Comparison of Control Volume Analysis and Porous Media Averaging for Formulation of Porous Media Transport

F. Civan

Abstract Although the porous media averaging is frequently used it creates many difficulties in handling of the closure problems. The control volume analysis on the other hand can avoid such difficulties in a practical manner. Thus, the outstanding and complementary features of these two approaches used in the macroscopic formulation of transport through porous media are critically reviewed, compared, and evaluated. Several instructive examples are presented demonstrating their applications with various improvements.

1 Introduction

The formulation of macroscopic equations of porous media transport phenomena continues to occupy the researchers because of various unresolved issues such as the necessity of using different representative elementary volumes (REV) for different quantities and proper methods required for closure problems and reducing the complexity of the resulting equations for applications of practical importance. Although some rules of averaging have now been well established, applications in different ways may result with different formulations and their inherent limiting conditions are frequently overlooked. This includes the use of the same representative elemental volume for different quantities and assuming the volume and area averages to be the same [12]. Volume averaging of microscopic equations of transport processes may lead to extremely complicated results to be of any practical value [10, 11, 14, 18, 20]. Nevertheless, the control volume analysis can be complementary in resolving some of the difficulties of porous media volume averaging [5–7]. This chapter investigates the formulation of macroscopic transport equations

F. Civan (✉)
Mewbourne School of Petroleum and Geological Engineering, The University of Oklahoma, 100 East Boyd, SEC Room 1210 Norman, OK 73019-1003, USA
e-mail: fcivan@ou.edu

J.A. Ferreira et al. (eds.), *Modelling and Simulation in Fluid Dynamics in Porous Media*, Springer Proceedings in Mathematics & Statistics 28, DOI 10.1007/978-1-4614-5055-9_2,

by means of the porous media averaging and control volume analysis approaches and comparison of the results to determine the complementary benefits of these two different approaches. The formulations of the spontaneous transport terms and the representative volume-averaging and control volume analysis approaches are reviewed and their applications are demonstrated by various examples. The discrepancies occurring between the results obtained here and the corresponding efforts reported in the literature and their reasons are explained.

2 Spontaneous Transport

Many processes in porous media occur spontaneously because of the nonequilibrium conditions prevailing over a distance in porous media and are expressed by the following empirical gradient law [6, 7]:

$$\mathbf{j}_{jb} = \varepsilon_j \mathbf{j}_j = -\frac{1}{\varepsilon_j}\mathbf{D}_{jb} \cdot (\nabla f_{jb} - \mathbf{f}_{Rjb}) = -\frac{1}{\varepsilon_j}\mathbf{D}_{jbe} \cdot \nabla f_{jb} = -\mathbf{D}_{je} \cdot \nabla f_{jb}, \qquad (1)$$

where ε_j is the volume fraction of the j-phase present in the bulk volume of porous media, f_{jb} and f_j denote the bulk- and phase-volume averages of a property of the j-phase, respectively, $\mathbf{j}_{jb}$ and $\mathbf{j}_j$ denote the flux vectors of property f_j of the j-phase transferred across the bulk-surface area and the open-pore-surface area, respectively, $\mathbf{f}_{Rjb}$ denotes a vector representing the net internal resistance to transfer of property f_j of the j-phase through the bulk volume of porous media, and $\mathbf{D}_{jb}$, $\mathbf{D}_{jbe}$, and $\mathbf{D}_{je}$ are the various types of transport coefficients of property f_j of the j-phase appearing in Eq. (1) which are related by the following relationships:

$$\mathbf{D}_{jbe} = \left(1 - \frac{\mathbf{f}_{Rjb} \cdot \nabla f_{jb}}{\nabla f_{jb} \cdot \nabla f_{jb}}\right)\mathbf{D}_{jb}, \quad \text{when } 1 - \frac{\mathbf{f}_{Rjb} \cdot \nabla f_{jb}}{\nabla f_{jb} \cdot \nabla f_{jb}} > 0,$$

$$\mathbf{D}_{jbe} = \mathbf{0}, \quad \text{when } 1 - \frac{\mathbf{f}_{Rjb} \cdot \nabla f_{jb}}{\nabla f_{jb} \cdot \nabla f_{jb}} \leqslant 0,$$

and

$$\mathbf{D}_{je} = \frac{1}{\varepsilon_j}\mathbf{D}_{jbe}. \qquad (2)$$

The gradient law in various forms similar to Eq. (1) can be used also as an empirical means of achieving the closure for the averages of the products of deviations from the individual averages of various quantities. The empirical coefficients of transport $\mathbf{D}_{jb}$, $\mathbf{D}_{jbe}$, and $\mathbf{D}_{je}$ of various types given in Eq. (2) are referred to as the dispersion coefficients.

The gradient term appearing in Eq. (1) and elsewhere in the various formulations of porous media transport inherently involves the application of the *extrapolated limit* concept of [13] as illustrated in Fig. 1. Therefore, the gradient is expressed with

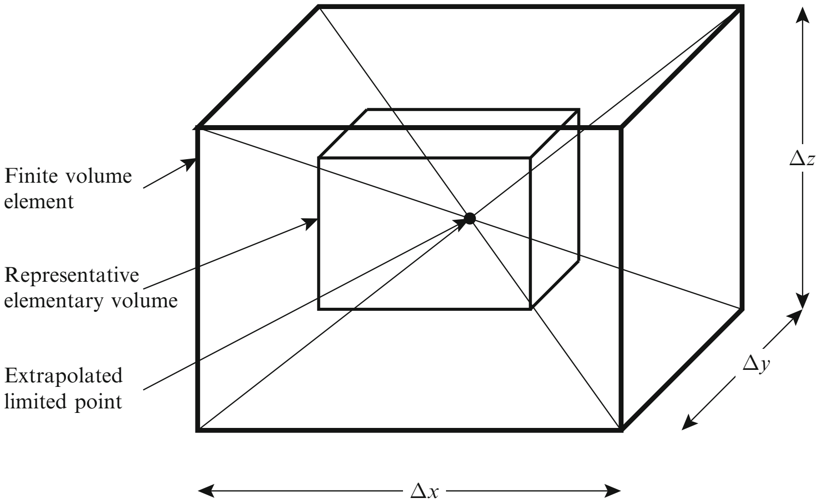

Fig. 1 Illustration of the *extrapolated limit* concept of [13]

respect to the amount of the property of interest contained in the bulk volume. This is a remedial measure because the basic definition of derivative which considers the limit as $\Delta x, \Delta y$, and $\Delta z \to 0$ is not rigorously applicable in porous media but nevertheless used in derivation of macroscopic equations. This issue is subtle and is mostly overlooked. For example, fluid tends to move from a high-density location to a low-density location at a rate proportional to the density gradient $\nabla\rho$ in the microscopic formulation of single-phase fluid systems. However, the gradient should be expressed as $\nabla(\varepsilon\rho)$ in terms of the mass of fluid contained in porous media for macroscopic formulation which is the product of the volume fraction ε and density ρ of the fluid present in porous media [4]. Consequently, the gradient law of porous media transport appears to have a problem in the special case when the fluid is incompressible. However, transport can occur in porous media averaged description even in the case of incompressible fluids when the volume fraction of the fluid varies with distance because the mass of fluid contained in porous media is different over a distance.

An application of Eq. (1) is demonstrated by derivation of Darcy's law. It should be emphasized that Darcy's law was derived with respect to the pressures applied at the outside surfaces of porous media. This bulk-volume average pressure is related to the intrinsic phase-volume average pressure by [6]

$$p_{jb} = [\alpha + (1-\alpha)c\varepsilon_j^{d/3}]p_j, \quad 0 \leqslant \alpha \leqslant 1. \tag{3}$$

Equation (3) assumes a fractal relationship between the cross-sectional area of flowing fluid and its volume fraction in porous media with a fractal coefficient of c and dimension of d, which may be replaced with other types of relationships, and

α denotes the effective stress coefficient of [1]. Equation (3) considers the effect of stress transmission through porous media depending on its elasticity.

Thus, the modified Darcy's law can be derived from Eq. (1) simply by substituting $f_{jb} \equiv p_{jb}$, $\mathbf{f}_{Rjb} \equiv \mathbf{p}_{Rjb}$, $\mathbf{j}_{jb} \equiv \mathbf{v}_{jb}$, and $\mathbf{D}_{jbe} \equiv \frac{1}{\mu_j}\mathbf{K}_{jbe}$ as

$$\begin{aligned}\mathbf{v}_{jb} = \varepsilon_j \mathbf{v}_j &= -\frac{1}{\varepsilon_j}\frac{1}{\mu_j}\mathbf{K}_{jb}\cdot(\nabla p_{jb} - \mathbf{p}_{Rjb}) \\ &= -\frac{1}{\varepsilon_j}\frac{1}{\mu_j}\mathbf{K}_{jbe}\cdot\nabla p_{jb} = -\frac{1}{\mu_j}\mathbf{K}_{je}\cdot\nabla p_{jb},\end{aligned} \tag{4}$$

where p_{jb} and p_j denote the bulk-volume average and the phase-volume average pressures, μ_j is the dynamic viscosity of the j-phase, respectively, $\mathbf{v}_{jb}$ and $\mathbf{v}_j$ denote the volumetric flux vectors of the j-phase transferred across the bulk-surface area and the open-pore-surface area, respectively, $\mathbf{p}_{Rjb}$ is a vector representing the threshold pressure gradient below which the j-phase cannot flow through porous media because of internal resistance [16], and $\mathbf{K}_{jb}$, $\mathbf{K}_{jbe}$, and $\mathbf{K}_{je}$ are the various types of effective permeability coefficients for the j-phase appearing in Eq. (4) which are related by the following relationships:

$$\mathbf{K}_{jbe} = \left(1 - \frac{\mathbf{p}_{Rjb}\cdot\nabla p_{jb}}{\nabla p_{jb}\cdot\nabla p_{jb}}\right)\mathbf{K}_{jb}, \quad \text{when } 1 - \frac{\mathbf{p}_{Rjb}\cdot\nabla p_{jb}}{\nabla p_{jb}\cdot\nabla p_{jb}} > 0,$$

$$\mathbf{K}_{jbe} = \mathbf{0}, \quad \text{when } 1 - \frac{\mathbf{p}_{Rjb}\cdot\nabla p_{jb}}{\nabla p_{jb}\cdot\nabla p_{jb}} \leqslant 0,$$

and

$$\mathbf{K}_{je} = \frac{1}{\varepsilon_j}\mathbf{K}_{jbe}. \tag{5}$$

In view of the above formulation, the law of motion introduced by [9] inherently assumes a perfectly rigid porous media for which case Biot's coefficient is given as $\alpha = 1$ [6, 15]. On the other hand, Biot's coefficient becomes $\alpha = 0$ for completely poroelastic porous media. However, its value needs to be determined empirically for natural porous materials. Further, the original Darcy's law assumes that flow occurs as long as a pressure differential is applied across porous media and therefore does not consider the need to overcome a threshold pressure gradient to initiate fluid flow.

3 Porous Media Averaging

The derivation of the macroscopic transport equations is often accomplished by averaging of the microscopic transport equations over REV [18]. The REV should be selected differently for different quantities as demonstrated by [8]. However, this issue is omitted in the following in order to avoid the additional complications so that the present discussion can focus on the issue of comparison and evaluation of results obtained from REV-averaging and control volume analysis approaches. First, a brief review and summary of the REV-averaging rules is presented and then several applications are illustrated by various examples. The scalar, vector, and tensor properties considered in the following sections are identified using the non-bold, lowercase bold, and uppercase bold symbols.

3.1 Representative Elementary Volume-Averaging Rules

Consider the j-phase in a multiphase system containing $j = 1, 2, , N$ phases (solid and fluid). The representative elementary bulk volume of porous media is V_b. The volume of the j-phase contained in the representative elementary bulk volume is V_j. The j-phase can interact with the other phases (solid and fluid) in various ways.

The bulk-volume (superficial) average of a property f of the j-phase is given by [11, 18, 20]

$$\langle f_j \rangle_b = \frac{1}{V_b} \int_{V_j} f_j \, dV. \tag{6}$$

The individual phase-volume (intrinsic) average of a property f of the j-phase is given by [11, 18, 20]

$$\langle f_j \rangle_j = \frac{1}{V_j} \int_{V_j} f_j \, dV. \tag{7}$$

The value of a property f of the j-phase at a certain point inside the REV is given as the sum of the intrinsic average value $\langle f_j \rangle_j$ and its deviation $\widehat{f_j}$ from the intrinsic average value as [11]

$$f_j = \langle f_j \rangle_j + \widehat{f_j}, \quad \left\langle \widehat{f_j} \right\rangle_j = 0 \text{ in } V_j. \tag{8}$$

The relationship between the superficial and intrinsic volume averages of a property f is given by

$$\langle f_j \rangle_b = \varepsilon_j \langle f_j \rangle_j . \tag{9}$$

Thus, the following expressions can be derived for the product of two properties and the averages of the products of two and three properties, respectively [7]:

$$\mathbf{v}_j f_j = \langle \mathbf{v}_j \rangle_j \langle f_j \rangle_j + 2\widehat{\mathbf{v}}_j \langle f_j \rangle_j + \widehat{\mathbf{v}}_j \widehat{f_j}, \tag{10}$$

$$\begin{aligned} \langle \mathbf{v}_j f_j \rangle_b &= \varepsilon_j \langle \mathbf{v}_j f_j \rangle_j = \varepsilon_j \left(\langle \mathbf{v}_j \rangle_j \langle f_j \rangle_j + \left\langle \widehat{\mathbf{v}}_j \widehat{f_j} \right\rangle_j \right) \\ &= \frac{1}{\varepsilon_j} \langle \mathbf{v}_j \rangle_b \langle f_j \rangle_b + \left\langle \widehat{\mathbf{v}}_j \widehat{f_j} \right\rangle_b, \end{aligned} \tag{11}$$

and

$$\begin{aligned}\langle\rho_j\mathbf{v}_jf_j\rangle_b &= \varepsilon_j\langle\rho_j\mathbf{v}_jf_j\rangle_j\\ &= \varepsilon_j\Big(\langle\rho_j\rangle_j\langle\mathbf{v}_j\rangle_j\langle f_j\rangle_j+\langle\rho_j\rangle_j\left\langle\widehat{\mathbf{v}}_j\widehat{f}_j\right\rangle_j\\ &\quad+\langle\mathbf{v}_j\rangle_j\langle\widehat{\rho}_j\widehat{f}_j\rangle_j+\langle f_j\rangle_j\langle\widehat{\rho}_j\widehat{\mathbf{v}}_j\rangle_j+\langle\widehat{\rho}_j\widehat{\mathbf{v}}_j\widehat{f}_j\rangle_j\Big)\\ &= \frac{1}{\varepsilon_j^2}\langle\rho_j\rangle_b\langle\mathbf{v}_j\rangle_b\langle f_j\rangle_b+\frac{1}{\varepsilon_j}\langle\rho_j\rangle_b\left\langle\widehat{\mathbf{v}}_j\widehat{f}_j\right\rangle_b+\frac{1}{\varepsilon_j}\langle\mathbf{v}_j\rangle_b\left\langle\widehat{\rho}_j\widehat{f}_j\right\rangle_b\\ &\quad+\frac{1}{\varepsilon_j}\langle f_j\rangle_b\langle\widehat{\rho}_j\widehat{\mathbf{v}}_j\rangle_b+\left\langle\widehat{\rho}_j\widehat{\mathbf{v}}_j\widehat{f}_j\right\rangle_b.\end{aligned}\tag{12}$$

The volume averages of the time derivative and gradient of a property f of the j-phase are given, respectively, by [10, 18, 20]

$$\begin{aligned}\left\langle\frac{\partial f_j}{\partial t}\right\rangle_b &= \frac{\partial\langle f_j\rangle_b}{\partial t}-\frac{1}{V_b}\int_{A_j}f_j\mathbf{v}_{Aj}\cdot\mathbf{n}_j\,dA\\ &= \frac{\partial(\varepsilon_j\langle f_j\rangle_j)}{\partial t}-\frac{1}{V_b}\int_{A_j}f_j\mathbf{v}_{Aj}\cdot\mathbf{n}_j\,dA,\end{aligned}\tag{13}$$

and

$$\langle\nabla f_j\rangle_b=\nabla\langle f_j\rangle_b+\frac{1}{V_b}\int_{A_j}f_j\mathbf{n}_j\,dA=\langle\nabla\rangle_j\left(\varepsilon_j\langle f_j\rangle_j\right)+\frac{1}{V_b}\int_{A_j}f_j\mathbf{n}_j\,dA.\tag{14}$$

Although frequently used in the literature, their derivations involve some assumptions which are not rigorously correct such as the consideration of the volumetric and area averages to be the same [12]. Here t is time and ∇ is the gradient operator. A_j is the surface area of the j-phase. $\mathbf{v}_{Aj}$ is the velocity of the surface area A_j of the j-phase at which it moves. A is the surface area variable. $\mathbf{n}_j$ is the outward unit normal vector. The symbols ∇ and $\langle\nabla\rangle_j$ denote the microscopic and macroscopic gradient operators, respectively [12]. However, in the rest of the development in this chapter simply ∇ will be used for convenience also for the macroscopic gradient operator.

However, the alternative procedure illustrated in the following yields different results than Eqs. (13) and (14), respectively, as

$$\begin{aligned}\left\langle\frac{\partial f_j}{\partial t}\right\rangle_b &= \varepsilon_j\left\langle\frac{\partial f_j}{\partial t}\right\rangle_j=\varepsilon_j\left(\frac{\partial\langle f_j\rangle_j}{\partial t}-\frac{1}{V_j}\int_{A_j}f_j\mathbf{v}_{Aj}\cdot\mathbf{n}_j\,dA\right)\\ &= \varepsilon_j\frac{\partial}{\partial t}\left(\frac{\langle f_j\rangle_b}{\varepsilon_j}\right)-\frac{1}{V_b}\int_{A_j}f_j\mathbf{v}_{Aj}\cdot\mathbf{n}_j\,dA\end{aligned}\tag{15}$$

and similarly

$$\left\langle \nabla f_j \right\rangle_b = \varepsilon_j \left\langle \nabla f_j \right\rangle_j = \varepsilon_j \left(\nabla \left\langle f_j \right\rangle_j + \frac{1}{V_j} \int_{A_j} f_j \mathbf{n}_j \, dA \right)$$

$$= \varepsilon_j \nabla \left(\frac{\langle f_j \rangle_b}{\varepsilon_j} \right) + \frac{1}{V_b} \int_{A_j} f_j \mathbf{n}_j \, dA \tag{16}$$

Notice that Eqs. (13) and (14) will be identical to Eqs. (15) and (16) only when the fluid phase-volume fraction ε_j is constant. This exercise illustrates the variation of the results by the ways of implementation of the basic rules of volume averaging.

Consider now the microscopic conservation equation for a property f of a system, given by

$$\frac{\partial f_j}{\partial t} + \nabla \cdot \mathbf{j}_{Tj} = q_j, \tag{17}$$

where q_j is a source term. The total transfer term for quantity f_j is described by

$$\mathbf{j}_{Tj} = \mathbf{v}_j f_j + \mathbf{j}_j, \; \mathbf{j}_j = -\mathbf{D}_j \cdot \nabla f_j, \tag{18}$$

where $\mathbf{j}_{Tj}$ is the total effective transport rate and $\mathbf{j}_j$ is the spontaneous transport of quantity f_j in the j-phase.

Thus, substituting Eq. (18) into Eq. (17) yields a transient-state convection/advection, dispersion, and source equation for transport of quantity f_j through porous media as

$$\frac{\partial f_j}{\partial t} + \nabla \cdot (\mathbf{v}_j f_j) = \nabla \cdot (\mathbf{D}_j \cdot \nabla f_j) + q_j. \tag{19}$$

Let ρ_j and f_j denote the density and an intensive property of the j-phase, respectively, $\mathbf{v}_j$ is the velocity vector, $\mathbf{D}_j$ is the diffusivity tensor, and q_j denotes the mass added per unit volume of the j-phase per unit time. In these equations, f_j is equal to ρ_j, $\rho_j \mathbf{v}_j$, and $H_j = \rho_j C_j T_j$ for the conservations of mass, momentum, and energy, respectively. H_j is the enthalpy, T_j denotes the temperature, and C_j is the specific heat capacity at constant pressure of the j-phase.

The volume average of Eq. (17) applying Eq. (6) yields

$$\left\langle \frac{\partial f_j}{\partial t} \right\rangle_b + \left\langle \nabla \cdot (\mathbf{v}_j f_j) \right\rangle_b = \left\langle \nabla \cdot (\mathbf{D}_j \cdot \nabla f_j) \right\rangle_b + \left\langle q_j \right\rangle_b. \tag{20}$$

The following relationships can be written using Eq. (13):

$$\left\langle \nabla \cdot (\mathbf{v}_j f_j) \right\rangle_b = \nabla \cdot \left\langle \mathbf{v}_j f_j \right\rangle_b + \frac{1}{V_b} \int_{A_j} (\mathbf{v}_j f_j) \cdot \mathbf{n}_j \, dA \tag{21}$$

and

$$\left\langle \nabla \cdot (\mathbf{D}_j \cdot \nabla f_j) \right\rangle_b = \nabla \cdot \left\langle \mathbf{D}_j \cdot \nabla f_j \right\rangle_b + \frac{1}{V_b} \int_{A_j} (\mathbf{D}_j \cdot \nabla f_j) \cdot \mathbf{n}_j \, dA. \tag{22}$$

Thus, the application of Eqs. (13), (21), and (22) to Eq. (20) yields

$$\frac{\partial}{\partial t}\left\langle f_j\right\rangle_b+\nabla\cdot\left\langle \mathbf{v}_j f_j\right\rangle_b=\nabla\cdot\left\langle \mathbf{D}_j\cdot\nabla f_j\right\rangle_b+\sum_l\frac{-1}{V_b}\int_{A_{jl}}f_j(\mathbf{v}_j-\mathbf{v}_{Ajl})\cdot\mathbf{n}_j\,dA$$
$$+\sum_l\frac{1}{V_b}\int_{A_{jl}}\mathbf{D}_j\cdot\nabla f_j\cdot\mathbf{n}_j\,dA+\left\langle q_j\right\rangle_b, \tag{23}$$

where l denotes the various phases present in porous media including the porous solid matrix which is in contact with the j-phase.

In the following, the expressions for $\left\langle \mathbf{D}_j\cdot\nabla f_j\right\rangle_b$ are derived by two alternative approaches and the results are compared. The details of the first approach are presented in the following, delineating simultaneously the application of the basic averaging rules:

$$\begin{aligned}
\left\langle \mathbf{D}_j\cdot\nabla f_j\right\rangle_b &= \varepsilon_j\left\langle \mathbf{D}_j\cdot\nabla f_j\right\rangle_j\\
&=\varepsilon_j\left\langle\left(\left\langle \mathbf{D}_j\right\rangle_j+\widehat{\mathbf{D}}_j\right)\cdot\nabla\left(\left\langle f_j\right\rangle_j+\widehat{f}_j\right)\right\rangle_j\\
&=\varepsilon_j\Big(\left\langle \mathbf{D}_j\right\rangle_j\cdot\left\langle\nabla\left\langle f_j\right\rangle_j\right\rangle_j\\
&\quad+\left\langle \mathbf{D}_j\right\rangle_j\cdot\left\langle\nabla\widehat{f}_j\right\rangle_j+\left\langle\widehat{\mathbf{D}}_j\right\rangle_j\cdot\left\langle\nabla\left\langle f_j\right\rangle_j\right\rangle_j+\left\langle\widehat{\mathbf{D}}_j\cdot\nabla\widehat{f}_j\right\rangle_j\Big)\\
&=\varepsilon_j\left(\left\langle \mathbf{D}_j\right\rangle_j\cdot\left[\nabla\left\langle\left\langle f_j\right\rangle_j\right\rangle_j+\frac{1}{V_j}\int_{A_j}\left\langle f_j\right\rangle_j\mathbf{n}_j\,dA\right]\right.\\
&\quad+\left\langle \mathbf{D}_j\right\rangle_j\cdot\left[\nabla\left\langle\widehat{f}_j\right\rangle_j+\frac{1}{V_j}\int_{A_j}\widehat{f}_j\mathbf{n}_j\,dA\right]\\
&\quad\left.+\left\langle\widehat{\mathbf{D}}_j\right\rangle_j\cdot\left\langle\nabla\left\langle f_j\right\rangle_j\right\rangle_j+\left\langle\widehat{\mathbf{D}}_j\cdot\nabla\widehat{f}_j\right\rangle_j\right)\\
&=\frac{1}{\varepsilon_j}\left\langle \mathbf{D}_j\right\rangle_b\cdot\left(\nabla\left\langle f_j\right\rangle_b+\frac{1}{V_b}\int_{A_j}f_j\mathbf{n}_j\,dA\right)+\left\langle\widehat{\mathbf{D}}_j\cdot\nabla\widehat{f}_j\right\rangle_b.
\end{aligned}\tag{24}$$

The details of the second approach are illustrated in the following:

$$\begin{aligned}
\left\langle \mathbf{D}_j\cdot\nabla f_j\right\rangle_b &= \varepsilon_j\left\langle \mathbf{D}_j\cdot\nabla f_j\right\rangle_j\\
&=\varepsilon_j\left(\left\langle \mathbf{D}_j\right\rangle_j\cdot\left\langle\nabla f_j\right\rangle_j+\left\langle\widehat{\mathbf{D}}_j\cdot\widehat{\nabla f_j}\right\rangle_j\right)\\
&=\left\langle \mathbf{D}_j\right\rangle_j\cdot\left\langle\nabla f_j\right\rangle_b+\left\langle\widehat{\mathbf{D}}_j\cdot\widehat{\nabla f_j}\right\rangle_b\\
&=\frac{1}{\varepsilon_j}\left\langle \mathbf{D}_j\right\rangle_b\cdot\left(\nabla\left\langle f_j\right\rangle_b+\frac{1}{V_b}\int_{A_j}f_j\mathbf{n}_j\,dA\right)+\left\langle\widehat{\mathbf{D}}_j\cdot\widehat{\nabla f_j}\right\rangle_b.
\end{aligned}\tag{25}$$

As can be seen, the results of Eqs. (24) and (25) are the same if $\widehat{\nabla f_j} = \nabla \widehat{f}_j$. However, the result will be different (incorrect) if the following approach is taken instead of the above approach:

$$\left\langle \mathbf{D}_j \cdot \nabla f_j \right\rangle_b = \left\langle \mathbf{D}_j \right\rangle_b \cdot \left\langle \nabla f_j \right\rangle_b + \left\langle \widehat{\mathbf{D}}_j \cdot \widehat{\nabla f_j} \right\rangle_b$$
$$= \left\langle \mathbf{D}_j \right\rangle_b \cdot \left(\nabla \left\langle f_j \right\rangle_b + \frac{1}{V_b} \int_{A_j} f_j \mathbf{n}_j \, dA \right) + \left\langle \widehat{\mathbf{D}}_j \cdot \widehat{\nabla f_j} \right\rangle_b . \quad (26)$$

Substituting Eqs. (11) and (25) into Eq. (23) yields the following macroscopic conservation equation for property f_j:

$$\frac{\partial}{\partial t} \left\langle f_j \right\rangle_b + \nabla \cdot \left(\frac{1}{\varepsilon_j} \left\langle \mathbf{v}_j \right\rangle_b \left\langle f_j \right\rangle_b + \left\langle \widehat{\mathbf{v}}_j \widehat{f}_j \right\rangle_b \right)$$
$$= \nabla \cdot \left(\frac{1}{\varepsilon_j} \left\langle \mathbf{D}_j \right\rangle_b \cdot \left(\nabla \left\langle f_j \right\rangle_b + \sum_l \frac{1}{V_b} \int_{A_{jl}} f_j \mathbf{n}_j \, dA \right) + \left\langle \widehat{\mathbf{D}}_j \cdot \widehat{\nabla f_j} \right\rangle_b \right)$$
$$+ \sum_l \frac{-1}{V_b} \int_{A_{jl}} f_j \left(\mathbf{v}_j - \mathbf{v}_{Ajl} \right) \cdot \mathbf{n}_j \, dA + \sum_l \frac{1}{V_b} \int_{A_{jl}} \mathbf{D}_j \cdot \nabla f_j \cdot \mathbf{n}_j \, dA + \left\langle q_j \right\rangle_b . \quad (27)$$

In the following, first the applications of Eq. (27) are demonstrated for derivation of the mass and momentum conservation equations. Then, their simplified forms are compared with the results presented by [19] by application of their simplifying conditions.

3.2 Mass Equation

The volume average of Eq. (17) is obtained as the following by applying Eq. (27), where f_j is equal to mass density ρ_j for mass conservation:

$$\frac{\partial}{\partial t} \left\langle \rho_j \right\rangle_b + \nabla \cdot \left(\frac{1}{\varepsilon_j} \left\langle \mathbf{v}_j \right\rangle_b \left\langle \rho_j \right\rangle_b \right) = \nabla \cdot \left(\frac{1}{\varepsilon_j} \left\langle \mathbf{D}_j \right\rangle_b \cdot \nabla \left\langle \rho_j \right\rangle_b \right)$$
$$- \nabla \cdot \left(\left\langle \widehat{\mathbf{v}}_j \widehat{\rho}_j \right\rangle_b \right) + \nabla \cdot \left(\frac{1}{\varepsilon_j} \left\langle \mathbf{D}_j \right\rangle_b \cdot \sum_l \frac{1}{V_b} \int_{A_{jl}} \rho_j \mathbf{n}_j \, dA \right) + \nabla \cdot \left(\left\langle \widehat{\mathbf{D}}_j \cdot \widehat{\nabla \rho_j} \right\rangle_b \right)$$
$$+ \sum_l \frac{-1}{V_b} \int_{A_{jl}} \rho_j \left(\mathbf{v}_j - \mathbf{v}_{Ajl} \right) \cdot \mathbf{n}_j \, dA + \sum_l \frac{1}{V_b} \int_{A_{jl}} \mathbf{D}_j \cdot \nabla \rho_j \cdot \mathbf{n}_j \, dA + \left\langle q_j \right\rangle_b . \quad (28)$$

Eq. (28) simplifies as the following when $\mathbf{D} = \mathbf{0}$ is substituted:

$$\frac{\partial}{\partial t}\left\langle \rho_j \right\rangle_b + \nabla \cdot \left(\frac{1}{\varepsilon_j} \left\langle \mathbf{v}_j \right\rangle_b \left\langle \rho_j \right\rangle_b + \left\langle \widehat{\mathbf{v}}_j \widehat{\rho}_j \right\rangle_b \right)$$
$$= \sum_l \frac{-1}{V_b} \int_{A_{jl}} \rho_j \left(\mathbf{v}_j - \mathbf{v}_{Ajl} \right) \cdot \mathbf{n}_j \, dA + \left\langle q_j \right\rangle_b . \quad (29)$$

Expressing Eq. (29) in terms of the intrinsic fluid properties only yields

$$\frac{\partial}{\partial t}\left(\varepsilon_j \left\langle \rho_j \right\rangle_j \right) + \nabla \cdot \left(\varepsilon_j \left\langle \mathbf{v}_j \right\rangle_j \left\langle \rho_j \right\rangle_j + \varepsilon_j \left\langle \widehat{\mathbf{v}}_j \widehat{\rho}_j \right\rangle_j \right)$$
$$= \sum_l \frac{-1}{V_b} \int_{A_{jl}} \rho_j \left(\mathbf{v}_j - \mathbf{v}_{Ajl} \right) \cdot \mathbf{n}_j \, dA + \varepsilon_j \left\langle q_j \right\rangle_j . \quad (30)$$

Substituting $\mathbf{v}_{Ajl} = 0$, $\left\langle q_j \right\rangle_b = 0$, assuming a no-slip condition ($\mathbf{v}_j = 0$) at the pore surface according to [19] into Eq. (28), and considering the fact that deviation quantities are zero for these constant values yields

$$\frac{\partial}{\partial t}\left\langle \rho_j \right\rangle_b + \nabla \cdot \left\langle \mathbf{v}_j \rho_j \right\rangle_b = 0. \quad (31)$$

3.3 Momentum Equation

The volume average of Eq. (17) where f_j is equal to $\rho_j \mathbf{v}_j$ for the conservation of momentum can be obtained as the following:

$$\frac{\partial}{\partial t}\left\langle \rho_j \mathbf{v}_j \right\rangle_b + \nabla \cdot \left(\frac{1}{\varepsilon_j} \left\langle \mathbf{v}_j \right\rangle_b \left\langle \rho_j \mathbf{v}_j \right\rangle_b + \left\langle \widehat{\mathbf{v}}_j \widehat{\rho_j \mathbf{v}_j} \right\rangle_b \right)$$
$$= \nabla \cdot \left(\frac{1}{\varepsilon_j} \left\langle \mathbf{D}_j \right\rangle_b \cdot \left(\nabla \left\langle \rho_j \mathbf{v}_j \right\rangle_b + \sum_l \frac{1}{V_b} \int_{A_{jl}} \rho_j \mathbf{v}_j \cdot \mathbf{n}_j \, dA \right) \right.$$
$$\left. + \left\langle \widehat{\mathbf{D}}_j \cdot \widehat{\nabla \rho_j \mathbf{v}_j} \right\rangle_b \right) + \sum_l \frac{-1}{V_b} \int_{A_{jl}} \rho_j \mathbf{v}_j \left(\mathbf{v}_j - \mathbf{v}_{Ajl} \right) \cdot \mathbf{n}_j \, dA$$
$$+ \sum_l \frac{1}{V_b} \int_{A_{jl}} \mathbf{D}_j \cdot \nabla \left(\rho_j \mathbf{v}_j \right) \cdot \mathbf{n}_j \, dA + \left\langle \mathbf{q}_j \right\rangle_b . \quad (32)$$

Note, applying Eq. (11)

$$\left\langle \rho_j \mathbf{v}_j \right\rangle_b = \frac{1}{\varepsilon_j} \left\langle \rho_j \right\rangle_b \left\langle \mathbf{v}_j \right\rangle_b + \left\langle \widehat{\rho}_j \widehat{\mathbf{v}}_j \right\rangle_b . \quad (33)$$

Applying Eq. (33) into Eq. (32) results in

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{1}{\varepsilon_j}\langle\rho_j\rangle_b\langle\mathbf{v}_j\rangle_b\right)+\nabla\cdot\left(\frac{1}{\varepsilon_j^2}\langle\mathbf{v}_j\rangle_b\langle\rho_j\rangle_b\langle\mathbf{v}_j\rangle_b\right)\\
&\quad+\frac{\partial}{\partial t}\left(\langle\widehat{\rho}_j\widehat{\mathbf{v}}_j\rangle_b\right)+\nabla\cdot\left(\frac{1}{\varepsilon_j}\langle\mathbf{v}_j\rangle_b\langle\widehat{\rho}_j\widehat{\mathbf{v}}_j\rangle_b+\langle\widehat{\mathbf{v}}_j\widehat{\rho_j\mathbf{v}_j}\rangle_b\right)\\
&\quad=\nabla\cdot\left(\frac{1}{\varepsilon_j}\langle\mathbf{D}_j\rangle_b\cdot\left(\nabla\langle\rho_j\mathbf{v}_j\rangle_b+\sum_l\frac{1}{V_b}\int_{A_{jl}}\rho_j\mathbf{v}_j\cdot\mathbf{n}_j\,dA\right)\right.\\
&\qquad\left.+\left\langle\widehat{\mathbf{D}}_j\cdot\widehat{\nabla\rho_j\mathbf{v}_j}\right\rangle_b\right)+\sum_l\frac{-1}{V_b}\int_{A_{jl}}\rho_j\mathbf{v}_j\left(\mathbf{v}_j-\mathbf{v}_{Ajl}\right)\cdot\mathbf{n}_j\,dA\\
&\qquad+\sum_l\frac{1}{V_b}\int_{A_{jl}}\mathbf{D}_j\cdot\nabla\left(\rho_j\mathbf{v}_j\right)\cdot\mathbf{n}_j\,dA+\langle\mathbf{q}_j\rangle_b. \qquad (34)
\end{aligned}$$

Alternatively, the general momentum conservation equation can be derived directly by substituting $f_j=\rho_j\mathbf{v}_j$ into Eq. (23) to obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\langle\rho_j\mathbf{v}_j\rangle_b+\nabla\cdot\left(\langle\rho_j\mathbf{v}_j\mathbf{v}_j\rangle_b\right)=\nabla\cdot\left(\langle\mathbf{D}_j\cdot\nabla\left(\rho_j\mathbf{v}_j\right)\rangle_b\right)\\
&\quad+\sum_l\frac{-1}{V_b}\int_{A_{jl}}\rho_j\mathbf{v}_j\left(\mathbf{v}_j-\mathbf{v}_{Ajl}\right)\cdot\mathbf{n}_j\,dA\\
&\quad+\sum_l\frac{1}{V_b}\int_{A_{jl}}\mathbf{D}_j\cdot\nabla\left(\rho_j\mathbf{v}_j\right)\cdot\mathbf{n}_j\,dA+\langle\mathbf{q}_j\rangle_b. \qquad (35)
\end{aligned}$$

Note that the gravity effect is included in the source terms as described later.

The stress tensor is given by the following expression for compressible Newtonian fluids [18]:

$$\mathbf{T}_j=\mathbf{D}_j\cdot\nabla\left(\rho_j\mathbf{v}_j\right)=\left(-p_j+\lambda_j\nabla\cdot\mathbf{v}_j\right)\mathbf{I}+\mu_j\left[\nabla\mathbf{v}_j+\left(\nabla\mathbf{v}_j\right)^T\right], \qquad (36)$$

where $\mathbf{I}$ is the unit tensor, $\lambda_j=\chi_j-\frac{2}{3}\mu_j$, and χ_j and μ_j denote the bulk and shear coefficients of viscosity of the j-phase, respectively. Thus, by means of Eq. (36), Eq. (35) can be written as [7]

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\langle\rho_j\mathbf{v}_j\rangle_b\right)+\nabla\cdot\left(\langle\rho_j\mathbf{v}_j\mathbf{v}_j\rangle_b\right)=\nabla\cdot\left(\langle\mathbf{T}_j\rangle_b\right)\\
&\quad+\sum_l\frac{-1}{V_b}\int_{A_{jl}}\rho_j\mathbf{v}_j\left(\mathbf{v}_j-\mathbf{v}_{Ajl}\right)\cdot\mathbf{n}_j\,dA+\sum_l\frac{1}{V_b}\int_{A_{jl}}\mathbf{T}_j\cdot\mathbf{n}_j\,dA+\langle\mathbf{q}_j\rangle_b. \qquad (37)
\end{aligned}$$

Note the following expressions can be obtained from Eqs. (11) and (12) by substituting $f_j = \rho_j$ and $f_j = \mathbf{v}_j$, respectively:

$$\left\langle \rho_j \mathbf{v}_j \right\rangle_b = \frac{1}{\varepsilon_j} \left\langle \rho_j \right\rangle_b \left\langle \mathbf{v}_j \right\rangle_b + \left\langle \widehat{\rho}_j \widehat{\mathbf{v}}_j \right\rangle_b \tag{38}$$

and

$$\begin{aligned}\left\langle \rho_j \mathbf{v}_j \mathbf{v}_j \right\rangle_b = {} & \frac{1}{\varepsilon_j^2} \left\langle \rho_j \right\rangle_b \left\langle \mathbf{v}_j \right\rangle_b \left\langle \mathbf{v}_j \right\rangle_b + \frac{1}{\varepsilon_j} \left\langle \rho_j \right\rangle_b \left\langle \widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j \right\rangle_b + \frac{1}{\varepsilon_j} \left\langle \mathbf{v}_j \right\rangle_b \left\langle \widehat{\rho}_j \widehat{\mathbf{v}}_j \right\rangle_b \\ & + \frac{1}{\varepsilon_j} \left\langle \mathbf{v}_j \right\rangle_b \left\langle \widehat{\rho}_j \widehat{\mathbf{v}}_j \right\rangle_b + \left\langle \widehat{\rho}_j \widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j \right\rangle_b .\end{aligned} \tag{39}$$

Thus, substituting Eq. (39) into Eq. (37) yields

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{1}{\varepsilon_j} \left\langle \rho_j \right\rangle_b \left\langle \mathbf{v}_j \right\rangle_b \right) + \nabla \cdot \left(\frac{1}{\varepsilon_j^2} \left\langle \rho_j \right\rangle_b \left\langle \mathbf{v}_j \right\rangle_b \left\langle \mathbf{v}_j \right\rangle_b \right) + \frac{\partial}{\partial t} \left(\left\langle \widehat{\rho}_j \widehat{\mathbf{v}}_j \right\rangle_b \right) \\ & + \nabla \cdot \left(\frac{1}{\varepsilon_j} \left\langle \rho_j \right\rangle_b \left\langle \widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j \right\rangle_b + \frac{1}{\varepsilon_j} \left\langle \mathbf{v}_j \right\rangle_b \left\langle \widehat{\rho}_j \widehat{\mathbf{v}}_j \right\rangle_b + \frac{1}{\varepsilon_j} \left\langle \mathbf{v}_j \right\rangle_b \left\langle \widehat{\rho}_j \widehat{\mathbf{v}}_j \right\rangle_b + \left\langle \widehat{\rho}_j \widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j \right\rangle_b \right) \\ & = \nabla \cdot \left(\left\langle \mathbf{T}_j \right\rangle_b \right) + \sum_l \frac{-1}{V_b} \int_{A_{jl}} \rho_j \mathbf{v}_j \left(\mathbf{v}_j - \mathbf{v}_{Ajl} \right) \cdot \mathbf{n}_j \, dA \\ & + \sum_l \frac{1}{V_b} \int_{A_{jl}} \mathbf{T}_j \cdot \mathbf{n}_j \, dA + \left\langle \mathbf{q}_j \right\rangle_b . \end{aligned} \tag{40}$$

Note that $\frac{\partial}{\partial t} \left\langle \widehat{\rho}_j \widehat{\mathbf{v}}_j \right\rangle_b = 0$ assuming that the deviations are spatial.

The following simplified expression for a single fluid phase flowing through porous media is obtained when the conditions of $\rho_j \equiv \left\langle \rho_j \right\rangle_j = const$, $\widehat{\rho}_j = 0$, $\varepsilon_j = const$, $\mu_j \equiv \left\langle \mu_j \right\rangle_j = const$, $\mathbf{v}_{Ajl} = \mathbf{0}$, and $\left\langle \mathbf{q}_j \right\rangle_b = \mathbf{0}$ according to [19] are substituted into Eq. (40) and considering the fact that deviation quantities are zero for these constant values

$$\begin{aligned} & \rho_j \left[\frac{\partial}{\partial t} \left\langle \mathbf{v}_j \right\rangle_b + \frac{1}{\varepsilon_j} \nabla \cdot \left(\left\langle \mathbf{v}_j \right\rangle_b \left\langle \mathbf{v}_j \right\rangle_b \right) + \nabla \cdot \left(\left\langle \widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j \right\rangle_b \right) + \frac{1}{V_b} \int_{A_{jl}} \mathbf{v}_j \mathbf{v}_j \cdot \mathbf{n}_j \, dA \right] \\ & = \nabla \cdot \left\langle \mathbf{T}_j \right\rangle_b + \frac{1}{V_b} \int_{A_{jl}} \mathbf{T}_j \cdot \mathbf{n}_j \, dA + \left\langle \mathbf{q}_j \right\rangle_b , \end{aligned} \tag{41}$$

where l denotes the porous solid matrix which is in contact with the j-phase.

Under the same above-mentioned simplifying conditions Eq. (36) simplifies as

$$\mathbf{T}_j = -p_j \mathbf{I} + \mu_j \left[\nabla \mathbf{v}_j + \left(\nabla \mathbf{v}_j \right)^T \right] . \tag{42}$$

Note according to Eq. (14)

$$\left\langle \nabla \mathbf{v}_j \right\rangle_b = \nabla \left\langle \mathbf{v}_j \right\rangle_b + \frac{1}{V_b} \int_{A_{jl}} \mathbf{v}_j \cdot \mathbf{n}_j \, dA. \tag{43}$$

Hence, the volume average of Eq. (42) and then application of Eq. (43) yield

$$\begin{aligned}
\left\langle \mathbf{T}_j \right\rangle_b &= \left\langle -p_j \mathbf{I} \right\rangle_b + \left\langle \mu_j \left[\nabla \mathbf{v}_j + (\nabla \mathbf{v}_j)^T \right] \right\rangle_b \\
&= \left\langle -p_j \right\rangle_b \mathbf{I} + \mu_j \left(\nabla \left\langle \mathbf{v}_j \right\rangle_b + \frac{1}{V_b} \int_{A_{jl}} \mathbf{v}_j \cdot \mathbf{n}_j \, dA \right. \\
&\quad \left. + \left(\nabla \left\langle \mathbf{v}_j \right\rangle_b + \frac{1}{V_b} \int_{A_{jl}} \mathbf{v}_j \cdot \mathbf{n}_j \, dA \right)^T \right).
\end{aligned} \tag{44}$$

Applying the no-slip boundary condition over the pore surface implies that

$$\int_{A_{jl}} \mathbf{v}_j \cdot \mathbf{n}_j \, dA = 0. \tag{45}$$

Thus, Eq. (44) simplifies by consideration of Eq. (45) as

$$\left\langle \mathbf{T}_j \right\rangle_b = \left\langle -p_j \right\rangle_b \mathbf{I} + \mu_j \left(\nabla \left\langle \mathbf{v}_j \right\rangle_b + \left(\nabla \left\langle \mathbf{v}_j \right\rangle_b \right)^T \right). \tag{46}$$

Then,

$$\begin{aligned}
\nabla \cdot \left\langle \mathbf{T}_j \right\rangle_b &= -\nabla \cdot \left(\left\langle p_j \right\rangle_b \mathbf{I} \right) + \mu_j \nabla \cdot \left(\nabla \left\langle \mathbf{v}_j \right\rangle_b + \left(\nabla \left\langle \mathbf{v}_j \right\rangle_b \right)^T \right) \\
&= -\nabla \cdot \left(\left\langle p_j \right\rangle_b \mathbf{I} \right) + \mu_j \left(\nabla \cdot \nabla \left\langle \mathbf{v}_j \right\rangle_b + \nabla \cdot \left(\nabla \left\langle \mathbf{v}_j \right\rangle_b \right)^T \right).
\end{aligned} \tag{47}$$

Exchanging the operators and then substituting $\nabla \cdot \left\langle \mathbf{v}_j \right\rangle_b = 0$ for incompressible fluids yields

$$\begin{aligned}
\nabla \cdot \left\langle \mathbf{T}_j \right\rangle_b &= -\nabla \cdot \left(\left\langle p_j \right\rangle_b \mathbf{I} \right) + \mu_j \left(\nabla \nabla \cdot \left\langle \mathbf{v}_j \right\rangle_b + \nabla \cdot \left(\nabla \left\langle \mathbf{v}_j \right\rangle_b \right)^T \right) \\
&= -\nabla \cdot \left(\left\langle p_j \right\rangle_b \mathbf{I} \right) + \mu_j \nabla \cdot \left(\nabla \left\langle \mathbf{v}_j \right\rangle_b \right)^T.
\end{aligned} \tag{48}$$

Applying Eqs. (8), (10), and (11) yields

$$\mathbf{v}_j = \left\langle \mathbf{v}_j \right\rangle_j + \widehat{\mathbf{v}}_j, \tag{49}$$

$$\mathbf{v}_j \mathbf{v}_j = \left\langle \mathbf{v}_j \right\rangle_j \left\langle \mathbf{v}_j \right\rangle_j + 2 \widehat{\mathbf{v}}_j \left\langle \mathbf{v}_j \right\rangle_j + \widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j, \tag{50}$$

and

$$\left\langle \mathbf{v}_j \mathbf{v}_j \right\rangle_b = \frac{1}{\varepsilon_j} \left\langle \mathbf{v}_j \right\rangle_b \left\langle \mathbf{v}_j \right\rangle_b + \left\langle \widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j \right\rangle_b . \tag{51}$$

Therefore, the following formulation can be carried out inferred by [19]

$$\begin{aligned} \frac{1}{V_b} \int_{A_{jl}} \mathbf{v}_j \mathbf{v}_j \cdot \mathbf{n}_j dA &= \frac{1}{V_b} \int_{A_{jl}} \left(\left\langle \mathbf{v}_j \right\rangle_j \left\langle \mathbf{v}_j \right\rangle_j + 2\widehat{\mathbf{v}}_j \left\langle \mathbf{v}_j \right\rangle_j + \widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j \right) \cdot \mathbf{n}_j \, dA \\ &= \frac{1}{V_b} \int_{A_{jl}} \widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j \cdot \mathbf{n}_j \, dA + \frac{1}{V_b} \left\langle \mathbf{v}_j \right\rangle_j \left\langle \mathbf{v}_j \right\rangle_j \cdot \int_{A_{jl}} \mathbf{n}_j \, dA \\ &\quad + \frac{2}{V_b} \left\langle \mathbf{v}_j \right\rangle_j \int_{A_{jl}} \left(\mathbf{v}_j - \left\langle \mathbf{v}_j \right\rangle_j \right) \cdot \mathbf{n}_j \, dA \\ &= -\frac{1}{V_b} \left\langle \mathbf{v}_j \right\rangle_j \left\langle \mathbf{v}_j \right\rangle_j \cdot \int_{A_{jl}} \mathbf{n}_j \, dA. \end{aligned} \tag{52}$$

Applying Eqs. (48) and (52) into Eq. (41) yields the following simplified momentum equation:

$$\begin{aligned} \rho_j \frac{\partial}{\partial t} \left\langle \mathbf{v}_j \right\rangle_b + \frac{\rho_j}{\varepsilon_j} \nabla \cdot \left(\left\langle \mathbf{v}_j \right\rangle_b \left\langle \mathbf{v}_j \right\rangle_b \right) = &- \nabla \left\langle p_j \right\rangle_b + \mu_j \nabla \cdot \left(\nabla \left\langle \mathbf{v}_j \right\rangle_b \right)^T \\ &- \rho_j \left\langle \nabla \cdot \left(\widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j \right) \right\rangle_b + \frac{1}{V_b} \int_{A_{jl}} \mathbf{T}_j \cdot \mathbf{n}_j \, dA \\ &+ \frac{\rho_j}{V_b} \left\langle \mathbf{v}_j \right\rangle_j \left\langle \mathbf{v}_j \right\rangle_j \cdot \int_{A_{jl}} \mathbf{n}_j \, dA + \left\langle \mathbf{q}_j \right\rangle_b . \end{aligned} \tag{53}$$

Note in the above a substitution of $\left\langle p_j \right\rangle_b = \left\langle p_j \right\rangle_j$ is made assuming rigid porous media matrix material and $\left\langle p_j \right\rangle_b = \varepsilon_j \left\langle p_j \right\rangle_j$ for fully elastic porous media matrix material in accordance with Eq. (3).

Eq. (53) can be expressed in terms of the intrinsic fluid properties as

$$\begin{aligned} &\varepsilon_j \rho_j \left[\frac{\partial}{\partial t} \left\langle \mathbf{v}_j \right\rangle_j + \nabla \cdot \left(\left\langle \mathbf{v}_j \right\rangle_j \left\langle \mathbf{v}_j \right\rangle_j \right) - \left\langle \nabla \cdot \left(\widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j \right) \right\rangle_j \right] \\ &\quad = \varepsilon_j \mu_j \nabla \cdot \left(\nabla \left\langle \mathbf{v}_j \right\rangle_j \right)^T + \frac{1}{V_b} \int_{A_{jl}} \mathbf{T}_j \cdot \mathbf{n}_j \, dA + \frac{\rho_j}{V_b} \left\langle \mathbf{v}_j \right\rangle_j \left\langle \mathbf{v}_j \right\rangle_j \cdot \int_{A_{jl}} \mathbf{n}_j \, dA \\ &\quad - \nabla \left\langle p_j \right\rangle_b + \left\langle \mathbf{q}_j \right\rangle_b . \end{aligned} \tag{54}$$

Note that Eq. (54) is different from the corresponding equation derived by [19]. Their equation in the nomenclature used in this chapter reads as

$$\rho_j \frac{\partial}{\partial t}\langle \mathbf{v}_j\rangle_b + \frac{\rho_j}{\varepsilon_j^2}\nabla\cdot\left(\langle \mathbf{v}_j\rangle_b \langle \mathbf{v}_j\rangle_b\right) = -\nabla\langle p_j\rangle_b + \mu_j \nabla\cdot\left(\nabla\langle \mathbf{v}_j\rangle_b\right)^T$$
$$-\rho_j\langle\nabla\cdot(\widehat{\mathbf{v}}_j\widehat{\mathbf{v}}_j)\rangle_b + \frac{1}{V_b}\int_{A_{jl}} \mathbf{T}_j\cdot\mathbf{n}_j\, dA$$
$$+\frac{\rho_j}{V_b}\langle \mathbf{v}_j\rangle_j \langle \mathbf{v}_j\rangle_j \cdot \int_{A_{jl}} \mathbf{n}_j\, dA. \tag{55}$$

The systematic error involved in the equation of [19] is explained in Appendix.

3.4 Energy Equation

Note the thermal diffusivity tensor is given by

$$\mathbf{D}_j = \frac{\boldsymbol{\kappa}_j}{\rho_j C_j}, \tag{56}$$

where $\boldsymbol{\kappa}$ denotes thermal conductivity tensor and C is specific heat capacity at constant pressure.

Applying Eq. (23) for the conservations of energy yields, where f_j is equal to the enthalpy given by $H_j = \rho_j C_j T_j$ and T_j denotes the temperature of the j-phase [3],

$$\frac{\partial}{\partial t}\left(\langle\rho_j C_j T_j\rangle_b\right) + \nabla\cdot\left(\langle\rho_j C_j T_j \mathbf{v}_j\rangle_b\right)$$
$$= \nabla\cdot\left(\langle \mathbf{D}_j\cdot\nabla(\rho_j C_j T_j)\rangle_b\right) + \sum_l \frac{-1}{V_b}\int_{A_{jl}} \rho_j C_j T_j\left(\mathbf{v}_j - \mathbf{v}_{Ajl}\right)\cdot\mathbf{n}_j\, dA$$
$$+\sum_l \frac{1}{V_b}\int_{A_{jl}} \mathbf{D}_j\cdot\nabla(\rho_j C_j T_j)\cdot\mathbf{n}_j\, dA + \langle q_j\rangle_b. \tag{57}$$

The averaging of the several terms appearing in Eq. (57) is explained in the following:

$$\langle \mathbf{D}_j\cdot\nabla(\rho_j C_j T_j)\rangle_b = \frac{1}{\varepsilon_j}\langle \mathbf{D}_j\rangle_b\cdot\langle\nabla(\rho_j C_j T_j)\rangle_b + \left\langle\widehat{\mathbf{D}}_j\cdot\widehat{\nabla H_j}\right\rangle_b$$
$$= \frac{1}{\varepsilon_j}\langle \mathbf{D}_j\rangle_b\cdot\left(\nabla\left(\frac{1}{\varepsilon_j}\langle\rho_j C_j\rangle_b\langle T_j\rangle_b\right) + \nabla\left(\left\langle\widehat{\rho_j C_j}\widehat{T}_j\right\rangle_b\right)\right.$$
$$\left.+\frac{1}{V_b}\int_{A_{jl}}(\rho_j C_j T_j)\mathbf{n}_j\, dA\right) + \left\langle\widehat{\mathbf{D}}_j\cdot\widehat{\nabla H_j}\right\rangle_b, \tag{58}$$

$$\langle(\rho_j C_j) T_j\rangle_b = \frac{1}{\varepsilon_j}\langle\rho_j C_j\rangle_b\langle T_j\rangle_b + \left\langle\widehat{\rho_j C_j}\widehat{T}_j\right\rangle_b, \tag{59}$$

and

$$\left\langle (\rho_j C_j)\,\mathbf{v}_j T_j \right\rangle_b = \frac{1}{\varepsilon_j^2}\left\langle \rho_j C_j \right\rangle_b \left\langle \mathbf{v}_j \right\rangle_b \left\langle T_j \right\rangle_b + \frac{1}{\varepsilon_j}\left\langle \rho_j C_j \right\rangle_b \left\langle \widehat{\mathbf{v}}_j \widehat{T}_j \right\rangle_b$$
$$+ \frac{1}{\varepsilon_j}\left\langle \mathbf{v}_j \right\rangle_b \left\langle \widehat{\rho_j C_j} \widehat{T}_j \right\rangle_b + \frac{1}{\varepsilon_j}\left\langle T_j \right\rangle_b \left\langle \widehat{\rho_j C_j} \widehat{\mathbf{v}}_j \right\rangle_b$$
$$+ \left\langle \widehat{\rho_j C_j} \widehat{\mathbf{v}}_j \widehat{T}_j \right\rangle_b . \tag{60}$$

Further, consider

$$\left\langle \rho_j C_j \right\rangle_b = \frac{1}{\varepsilon_j}\left\langle \rho_j \right\rangle_b \left\langle C_j \right\rangle_b + \left\langle \widehat{\rho}_j \widehat{C}_j \right\rangle_b . \tag{61}$$

Thus, substituting Eqs. (58)–(61) into Eq. (57) yields the volume averaged equation as

$$\frac{\partial}{\partial t}\left(\frac{1}{\varepsilon_j}\left(\frac{1}{\varepsilon_j}\left\langle \rho_j \right\rangle_b \left\langle C_j \right\rangle_b + \left\langle \widehat{\rho}_j \widehat{C}_j \right\rangle_b \right) \left\langle T_j \right\rangle_b + \left\langle \widehat{\rho_j C_j} \widehat{T}_j \right\rangle_b \right)$$
$$+\nabla \cdot \left(\left(\frac{1}{\varepsilon_j}\left\langle \rho_j \right\rangle_b \left\langle C_j \right\rangle_b + \left\langle \widehat{\rho}_j \widehat{C}_j \right\rangle_b \right) \left(\frac{1}{\varepsilon_j^2}\left\langle \mathbf{v}_j \right\rangle_b \left\langle T_j \right\rangle_b + \frac{1}{\varepsilon_j}\left\langle \widehat{\mathbf{v}}_j \widehat{T}_j \right\rangle_b \right) \right.$$
$$\left. + \frac{1}{\varepsilon_j}\left\langle \mathbf{v}_j \right\rangle_b \left\langle \widehat{\rho_j C_j} \widehat{T}_j \right\rangle_b + \frac{1}{\varepsilon_j}\left\langle T_j \right\rangle_b \left\langle \widehat{\rho_j C_j} \widehat{\mathbf{v}}_j \right\rangle_b + \left\langle \widehat{\rho_j C_j} \widehat{\mathbf{v}}_j \widehat{T}_j \right\rangle_b \right)$$
$$= \nabla \cdot \left[\frac{1}{\varepsilon_j}\left\langle \mathbf{D}_j \right\rangle_b \cdot \left(\nabla \left(\frac{1}{\varepsilon_j}\left(\frac{1}{\varepsilon_j}\left\langle \rho_j \right\rangle_b \left\langle C_j \right\rangle_b + \left\langle \widehat{\rho}_j \widehat{C}_j \right\rangle_b \right) \left\langle T_j \right\rangle_b \right) \right. \right.$$
$$\left. \left. + \nabla \left(\left\langle \widehat{\rho_j C_j} \widehat{T}_j \right\rangle_b \right) + \frac{1}{V_b}\int_{A_j} (\rho_j C_j T_j)\,\mathbf{n}_j\, dA \right) + \left\langle \widehat{\mathbf{D}}_j \cdot \widehat{\nabla H_j} \right\rangle_b \right]$$
$$+ \sum_l \frac{-1}{V_b}\int_{A_{jl}} \rho_j C_j T_j \left(\mathbf{v}_j - \mathbf{v}_{Ajl} \right) \cdot \mathbf{n}_j\, dA$$
$$+ \sum_l \frac{1}{V_b}\int_{A_{jl}} \mathbf{D}_j \cdot \nabla (\rho_j C_j T_j) \cdot \mathbf{n}_j\, dA + \left\langle q_j \right\rangle_b . \tag{62}$$

Consequently, invoking $C_j \equiv \left\langle C_j \right\rangle_j = const$, $\rho_j \equiv \left\langle \rho_j \right\rangle_j = const$, $\varepsilon_j = const$, $\boldsymbol{\kappa}_j = const$, and $\mathbf{v}_j = \mathbf{v}_{Ajl} = 0$ according to [19] and considering the deviation quantities to be zero for these constant values yield

$$\varepsilon_j \left\langle \rho_j \right\rangle_j \left\langle C_j \right\rangle_j \left[\frac{\partial}{\partial t} \left\langle T_j \right\rangle_j + \frac{1}{\varepsilon_j} \nabla \cdot \left(\left\langle \mathbf{v}_j \right\rangle_b \left\langle T_j \right\rangle_j \right) + \nabla \cdot \left(\left\langle \widehat{\mathbf{v}}_j \widehat{T}_j \right\rangle_j \right) \right]$$
$$= \varepsilon_j \boldsymbol{\kappa}_j \cdot \nabla \cdot \nabla \left\langle T_j \right\rangle_j + \frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \nabla \cdot \left(\int_{A_{jl}} T_j \mathbf{n}_j \, dA \right)$$
$$+ \frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \int_{A_{jl}} \nabla T_j \cdot \mathbf{n}_j \, dA + \left\langle q_j \right\rangle_b . \tag{63}$$

But applying Eq. (8)

$$T_j = \left\langle T_j \right\rangle_j + \widehat{T}_j. \tag{64}$$

Therefore, the following expression can be written inferred by [19]

$$\frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \nabla \cdot \left(\int_{A_{jl}} T_j \mathbf{n}_j \, dA \right)$$
$$= \frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \nabla \cdot \left(\int_{A_{jl}} \left\langle T_j \right\rangle_j \mathbf{n}_j \, dA \right) + \frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \nabla \cdot \left(\int_{A_{jl}} \widehat{T}_j \mathbf{n}_j \, dA \right). \tag{65}$$

Then, consider the divergence theorem, given by

$$\int_{A_{jl}} \left\langle T_j \right\rangle_j \mathbf{n}_j \, dA = \int_{V_b} \nabla \left\langle T_j \right\rangle_j \, dV = 0. \tag{66}$$

Thus, substituting Eq. (66) into Eq. (65) yields [19]

$$\frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \nabla \cdot \left(\int_{A_{jl}} T_j \mathbf{n}_j \, dA \right) = \frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \nabla \cdot \left(\int_{A_{jl}} \widehat{T}_j \mathbf{n}_j \, dA \right). \tag{67}$$

Consequently, the energy conservation expressed only in terms of the intrinsic fluid properties is obtained as

$$\varepsilon_j \left\langle \rho_j \right\rangle_j \left\langle C_j \right\rangle_j \left[\frac{\partial}{\partial t} \left\langle T_j \right\rangle_j + \nabla \cdot \left(\left\langle \mathbf{v}_j \right\rangle_j \left\langle T_j \right\rangle_j \right) + \nabla \cdot \left\langle \widehat{\mathbf{v}}_j \widehat{T}_j \right\rangle_j \right]$$
$$= \varepsilon_j \boldsymbol{\kappa}_j \cdot \nabla \cdot \nabla \left\langle T_j \right\rangle_j + \frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \nabla \cdot \left(\int_{A_{jl}} \widehat{T}_j \mathbf{n}_j \, dA \right)$$
$$+ \frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \int_{A_{jl}} \nabla T_j \cdot \mathbf{n}_j \, dA + \left\langle q_j \right\rangle_j . \tag{68}$$

Note that this result is different from the corresponding equation derived by [19]. Their equation in the nomenclature used in this chapter reads as

$$\varepsilon_j \rho_j C_j \left[\frac{\partial}{\partial t} \left\langle T_j \right\rangle_j + \frac{1}{\varepsilon_j} \nabla \cdot \left(\left\langle \mathbf{v}_j \right\rangle_j \left\langle \mathbf{T}_j \right\rangle_j \right) + \nabla \cdot \left\langle \widehat{\mathbf{v}}_j \widehat{T}_j \right\rangle_j \right]$$
$$= \varepsilon_j \boldsymbol{\kappa}_j \cdot \nabla \cdot \nabla \left\langle T_j \right\rangle_j + \frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \nabla \cdot \left(\int_{A_{jl}} \widehat{T}_j \mathbf{n}_j \, dA \right) + \frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \int_{A_{jl}} \nabla T_j \cdot \mathbf{n}_j \, dA. \tag{69}$$

The systematic error involved in the equation of [19] is explained in Appendix.

4 Control Volume Analyses

The control volume of the j-phase is considered to be the portion of the void space occupied by this phase in porous media. Therefore, when an element of bulk porous media $(\Delta x \Delta y \Delta z)$ with the dimensions of Δx, Δy, and Δz is considered in the x-, y-, and z- Cartesian coordinates, the j-phase contained in this element will have the *external control volume boundaries* over the external surface of this element and the *internal control volume boundaries* through which the fluid interacts at the interface with the pore surface and other phases. Thus, a general balance equation can be derived by considering the net flow through the open boundaries and the interactions at the internal pore volume and interface with other fluid phases. Thus, the following general macroscopic equation of conservation is obtained for the j-phase in the *extrapolated limit* Δx, Δy, and $\Delta z \to 0$ of [5, 6, 13]:

$$\nabla \cdot \mathbf{j}_{Tjb} + \frac{\partial f_{jb}}{\partial t} = \sum_{CV} \dot{\mathbf{r}}_{jb}, \tag{70}$$

where $\dot{\mathbf{r}}_{jb}$ denotes the source of a property f_{jb} supplied to the j-phase per unit bulk volume of porous media and the total flux $\mathbf{j}_{Tjb}$ of the same property is expressed by the sum of transport by convection and dispersion as

$$\mathbf{j}_{Tjb} = \mathbf{v}_j f_{jb} + \mathbf{j}_{jb}, \;\; \mathbf{j}_{jb} = -\frac{1}{\varepsilon_j} \mathbf{D}_{jb} \cdot \nabla f_{jb}, \tag{71}$$

where $\mathbf{D}_{jb}$ is a bulk-dispersion tensor.

Combining Eqs. (71) and (70) the following macroscopic transport equation is derived:

$$\nabla \cdot \left(\mathbf{v}_j f_{jb} \right) + \frac{\partial f_{jb}}{\partial t} = \nabla \cdot \left(\frac{1}{\varepsilon_j} \mathbf{D}_{jb} \cdot \nabla f_{jb} \right) + \sum_{CV} \dot{\mathbf{r}}_{jb}. \tag{72}$$

In the following, the applications of Eq. (72) are demonstrated for derivation of the mass, momentum, and energy conservation equations.

4.1 Mass Equation

The porous media mass balance equation can be derived by substituting $f_{jb} = \rho_{jb}$ in Eq. (72) to obtain

$$\nabla \cdot \left(\mathbf{v}_j \rho_{jb}\right) + \frac{\partial \rho_{jb}}{\partial t} = \nabla \cdot \left(\frac{1}{\varepsilon_j}\mathbf{D}_{jb} \cdot \nabla \rho_{jb}\right) + \sum_{CV} \dot{\mathbf{r}}_{jb}. \tag{73}$$

This equation can be manipulated as

$$\nabla \cdot \left(\frac{\mathbf{v}_{jb}\rho_{jb}}{\varepsilon_j}\right) + \frac{\partial \rho_{jb}}{\partial t} = \nabla \cdot \left(\frac{1}{\varepsilon_j}\mathbf{D}_{jb} \cdot \nabla \rho_{jb}\right) + \sum_{CV} \dot{\mathbf{r}}_{jb}. \tag{74}$$

Then, expressing in terms of the intrinsic fluid properties only yields

$$\frac{\partial}{\partial t}\left(\varepsilon_j \left\langle \rho_j \right\rangle_j\right) + \nabla \cdot \left(\varepsilon_j \left\langle \rho_j \right\rangle_j \left\langle \mathbf{v}_j \right\rangle_j\right) = \nabla \cdot \left[\frac{1}{\varepsilon_j}\mathbf{D}_{jb} \cdot \nabla\left(\varepsilon_j \left\langle \rho_j \right\rangle_j\right)\right] + \varepsilon_j \left\langle q_j \right\rangle_j. \tag{75}$$

4.2 Momentum Equation

Consider that $\mathbf{v}$ is the volume flux, μ is viscosity, p is pressure, and ρ is density of fluid. The subscripts jb and j refer to the bulk- and fluid-volume averages of the properties of the j-phase. $\mathbf{K}$ and ϕ denote the permeability tensor and porosity of porous media and $\mathbf{g}$ is the gravitational acceleration vector. ∇p_{jb} is the fluid pressure gradient, ∇p_{jbth} is the threshold or minimum fluid pressure gradient required to overcome the resistance of porous media to fluid flow [16], ϕ, D_{hx}, and τ_{hx} denote the porosity, mean-hydraulic diameter, and tortuosity of porous media, respectively, f_{kj} is a shear factor, $\mathbf{K}_{bk}$ and $\boldsymbol{\beta}_{bk}$ are the tensor effective fluid permeability (a product of relative and absolute permeability) and inertial flow coefficient, respectively, $\mathbf{T}$ is the shear stress tensor, and $\boldsymbol{\Phi}$ is the flow potential, defined later.

The porous media momentum equation can be derived by substituting $f_{jb} = \varepsilon_j f_j = \varepsilon_j \rho_j \mathbf{v}_j = \frac{\rho_{jb}\mathbf{v}_{jb}}{\varepsilon_j}$, $\mathbf{v}_j = \frac{\mathbf{v}_{jb}}{\varepsilon_j}$ in Eq. (72) to obtain

$$\nabla \cdot \left(\frac{\mathbf{v}_{jb}}{\varepsilon_j}\frac{\rho_{jb}\mathbf{v}_{jb}}{\varepsilon_j}\right) + \frac{\partial}{\partial t}\left(\frac{\rho_{jb}\mathbf{v}_{jb}}{\varepsilon_j}\right) = \nabla \cdot \left(\frac{1}{\varepsilon_j}\mathbf{D}_{jb} \cdot \nabla\left(\frac{\rho_{jb}\mathbf{v}_{jb}}{\varepsilon_j}\right)\right) + \sum_{CV} \dot{\mathbf{r}}_{jb}. \tag{76}$$

Alternatively,

$$\nabla \cdot \left(\rho_j \frac{\mathbf{v}_{jb}\mathbf{v}_{jb}}{\varepsilon_j}\right) + \frac{\partial}{\partial t}\left(\rho_j \mathbf{v}_{jb}\right) = \nabla \cdot \left(\frac{1}{\varepsilon_j}\mathbf{D}_{jb} \cdot \nabla\left(\rho_j \mathbf{v}_{jb}\right)\right) + \varepsilon_j \sum_{CV} \dot{\mathbf{r}}_j. \tag{77}$$

The source term is expressed as a sum of the external and internal sources as [5]

$$\sum_{CV}\dot{\mathbf{r}}_{jb} = \sum_{CV-External}\dot{\mathbf{r}}_{jb} + \sum_{CV-Internal}\dot{\mathbf{r}}_{jb}$$
$$= [\mathbf{F}_N - \mathbf{F}_{TH} + \mathbf{F}_T + \mathbf{F}_B]_{External} + [-\mathbf{F}_S - \mathbf{F}_O - \mathbf{F}_{IF}]_{Internal}. \quad (78)$$

The various internal and external forces acting on the fluid can be expressed as the following based on the capillary-orifice model [2,5].

The forces associated with the normal and tangential stresses are given by $\mathbf{F}_N = -\nabla p_{jb}$ and $\mathbf{F}_T = -\nabla\cdot\mathbf{T}_{jb}$, respectively. $\mathbf{F}_{TH} = -\nabla p_{jbth}$ denotes the resistive force associated with the threshold pressure gradient that must be overcome to initiate flow through porous media because of the internal resistance to motion of the j-phase through porous media [16].

The gravitational body force is given by

$$\mathbf{F}_B = \varepsilon_j\rho_j\mathbf{g}. \quad (79)$$

The pore surface friction force is given by

$$\mathbf{F}_S = \varepsilon_j^2\mu_j\mathbf{K}_{jb}^{-1}\cdot\mathbf{v}_j = \varepsilon_j\mu_j\mathbf{K}_{jb}^{-1}\cdot\mathbf{v}_{jb}. \quad (80)$$

The components of the permeability tensor are similarly expressed using the capillary-orifice model. For example, K_x is the permeability of porous medium in the x-principal direction, expressed by

$$K_x = \frac{\phi D_{hx}^2}{32\tau_{hx}}. \quad (81)$$

The pore throat orifice effect drag force is given by

$$\mathbf{F}_O = \varepsilon_j^3\langle\rho_j\rangle_j\boldsymbol{\beta}_{jb}\cdot\left|\langle\mathbf{v}_j\rangle_j\right|\langle\mathbf{v}_j\rangle_j = \varepsilon_j\rho_j\boldsymbol{\beta}_{jb}\cdot\left|\mathbf{v}_{jb}\right|\mathbf{v}_{jb}. \quad (82)$$

The components of the inertial flow coefficient tensor are similarly expressed using the capillary-orifice model. For example, β_x is an inertial flow coefficient in the x-principal direction, given by

$$\beta_x = \frac{c_{Dx}\tau_{hx}^2}{2\phi^2 D_{hx}}, \quad (83)$$

where c_{Dx} is the drag coefficient in the x-direction.

Eliminating D_{hx} between Eqs. (81) and (83) yields

$$\beta_x = c_f\frac{1}{K_x^{1/2}}, \quad \text{or } \boldsymbol{\beta} = \mathbf{c}_f\cdot\mathbf{K}^{-1/2} \quad (84)$$

and

$$c_f = \frac{c_{Dx}}{8\sqrt{2}}\left(\frac{\tau_{hx}}{\phi}\right)^{3/2}, \tag{85}$$

where c_f is the pressure coefficient. Therefore, the pore throat drag coefficient is expressed as

$$\mathbf{F}_O = \varepsilon_j \rho_j \boldsymbol{\beta}_{jb} \cdot \left|\mathbf{v}_{jb}\right| \mathbf{v}_{jb} = \varepsilon_j \rho_j \mathbf{c}_f \cdot \mathbf{K}_{jb}^{-1/2} \cdot \left|\mathbf{v}_{jb}\right| \mathbf{v}_{jb}. \tag{86}$$

The interfacial drag force is given by, modifying the equation of [17],

$$\mathbf{F}_{IF} = \varepsilon_j \sum_{j=1, j\neq k}^{N} f_{jk}\left(\mathbf{T}_{hj} \cdot \left\langle \mathbf{v}_j \right\rangle_j - \mathbf{T}_{hk} \cdot \left\langle \mathbf{v}_k \right\rangle_k\right), \tag{87}$$

where $\mathbf{T}_{hk}$ is the tortuosity tensor.

Thus, using Eqs. (78)–(87), the source term is expressed as

$$\begin{aligned}
\sum_{CV} \dot{\mathbf{r}}_{jb} = & \left[-\nabla p_{jb} + \nabla p_{jbth} - \nabla \cdot \mathbf{T}_{jb} + \varepsilon_j \rho_j \mathbf{g}\right]_{External} \\
& + \left[-\varepsilon_j^2 \mu_j \mathbf{K}_{jb}^{-1} \cdot \mathbf{v}_j - \varepsilon_j^3 \rho_j \boldsymbol{\beta}_{jb} \cdot \left|\mathbf{v}_j\right| \mathbf{v}_j \right.\\
& \left. + \varepsilon_j \sum_{j=1, j\neq k}^{N} f_{jk}\left(\mathbf{T}_{hj} \cdot \mathbf{v}_j - \mathbf{T}_{hk} \cdot \mathbf{v}_k\right)\right]_{Internal} \\
= & -\rho_j \nabla \Phi_{jb} - \varepsilon_j \mu_j \mathbf{K}_{jb}^{-1} \cdot \mathbf{v}_{jb} - \varepsilon_j \rho_j \boldsymbol{\beta}_{jb} \cdot \left|\mathbf{v}_{jb}\right| \mathbf{v}_{jb} \\
& + \sum_{j=1, j\neq k}^{N} f_{jk}\left(\mathbf{T}_{hj} \cdot \mathbf{v}_{jb} - \mathbf{T}_{hk} \cdot \mathbf{v}_{kb}\right),
\end{aligned} \tag{88}$$

where a flow potential is defined as [7]

$$\Psi_{jb} = \varepsilon_j \Psi_j = \int_{P_o}^{p} \frac{d\left(p_{jb} - p_{jbth}\right)}{\rho_j} + g\left(\varepsilon_j z - \varepsilon_{jo} z_o\right). \tag{89}$$

Thus, the following are written:

$$-\rho_j \nabla \Psi_{jb} = -\nabla p_{jb} + \nabla p_{jbth} + \varepsilon_j \rho_j \mathbf{g} \tag{90}$$

and

$$-\rho_j \nabla \Phi_{jb} = -\rho_j \nabla \Psi_{jb} - \nabla \cdot \mathbf{T}_{jb}. \tag{91}$$

The source term can be expressed in the volume-averaging nomenclature as

$$\sum_{CV}\dot{\mathbf{r}}_{jb} = -\nabla p_{jb} + \nabla p_{jbth} - \nabla\cdot\mathbf{T}_{jb} + \varepsilon_j\rho_j\mathbf{g}$$
$$-\varepsilon_j^2\left\langle\mu_j\right\rangle_j\mathbf{K}_{jb}^{-1}\cdot\left\langle\mathbf{v}_j\right\rangle_j - \varepsilon_j^3\left\langle\rho_j\right\rangle_j\boldsymbol{\beta}_{jb}\cdot\left|\left\langle\mathbf{v}_j\right\rangle_j\right|\left\langle\mathbf{v}_j\right\rangle_j$$
$$+\varepsilon_j\sum_{j=1, j\neq k}^{N} f_{jk}\left(\mathbf{T}_{hj}\cdot\left\langle\mathbf{v}_j\right\rangle_j - \mathbf{T}_{hk}\cdot\left\langle\mathbf{v}_k\right\rangle_k\right). \quad (92)$$

Substituting Eq. (88) into Eq. (77) yields

$$\nabla\cdot\left(\rho_j\frac{\mathbf{v}_{jb}\mathbf{v}_{jb}}{\varepsilon_j}\right) + \frac{\partial}{\partial t}\left(\rho_j\mathbf{v}_{jb}\right) = \nabla\cdot\left(\frac{1}{\varepsilon_j}\mathbf{D}_{jb}\cdot\nabla\left(\rho_j\mathbf{v}_{jb}\right)\right) - \nabla p_{jb} + \nabla p_{jbth}$$
$$+\varepsilon_j\rho_j\mathbf{g} - \nabla\cdot\mathbf{T}_{jb} - \varepsilon_j\mu_j\mathbf{K}_{jb}^{-1}\cdot\mathbf{v}_{jb}$$
$$-\varepsilon_j\rho_j\boldsymbol{\beta}_{jb}\cdot\left|\mathbf{v}_{jb}\right|\mathbf{v}_{jb}$$
$$+\sum_{j=1, j\neq k}^{N} f_{jk}\left(\mathbf{T}_{hj}\cdot\mathbf{v}_{jb} - \mathbf{T}_{hk}\cdot\mathbf{v}_{kb}\right). \quad (93)$$

4.3 Energy Equation

The porous media energy equation can be derived by substituting $f_{jb} = \varepsilon_j f_j = \varepsilon_j\rho_j C_j T_j = \frac{\rho_{jb}C_{jb}T_{jb}}{\varepsilon_j^2}$, $\mathbf{v}_j = \frac{\mathbf{v}_{jb}}{\varepsilon_j}$ into Eq. (72) to obtain

$$\nabla\cdot\left(\frac{\mathbf{v}_{jb}}{\varepsilon_j}\frac{\rho_{jb}C_{jb}T_{jb}}{\varepsilon_j^2}\right) + \frac{\partial}{\partial t}\left(\frac{\rho_{jb}C_{jb}T_{jb}}{\varepsilon_j^2}\right) = \nabla\cdot\left[\frac{1}{\varepsilon_j}\mathbf{D}_{jb}\cdot\nabla\left(\frac{\rho_{jb}C_{jb}T_{jb}}{\varepsilon_j^2}\right)\right]$$
$$+\sum_{CV}\dot{\mathbf{r}}_{jb}. \quad (94)$$

This equation can be expressed for constant fluid properties as

$$\varepsilon_j\rho_j C_j\left[\frac{\partial T_j}{\partial t} + \frac{1}{\varepsilon_j}\nabla\cdot\left(\mathbf{v}_{jb}T_j\right)\right] = \varepsilon_j\rho_j C_j\mathbf{D}_j\cdot\nabla\cdot\nabla T_j + \sum_{CV}\dot{\mathbf{r}}_{jb}$$
$$= \varepsilon_j\boldsymbol{\kappa}_j\cdot\nabla\cdot\nabla T_j + \sum_{CV}\dot{\mathbf{r}}_{jb}. \quad (95)$$

5 Comparison of Porous Media Averaging and Control Volume Analysis

Comparison of the equations derived above by means of the porous media averaging and control volume analysis approaches reveals several plausible closure methods for various terms as described in the following.

5.1 *Mass Equation*

Comparing Eqs. (28) and (74) the following relationships can be obtained for interpretation of the two terms appearing on the right of Eq. (74):

$$\frac{1}{\varepsilon_j}\mathbf{D}_{jb}\cdot\nabla\rho_{jb} = \frac{1}{\varepsilon_j}\left\langle \mathbf{D}_j\right\rangle_b\cdot\nabla\left\langle \rho_j\right\rangle_b - \left\langle \widehat{\mathbf{v}}_j\widehat{\rho}_j\right\rangle_b + \left\langle \widehat{\mathbf{D}}_j\cdot\widehat{\nabla\rho_j}\right\rangle_b \tag{96}$$

and

$$\begin{aligned}\sum_{CV}\dot{\mathbf{r}}_{jb} &= \nabla\cdot\left(\frac{1}{\varepsilon_j}\left\langle \mathbf{D}_j\right\rangle_b\cdot\sum_l\frac{1}{V_b}\int_{A_{jl}}\rho_j\mathbf{n}_j\,dA\right)\\ &\quad+\sum_l\frac{-1}{V_b}\int_{A_{jl}}\rho_j\left(\mathbf{v}_j-\mathbf{v}_{Ajl}\right)\cdot\mathbf{n}_j\,dA\\ &\quad+\sum_l\frac{1}{V_b}\int_{A_{jl}}\mathbf{D}_j\cdot\nabla\rho_j\cdot\mathbf{n}_j\,dA+\left\langle \mathbf{q}_j\right\rangle_b.\end{aligned} \tag{97}$$

5.2 *Momentum Equation*

Note the first term on the right of Eq. (76) is given by Eq. (36). Comparing Eqs. (76) and (40) the following can be obtained:

$$\begin{aligned}\nabla\cdot\left[\frac{1}{\varepsilon_j}\mathbf{D}_{jb}\cdot\nabla\cdot\left(\frac{\rho_{jb}\mathbf{v}_{jb}}{\varepsilon_j}\right)\right] &= \nabla\cdot\left\langle \mathbf{T}_j\right\rangle_b-\nabla\cdot\left(\frac{1}{\varepsilon_j}\left\langle \rho_j\right\rangle_b\left\langle \widehat{\mathbf{v}}_j\widehat{\mathbf{v}}_j\right\rangle_b\right.\\ &\quad\left.+\frac{2}{\varepsilon_j}\left\langle \mathbf{v}_j\right\rangle_b\left\langle \widehat{\rho}_j\widehat{\mathbf{v}}_j\right\rangle_b+\left\langle \widehat{\rho}_j\widehat{\mathbf{v}}_j\widehat{\mathbf{v}}_j\right\rangle_b\right)\end{aligned} \tag{98}$$

and

$$\sum_{CV}\dot{\mathbf{r}}_{jb} = \sum_l\frac{-1}{V_b}\int_{A_{jl}}\rho_j\mathbf{v}_j\left(\mathbf{v}_j-\mathbf{v}_{Ajl}\right)\cdot\mathbf{n}_j\,dA+\sum_l\frac{1}{V_b}\int_{A_{jl}}\mathbf{T}_j\cdot\mathbf{n}_j\,dA+\left\langle \mathbf{q}_j\right\rangle_b. \tag{99}$$

Comparing Eqs. (93) and (53) for single-phase flow with constant ε_j and ρ_j, the following expressions can be determined as a method of closure based on the control volume analysis approach:

$$-\rho_j \left\langle \nabla \cdot (\widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j) \right\rangle_b + \frac{\rho_j}{V_b} \left\langle \mathbf{v}_j \right\rangle_j \left\langle \mathbf{v}_j \right\rangle_j \cdot \int_{A_{jl}} \mathbf{n}_j \, dA = -\varepsilon_j \rho_j \boldsymbol{\beta}_{jb} \cdot \left| \mathbf{v}_{jb} \right| \mathbf{v}_{jb}, \tag{100}$$

$$\frac{1}{V_b} \int_{A_{jl}} \mathbf{T}_j \cdot \mathbf{n}_j \, dA = -\varepsilon_j \mu_j \mathbf{K}_{jb}^{-1} \cdot \mathbf{v}_{jb} \tag{101}$$

and

$$\left\langle \mathbf{q}_j \right\rangle_b = \nabla \cdot \left[\frac{1}{\varepsilon_j} \mathbf{D}_{jb} \cdot \nabla \cdot \left(\rho_j \mathbf{v}_{jb} \right) \right] + \varepsilon_j \rho_j \mathbf{g}. \tag{102}$$

5.3 Energy Equation

Comparing Eq. (95) with Eq. (63) yields

$$\sum_{CV} \dot{\mathbf{r}}_{jb} = -\varepsilon_j \left\langle \rho_j \right\rangle_j \left\langle C_j \right\rangle_j \nabla \cdot \left\langle \widehat{\mathbf{v}}_j \widehat{T}_j \right\rangle_j + \frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \nabla \cdot \left(\int_{A_j} T_j \mathbf{n}_j \, dA \right)$$
$$+ \frac{1}{V_b} \boldsymbol{\kappa}_j \cdot \int_{A_j} \nabla T_j \cdot \mathbf{n}_j \, dA + \left\langle q_{je} \right\rangle_b . \tag{103}$$

Note the source term on the right of Eq. (103) can be expressed as the following:

$$\begin{aligned} \left\langle q_{je} \right\rangle_b &= \left\langle \mathbf{v}_j \cdot \nabla p_j + \frac{\partial p_j}{\partial t} \right\rangle_b \\ &= \frac{\left\langle \mathbf{v}_j \right\rangle_b \cdot \left\langle \nabla p_j \right\rangle_b}{\varepsilon_j} + \left\langle \widehat{\mathbf{v}}_j \cdot \widehat{\nabla p_j} \right\rangle_b + \frac{\partial \left\langle p_j \right\rangle_b}{\partial t} - \frac{1}{V_b} \int_{A_{jl}} p_j \mathbf{v}_{Aj} \cdot \mathbf{n}_j \, dA \\ &= \frac{\left\langle \mathbf{v}_j \right\rangle_b \cdot \nabla \left\langle p_j \right\rangle_b}{\varepsilon_j} + \frac{\partial \left\langle p_j \right\rangle_b}{\partial t} + \left\langle \widehat{\mathbf{v}}_j \cdot \widehat{\nabla p_j} \right\rangle_b \\ &\quad + \frac{\left\langle \mathbf{v}_j \right\rangle_b}{\varepsilon_j} \cdot \frac{1}{V_b} \int_{A_{jl}} p_j \mathbf{n}_j \, dA - \frac{1}{V_b} \int_{A_{jl}} p_j \mathbf{v}_{Aj} \cdot \mathbf{n}_j \, dA. \end{aligned} \tag{104}$$

6 Discussions and Conclusions

The methodology and formulations presented in this chapter for derivation of the porous media macroscopic transport equations have emphasized and demonstrated the following issues:

- The gradient law used in porous media is based on the extrapolated limit concept. This extrapolation assumes the applicability of the basic definition of derivative in the limit as Δx, Δy, and $\Delta z \to 0$. This is simply a remedial measure considered for convenience in derivation of macroscopic equations. Frequently, the point of location is mistakenly assumed to be in a certain phase when the size of the elementary volume is reduced to a point of zero volume and therefore the volume fraction of that phase at that point of location is assumed to be 100%, whereas this point of location is nothing more than an imaginary extrapolation point from the elementary finite volume of bulk porous media where the amount of a certain phase is equal to its volume fraction in the bulk volume which is less than or equal to the porosity of porous media.
- The frequently used gradient law of spontaneous transport is incorrectly defined proportionally to the gradient of the intensive property expressed at a point in a given phase which is therefore identical to the gradient considered in microscopic transport formulation. This implies that the gradient becomes zero when the phase property is constant, whereas such condition applies only for microscopic transport formulation and is not applicable in porous media averaged formulation. Thus, the correct expression of the spontaneous transport in porous media should be taken with respect to the gradient of the driving (inducing) property contained per unit bulk volume which is equal to the property multiplied by the volume fraction of the phase. Consequently, the porous media gradient becomes zero only when both the phase property and volume fraction are constant for macroscopic transport formulation. Therefore, the porous media gradient is not zero and transport can still occur if either the phase property or volume fraction, or both vary with distance in porous media.
- Control volume analysis is proven to be complementary to porous media averaging in the formulation of porous media macroscopic transport equations. Hence, comparison of the results obtained by control volume analysis and porous media averaging can be beneficial in resolving some difficulties such as developing methods for closure of various complicated terms involving the deviations of properties from their intrinsic fluid-volume averages. Proper closure methods inferred by control volume analysis can reduce the complexity of the equations obtained by porous media averaging. The control volume analysis requires creative applications to resolve various issues such as based on the capillary-orifice model used for expressing the pore surface wall friction forces and pore throat drag. This, in turn, provides valuable insights of practical importance.
- Improper applications of the basic rules of averaging may result with erroneous formulations. This explains the discrepancies between the results obtained here and the corresponding efforts reported in the literature. However, this issue is subtle and continues to be a major source of errors in porous media averaging. It is essential to apply the fluid intrinsic volume and bulk-volume-averaging rules in the derivation of macroscopic formulations in a manner such that the resulting expressions can in turn conform to these rules. This issue was illustrated for expressing the bulk-volume averages of the products of two and three properties in terms of the bulk-volume averages of the individual properties. It was shown

that only the approach based on decomposing the fluid property into its intrinsic fluid-volume average and its deviation from this average leads to a formulation which is reversibly consistent with the averaging rules.

Appendix

The equations of [19] are different from those presented in this chapter because of a systematic error involved in their derivations of the momentum and energy equations. This issue is subtle and will be explained by the following example in their choice of symbols which correspond to those used in this chapter as $\langle u\rangle^f = \langle u\rangle_j$ and $\langle u\rangle = \langle u\rangle_b$ representing the intrinsic fluid j-phase-volume and REV bulk-volume averages, respectively.

For example, the first terms in Eq. (20) of [19] read as

$$\left\langle \partial_j u_i u_j \right\rangle = \partial_j \left\langle \langle u_i\rangle^f \left\langle u_j \right\rangle^f \right\rangle + \text{ other terms.} \tag{105}$$

They arrive at the following expression of their Eq. (21) after processing Eq. (105) by considering that the average of $\langle u_i\rangle^f \left\langle u_j\right\rangle^f$ is identical to itself only over the REV bulk volume because $\langle u_i\rangle^f \left\langle u_j\right\rangle^f$ is constant:

$$\left\langle \partial_j u_i u_j \right\rangle = \partial_j \langle u_i\rangle^f \left\langle u_j \right\rangle^f + \text{ other terms.} \tag{106}$$

Therefore, their Eq. (21) leads to the following expression in the rest of their formulations, for example, in their Eq. (26), when expressed in terms of the superficial or REV bulk-volume average fluid velocity:

$$\left\langle \partial_j u_i u_j \right\rangle = \partial_j \left(\frac{\langle u_i\rangle \left\langle u_j \right\rangle}{\varepsilon^2} \right) + \text{ other terms.} \tag{107}$$

This procedure is not correct because it is not consistent with the rules of averaging. The proper procedure is described in the following.

The first terms in Eq. (20) of [19] should be processed as

$$\left\langle \partial_j u_i u_j \right\rangle = \partial_j \left\langle \langle u_i\rangle^f \left\langle u_j \right\rangle^f \right\rangle + \text{ other terms} = \partial_j \left(\varepsilon \left\langle \langle u_i\rangle^f \left\langle u_j \right\rangle^f \right\rangle^f \right) + \text{ other terms.} \tag{108}$$

Then, their Eq. (21) takes the following form after processing Eq. (105) by considering that the average of $\langle u_i\rangle^f \left\langle u_j\right\rangle^f$ is identical to itself over the fluid volume contained inside the REV bulk volume:

$$\left\langle \partial_j u_i u_j \right\rangle = \partial_j \left(\varepsilon \langle u_i\rangle^f \left\langle u_j \right\rangle^f \right) + \text{ other terms.} \tag{109}$$

Therefore, Eq. (109) takes the following form when expressed in terms of the superficial or REV bulk-volume average fluid velocity:

$$\left\langle \partial_j u_i u_j \right\rangle = \partial_j \left(\frac{\langle u_i \rangle \left\langle u_j \right\rangle}{\varepsilon} \right) + \text{ other terms.} \tag{110}$$

In view of the above explanation and illustration, the formulations of [19] require corrections of their systematic errors.

References

1. Biot, M.A.: General theory of three dimensional consolidation. J. Appl. Phys. **12**, 155–164 (1941)
2. Blick, E.F., Civan, F.: Porous media momentum equation for highly accelerated flow. SPE Reserv. Eng. **3**, 1048–1052 (1988)
3. Civan, F.: Unfrozen water content in freezing and thawing soils—kinetics and correlation. J. Cold Reg. Eng. **14**, 146–156 (2000)
4. Civan, F.: Implications of alternative macroscopic descriptions illustrated by general balance and continuity equations. J. Porous Media. **5**, 271–282 (2002)
5. Civan, F.: Generalized Darcy's law by control volume analysis including capillary and orifice effects. J. Can. Petrol. Tech. **47**, 1–7 (2008)
6. Civan, F.: Modeling transport in porous media by control volume analysis. J. Porous Media **13**, 855–873 (2010)
7. Civan, F.: Porous Media Transport Phenomena. Wiley, Hoboken, New Jersey (2011)
8. Civan, F., Evans, R.D.: Determining the parameters of the Forchheimer equation from pressure-squared vs. pseudo-pressure Formulations. SPE Reserv. Eval. Eng. **1**, 43–46 (1998)
9. Darcy, H.: Les Fontaines Publiques de la Ville de Dijon. Victor Dalmont, Paris (1856)
10. de Lemos, M.J.S.: Analysis of turbulent flows in fixed and moving permeable media. Acta Geophys. **56**, 562–583 (2008)
11. Gray, W.G.: A derivation of the equations for multiphase transport. Chem. Eng. Sci. **30**, 229–233 (1975)
12. Hsu, C.-T.: Dynamic modeling of convective heat transfer. In: Vafai, K. (ed.) Porous Media, Handbook of Porous Media, 2nd edn., pp. 39–80. CRC Press, Taylor & Francis, LLC, Boca Raton (2005)
13. Hubbert, M.K.: Darcy's law and the field equations of the flow of underground fluids. Petrol. Trans. AIME **207**, 222–239 (1956)
14. Lasseux, D., Ahmadi, A., Arani, A.A.A.: Two-phase inertial flow in homogeneous porous media: a theoretical derivation of a macroscopic model. Transport Porous Med. **75**, 371–400 (2008)
15. Liu, S., Masliyah, J.H.: Dispersion in porous media. In: Vafai, K. (ed.) Porous Media, Handbook of Porous Media, 2nd edn., pp. 81–140. CRC Press, Taylor & Francis Group, LLC, Boca Raton (2005)
16. Prada, A., Civan, F.: Modification of Darcy's law for the threshold pressure gradient. J. Petrol. Sci. Eng. **22**, 237–240 (1999)
17. Schulenberg, T., Muller U.: An improved model for two-phase flow through beds of coarse particles. Int. J. Multiphase Flow. **13**, 87–97 (1987)
18. Slattery, J.C.: Momentum, Energy and Mass Transfer in Continua. McGraw-Hill, New York (1972)
19. Sozer, E., Shyy, W.: Multi-scale thermo-fluid transport in porous media. Int. J. Numer. Meth. Heat Fluid Flow **18**, 883–899 (2008)
20. Whitaker, S.: The Method of Volume Averaging. Kluwer Academic Publishers, Boston (1999)

Springer
http://www.springer.com/978-1-4614-5054-2

Modelling and Simulation in Fluid Dynamics in Porous Media
Ferreira, J.A.; Barbeiro, S.; Pena, G.; Wheeler, M.F. (Eds.)
2013, X, 210 p., Hardcover
ISBN: 978-1-4614-5054-2