# Chapter 2
# Galileo's Great Discovery: How Things Fall

Galileo Galilei (1564–1642), the famous Italian mathematician at the leading edge of the scientific revolution that was to sweep Europe, was curious about motion. He was an experimentalist who for the first time had the insight and talent to link theory with experiment. He rolled balls down an inclined plane in order to see how things fell toward the Earth. He discovered in this way that objects of any weight fell toward the Earth at the same rate – that they had a uniform acceleration. He surmised that if they fell in a vacuum, where there was no air resistance to slow some objects more than others, even a feather and a cannon ball would descend at the same rate and reach the ground at the same time. He also explored the motion of pendulums and other phenomena. He is perhaps most famous for his 1610 telescopic discoveries of the moving moons of Jupiter, the phases of Venus, and the craters of the moon, all of which convinced him, against the ages-old wisdom of Aristotle and of the Catholic Church, of the rightness of the Copernican heliocentric view of the solar system.

In his investigations of motion, Galileo was the first clearly to understand that the forces acting upon objects could be broken into *independent components*; that a thrown stone had a force pulling it down as well as the force throwing it horizontally outward. These insights would be of great use to Isaac Newton, born the year Galileo died, in devising the calculus and his universal laws of gravity and motion.

## The Distance a Thing Falls

Galileo was interested in understanding how things moved and fell. What laws of motion governed them? Determining physical laws from experiment was completely new in early Seventeenth Century Italy, but Galileo was intellectually adventurous enough to try. One of Galileo's most famous experiments, in 1604, was his inclined plane experiment, where he measured the distance a ball rolled down a ramp in each unit of time. Since forces in different directions act independently,

D.W. MacDougal, *Newton's Gravity: An Introductory Guide to the Mechanics of the Universe,* Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-1-4614-5444-1_2, 

he could time the descent of the ball and learn how forces act on the ball as if it were only moving in the down direction. From that he could deduce how a freely falling object would move.

Why did he use a ramp to measure fall? It is because with the limited technical means then available to Galileo, he could not possibly have timed, with any reasonable accuracy, the rapid descent of a vertically dropped ball. Using an inclined plane allowed him to dilute the force of gravity and slow the ball down so he could time it with a water clock, where he could then compare the weight of water that poured out before and after each event. This told him the time intervals elapsed during the ball's fall down the ramp, which were not necessarily the equivalent of the seconds of a modern clock.[1] Galileo repeated this experiment many times to help remove some of the subjectivity in his measurements and thereby gain greater accuracy of the result.

Galileo's idea of using an inclined plane to accurately measure free fall as noted took advantage of his insight that forces act independently in each dimension. As we saw in Chap. 1, the downward, vertical force on the rolling ball (the gravitational force) can be analyzed separately from the horizontal force that moves it laterally along the plane. The movement down the plane is the result of the *combination* of the downward gravitational force on the ball and the force of the plane on the ball, acting perpendicular or normal to the plane, resisting it and deflecting it horizontally. Recall our example of a sailboat where the resultant path is the combination of the forces of wind on the sail and resisting water on the keel. Studying the downward *component* of the forces acting on the rolling ball, then, is equivalent (if we disregard the frictional forces acting on the ball) to studying the force drawing down a freely falling object, where all the forces act downward, in

[1] It is quite possible that Galileo initially timed the rolling ball by means of musical beats and later confirmed their accuracy by means of the water clock. Galileo was musically inclined, accomplished on the lute, and his father and brother were musicians. Setting adjustable gut "frets" on the inclined plane would enable the ball to make an audible bumping sound as it passed over the frets. Adjusting the spacing of these frets so the bumps occurred at exactly even intervals, according to his internal sense of rhythm, could easily have been done. Indeed, that method was likely far more accurate than any clocks of the day, which could not measure times shorter than a second. This idea was advanced by the late Stillman Drake, Canadian historian of science and Galileo expert. See Stillman Drake, "The Role of Music in Galileo's Experiments", *Scientific American*, June, 1975. The important thing is that the time intervals be deemed to be equal, whatever those intervals may be:

> The phrase "measure time" makes us think at once of some standard unit such as the astronomical second. Galileo could not measure time with that kind of accuracy. His mathematical physics was based entirely on ratios, not on standard units as such. In order to compare ratios of times it is necessary only to divide time equally; it is not necessary to name the units, let alone measure them in seconds. The conductor of an orchestra, moving his baton, divides time evenly with great precision over long periods without thinking of seconds or any other standard unit. He maintains a certain even beat according to an internal rhythm, and he can divide that beat in half again with an accuracy rivaling that of any mechanical instrument. Ibid., 98.

the $y$ direction. This was a fundamental intuition, and helped lay the conceptual foundation for Newton's work on the action of forces, and the concept of vectors.

Using the ramp, however, came with a price. Any object falling through the atmosphere will experience friction, whose effects will vary with the weight and shape of the object. Using an inclined plane introduces a whole new element of friction into the experiment. Galileo tried to minimize it by covering the plane with parchment. Air resistance on a heavy, slow-moving ball would probably have been negligible. Energy went into rotating the mass, though, and with the remaining friction the ball would descend at some fraction of the speed with which it would roll freely on a frictionless surface. But in this experiment Galileo was seeking only proportions: the relationship between time and distance of fall. Even if the ball was retarded by some unknown amount of friction, the relationship between distance and time after it got going should be affected more or less equally in each interval.

Galileo's famous *Dialog Concerning Two New Sciences*[2] of 1638 is his rich and delightful inquiry into fundamental physical questions about the strength of materials and motion. It is highly readable, even after more than three and a half centuries. Here is Galileo's description of his inclined plane experiment, stating his findings regarding the relation between distance and the time of fall:

> A piece of wooden moulding or scantling, about 12 cubits long, half a cubit wide, and three finger-breadths thick, was taken; on its edge was cut a channel a little more than one finger in breadth; having made this groove very straight, smooth, and polished, and having lined it with parchment, also as smooth and polished as possible, we rolled along it a hard, smooth, and very round bronze ball. Having placed this board in a sloping position, by lifting one end some one or two cubits above the other, we rolled the ball, as I was just saying, along the channel, noting, in a manner presently to be described, the time required to make the descent. We . . . now rolled the ball only one-quarter the length of the channel; and having measured the time of its descent, we found it precisely one-half of the former. Next we tried other distances, comparing the time for the whole length with that for the half, or with that for two-thirds, or three-fourths, or indeed for any fraction; in such experiments, repeated a full hundred times, *we always found that the spaces traversed were to each other as the squares of the times, and this was true for all inclinations of the plane*, i.e., of the channel, along which we rolled the ball.[3]

We can confirm this mathematical relationship found by Galileo between the distance the ball went and its time of descent. Below are some of Galileo's measured times and distances for the rolling ball as it progressed down the inclined plane. Galileo used a measuring unit called points, each of which was about 29/30 mm. As shown on the chart, the ball rolled about 2104 $\times$ 29/30 mm or about 2,034 mm (2.34 m), less than 7 ft, in eight intervals (again, not necessarily seconds):

[2] *Dialog Concerning Two New Sciences* (translated by Henry Crew and Alfonso de Salvio, Macmillan 1914). This classic translation is also available online: http://galileoandeinstein.physics.virginia.edu/tns_draft/index.html. See also Hawking [1] (Short title) *Dialogs*, which uses the same translation.

[3] *Dialogs,* 136–37 (my italics).

| Time (equal intervals) | Distance (in points) | Distance divided by 33 |
|---|---|---|
| 1 | 33 | 1.00 |
| 2 | 130 | 3.94 |
| 3 | 298 | 9.03 |
| 4 | 526 | 15.94 |
| 5 | 824 | 24.97 |
| 6 | 1,192 | 36.12 |
| 7 | 1,620 | 49.09 |
| 8 | 2,104 | 63.76 |

The last column we added to show the multiples of the initial distance. The idea is to try to intuit a pattern in the numbers, by dividing the initial distances by the first distance number (33) and seeing if the other numbers are multiples of it. It is apparent that, after dividing each distance (which we will call $s$) by 33, the last column increases approximately as the square of the time. Indeed it does not take a great deal of rounding of this experimental data to make the relationship just that, as Galileo saw. The exact relationship is confirmed by many repeated observations. It can be expressed by a simple equation:

$$t^2 = \frac{s}{k}$$

Where $k$ is 33, hence,

$$s = kt^2$$

We have not explained what the $k$ means in this equation. It is a "proportionality constant" whose value determines the distance in the equation, in units of our choice. As we shall see, the constant is a constant of acceleration, the gradual incremental addition to velocity in the presence of a force inducing it.

## The Meaning of Constant Acceleration

To understand uniformly accelerated motion, we shall return to the man who articulated it in simple terms. In his *Dialogs*, Galileo reasoned that constant acceleration implied steady, incremental additions of velocity evenly in proportion to time:

> When, therefore, I observe a stone initially at rest falling from an elevated position and continually acquiring new increments of speed, why should I not believe that such increases take place in a manner which is exceedingly simple and rather obvious to everybody? If now we examine the matter carefully we find no addition or increment more simple than that which repeats itself always in the same manner. This we readily understand when we consider the intimate relationship between time and motion; for just as uniformity of motion is defined by and conceived through equal times and equal spaces (thus we call a

> motion uniform when equal distances are traversed during equal time-intervals), so also we may, in a similar manner, through equal time-intervals, conceive additions of speed as taking place without complication; *thus we may picture to our mind a motion as uniformly and continuously accelerated when, during any equal intervals of time whatever, equal increments of speed are given to it*. Thus if any equal intervals of time whatever have elapsed, counting from the time at which the moving body left its position of rest and began to descend, the amount of speed acquired during the first two time-intervals will be double that acquired during the first time-interval alone; so the amount added during three of these time-intervals will be treble; and that in four, quadruple that of the first time interval. To put the matter more clearly, if a body were to continue its motion with the same speed which it had acquired during the first time-interval and were to retain this same uniform speed, then its motion would be twice as slow as that which it would have if its velocity had been acquired during two time intervals.[4]

In other words, in *uniformly* accelerated motion, the velocity will be proportional to time,

$$v \propto t$$

The proportionality constant to convert the proportion $v \propto t$ into an equation must be the amount by which increases in velocity occur steadily with time. This is the definition of constant acceleration. In a freely falling object, acceleration in falling due to the Earth's gravitational field is known from experiment (and theory) to be 9.8 m/s per second.[5] Hence for such cases the equation may be written $v = 9.8\ t$.

To make this clearer, suppose a rocket is boosting a probe into deep space at a constant acceleration of 10 m/s, every second. Constant acceleration means constant increase in velocity – in heaps of 10 m/s velocity in every new second.

| Time in seconds | Increments of velocity (m/s) | Total velocity (m/s) |
|---|---|---|
| 1 | 10 | 10 |
| 2 | 10 + 10 | 20 |
| 3 | 20 + 10 | 30 |
| 4 | 30 + 10 | 40 |
| 5 | 40 + 10 | 50 |

Every second begins with the velocity the object had at the end of the previous second. The incremental addition to velocity in each second – the constant by which the time is multiplied – is 10, which is the probe's constant acceleration. Where acceleration (again symbolized by $a$) is constant, the equation for velocity is,

$$v = at$$

[4] *Dialogs*, 123 (my italics).

[5] The acceleration of a falling body near the surface of the Earth due to gravity is 9.8 m/s per second, written usually as 9.8 m/s$^2$. This means it gains 9.8 m/s in velocity each second of its fall. As noted in Chap. 1, this value for acceleration, or "g" as it is called, will diminish as we move farther away from the Earth's surface (diminishing, in fact, with the square of the increasing distance from the center of the Earth).

The *distance* the spacecraft will go in a given time at this rate of velocity increase is also some function of time. Let's see what it is. If an object moves at *constant velocity* (zero acceleration), the distance it travels is,

$$s = vt$$

where $s$ is our symbol for distance. But the velocity in our case keeps increasing every second, at a constant rate. For example, it goes 10 m/s in the first second and 20 m/s in the next second. So the overall velocity during these 2 s will not be 10 or 20 but the *average* of them, or 15 m/s. In 2 s, it will actually traverse 30 m. Given an average velocity $\bar{v}$, the distance equation becomes,

$$s = \bar{v}t$$

The distance the spacecraft travels will equal its average velocity times time. But is there a simpler way of computing average velocity when we know acceleration? To compute the average of first two velocities, as we did above, we mentally did this:

$$\bar{v} = \frac{v_2 - v_1}{2}$$

To compute the average velocity over a span of time, we need to know how many seconds elapsed, from the time the motion started, $t_s$ to the time it ended, $t_e$. Since velocity will in each case be equal to acceleration times time, the above equation becomes,

$$\bar{v} = \frac{at_e - at_s}{2}$$

Or, to simplify, assuming the start time is zero seconds, letting $t$ just be elapsed time,

$$\bar{v} = \frac{1}{2}at$$

So in our example the average velocity in 5 s will be 25 m/s. How far will it go? To find out, substitute the above equation for average velocity into the distance equation, $s = \bar{v}t$:

$$s = \frac{1}{2}at^2$$

In 5 s the spacecraft in our example with constant acceleration of 10 m/s, every second (expressed as 10 m/s$^2$) will therefore travel ($\frac{1}{2} \times 10 \times 5^2 =$) 125 m.

This is the relationship between distance and uniformly accelerated motion which we saw was discovered by Galileo from his experiments with rolling balls. The $k$ in the relation $s = kt^2$ must therefore equal half the acceleration: $k = \frac{1}{2}a$, where the constant acceleration $g$ (of 9.8 m/s$^2$) was furnished by the Earth's gravity. Rewriting the equation specific to Earth's gravitational acceleration we have,

$$s = \frac{1}{2}gt^2$$

Notice that there is no horizontal component to this equation. The vertical distance is, again, independent of any additional thrust, push, shove or shot in a direction perpendicular to line of descent. This equation applies to the little man we encountered on the cliff in Chap. 1 dropping and throwing the rocks. If it takes 3 s for the rocks to reach the ground, the balls were dropped from about 44 m (almost 145 ft):

$$s = \frac{1}{2}(9.8)(3)^2$$

$$s = 44.1 \text{ meters}$$

## *Graphing Velocity and Time in Uniformly Accelerated Motion*

**Problem** Show *graphically* that the distance traveled by a *freely falling* object under uniform gravitational acceleration equals half the velocity times the time of the fall. In doing so, use the value of the Earth's gravitational acceleration as the uniform acceleration constant.

**Given**

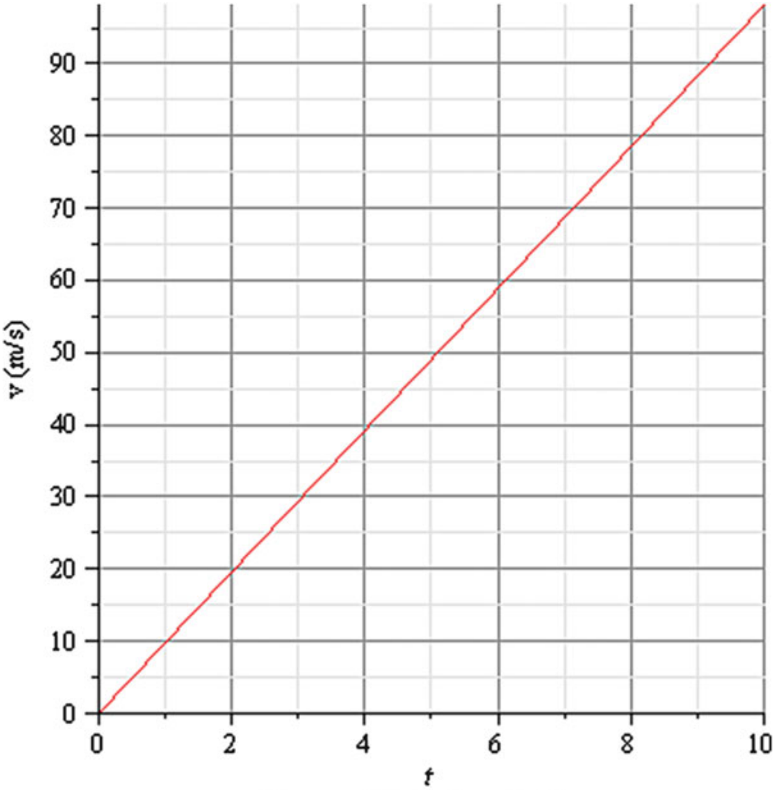
| | |
|---|---|
| $v = at$ | Velocity in uniformly accelerated motion |
| $a = 9.8\ \text{m/s}^2$ | The acceleration due to Earth's gravity at its surface |

**Assumptions** We will assume that the object moves without resistance, and disregard any slowing effects of friction, as if it were falling in a vacuum.

**Method** By inspection of the equation $v = 9.8t$, one can see that it must be a straight line, with a slope (or rate of change of *s/t*) of 9.8 and an s intercept (on the y axis) of 0. This tells us that the velocity increases at a constant rate with time. The change of velocity with time is by definition acceleration. Velocity is changing at each instant, so to capture the true $s = vt$ relationship for every changing increment of time, we need to find the total area under the graph.

**Calculations** The relationship between time (in seconds) and the ever-increasing velocity during fall is an upwardly sloping straight line, apparent on the graph below.

It is evident that a falling object speeds up with a *constant* or uniform acceleration, with a *velocity proportional to time of the object's descent*. That the slope is 9.8 to 1 means that the velocity increases 9.8 m/s every second, in the progression, 9.8, 19.6, 29.4, 39.2… That is the "the amount of speed acquired" in each interval of time, where the intervals are 1 s each.



Change of Velocity with Time

**Observations**

1. How can we find the distance when the time and velocity at each point are known? It may be tempting to conclude that the total distance traveled by the object is found merely by multiplying *v* times *t* in the chart, under the assumption that $v = st$. But here velocity is not constant, it is changing. The object is falling at 98 m/s after 10 s – but the total distance it fell is not 980 m. For the first half of the trip down (zero to 5 s), it was falling at *less* than 50 m/s, and for the second half (after 5 s) it was falling at *more* than 50 m/s. So we must multiply the *average* velocity of the voyage by time.[6] Average velocity is just $v_{avg} = (0 + 9.8)/2$, or 49 m/s. This multiplied by the full time of fall of 10 s is 490 m, the

[6] Galileo's first theorem in the section of his *Dialogs* titled "Naturally Accelerated Motion" states: "The time in which any space is traversed by a body starting from rest and uniformly accelerated is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed and the speed just before acceleration began." *Dialogs*, 132.

total distance of fall. The equation for finding distance, when the elapsed time and uniformly changing velocity from rest are known, is $s = ½vt$. This procedure is equivalent to finding the area of the triangle under the sloped line. The area of a triangle is half the base times the height, so

$$s = \frac{1}{2}vt.$$

2. In his investigations of accelerated motion, Galileo utilized both experimental and theoretical methods. Since measuring time precisely and easily was not possible in his day (the water clock as we mentioned above was probably used by him to confirm, rather than discover, these laws), he sought to understand his experimental results by mathematical reasoning, using geometry to illustrate *proportional* relationships.[7] The rolling ball experiment was his attempt to find the proportional relationship between distance traveled and time. This yielded the law that distance was proportional to time squared.[8] Since he had concluded that in uniformly accelerated motion an object's velocity increased in a manner that was proportional to time, he could intuit the correctness of the time-squared law for distance mathematically. Here is an intuitive, non-geometrical shorthand version of his reasoning: The distance covered by a non-accelerating object is proportional to velocity times time. But in a uniformly accelerating body, that velocity is *itself* proportional to time. Hence the distance is proportional to times squared. Expressed symbolically:

$$s \propto vt \text{ in uniform motion; but}$$

$$v \propto t \text{ in a uniformly accelerating body, so}$$

$$s \propto (t)\ t \text{ in a uniformly accelerating body, or}$$

$$s \propto t^2 \ \leftarrow$$

This result corresponds to the relation Galileo obtained in his rolling ball experiment. Hence he could feel confidence in the truth of his theorem: "The spaces described by a body falling from rest with a uniformly accelerated motion are to each other as the squares of the time-intervals employed in traversing these distances." [9]

3. Galileo's own reasoning is a good example of geometrical thinking in an era where quantifiable results were difficult to come by. Below is his line of argument and diagram proving the same problem we worked through above[10]:

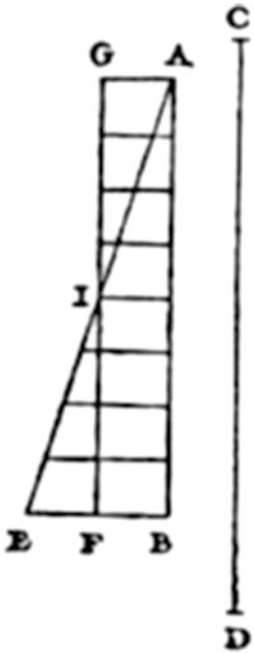
[7] The method of finding the limit of a function to quantitatively derive velocity or acceleration was unknown, and lay three-quarters of a century in the future.

[8] *Dialogs*, 133: Theorem II, Proposition II in "Naturally Accelerated Motion."

[9] Ibid.

[10] Galileo is here proving his Theorem I, Proposition I: "The time in which any space is traversed by a body starting from rest and uniformly accelerated is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed and the speed just before acceleration began." *Dialogs*, 132.

Let us represent by the line AB the time in which the space CD is traversed by a body which starts from rest at C and is uniformly accelerated; let the final and highest value of the speed gained during the interval AB be represented by the line EB, drawn at right angles to AB; draw the line AE, then all lines drawn from equidistant points on AB and parallel to BE will represent the increasing values of the speed, beginning with the instant A. Let the point F bisect the line EB; draw FG parallel to BA, and GA parallel to FB, thus forming a parallelogram AGFB which will be equal in area to the triangle AEB, since the side GF bisects the side AE at the point I; for if the parallel lines in the triangle AEB are extended to GI, then the sum of all the parallels contained in the quadrilateral is equal to the sum of those contained in the triangle AEB; for those in the triangle IEF are equal to those contained in the triangle GIA, while those included in the trapezium AIFB are common. Since each and every instant of time in the time-interval AB has its corresponding point on the line AB, from which points parallels drawn in and limited by the triangle AEB represent the increasing values of the growing velocity, and since parallels contained within the rectangle represent the values of a speed which is not increasing, but constant, it appears, in like manner, that the momenta assumed by the moving body may also be represented, in the case of the accelerated motion, by the increasing parallels of the triangle AEB, and, in the case of the uniform motion, by the parallels of the rectangle GB. For, what the momenta may lack in the first part of the accelerated motion (the deficiency of the momenta being represented by the parallels of the triangle AGI) is made up by the momenta represented by the parallels of the triangle IEF. Hence it is clear that equal spaces will be traversed in equal times by two bodies, one of which, starting from rest, moves with a uniform acceleration, while the momentum of the other, moving with uniform speed, is one-half its maximum momentum under accelerated motion. Q.E.D.[11]




[11] Ibid. 132–33.

## *If Galileo Only Knew Calculus: A Quick Look at Instantaneous Velocity*

We noted above that Galileo realized that the forces acting upon objects could be broken into independent components, which we now typically represent as vectors. An object is pulled directly downward by gravity with uniform acceleration whether or not it is flung outward or just dropped. The idea of separating forces and motions into components and analyzing them separately is a method of analysis now commonplace in the study of dynamics. The path of a falling object can be analyzed using a graph in terms of change along the y axis and change along the $x$ axis. The ratio of those quantities, called the slope, tells about the rate of change of the curve at the point in question. Let us look again at the simple case of objects falling – or rolling – freely in a gravitational field. It will help us understand how the ever-changing velocity entailed in accelerated motion ultimately required Newton to explore what happens when segments under investigation are broken down into smaller and smaller parts, which was the heart of his remarkable discovery of the calculus. The principles are best conveyed through the use of a few simple problems.

**Problem** The actual initial distance a dropped object falls to Earth in the first second is about 4.9 m (16 ft). Using only the distance-time-squared relationship discovered by Galileo, expressed in the equation $s = kt^2$, where the constant $k = 4.9$ (since at $t = 1$, $k = s = 4.9$), and the distance and time data plotted below, derive an equation for the instantaneous velocity of a freely-falling boulder, during any moment of its descent.

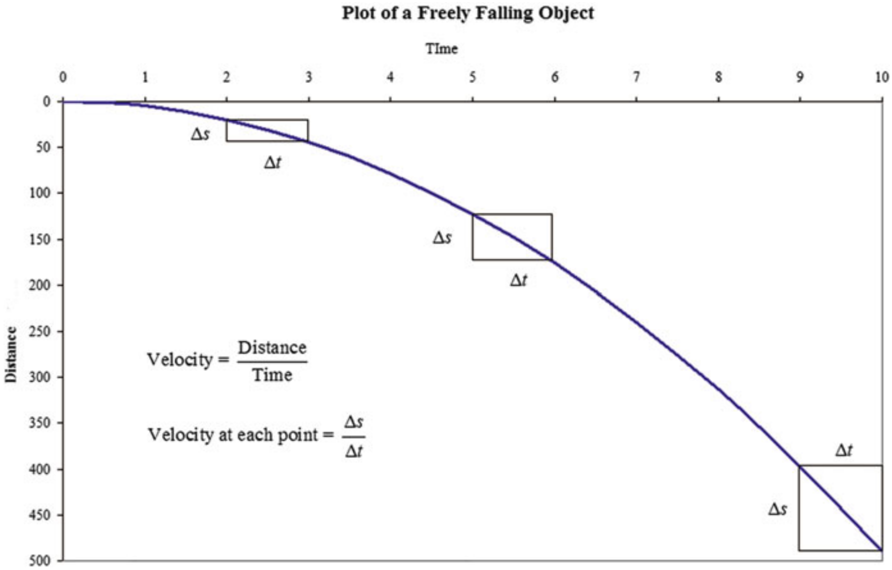
**Given** Below is the distance in meters the freely-falling boulder covers in 10 s.

| Time (s) | Time squared | Distance (m) $s = kt^2$ |
|---|---|---|
| 1 | 1 | 4.9 |
| 2 | 4 | 19.6 |
| 3 | 9 | 44.1 |
| 4 | 16 | 78.4 |
| 5 | 25 | 122.5 |
| 6 | 36 | 176.4 |
| 7 | 49 | 240.1 |
| 8 | 64 | 313.6 |
| 9 | 81 | 396.9 |
| 10 | 100 | 490 |

**Assumptions** We will as before ignore the role air friction plays, and assume it falls in a vacuum toward the Earth.

**Method** Examining the plot of data below, we note that the rate of descent appears non-uniform. Earlier parts of the curve are shallow, gradually becoming steeper and steeper with increasing time. Steeper means faster, since there is increasingly more change along the y axis – distance – in a unit of time than where the curve is more horizontal. We can see from both charts, too, that the numbers grow faster with each second. This conforms to our intuition, since things do fall faster and faster as they drop, as one can see by looking at a waterfall, where the water descends with increasing rapidity as it nears the bottom.

The *rate* of this descent – the velocity of the object – will differ at each point along the curved graph. How can we find the velocity of our rock at any particular point? We must pick a small place on the curve beginning with that point, illustrated below by small boxes, and find the *slope* of the line there. The slope is the difference in the boxes between the two $s$ points on the $y$ axis (how distance has changed), which is $\Delta s$, divided by the difference between the two $t$ points on the $x$ axis (how time has changed), which is $\Delta t$. This relationship $\Delta s/\Delta t$, increment of distance per increment of time, is the mean velocity within the box. We can see how this looks on a graph of the 10 s of data:




Little rectangles are drawn at arbitrary times (2, 5 and 9 s) to illustrate that that the slope gets increasingly steep with time. The first rectangle is elongated horizontally, and the second less so, and the last is beginning to be elongated vertically. Since velocity is the ratio of distance to time, we can see the velocity of the rock (and the slope, $\Delta s/\Delta t$) is increasing as time goes on. If we call these boxes Box A, B and C respectively, and compute $\Delta s/\Delta t$ for each, we get the following results:

| Box | Time $t$ when entering box | Distance $s$ when entering box | Distance $s'$ when leaving box | Difference in s values: $\Delta s$ | $\Delta t$ | Average velocity: $\Delta s/\Delta t$ |
|---|---|---|---|---|---|---|
| A | 2 | 19.6 | 44.1 | 24.5 | 1 | 24.5 |
| B | 5 | 122.5 | 176.4 | 53.9 | 1 | 53.9 |
| C | 9 | 396.9 | 490 | 93.1 | 1 | 93.1 |

The velocity increases dramatically in each box. But what do those represent? They are, again, the *average* velocities of the object within each box. The rock took 1 s to drop through each box, and the length of that drop was the $\Delta s$ value applicable to each. If $\Delta t$ is as big as it is, then we must be content with this approximation of velocity at the location of each box. Within the boxes there is an approximation: the curve changes, albeit slightly inside each box, so the slope in each is really not the same throughout the whole box. We still don't know the actual velocity at any particular point. If one were trying to determine exactly the force on a bungee cord, or the speed for a soft landing on the Moon, however, one would need to know the exact, not merely average, velocity. How can we do this? We can try by making $\Delta t$ smaller. Since Galileo's equation for this curve is known, we can see what happens to $s$ when $\Delta t$ is added to the equation; specifically, how the equation may change when we reduce the value of that $\Delta t$ to near zero. This is called finding the *limit* of a function, as $\Delta t$ *approaches* (but does not actually reach) zero. It is equivalent to reducing the time intervals to ever smaller units, as if the boxes were made smaller and smaller, and seeing how the distance changes in those smaller units of time. If the boxes are made almost *infinitely* small, such that they become virtually *points*, the velocity at those points essentially becomes the *instantaneous* velocity we seek instead of the average velocity.

**Calculations** The average velocity in any box is given by this relation:

$$\bar{v} = \frac{\Delta s}{\Delta t}$$

We eventually want to find what the instantaneous velocity is as $\Delta t$ approaches a limit, in this case zero, expressed mathematically by this notation:

$$\lim_{\Delta t \to 0} v = \frac{\Delta s}{\Delta t}$$

Calling the distance at entry into the box $s$ and the distance at exit $s'$, and the small increment of time $\Delta t$, the velocity equation before we limit the size of $\Delta t$ is,

$$v = \frac{s' - s}{\Delta t}$$

Now let us modify Galileo's equation by adding the $\Delta t$ to the time increment:

$$s = kt^2$$

$$s' = k(t + \Delta t)^2$$

and the equation for velocity in the box becomes,

$$v = \frac{k(t + \Delta t)^2 - kt^2}{\Delta t}$$

This can be expanded, then simplified,

$$v = \frac{k(t^2 + 2t\Delta t + \Delta t^2) - kt^2}{\Delta t}$$

$$v = \frac{k\Delta t(2t + \Delta t)}{\Delta t}$$

$$v = k(2t + \Delta t)$$

From this equation the velocity through any size box can be computed. All we do is adjust the size of $\Delta t$ and make the box as small as we choose. Now we can see what happens when we make $\Delta t$ approach zero:

$$\lim_{\Delta t \to 0} v = k(2t + \Delta t)$$

$$v = 2kt$$

which, under our initial assumption that an object falls toward Earth a distance of 4.9 m in the first second (that is, $k = 4.9$) reduces to,

$$v = 9.8t$$

**Observations**

1. This of course is the familiar $v = at$ equation discussed earlier, where $a$ is the value of acceleration due to gravity near the Earth's surface of 9.8 m/s$^2$.
2. Inserting the values of 1 at 2, 5 and 9 s, respectively, for $\Delta t$ in the equation $v = k$ (2 t + $\Delta t$) yields 24.5, 53.9 and 93.1 m/s. These are the same values for average velocity given by the numerical method in the chart above. When we reduce $\Delta x$ to zero, to find instantaneous velocity, the equation $v = 9.8\ t$ applies. The *instantaneous* velocities at those times are 19.6, 49 and 88.2 m/s, respectively.
3. To see how the initial values of velocity *converge* on these final values as $\Delta t$ gets smaller, take the middle box as an example, where the object has fallen for 5 s and traveled 122.5 m. Beginning with the equation $v = k$ (2 t + $\Delta t$) with $\Delta t$ at 1, we reduce the size of $\Delta t$ by increments before finally bringing it to zero. The average velocity of the object through the box between 5 and 6 s becomes the instantaneous velocity of 49 m/s at exactly 5 s:

| $\Delta t$ | Velocity at 5 s (m/s) |
|---|---|
| 1 | 53.9 |
| .5 | 51.45 |
| .25 | 50.225 |
| .1 | 49.49 |
| .01 | 49.049 |

(continued)

| $\Delta t$ | Velocity at 5 s (m/s) |
|---|---|
| .001 | 49.0049 |
| .0001 | 49.00049 |
| 0 | 49 |

4. The method of finding the limit of the equation as we shrink $\Delta t$ is at the heart of the differential calculus, invented independently at about the same time by Englishman Newton and the German Gottfried Wilhelm Leibniz (1646–1716). It sparked a revolution in mathematics and was Newton's prime tool for his seminal analysis of the motion of moving bodies, published in 1687 as *Philosophiae Naturalis Principia Mathematica*, or *The Mathematical Principles of Natural Philosophy*, (often referred to as the *Principia*). The increment $\Delta t$ (or any such small increment of a value, be it time, velocity, distance, etc.) was called by Newton a "fluxion" and a "differential" by Leibniz. Leibniz's terminology and symbols have survived.

## *Using Limits to Find the Acceleration of a Freely-Falling Body*

We have seen that the velocity of a falling object near the Earth increases at a constant 9.8 m/s$^2$. That is, the rate of change of velocity (the acceleration) is 9.8 m/s$^2$. This result we *infer* from the equation for velocity, $v = 9.8\ t$. Let us now use some of the same limit concepts to find it directly.

**Problem** Derive the acceleration of a freely falling object using the same techniques as were used in the above discussion.

**Given**

| | |
|---|---|
| $v = 9.8t$ | The equation for velocity of a freely falling object toward Earth |
| $\lim\limits_{\Delta t \to 0} a = \Delta v/\Delta t$ | Acceleration (change of velocity with time) as $\Delta t$ approaches zero |

**Assumptions** We will again ignore the role air friction plays, and assume the object falls in a vacuum toward the Earth.

**Method** To assist our analysis, we consider the object's velocity at entry into any arbitrarily small box and the velocity at exit, after a small increment of time has passed. The difference may then be found between the velocity after the small time increment and the velocity before it, as the numerator in the above limit. We then calculate the expression when $\Delta t$ goes to zero. Graphically, the result is the same as before: Recalling the straight-line graph of velocity (on the $y$ axis) and time (on the $x$ axis) we can again visualize smaller and smaller increments of time, and correspondingly smaller and smaller increments of velocity as a function of time.

**Calculations** If we let the velocity be $v$ at entry into any arbitrarily small box and the velocity at exit be $v'$, then $\Delta v = v' - v$. The small increment of time we again represent as $\Delta t$, so

$$\lim_{\Delta t \to 0} a = \frac{\Delta v}{\Delta t}$$

Since $\Delta v = v' - v$, the acceleration equation before we limit the size of $\Delta t$ is,

$$a = \frac{v' - v}{\Delta t}$$

Now let us modify the velocity equation by adding the $\Delta t$ to the time increment:

$$v = 9.8t$$

$$v' = 9.8(t + \Delta t)$$

Substituting these values into the equation for acceleration, the expression becomes,

$$a = \frac{9.8(t + \Delta t) - 9.8t}{\Delta t}$$

$$a = \frac{9.8t + 9.8\Delta t - 9.8t}{\Delta t}$$

Simplifying and dividing by $\Delta t$, we find,

$$a = 9.8\,\mathrm{m/s^2}$$

**Observations**

1. The approach is called differentiation, and with it we arrive at the same result as we found before: that the acceleration is indeed a constant. The important point is the method, unknown to Galileo, by which Newton was able to unlock the secrets of lunar and planetary motions.
2. In summary, we used the method of differentials on two equations above. It was applied to the distance-time-squared equation $y = kt^2$ and then the other was $y = 9.8\ t$. The method yielded $2kt$ and 9.8, respectively. The first produced the rate of change of distance with respect to time, or velocity. The second yielded the rate of change of velocity with time, or acceleration. It looks as if the same operation was applied to each equation. Can we generalize the method? Indeed, it can be shown that, in most cases, all one needs to do to find the rate of change of such a function is first to multiply the function by its exponent and then reduce

the power of the exponent by one.[12] Suppose we try it on a cubic equation, say $y = Zx^3$, where $Z$ is any constant, returning to the conventional $x$, $y$ notation for the abscissa and ordinate. If the general rule is correct, then the rate of change of $y$ with respect to $x$, written as $\Delta y/\Delta x$, should simply be $3Zx^2$. Using the above methods, try this for yourself to see if you agree.

## A Summary of Galilean Equations

It is worthwhile thoroughly to know the basic equations of motion. Below is a summary of the relationships among distance, velocity, acceleration and time, which we will sometimes call the "Galilean equations" even though Galileo himself never actually expressed these relationships in this algebraic form:

Summary of Galilean equations

| Condition | Proportion | Proportionality constant | Equation |
|---|---|---|---|
| Distance at time $t$ assuming continuous, uniform motion (i.e., where velocity has been constant) | $s \propto vt$ | 1 | $s = vt$ |
| Distance at time $t$ after uniform acceleration from rest (i.e., where initial velocity was zero) | $s \propto vt$ | ½ | $s = \frac{1}{2}vt$ |
| Velocity at time $t$ after uniform acceleration from rest | $v \propto t$ | $a$ | $v = at$ |
| Distance at time $t$ after uniform acceleration from rest | $s \propto t^2$ | $\frac{1}{2}a$ | $s = \frac{1}{2}at^2$ |

### *Notes on Using the Law of Conservation of Mechanical Energy to Derive the Galilean Distance Equation*

We may use energy concepts to derive the Galilean distance-time-squared relation. The law of conservation of mechanical energy was not well-understood until the nineteenth Century. The subject is treated extensively in most physics textbooks; we will give only a brief summary here of the key concepts relevant to our discussion. Energy is the ability to do *work*, which is the ability of a force to move an object through a distance. *Kinetic energy* is the energy of motion. It has the ability to do work upon impact with another object. It is expressed by the relation: $KE = \frac{1}{2}\, mv^2$. That is, the kinetic energy of a mass $m$ varies as half the square of its velocity. The kinetic energy per unit mass $m$ thus equals $v^2/2$. *Potential energy* is a kind of stored energy. When a spring is wound, it can do work when unwound. When an object is raised above the ground it acquires potential energy, the greater

[12] A calculus course is advised to properly examine the many nuances and variations of this idea to more complex functions, and its inapplicability to certain classes of functions not considered here.

with height, which is converted to kinetic energy as it falls and gains speed. It is expressed by the following relation near the surface of the Earth: $PE = mgh$. That is, the kinetic energy of a mass $m$ increases as it is raised a distance $h$ against the accelerative force of gravity $g$.[13] The potential energy per unit mass $m$ thus equals $gh$. It is a fundamental law of physics that total mechanical energy in a system can neither be created nor destroyed. So in any "closed" mechanical system, where we assume no gains or losses of energy from external sources (for example, in the form of heat due to friction) then $KE + PE =$ constant. The total energy in the system is $E_{total} = KE + PE$.

Now with these ideas in mind, imagine a rock high up on a cliff, whose height is $h$. You are about to push it off. Before the rock is pushed, its velocity is zero, so $v = 0$, and its total energy is $mgh$, which is entirely potential. After you shove it off, it loses height but gains velocity: the potential energy is "exchanged" for kinetic energy as it falls. As it hits the ground, where $h = 0$, potential energy has diminished to zero,[14] and its total energy, now all kinetic, is $\frac{1}{2}mv^2$. The total energy from top to bottom, however, has always remained constant:

$$E_{total} = \frac{mv^2}{2} + mgh$$

The kinetic energy at the bottom of the cliff thus equals $E_{total}$ and that is the same value as the potential energy at the top. Equating them, cancelling the mass terms, relabeling acceleration $a$ and the height $s$, and isolating the velocity term, we have

$$v^2 = 2as$$

Given that in the case of uniform acceleration, the velocity at time $t$ is equal to acceleration times time elapsed, or $v = at$, we can square that expression (to become $v^2 = a^2t^2$) and substitute the right-hand side for $v^2$ in the previous equation:

$$a^2t^2 = 2as$$

Solving for distance yields,

$$s = \frac{1}{2}at^2$$

We have derived the Galilean relationship from purely energy concepts unknown in Galileo's time. Had he known, how much time it would have saved him!

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[13] As one ventures far from Earth the value of $g$ changes with the inverse square of the object from the center of the Earth, so in those cases the equation must be modified to take that into account, and the equation is $PE = -GMm/r$.

[14] We here adopt the surface of the Earth as the arbitrary zero reference point for potential energy, since the rock can descend no lower.

**Exercises: Playing Around with Gravity** Galileo undertook his experiments with early nineteenth Century materials and concepts. We now have modern mathematics and far greater knowledge of the motion of bodies both on this and other worlds. We also have experience in doing thought experiments using mathematical models to study and predict outcomes. The problems below invite you to model simple dynamic situations in various contexts to help you gain greater familiarity with the basic Galilean concepts.

## Problems

1. An amateur rocket test-fired from rest travels horizontally with constant acceleration. After 30 s of flight it has achieved a velocity of 900 m/s. What is its horizontal acceleration and distance travelled?
2. Ignoring friction, at what height must the horizontally-aimed rocket in the previous problem be launched in order for it not to hit the ground before 30 s of flight?
3. A rocket has been fired vertically from the ground with steady acceleration. Its velocity at height $h$ is 300 m/s. What is height $h$?
4. A small balloon ascends to an altitude of 1 km above the surface of the earth, whereupon a 1 kg ball is dropped. What potential energy has the ball gained at the top of its ascent? Ignoring friction, (a) what would be the ball's kinetic energy and velocity be when it hits the earth; and (b) assuming a gravitational acceleration of 9.8 m/s per second, how long would the fall take?
5. Mars has a surface gravity that is 3.71 m/s$^2$. If Galileo's rolling ball experiment were set up on Mars, estimate the numerical results that would be obtained. Ignore friction. Explain your reasoning.
6. Neptune has surface gravity that is about 1.138 times that of the Earth. If Galileo's rolling ball experiment were set up on Neptune, estimate the numerical results that would be obtained on that planet. Ignore friction. Explain your reasoning.
7. Galileo's rolling ball experiment is primarily an illustration of: (a) how fast things fall; (b) how friction affects the speed of moving objects; (c) how objects in motion tend to stay in motion; (d) how vectors work.
8. Suppose you drop a tennis ball from a height of 1.5 m while your friend holds a gun and, at the moment you drop the ball, shoots it perfectly horizontally from the same height. Ignoring friction, which should hit the ground first: the ball or the bullet? Explain your answer.
9. Galileo discovered that a dropped object will fall (a) equal distances in equal times; (b) twice as fast in the second second as in the first second; (c) as far proportionally as the square of the time; (d) half the acceleration times the time.
10. Imagine a 1 kg ball is rolled down a frictionless ramp 10 m long whose high end is 1 m off the ground. The ramp merges into a perfectly flat pathway at ground level. Describe and compute the forces, accelerations and velocities of the ball at each second of its motion down the ramp and along the level plane.

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http://www.springer.com/978-1-4614-5443-4

Newton's Gravity
An Introductory Guide to the Mechanics of the Universe
MacDougal, D.W.
2012, XIX, 433 p. 78 illus., 69 illus. in color., Softcover
ISBN: 978-1-4614-5443-4