Chapter 2
Observability for the $1-d$ Finite-Difference Wave Equation

2.1 Objectives

In this chapter, we discuss the observability properties for the $1-d$ finite-difference wave equation.

For the convenience of the reader, let us recall the equations, already introduced in Eq. (1.80).

Let $N \in \mathbb{N}$, $h = 1/(N+1)$. Given $(\varphi_{0h}, \varphi_{1h})$, compute the solution φ_h of the following system:

$$\begin{cases} \partial_{tt}\varphi_{j,h} - \dfrac{1}{h^2}\left(\varphi_{j+1,h} - 2\varphi_{j,h} + \varphi_{j-1,h}\right) = 0, & (t,j) \in (0,T)\times\{1,\cdots,N\}, \\ \varphi_{0,h}(t) = \varphi_{N+1,h}(t) = 0, & t \in (0,T), \\ (\varphi_h(0), \partial_t\varphi_h(0)) = (\varphi_{0h}, \varphi_{1h}). \end{cases} \tag{2.1}$$

Here, we will not be interested in any convergence process, but rather try to prove some estimates uniformly with respect to $h > 0$, and in particular uniform admissibility and observability results. Before going further, let us also emphasize that this uniform admissibility result will be an important step in the proof of the convergence of the discrete waves towards the continuous ones when working with boundary data in $L^2(0,T)$.

Note that the discrete equation (2.1), as its continuous counterpart, is conservative in the sense that its energy

$$E_h[\varphi_h](t) = h\sum_{j=1}^{N} |\partial_t\varphi_j(t)|^2 + h\sum_{j=0}^{N}\left(\frac{\varphi_{j+1}(t) - \varphi_j(t)}{h}\right)^2, \tag{2.2}$$

sometimes simply denoted by $E_h(t)$ when no confusion may occur, is constant in time:

$$\forall t \geq 0, \quad E_h(t) = E_h(0). \tag{2.3}$$

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2.2 Spectral Decomposition of the Discrete Laplacian

In this section, we briefly recall the spectral decomposition of the discrete Laplacian.

To be more precise, we consider the eigenvalue problem associated with the 3-point finite-difference scheme for the $1-d$ Laplacian:

$$\begin{cases} -\dfrac{w_{j+1}+w_{j-1}-2w_j}{h^2} = \lambda w_j, \quad j=0,\cdots,N+1, \\ w_0 = w_{N+1} = 0. \end{cases} \tag{2.4}$$

A simple iteration process shows that if $w_1 = 0$ and w solves (2.4), then $w_j = 0$ for all $j \in \{0,\cdots,N+1\}$. Hence all the eigenvalues are simple.

Furthermore the spectrum of the discrete Laplacian is given by the sequence of eigenvalues

$$0 < \lambda_1(h) < \lambda_2(h) < \cdots < \lambda_N(h),$$

which can be computed explicitly

$$\lambda_k(h) = \frac{4}{h^2}\sin^2\left(\frac{\pi k h}{2}\right), \quad k=1,\cdots,N. \tag{2.5}$$

The eigenvector $w^k = \left(w_1^k,\cdots,w_N^k\right)$ associated to the eigenvalue $\lambda_k(h)$ can also be computed explicitly:

$$w_j^k = \sqrt{2}\sin\left(\pi k jh\right), \quad j=1,\cdots,N. \tag{2.6}$$

Observe in particular that the eigenvectors of the discrete system do not depend on $h>0$ and coincide with the restriction of the continuous eigenfunctions $w^k(x) = \sqrt{2}\sin\left(k\pi x\right)$ of the Laplace operator on $(0,1)$ to the discrete mesh.

Let us now compare the eigenvalues of the discrete Laplace operator Δ_h and the continuous one ∂_{xx}:

- For fixed k, $\lim_{h\to 0}\lambda_k(h) = \pi^2 k^2$, which is the k-th eigenvalue of the continuous Laplace operator $-\partial_{xx}$ on $(0,1)$.
- We have the following bounds:

$$\frac{4}{\pi^2}k^2\pi^2 \le \lambda_k(h) \le k^2\pi^2 \quad \text{for all} \quad 0<h<1, \quad 1\le k\le N. \tag{2.7}$$

- The discrete eigenvalues $\sqrt{\lambda_k(h)}$ uniformly converge to the corresponding continuous ones $k\pi$ when $k = o(1/h^{2/3})$ since, at first order,

$$\left|\sqrt{\lambda_k(h)} - k\pi\right| \sim Ck^3h^2. \tag{2.8}$$

Let us now recall some orthogonality properties of the eigenvectors that can be found, e.g., in [28]:

Lemma 2.1. *For any eigenvector w with eigenvalue λ of Eq. (2.4) the following identity holds:*

$$h\sum_{j=0}^{N}\left|\frac{w_{j+1}-w_j}{h}\right|^2 = \lambda h\sum_{j=1}^{N}|w_j|^2. \tag{2.9}$$

The eigenvectors $(w^k)_{k\in\{1,\cdots,N\}}$ in (2.6) satisfy:

$$h\sum_{j=1}^{N} w_j^k w_j^\ell = \delta_{k\ell}, \tag{2.10}$$

and

$$h\sum_{j=0}^{N}\left(\frac{w_{j+1}^k-w_j^k}{h}\right)\left(\frac{w_{j+1}^\ell-w_j^\ell}{h}\right) = \lambda_k\delta_{k\ell}, \tag{2.11}$$

where $\delta_{k\ell}$ is the Kronecker symbol.

2.3 Uniform Admissibility of Discrete Waves

For convenience and later use, we begin by stating a uniform admissibility result, which can also be found in [28] and will be useful for studying the convergence of the discrete normal derivatives of the solutions of Eq. (2.1) towards the continuous ones.

Theorem 2.1. *For all time $T > 0$ there exists a finite positive constant $C(T) > 0$ such that*

$$\int_0^T \left|\frac{\varphi_N(t)}{h}\right|^2 \mathrm{d}t \leq C(T)E_h(0), \tag{2.12}$$

for all solution φ_h of the adjoint equation (2.1) and for all $h > 0$. Besides, we can take $C(T) = T + 2$.

The proof of Theorem 2.1 is briefly given in Sect. 2.3.2. It is based on a multiplier identity given in the next section.

2.3.1 The Multiplier Identity

Our results are based on the following multiplier identity that can be found in [28]:

Theorem 2.2. *For all $h > 0$ and $T > 0$ any solution φ_h of Eq. (2.1) satisfies*

$$TE_h(0) + X_h(t)\Big|_0^T = \int_0^T \left|\frac{\varphi_N(t)}{h}\right|^2 \mathrm{d}t + \frac{h^3}{2}\sum_{j=0}^{N}\int_0^T \left|\frac{\partial_t\varphi_{j+1}-\partial_t\varphi_j}{h}\right|^2 \mathrm{d}t, \tag{2.13}$$

with

$$X_h(t) = 2h \sum_{j=1}^{N} jh \left(\frac{\varphi_{j+1} - \varphi_{j-1}}{2h} \right) \partial_t \varphi_j. \tag{2.14}$$

The proof of Theorem 2.2 uses the multiplier $j(\varphi_{j+1} - \varphi_{j-1})$, which is the discrete counterpart of $x\partial_x \varphi$. Integrating by parts in space (in a discrete manner) and time, we obtain (2.13). We refer to [28] for the details of the computations. We only sketch it below since it will be useful later on in Chap. 4.

Proof (Sketch). Multiplying the Eq. (2.1) by $jh(\varphi_{j+1} - \varphi_{j-1})/h$, we have

$$\begin{aligned} &h \sum_{j=1}^{N} \int_0^T \partial_{tt} \varphi_j \, jh \left(\frac{\varphi_{j+1} - \varphi_{j-1}}{h} \right) \mathrm{d}t \\ &\quad = h \sum_{j=1}^{N} \int_0^T \frac{1}{h^2} \left(\varphi_{j+1} - 2\varphi_j + \varphi_{j-1} \right) jh \left(\frac{\varphi_{j+1} - \varphi_{j-1}}{h} \right) \mathrm{d}t. \end{aligned}$$

After tedious computations, one shows (cf. [28])

$$\begin{aligned} h \sum_{j=1}^{N} \int_0^T \partial_{tt} \varphi_j \, jh \left(\frac{\varphi_{j+1} - \varphi_{j-1}}{h} \right) \mathrm{d}t = X_h(t) \Big|_0^T + h \sum_{j=1}^{N} \int_0^T |\partial_t \varphi_j|^2 \, \mathrm{d}t \\ - \frac{h^3}{2} \sum_{j=0}^{N} \int_0^T \left| \frac{\partial_t \varphi_{j+1} - \partial_t \varphi_j}{h} \right|^2 \mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} &h \sum_{j=1}^{N} \int_0^T \frac{1}{h^2} \left(\varphi_{j+1} - 2\varphi_j + \varphi_{j-1} \right) jh \left(\frac{\varphi_{j+1} - \varphi_{j-1}}{h} \right) \mathrm{d}t \\ &\quad = \int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 \mathrm{d}t - h \sum_{j=0}^{N} \int_0^T \left(\frac{\varphi_{j+1} - \varphi_j}{h} \right)^2 \mathrm{d}t. \end{aligned}$$

Putting these identities together yields (2.13). □

2.3.2 Proof of the Uniform Hidden Regularity Result

Proof (Theorem 2.1). This is an immediate consequence of Theorem 2.2. It suffices to bound the time boundary terms $X_h(T) - X_h(0)$ by the energy E_h to get the result:

$$
\begin{aligned}
|X_h| &\leq 2\left[h\sum_{j=1}^{N}\left|\partial_t\varphi_j\right|^2\right]^{1/2}\left[h\sum_{j=1}^{N}\left|jh\left(\frac{\varphi_{j+1}-\varphi_{j-1}}{2h}\right)\right|^2\right]^{1/2}\\
&\leq 2\left[h\sum_{j=1}^{N}\left|\partial_t\varphi_j\right|^2\right]^{1/2}\left[h\sum_{j=1}^{N}\left(\frac{\varphi_{j+1}-\varphi_{j-1}}{2h}\right)^2\right]^{1/2} \leq E_h. \qquad (2.15)
\end{aligned}
$$

This concludes the proof of Theorem 2.1. □

2.4 An Observability Result

The goal of this section is to show the following result:

Theorem 2.3. *Assume that $\gamma < 1$. Then for all T such that*

$$
T > T(\gamma) = 2/\cos(\pi\gamma/2), \qquad (2.16)
$$

for every solution φ_h of Eq. (1.80) *in the class*

$$
\mathscr{V}_h(\gamma/h) = Span\left\{w^k, \quad kh \leq \gamma\right\}
$$

uniformly as $h \to 0$, we have

$$
\left(T\cos^2\left(\frac{\gamma\pi}{2}\right) - 2\cos\left(\frac{\pi\gamma}{2}\right) - \frac{h}{2}\right)E_h(0) \leq \int_0^T \left|\frac{\varphi_N}{h}\right|^2 \mathrm{d}t, \qquad (2.17)
$$

where E_h is the discrete energy of solutions of Eq. (2.1) *defined in Eq.* (2.2).

The proof of Theorem 2.3 is based on the discrete multiplier identity in Theorem 2.2 (and developed in [28]). However, the estimates we explain below yield a sharp result on the uniform time of observability for discrete waves with an explicit uniform observability constant, thus improving the estimates in [28].

2.4.1 Equipartition of the Energy

We also recall the following proof of the so-called property of equipartition of the energy for discrete waves:

Lemma 2.2 (Equipartition of the energy). *For $h > 0$ and φ_h solution of Eq.* (2.1),

$$
-h\sum_{j=1}^{N}\int_0^T |\partial_{tt}\varphi_j|^2 \mathrm{d}t + h\sum_{j=0}^{N}\int_0^T\left|\frac{\partial_t\varphi_{j+1}-\partial_t\varphi_j}{h}\right|^2\mathrm{d}t + Y_h(t)\Big|_0^T = 0, \qquad (2.18)
$$

where

$$Y_h(t) = h \sum_{j=1}^{N} \partial_{tt} \varphi_j \partial_t \varphi_j. \tag{2.19}$$

Again, for the proof of Lemma 2.2, we refer to [28].

2.4.2 The Multiplier Identity Revisited

From now on, we do not follow anymore the proofs of [28] but rather try to optimize them to improve the obtained estimates.

We introduce a modified energy $\tilde{E}_h$ for solutions φ_h or Eq. (2.1). First, remark that any φ_h solution of Eq. (2.1) can be developed on the basis of eigenfunctions of $-\Delta_h$ as follows:

$$\varphi_h(t) = \sum_{|k|\le N} \hat{\varphi}_k e^{i\mu_k(h)t} w^{|k|} \tag{2.20}$$

with $\mu_k(h) = \sqrt{\lambda_k(h)}$ for $k > 0$ and $\mu_{-k}(h) = -\mu_k(h)$.

According to Lemma 2.1, its energy reads as

$$E_h[\varphi_h] = 2 \sum_{|k|\le N} |\hat{\varphi}_k|^2 \lambda_k(h). \tag{2.21}$$

Similarly, the energy of $\partial_t \varphi_h$, which is also a solution of Eq. (2.1), and that we shall denote by $E_h[\partial_t \varphi_h]$ to avoid confusion, can be rewritten as

$$E_h[\partial_t \varphi_h] = 2 \sum_{|k|\le N} |\hat{\varphi}_k|^2 \lambda_k(h)^2.$$

Note that, of course, $E_h[\varphi_h]$ and $E_h[\partial_t \varphi_h]$ are independent of time since φ_h and $\partial_t \varphi_h$ are solutions of Eq. (2.1).

We then introduce

$$\tilde{E}_h[\varphi_h] = E_h[\varphi_h] - \frac{h^2}{4} E_h[\partial_t \varphi_h]. \tag{2.22}$$

This modified energy is thus constant in time and satisfies

$$\tilde{E}_h[\varphi_h] = 2 \sum_{|k|\le N} |\hat{\varphi}_k|^2 \lambda_{|k|}(h) \cos^2\left(\frac{k\pi h}{2}\right). \tag{2.23}$$

We are now in position to state the following multiplier identity:

Theorem 2.4. *For all $h > 0$ and $T > 0$, any solution φ_h of Eq. (2.1) satisfies*

$$T\tilde{E}_h[\varphi_h] + Z_h(t)\Big|_0^T = \int_0^T \left|\frac{\varphi_N(t)}{h}\right|^2 \mathrm{d}t \tag{2.24}$$

with

$$Z_h(t) = X_h(t) + \frac{h^2}{4} Y_h(t), \quad \textit{with } Y_h(t) = h \sum_{j=1}^{N} \partial_t \varphi_j \partial_{tt} \varphi_j. \tag{2.25}$$

Proof. To simplify the notations, we do not make explicit the dependence in $h > 0$, which is assumed to be fixed along the computations.

According to Lemma 2.2, since $\partial_t \varphi_h$ is a solution of Eq. (2.1), the following identity holds:

$$h \sum_{j=0}^{N} \int_0^T \left| \frac{\partial_t \varphi_{j+1} - \partial_t \varphi_j}{h} \right|^2 \mathrm{d}t = \frac{h}{2} \sum_{j=1}^{N} \int_0^T |\partial_{tt} \varphi_j|^2 \mathrm{d}t + \frac{h}{2} \sum_{j=0}^{N} \int_0^T \left| \frac{\partial_t \varphi_{j+1} - \partial_t \varphi_j}{h} \right|^2 \mathrm{d}t - \frac{Y_h(t)}{2} \Big|_0^T, \tag{2.26}$$

where Y_h is as in Eq. (2.25).

Of course,

$$h \sum_{j=1}^{N} \int_0^T |\partial_t \varphi_j|^2 \mathrm{d}t + h \sum_{j=0}^{N} \int_0^T \left| \frac{\partial_t \varphi_{j+1} - \partial_t \varphi_j}{h} \right|^2 \mathrm{d}t = T E_h[\partial_t \varphi_h],$$

and then Eq. (2.24) follows from Eqs. (2.26) and (2.13). □

2.4.3 *Uniform Observability for Filtered Solutions*

We now focus on the proof of Theorem 2.3. It mainly consists in estimating the terms in Eq. (2.24) and in particular $Z_h(t)$.

2.4.3.1 Estimates on $Y_h(t)$

Let us begin with the following bound on Y_h:

Lemma 2.3. *For all $h > 0$ and $t \geq 0$, for any solution φ_h of Eq. (2.1),*

$$h^2 |Y_h(t)| \leq h E_h[\varphi_h]. \tag{2.27}$$

Proof. Computing $h^2 Y_h$ we get

$$\begin{aligned} h^2 Y_h(t) &= h \sum_{j=1}^{N} \partial_t \varphi_j (h^2 \partial_{tt} \varphi_j) \\ &= h \sum_{j=1}^{N} \partial_t \varphi_j (\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}) \\ &= h^2 \sum_{j=1}^{N} \partial_t \varphi_j \left(\frac{\varphi_{j+1} - \varphi_j}{h} \right) - h^2 \sum_{j=1}^{N} \partial_t \varphi_j \left(\frac{\varphi_j - \varphi_{j-1}}{h} \right). \end{aligned}$$

But

$$2\left|h\sum_{j=1}^{N}\partial_t\varphi_j\left(\frac{\varphi_{j+1}-\varphi_j}{h}\right)\right|\leq E_h(t),$$

and thus estimate (2.27) follows immediately. □

2.4.3.2 Estimates on $X_h(t)$

This is the most technical step of our proof. The idea is to use the Fourier decomposition of solutions φ_h of Eq. (2.1) to bound X_h conveniently.

Proposition 2.1. *For all $h>0$, $t\geq 0$, and $\gamma\in(0,1)$, any solution φ_h of Eq. (2.1) with data in $\mathscr{V}_h(\gamma/h)$ satisfies*

$$|X_h(t)|\leq\frac{\tilde{E}_h[\varphi_h]}{\cos\left(\dfrac{\gamma\pi}{2}\right)}. \tag{2.28}$$

Proof. Let us begin by computing $\tilde{E}_h[\varphi_h]$ at some time t, for instance, $t=0$, in terms of the Fourier coefficients of $\varphi_h(t),\partial_t\varphi_h(t)$. If

$$\varphi_h^0=\sum_{k=1}^{N}\hat{a}_k w^k,\quad \varphi_h^1=\sum_{\ell=1}^{N}\hat{b}_\ell w^\ell,$$

then $\tilde{E}_h$ can be written as

$$\tilde{E}_h=\sum_{k=1}^{N}|\hat{a}_k|^2\lambda_k(h)\cos^2\left(\frac{k\pi h}{2}\right)+\sum_{\ell=1}^{N}|\hat{b}_\ell|^2\cos^2\left(\frac{\ell\pi h}{2}\right). \tag{2.29}$$

Proposition 2.1 is then a direct consequence of the following lemma:

Lemma 2.4. *Let a_h and b_h be two discrete functions which can be written as*

$$a_h=\sum_{k=1}^{N}\hat{a}_k w^k,\quad b_h=\sum_{\ell=1}^{N}\hat{b}_\ell w^\ell.$$

Then, setting

$$X_h(a_h,b_h)=2h\sum_{j=1}^{N}jh\left(\frac{a_{j+1}-a_{j-1}}{2h}\right)b_j,$$

we have

$$|X_h(a_h,b_h)|\leq 2\left(\sum_{k=1}^{N}|\hat{a}_k|^2\lambda_k(h)\cos^2\left(\frac{k\pi h}{2}\right)\right)^{1/2}\left(\sum_{\ell=1}^{N}|\hat{b}_\ell|^2\right)^{1/2}. \tag{2.30}$$

In particular, if we assume that, for some $\gamma\in(0,1)$,

$$\hat{a}_k = \hat{b}_\ell = 0, \quad \forall k, \ell \geq \gamma(N+1), \tag{2.31}$$

then

$$|X_h(a_h, b_h)| \leq \frac{1}{\cos\left(\frac{\gamma\pi}{2}\right)} \left[\sum_{k=1}^{N} |\hat{a}_k|^2 \lambda_k(h) \cos^2\left(\frac{k\pi h}{2}\right) + \sum_{\ell=1}^{N} |\hat{b}_\ell|^2 \cos^2\left(\frac{\ell\pi h}{2}\right) \right]. \tag{2.32}$$

Of course, Lemma 2.4 and in particular estimate (2.32), proved hereafter, immediately yield (2.28). □

Proof (Lemma 2.4). For all $j \in \{1, \cdots, N\}$,

$$\frac{a_{j+1} - a_{j-1}}{2h} = \sqrt{2} \sum_{k=1}^{N} \hat{a}_k \cos(k\pi jh) \frac{\sin(k\pi h)}{h}.$$

Thus,

$$X_h(a_h, b_h) = 4h \sum_{j=1}^{N} jh \left(\sum_{k=1}^{N} \hat{a}_k \cos(k\pi jh) \frac{\sin(k\pi h)}{h} \right) \left(\sum_{\ell=1}^{N} \hat{b}_\ell \sin(\ell\pi jh) \right).$$

Therefore, by orthogonality properties of the discrete cosine functions (the counterpart of Lemma 2.1 with the cosine functions),

$$\begin{aligned} &|X_h(a_h, b_h)|^2 \\ &\leq 4 \left(2h \sum_{j=1}^{N} \left(\sum_{k=1}^{N} \hat{a}_k \cos(k\pi jh) \frac{\sin(k\pi h)}{h} \right)^2 \right) \left(2h \sum_{j=1}^{N} \left(\sum_{\ell=1}^{N} \hat{b}_\ell \sin(\ell\pi jh) \right)^2 \right) \\ &\leq 4 \left(\sum_{k=1}^{N} |\hat{a}_k|^2 \left(\frac{\sin(k\pi h)}{h} \right)^2 \right) \left(\sum_{\ell=1}^{N} |\hat{b}_\ell|^2 \right), \end{aligned}$$

where we used that, for all sequence $(\alpha_k)_{1 \leq k \leq N}$,

$$2h \sum_{j=1}^{N} \left(\sum_{k=1}^{N} \alpha_k \cos(k\pi jh) \right)^2 = \sum_{k=1}^{N} |\alpha_k|^2.$$

Note then that

$$\left(\frac{\sin(k\pi h)}{h} \right)^2 = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2}\right) \cos^2\left(\frac{k\pi h}{2}\right) = \lambda_k(h) \cos^2\left(\frac{k\pi h}{2}\right).$$

The bound (2.30) immediately follows.

If we assume (2.31), then by Cauchy–Schwarz inequality, Eq. (2.30) implies

$$|X_h(a_h,b_h)| \leq \frac{1}{\cos\left(\frac{\gamma\pi}{2}\right)} \sum_{k=1}^{N} |\hat{a}_k|^2 \lambda_k(h) \cos^2\left(\frac{k\pi h}{2}\right) + \cos\left(\frac{\gamma\pi}{2}\right) \sum_{\ell=1}^{N} |\hat{b}_\ell|^2,$$

and the last term satisfies:

$$\cos\left(\frac{\gamma\pi}{2}\right) \sum_{\ell=1}^{N} |\hat{b}_\ell|^2 \leq \frac{1}{\cos\left(\frac{\gamma\pi}{2}\right)} \sum_{\ell=1}^{N} |\hat{b}_\ell|^2 \cos^2\left(\frac{\ell\pi h}{2}\right),$$

and estimate (2.32) follows immediately. □

2.4.4 Proof of Theorem 2.3

Proof (Theorem 2.3). Identity (2.24) and estimates (2.27) and (2.28) imply that any solution φ_h of Eq. (2.1) in the class $\mathscr{V}_h(\gamma/h)$ satisfies

$$\left| T\tilde{E}_h[\varphi_h] - \int_0^T \left|\frac{\varphi_N}{h}\right|^2 \mathrm{d}t \right| \leq \frac{2}{\cos\left(\frac{\gamma\pi}{2}\right)} \tilde{E}_h(\varphi_h) + \frac{h}{2} E_h(\varphi_h). \tag{2.33}$$

Therefore,

$$\left(T - \frac{2}{\cos\left(\frac{\gamma\pi}{2}\right)}\right) \tilde{E}_h[\varphi_h] - \frac{h}{2} E_h[\varphi_h] \leq \int_0^T \left|\frac{\varphi_N}{h}\right|^2 \mathrm{d}t.$$

But, since φ_h belongs to the class $\mathscr{V}_h(\gamma/h)$, the Fourier expressions of the energy $E_h[\varphi_h]$ in Eq. (2.21) and $\tilde{E}_h[\varphi_h]$ in Eq. (2.23) yield

$$\cos\left(\frac{\gamma\pi}{2}\right)^2 E_h[\varphi_h] \leq \tilde{E}_h[\varphi_h], \tag{2.34}$$

which concludes the proof of Theorem 2.3. □

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