$ are the right-/left-handed limits of $f$ at $x$, which may take different values when $f$ is discontinuous at $x$. Similar definitions can be given in the multidimensional case (cf. [6]).

Let $\mathcal{P}^h_1$ be the set of polynomial functions of degree at most one in each interval $I_j$, $j \in \mathbb{Z}$. The finite element space we consider is $\mathcal{V}^h := L^2(\mathbb{R}) \cap \mathcal{P}^h_1$. Observe that $\mathcal{V}^h$ admits the decomposition $\mathcal{V}^h = \mathcal{V}^{h,\{\cdot\}} \oplus \mathcal{V}^{h,[\cdot]}$, with $\mathcal{V}^{h,\{\cdot\}} := \text{span}\{\phi^\{\cdot\}_j, j \in \mathbb{Z}\}$, and $\mathcal{V}^{h,[\cdot]} := \text{span}\{\phi^[[\cdot]]_j, j \in \mathbb{Z}\}$, where (see Fig. 2.1)

\[ \phi^\{\cdot\}_j(x) = \left(1 - \frac{1}{h}|x - x_j|\right)^+ \quad \text{and} \quad \phi^[[\cdot]]_j(x) = \frac{1}{2} \text{sign}(x_j - x) \left(1 - \frac{1}{h}|x - x_j|\right)^+ . \]

Here, the superscript $+$ denotes the positive part of a function.
Remark that $\phi^j_i$ is the typical basis function used in the $P_1$-FEM, whereas $\phi^{[j]}_i$, having zero average and unit jump at $x_j$, is designed to represent the jump at the nodal point $x_j$.

Each element $f^h \in V^h$ has a unique representation as a linear combination of the form

$$f^h(x) = f^{h,\{j\}}(x) + f^{h,[j]}(x) := \sum_{j \in \mathbb{Z}} \{f^h\}(x_j)\phi^j_i(x) + \sum_{j \in \mathbb{Z}} [f^h](x_j)\phi^{[j]}_i(x),$$

where $f^{h,\{j\}}$ and $f^{h,[j]}$ are the continuous (average)/jump components of $f^h$, respectively. In particular, the piecewise linear discontinuous functions under consideration are perturbations by the jumps $f^{h,[j]}$ added at each nodal point of the classical piecewise linear and continuous ones, represented by the averages $f^{h,\{j\}}$.

Let us introduce some notations: the superscript $\ast$ is the matrix transposition. By bold lower-case characters having the superscript $h$ (e.g., $f^h := (f_j)_{j \in \mathbb{Z}}$) we denote a sequence (an infinite vector) which associates to any grid point $x_j \in \mathcal{G}^h$ a unique value $f_j$. By changing the superscript $h$ into $h$ (to have $f^h$), we indicate a vector associating to any grid point $x_j$ a pair of two values. In our case, any sequence $f^h$ contains two subsequences $\{f^h\} := (\{f^h\}(x_j))_{j \in \mathbb{Z}}$ and $[f^h] := ([f^h](x_j))_{j \in \mathbb{Z}}$, storing all the averages/jumps of $f^h \in V^h$. All vectors under consideration are column vectors. Standard finite matrices are denoted by capital letters (e.g., $A^h$), while the bold capital letter $A^h$ (or $A^h$) stands for infinite ($2 \times 2$-block) matrices. Often we use Toeplitz matrices $A^h$ with the property that all the elements along each diagonal are identical, i.e., $a^h_{i,j} = a^h_{i+1,j+1}$ for all $i,j \in \mathbb{Z}$ (cf. [34]). They arise in the numerical approximations of constant coefficients PDEs on uniform meshes. The infinite block Toeplitz matrices $A^h$, for which the elements on each diagonal of blocks are identical, appear, as before, in the numerical approximation of constant coefficients PDEs on uniform meshes, when using more complex methods associating to each node several degrees of freedom.

The discrete first-order derivative of the piecewise regular function $f^h$ having possible discontinuities only at the grid points $x_j$, $j \in \mathbb{Z}$, is denoted by $\partial_x f^h$ and coincides with the classical first-order derivative $\partial_x f$ in all the open intervals $I_j$, $j \in \mathbb{Z}$. The inner product on the space $L^2(\mathcal{T}^h)$ containing piecewise $L^2$-functions with possible discontinuities on the mesh points is denoted by $(\cdot, \cdot)_{L^2(\mathcal{T}^h)} := \sum_{j \in \mathbb{Z}} (\cdot, \cdot)_{L^2(I_j)}$. Furthermore, the inner products on the spaces $L^2(\mathcal{G}^h)$ and $L^2(\mathcal{G}^h)$ of square summable sequences associating one/two values to each grid point in $\mathcal{G}^h$ are given by

$$(f^h, g^h)_{L^2(\mathcal{G}^h)} = \sum_{j \in \mathbb{Z}} f^h_j \overline{g^h}_j, \quad (f^h, g^h)_{L^2(\mathcal{G}^h)} = \sum_{j \in \mathbb{Z}} (\{f^h\}(x_j)\overline{\{g^h\}}(x_j) + [f^h](x_j)\overline{[g^h]}(x_j)), $$

while the corresponding norms are $\| \cdot \|_{L^2(\mathcal{T}^h)}$, $\| \cdot \|_{L^2(\mathcal{G}^h)}$, and $\| \cdot \|_{L^2(\mathcal{G}^h)}$. 
The DG approximation we analyze in this book is the so-called symmetric interior penalty (SIPG) method [5, 6], whose bilinear form \( A_h \) is defined for all penalty parameters \( s > 1\) as follows:

\[
A_h^s(u^h, v^h) := (\partial_x u^h, \partial_x v^h)_{L^2(\mathcal{F}^h)} - ([\partial_x u^h], [v^h])_{L^2(\mathcal{G}^h)} - ([u^h], \partial_x [v^h])_{L^2(\mathcal{G}^h)} + \frac{s}{h} ([u^h], [v^h])_{L^2(\mathcal{G}^h)}.
\]

(2.1)

We then consider the variational formulation of the SIPG semi-discretization of the 1-d wave equation

\[
u^h(s, t) \in \mathcal{V}^h, \quad (\partial_{tt} u^h(s, t), v^h)_{L^2(\mathbb{R})} + A_h^s(u^h(s, t), v^h) = 0
\]

(2.2)

and of the 1-d Klein–Gordon equation

\[
u^h(s, t) \in \mathcal{V}^h, \quad (\partial_{tt} u^h(s, t), v^h)_{L^2(\mathbb{R})} + A_h^s(u^h(s, t), v^h) + (u^h(s, t), v^h)_{L^2(\mathbb{R})} = 0
\]

(2.3)

for all \( v^h \in \mathcal{V}^h \) and \( t > 0\), complemented with the initial data \( u^h(s, 0) = u^{h,0} \in \mathcal{V}^h \) and \( \partial_t u^h(s, 0) = u^{h,1} \in \mathcal{V}^h \).

By the Hille–Yosida theorem, there exists a unique solution of (2.2) and of (2.3), \( u^h(s, \cdot) \in C([0, \infty), \mathcal{V}^h) \cap C^1([0, \infty), L^2(\mathbb{R})) \). Moreover, the total energy of solutions corresponding to the semi-discrete wave equation,

\[
ed_s^{W,h}(u^{h,0}, u^{h,1}) := \frac{1}{2} (||\partial_t u^h(s, t)||_{L^2} + \mathcal{A}_h^s(u^h(s, t), u^h(s, t))),
\]

(2.4)

or to the discrete Klein–Gordon equation,

\[
ed_s^{K,h}(u^{h,0}, u^{h,1}) := \frac{1}{2} (||\partial_t u^h(s, t)||_{L^2}^2 + \mathcal{A}_h^s(u^h(s, t), u^h(s, t)) + ||u^h(s, t)||_{L^2}^2),
\]

(2.5)

is conserved in time.

The unknown \( u^h(s, \cdot) \), being an element of \( \mathcal{V}^h \) for each \( t > 0 \), can be decomposed as follows:

\[
u^h(x, t) = \sum_{j \in \mathbb{Z}} u^h_j(x_j, t) \phi^{(1)}_j(x) + \sum_{j \in \mathbb{Z}} u^h_j(x_j, t) \phi^{(1)}_j(x).
\]

Using \( \phi^{(1)}_k \) and \( \phi^{(1)}_k \), for all \( k \in \mathbb{Z} \), as test functions in (2.2) and in (2.3), we obtain that the sequence of coefficients \( u^h_j(t) = (u^h_j(x_j, t), [u^h_j](x_j, t))_{j \in \mathbb{Z}} \) of the approximation \( u^h(s, t) \in \mathcal{V}^h \) is the solution of the following infinite system of second-order linear ordinary differential equations (ODEs) for the approximate wave equations for all \( t > 0 \):

\[
M^h \partial_{tt} u^h_x(t) + R^h u^h_x(t) = 0, \quad u^h_x(0) = u^{h,0}, \quad \partial_t u^h_x(0) = u^{h,1}.
\]

(2.6)
For the Klein–Gordon equation, we obtain the following approximating ODE system for all $t > 0$:

$$
M_h \partial_{tt} u_h^s(t) + R_h^s u_h^s(t) + M_h u_h^s(t) = 0, \quad u_h^s(0) = u_h^{0,s}, \quad \partial_t u_h^s(0) = u_h^{1,s}.
$$

The infinite mass and stiffness matrices $M_h$ and $R_h^s$ are block tridiagonal, generated by the stencils $M_h$ and $R_h^s$ below

$$
M_h = \begin{pmatrix}
\frac{h}{6} & \frac{h}{6} & 0 & \frac{h}{6} \\
\frac{h}{6} & \frac{h}{6} & \frac{h}{6} & \frac{h}{6} \\
0 & \frac{h}{6} & \frac{h}{6} & \frac{h}{6} \\
\frac{h}{6} & 0 & \frac{h}{6} & \frac{h}{6}
\end{pmatrix}, \quad R_h^s = \begin{pmatrix}
-\frac{1}{h} & 0 & 0 & -\frac{1}{h} \\
0 & -\frac{1}{4h} & 0 & 0 \\
-\frac{1}{4h} & 0 & -\frac{1}{4h} & 0 \\
0 & 0 & 0 & -\frac{1}{4h}
\end{pmatrix}.
$$

Each block in the stencils $M_h$ and $R_h^s$ is of dimension $2 \times 2$ since it stores the interaction between two nodes and the dynamics at each node $x_j$ is modeled by two basis functions, $\phi_j^{(s)}$ and $\phi_j^{(c)}$. Moreover, $M_h$ and $R_h^s$ contain three $2 \times 2$-blocks since each node $x_j$ has nontrivial interactions with itself and with the two neighboring nodes, $x_{j-1}$ and $x_{j+1}$.

Observe that (2.6) or (2.7) is a coupled system of two different kinds of ODEs, each of them being generated by one of the two rows of the stencils $M_h$ and $R_h^s$.

The total energy $E_{W,h}(u_h^{0,s}, u_h^{1,s})$ in (2.4) of the solution $u_h^s(t)$ of (2.6) can be written in terms of the mass and stiffness matrices $M_h$ and $R_h^s$ as follows:

$$
E_{W,h}(u_h^{0,s}, u_h^{1,s}) := \frac{1}{2} (M_h \partial_t u_h^s(t), \partial_t u_h^s(t))_{L^2(\Omega_h)} + \frac{1}{2} (R_h^s u_h^s(t), u_h^s(t))_{L^2(\Omega_h)}.
$$

Also,

$$
E_{K,h}(u_h^{0,s}, u_h^{1,s}) := E_{W,h}(u_h^{0,s}, u_h^{1,s}) + \frac{1}{2} (M_h u_h^s(t), u_h^s(t))_{L^2(\Omega_h)}.
$$

The aim of this book is to analyze the propagation properties of these DG methods. More precisely, for $\Omega := \mathbb{R} \setminus (-1, 1)$, we investigate the conditions on the observability time $T > 0$ and on the class of initial data $(u_h^{0,s}, u_h^{1,s})$ under consideration in (2.6) or (2.7), for which the following DG version of the observability inequality (1.20) holds uniformly as $h \to 0$ (here, $\varsigma \in \{W, K\}$):

$$
E_{\varsigma,h}(u_h^{0,s}, u_h^{1,s}) \leq C_{\varsigma,h}^h(T) \int_0^T E_{\varsigma,h}(u_h^s(t), \partial_t u_h^s(t)) dt.
$$

As it occurs for classical finite difference and finite element methods, the observability inequality will fail to be uniform for the DG methods under consideration when considering all possible discrete solutions. Thus, in order to make the inequality uniform, one will need to filter the class of initial data to be considered.

The initial data $(u_h^{0,s}, u_h^{1,s})$ in (2.6) or (2.7) can be chosen in various different ways:

- The first one, proposed in [35], consists on taking $(u_h^{0,s}, u_h^{1,s})$ as the $L^2$-projection on the space $V_h$ of the more regular initial data $(u^0, u^1) \in \dot{H}^{1+\sigma}(\mathbb{R}) \times \dot{H}^{\sigma}(\mathbb{R})$ in (1.18) or $(u^0, u^1) \in H^{1+\sigma}(\mathbb{R}) \times H^{\sigma}(\mathbb{R})$ in (1.26), with $\sigma > 0$. Namely, for each
i = 0, 1, the sequence $\mathbf{u}^{h,i}$ is the solution of the system $M^h \mathbf{u}^{h,i} = \mathbf{u}^{h,i}$ where $\mathbf{u}^{h,i} = (u^{h,i}_p(x_j), [u^{h,i}_p](x_j))_{j \in \mathbb{Z}}$, $\{u^{h,i}_p\} = (u^i, \phi_j^i)_{L^2(\mathbb{R})}$ \textit{and $\{u^{h,i}_p\} = (u^i, \phi_j^i)_{L^2(\mathbb{R})}$}. In this way, the error between the solution of the continuous model (1.18) and its DG approximation in the energy norm is of order $h^{\min(0,1)}$ and the following estimates between the continuous and the discrete data hold (cf. [35]):

$$||u^1 - u^{h,1}||_{L^2} \lesssim h^{2\min(0,2)} ||u^1||_{H^2}, \quad A_s^h (u^0 - u^{h,0}, u^0 - u^{h,0}) \lesssim h^{2\min(0,1)} ||u^0||_{H^3+1}.$$

- Another possible choice for the initial data $(u^{h,0}, u^{h,1})$ in the discrete systems (2.6) and (2.7) is to take $(\{u^{h,i}\}(x_j), [u^{h,i}](x_j)) = (u^i(x_j), 0)$, which imposes \textit{null jump conditions} at the initial time. This choice of the discrete initial data also requires $u^i$ to be \textit{continuous} at the grid points. This condition is fulfilled for $u^0 \in H^1(\mathbb{R}) \subset C^0(\mathbb{R})$. For the component $u^1 \in L^2(\mathbb{R})$ we can take $\{u^{h,1}\}(x_j)$ to be the average of $u^1$ on the cell $(x_j-1/2, x_j+1/2)$.

For any two sequences $f^{h,0}, f^{h,1} \in \ell^2(\mathcal{G}^h)$, the discrete \textit{energies concentrated in} $\Omega$, entering on the right-hand side of (2.9), are given by

$$E^{W,h}_{s,\Omega}(f^{h,0}, f^{h,1}) := \frac{1}{2} (M^h f^{h,1}, f^{h,1})_{\ell^2(\mathcal{G}^h \cap \Omega)} + \frac{1}{2} (R^h f^{h,0}, f^{h,0})_{\ell^2(\mathcal{G}^h \cap \Omega)}$$

and

$$E^{K,h}_{s,\Omega}(f^{h,0}, f^{h,1}) := E^{W,h}_{s,\Omega}(f^{h,0}, f^{h,1}) + \frac{1}{2} (M^h f^{h,0}, f^{h,0})_{\ell^2(\mathcal{G}^h \cap \Omega)}.$$

Here, $\ell^2(\mathcal{G}^h \cap \Omega)$ is the space of \textit{square summable} functions endowed with the norm $|| \cdot ||_{\ell^2(\mathcal{G}^h \cap \Omega)}$ and the inner product

$$(f^h, g^h)_{\ell^2(\mathcal{G}^h \cap \Omega)} := \sum_{x_j \in \mathcal{G}^h \cap \Omega} (\{f^h\}(x_j), \{g^h\}(x_j) + [f^h](x_j)) [\{g^h\}(x_j)].$$

\section*{2.2 Presentation of the Main Results}

To the best of our knowledge, the present book is the first one containing a rigorous analysis of the DG methods for wave control problems, their pathologies and remedies. In what follows, we briefly describe the main results of this book.

In Chap. 3 we present a brief historical development of the DG methods under consideration and the connection between our results and the existing literature. This will allow us to compare these well-known results with the ones we shall get for the more sophisticated DG methods.
Given any sequence \( f^h = (f_j)_{j \in \mathbb{Z}} \in \ell^2(\mathcal{G}^h) \), we define its semi-discrete Fourier transform (SDFT) at scale \( h \) as follows (see, e.g., [38, 67]):

\[
\hat{f}^h(\xi) := h \sum_{j \in \mathbb{Z}} f_j \exp(-i\xi x_j), \quad \text{with } \xi \in \Pi^h := [-\pi/h, \pi/h].
\] (2.10)

In what follows, by bold lower-case/capital letters with hat symbol and superscript \( h \) [e.g., \( \mathbf{f}^h \) and \( \mathbf{F}^h(\xi) \)], we denote vectors/matrices of SDFTs, while the scalar SDFTs are represented by standard lower-case characters with hat symbol and superscript \( h \) [e.g., \( \hat{f}^h(\xi) \)].

One of the most important properties of the SDFT at scale \( h \) is its periodicity of principal period \( 2\pi/h \), i.e., \( \hat{f}^h(\xi) = \hat{f}^h(\xi + 2k\pi/h) \) for all \( \xi \in \Pi^h \) and \( k \in \mathbb{Z} \). This explains the choice of the domain of definition for the SDFT to be the interval \( \Pi^h \) of length \( 2\pi/h \). Although any interval of the form \( [\xi^0, \xi^0 + 2\pi/h] \), with \( \xi^0 \in \mathbb{R} \), could be chosen as domain of definition for the SDFT, \( \Pi^h \) is the only possible interval of the above form which is symmetric with respect to \( \xi = 0 \), a property that the domain \( \mathbb{R} \) of the continuous Fourier transform also has. In fact, as \( h \to 0 \), the interval \( \Pi^h \) covers the whole real line.

The second important property of the SDFT is the Parseval identity:

\[
||f^h||^2_{\ell^2(\mathcal{G}^h)} = \frac{1}{2\pi} ||\hat{f}^h||^2_{L^2(\Pi^h)}, \quad \forall f^h \in \ell^2(\mathcal{G}^h).
\] (2.11)

The third property of the SDFT useful for our analysis is the fact that the SDFT can be inverted. Indeed, for any \( f^h \in \ell^2(\mathcal{G}^h) \) and for any \( j \in \mathbb{Z} \), the following identity holds:

\[
f_j = \frac{1}{2\pi} \int_{\Pi^h} \hat{f}^h(\xi) \exp(i\xi x_j) \, d\xi.
\] (2.12)

Chapter 4 is devoted to a careful Fourier analysis of the systems (2.6) and (2.7). Their solutions \( u^h_s(t) \) can also be written as a pair \( (\mathbf{u}^h_s(\cdot, t), [\mathbf{u}^h_s(\cdot, t)]) \) composed by the two sequences of different nature containing the averages/jumps of the numerical solutions. Let \( \mathbf{u}^h_s(\xi, t) := (\hat{u}^h_s(\xi, t), \hat{u}^h_s[\xi, t]) \) be the column vector of the SDFTs of the sequence of averages \( \{\mathbf{u}^h_s(\cdot, t)\} \) and of jumps \( [\mathbf{u}^h_s(\cdot, t)] \). When dealing with system (2.6), the vector function \( \hat{\mathbf{u}}^h_s(\xi, t) \) satisfies the following system of second-order ODEs in time, depending on the wave number parameter \( \xi \in \Pi^h \):

\[
\begin{aligned}
\hat{\mathbf{M}}^h(\xi) \partial_t \hat{\mathbf{u}}^h_s(\xi, t) + \hat{\mathbf{K}}^h(\xi) \hat{u}^h_s(\xi, t) &= 0, \\
\hat{u}^h_s(\xi, 0) &= \hat{u}^{h,0}_s(\xi), \quad \partial_t \hat{u}^h_s(\xi, 0) = \hat{u}^{h,1}_s(\xi),
\end{aligned}
\] (2.13)

where \( \xi \in \Pi^h \), \( t > 0 \), and \( \hat{\mathbf{M}}^h(\xi) \) and \( \hat{\mathbf{K}}^h_s(\xi) \) are the matrix Fourier symbols of the mass and stiffness matrices \( \mathbf{M}^h \) and \( \mathbf{K}^h_s \) given by

\[
\hat{\mathbf{M}}^h(\xi) = \begin{pmatrix}
\frac{2+\cos(\xi h)}{3} & \frac{i \sin(\xi h)}{6} \\
\frac{-i \sin(\xi h)}{6} & \frac{2-\cos(\xi h)}{12}
\end{pmatrix}, \quad \hat{\mathbf{K}}^h_s(\xi) = \begin{pmatrix}
\frac{4}{h^2} \sin^2 \left( \frac{\xi h}{2} \right) & 0 \\
0 & \frac{s-\cos^2 \left( \frac{\xi h}{2} \right)}{h^2}
\end{pmatrix}.
\] (2.14)
Using the Parseval identity (2.11), we find the following Fourier representation of the total energy (2.4) (in which $(\cdot, \cdot)_{C}$ is the inner product in $C^n$, with $n \in \mathbb{N}$):

$$
E_{h}^{W,h}(u^{h,0}, u^{h,1}) = \frac{1}{4\pi} \int_{h} \left[ (\hat{R}^{h}_{s}(\xi)\hat{u}^{h,0}(\xi), \hat{u}^{h,0}(\xi))_{C^2} + (\hat{M}^{h}(\xi)\hat{u}^{h,1}(\xi), \hat{u}^{h,1}(\xi))_{C^2} \right] d\xi. \quad (2.15)
$$

Remark that, unlike (2.6), which is an infinite system, (2.13) can be explicitly solved in terms of the eigenvalues $\hat{\lambda}^{h}_{s,\alpha}(\xi)$ of the matrix $\hat{S}^{h}_{s}(\xi)$ and of its eigenvectors $\hat{v}^{h}_{s}(\xi)$. The two families of eigensolutions $(\hat{\lambda}^{h}_{s,\alpha}(\xi), \hat{v}^{h}_{s}(\xi))$ of the matrix $\hat{S}^{h}_{s}(\xi)$ are denoted by $(\hat{\lambda}^{h}_{s,\alpha}(\xi), \hat{v}^{h}_{s}(\xi)), \alpha \in \{\text{ph, sp}\}$, where the subscripts ph/sp stand for physical/spurious. Set

$$
\hat{\lambda}^{h}_{s,\alpha}(\xi) := \text{sign}(\xi) \sqrt{\hat{\lambda}^{h}_{s,\alpha}(\xi)}, \quad (2.16)
$$

with $\alpha \in \{\text{ph, sp}\}$, to be the so-called dispersion relations of the SIPG method under consideration.

As we can see in Fig. 2.2 and rigorously in Chap. 4, the physical dispersion relation $\hat{\lambda}^{h}_{s,\text{ph}}(\xi)$ lies between the two diagrams corresponding to the FD and $P_{1}$-FEM semi-discretizations, $\hat{\lambda}^{h}_{1,\text{ph}}(\xi)$ and $\hat{\lambda}^{h}_{\infty,\text{ph}}(\xi)$. The spurious dispersion relation, $\hat{\lambda}^{h}_{s,\text{sp}}(\xi)$, tends to infinity for large values of $s$. One can prove that, for all $s \in (1, \infty) \setminus \{3\}$, the physical dispersion relation is strictly increasing for all the wave numbers $\xi \in (0, \pi/h)$, with only one critical point located at $\xi = \pi/h$, at which its first-order derivative $\partial_{\xi} \hat{\lambda}^{h}_{s,\text{ph}}(\xi)$, the so-called physical group velocity, vanishes. Concerning the monotonicity of the spurious dispersion relation, several ranges of $s$ can be identified according to the stabilization parameter $s$. Excepting for the values $s \in (5/3, 5/2)$, for which there are three critical points on $\hat{\lambda}^{h}_{s,\text{ph}}(\xi)$, and $s = 3$, for which there exists only the critical point $\xi = 0$, for all the other values of $s$, the group velocity $\partial_{\xi} \hat{\lambda}^{h}_{s,\text{sp}}(\xi)$ vanishes at two wave numbers: $\xi = 0$ and $\xi = \pi/h$.

A similar Fourier analysis to the one in Chap. 4 can be also applied to the discrete Klein–Gordon equation (2.7). By taking SDFT in the approximate Klein–Gordon equation (2.7), we obtain

$$
\begin{cases}
\hat{M}^{h}(\xi)\partial_{tt}\hat{u}^{h}(\xi, t) + \hat{R}^{h}(\xi)\hat{u}^{h}(\xi, t) + \hat{M}^{h}(\xi)\hat{u}^{h}(\xi, t) = 0, \\
\hat{u}^{h}(\xi, 0) = \hat{u}^{h,0}(\xi), \quad \partial_{t}\hat{u}^{h}(\xi, 0) = \hat{u}^{h,1}(\xi),
\end{cases} \quad (2.18)
$$

so that the two eigenvalues of matrix $(\hat{M}^{h}(\xi))^{-1}\hat{S}^{h}_{s}(\xi) + I_{2}$ corresponding to system (2.18) are one unit higher than the ones of matrix $(\hat{M}^{h}(\xi))^{-1}\hat{S}^{h}_{s}(\xi)$ in system (2.13), while the eigenvectors are the same.
Similarly to (2.15), the Fourier representation of the total energy (2.5) is

$$
\mathcal{E}_s^K(u^{h,0}, u^{h,1}) = \mathcal{E}_s^W(u^{h,0}, u^{h,1}) + \frac{1}{4\pi} \int_{\Omega_h} (\hat{M}^h(\xi) \hat{u}^{h,0}(\xi), \hat{u}^{h,0}(\xi))_{C^2} d\xi. \quad (2.19)
$$

The existence of wave numbers $\xi$, where one of the two group velocities (the physical or the spurious one) vanishes, necessarily implies that the observability constant $C^h_s(T)$ in (2.9) blows up at least polynomially at any order as the mesh size parameter $h$ tends to zero. The behavior of the constant $C^h_s(T)$ will be analyzed in Chap. 5 by adapting well-known constructions of high-frequency Gaussian wave packets previously implemented for the classical approximations (FD or $P_1$-FEM) in [28] or [49].

Consequently, the effect of these pathological wave numbers and components of the dispersion diagram need to be attenuated or filtered out. This needs to be done,
in particular, with the high-frequency components at $\xi = \pi/h$ on the physical dispersion diagram or the ones at $\xi = 0$ and at $\xi = \pi/h$ on the spurious one. Chapter 6 is devoted to designing and analyzing appropriate filtering techniques for the SIPG method aimed to face the pathologies introduced by the singularities of the two dispersion diagrams, i.e., to reestablish the discrete observability inequality (2.9) with a constant $C_h(\mathbf{T})$ independent of $h$. All these filtering mechanisms consist in fact in imposing restrictions on the initial data $(\mathbf{u}^{h,0}, \mathbf{u}^{h,1})$ in (2.6) so that the observability result (2.9) is uniform with respect to $h$. Due to the complex structure of the Fourier decomposition of the systems under consideration, each one of these filtering algorithms combines two techniques previously proved to be efficient for the FD or the $P_1$-FEM approximations. Let us briefly present the four filtering strategies we analyze.

**Algorithm A.** The algorithm in Sect. 6.1 is based on the use of numerical initial data in (2.6) or (2.7) involving only the physical mode and chosen to be of the form 
$$\hat{\mathbf{u}}^{h,i}(\xi) := \mathbf{v}_{s,ph}^h(\xi)\hat{u}^{h,i}(\xi)$$
for all $i = 0, 1$, where $\mathbf{v}_{s,ph}^h(\xi)$ is the physical eigenvector of the SIPG method and $\hat{u}^{h,i}(\xi)$ is a scalar function. This choice yields solutions of (2.13) involving only the physical mode. Then we apply the Fourier truncation method, so that the support of the Fourier transform of the initial data is contained in $\mathbb{Q}_h(\mathbf{T}) := [-\pi\delta/h, \pi\delta/h]$ for some $\delta \in (0, 1)$ and it does not contain the critical point of the physical dispersion relation.

**Algorithm B.** The strategy in Sect. 6.2 is also based on data involving only the physical mode as in Algorithm A, but with its odd components obtained as SDFTs of two sequences $\mathbf{u}^{h,0}$ and $\mathbf{u}^{h,1}$ produced by the classical bi-grid algorithm, i.e., only its odd components $(u_{2j+1}^i)^{h,i}$ are given, while the even ones are defined as linear interpolation of the two neighboring odd ones. We refer to [37, 59] for the basic properties of data in the bi-grid class.

This algorithm is of purely theoretical interest, since the practical utility of the bi-grid algorithm consists precisely in the fact that it does not need to use the Fourier transform and its inverse to compute the numerical approximation, but rather that it can be fully implemented on the physical grid. The goal of our analysis in Sect. 6.2 is to highlight that, at least from a theoretical point of view, once a dispersion relation having only a critical point at $\xi = \pi/h$ is generated, independently of the complexity of its formula, the bi-grid algorithm is still efficient to attenuate the high-frequency pathologies of the underlying solutions.

**Algorithm C.** The filtering algorithm in Sect. 6.3 is based on the use of initial data with null jump components, i.e., their jump components $[\mathbf{u}^{h,i}]$ vanish for both $i = 0$ and $i = 1$. The average components $\{\mathbf{u}^{h,i}\}$ are then given by the Fourier truncation method with parameter $\delta \in (0, 1)$ as in the Algorithm A, to avoid the critical point on the physical diagram at $\xi = \pi/h$.

**Algorithm D.** The strategy in Sect. 6.4 is also based on initial data with null jump components as in Algorithm C, but now the averages $\{\mathbf{u}^{h,i}\}$ are defined by a bi-grid algorithm of mesh ratio $1/2$, as in Algorithm B.
Chapter 7 is devoted to briefly analyze some other closely related methods such as the classical quadratic finite element method ($P_2$-FEM), the so-called $P_1$-local discontinuous Galerkin ($P_1$-LDG) method and a version of the SIPG method in which both the jumps of the numerical solution and of its normal derivative along the interfaces are penalized (SIPG-n). As we shall see, the analysis of all these methods can be developed using the tools presented in this book.
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