Preface

A famous Indian saying can be approximatively phrased in the following way: "Our earth is not just a legacy from our parents; it is a loan from our children."

In mathematical analysis, a precious legacy has been given to us: differential calculus and integral calculus are tools that play an important role in the present state of knowledge and technology. They even gave rise to a philosophical opinion, often called determinism, that amounts to saying that any phenomenon can be predicted, provided one knows its rules and the initial conditions. Such a triumphant claim has been mitigated by modern theories such as quantum mechanics. The "fuzziness" one meets in this book presents some analogy with modern mechanics. In some sense, it is the best we can leave to our children in case they have to deal with rough data.

In the middle of the nineteenth century, Weierstrass made clear the fact that not all functions are differentiable. He even proved that there are continuous functions of one real variable that are nowhere differentiable. Although such "exotic" functions are not negligible, it appears that most nonsmooth functions that are met in concrete mathematical problems have a behavior that is not beyond the reach of analysis.

It is the purpose of the present book to show that an organized bundle of knowledge can be applied to situations in which differentiability is not present.

In favorable cases, such as pointwise maxima of finite families of differentiable functions or sums of convex functions with differentiable functions, a rather simple apparatus allows us to extend in a unified way the rules known in the realms of convex analysis and differentiable analysis. The pioneers in this restricted framework were Pshenichnii, Ioffe, and Tikhomirov (and later on, Demy'anov, Janin, among others). For general functions, more subtle constructions must be devised.

Already at this elementary stage, a combination of geometrical and analytical viewpoints gives greater and more incisive insight. Such a unified viewpoint is one of the revolutionary characteristics of nonsmooth analysis: functions, sets, mappings, and multimappings (or correspondences) can be considered to be equally important, and the links between them allow us to detect fruitful consequences. Historically, geometrical concepts (tangent and normal cones with Bouligand, Severi, Choquet, Dubovitskii-Milyutin, ...) appeared earlier than analytical notions (generalized directional derivatives, subdifferentials with Clarke, Ioffe, Kruger, Mordukhovich, . . .).

On the other hand, the variety of situations and needs has led to different approaches. In our opinion, it would not be sensible to leave the reader with the impression that a single type of answer or construction can meet all the needs one may encounter (it is not even the case with smooth calculus). It is our purpose to give the reader the ability to choose an appropriate scheme depending on the specificities of the problem at hand. Quite often, the problem itself leads to an adapted space. In turn, the space often commands the choice of the subdifferential as a manageable substitute for the derivative.

In this book we endeavor to present a balanced picture of the most elementary attempts to replace a derivative with a one-sided generalized derivative called a subdifferential. This means that instead of associating a linear form to a function at some reference point in order to summarize some information about the behavior of the function around that point, one associates a bunch of linear forms. Of course, the usefulness of such a process relies on the accurateness of the information provided by such a set of linear forms. It also relies on the calculus rules one can design. These two requirements appear to be somewhat antagonistic. Therefore, it may be worthwhile to dispose of various approaches satisfying at least one of these two requirements.

In spite of the variety of approaches, we hope that our presentation here will give an impression of unity. We do not consider the topic as a field full of disorder. On the contrary, it has its own methods, and its various achievements justify a comprehensive approach that has not yet been presented. Still, we do not look for completeness; we rather prefer to present significant tools and methods. The references, notes, exercises, and supplements we present will help the reader to get a more thorough insight into the subject.

In writing a book, one has to face a delicate challenge: either follow a tradition or prepare for a more rigorous use. Our experience with texts that were written about a lifetime ago showed us that the need for rigor and precision has increased and is likely to increase more. Thus, we have avoided some common abuses such as confusing a function with its value, a sequence with its general term, a space with its dual, the gradient of a function with its derivative, the adjoint of a continuous linear map with its transpose. That choice may lead to unusual expressions. But in general, we have made efforts to reach as much simplicity as possible in proofs, terminology and notation, even if some proofs remain long. Moreover, we have preferred suggestive names (such as allied, coherence, gap, soft) to complicated expressions or acronyms, and we have avoided a heavy use of multiple indices, of Greek letters (and also of Cyrillic, Gothic, Hebrew fonts). It appears to us that sophisticated notation blossoms when the concepts are fresh and still obscure; as soon as the concepts appear as natural and simple, the notation tends to get simpler too. Of course, besides mathematicians who are attached to traditions, there are some others who implicitly present themselves as magicians or learned people and like to keep sophisticated notation.

Let us present in greater detail the analysis that served as a guideline for this book.

The field of mathematics offers a number of topics presenting beautiful results. However, many of them are rather remote from practical applications. This fact makes them not too attractive to many students. Still, they are proposed in many courses because they are considered either as important from a theoretical viewpoint or precious for the formation of minds.

It is the purpose of this book to present fundamental aspects of analysis that have close connections with applications. There is no need to insist on the success of analysis. So many achievements of modern technology rely on methods or results from mathematical analysis that it is difficult to imagine what our lives would be like if the consequences of the so-called infinitesimal analysis of Fermat, Leibniz, Newton, Euler and many others would be withdrawn from us.

However, the classical differential calculus is unable to handle a number of problems in which order plays a key role; J.-J. Moreau called them “unilateral problems,” i.e., one-sided problems. Usually, they are caused by constraints or obstacles.

A few decades ago, some tools were designed to study such problems. They are applied in a variety of fields, such as economics, mechanics, optimization, numerical analysis, partial differential equations. We believe that this rich spectrum of applications can be attractive for the reader and deserves a sequel to this book with complementary references, since here we do not consider applications as important as those in optimal control theory and mathematical analysis. Also, we do not consider special classes of functions or sets, and we do not even evoke higher-order notions, although considering second-order generalized derivatives of nondifferentiable functions can be considered a feat!

Besides some elements of topology and functional analysis oriented to our needs, we gather here three approaches: differential calculus, convex analysis, and nonsmooth analysis. The third of these is the most recent, but it is becoming a classical topic encompassing the first two.

The novelty of a joint presentation of these topics is justified by several arguments. First of all, since nonsmooth analysis encompasses both convex analysis and differential calculus, it is natural to present these two subjects as the two basic elements on which nonsmooth analysis is built. They both serve as an introduction to the newest topic. Moreover, they are both used as ingredients in the proofs of calculus rules in the nonsmooth framework. On the other hand, nonsmooth analysis represents an incentive to enrich convex analysis (and maybe differential calculus too, as shown here by the novelty of incorporating directional smoothness in the approach). As an example, we mention the relationship between the subdifferential of the distance function to a closed convex set C at some point z out of C and the normal cone to the set C at points of C that almost minimize the distance to z (Exercises 6 and 7 of Sect. 7.1 of the chapter on convex analysis). Another example is the fuzzy calculus that is common to convex analysis and nonsmooth analysis and was prompted by the last domain.

In this book, we convey some ideas that are simple enough but important. First we want to convince the reader that approximate calculus rules are almost as useful as exact calculus rules. They are more realistic, since from a numerical viewpoint, only approximate values of functions and derivatives can be computed (apart from some special cases).

Second, we stress the idea that basic notions, methods, or results such as variational principles, methods of error bounds, calmness, and metric regularity properties offer powerful tools in analysis. They are of interest in themselves, and we are convinced that they may serve as a motivated approach to the study of metric spaces, whereas such a topic is often considered very abstract by students.

The penalization method is another example illustrating our attempt. It is a simple idea that in order to ensure that a constraint (for instance a speed limit or an environmental constraint) is taken into account by an agent, a possible method consists in penalizing the violation of this constraint. The higher the penalty, the better the behavior. We believe that such methods related to the experience of the reader may enhance his or her interest in mathematics. They are present in the roots of nonsmooth calculus rules and in the study of partial differential equations.

Thus, the contents of the first part can be used for at least three courses besides nonsmooth analysis: metric and topological notions, convex analysis, and differential calculus. These topics are also deeply linked with optimization questions and geometric concepts.

In the following chapters dealing with nonsmooth analysis, we endeavor to present a view encompassing the main approaches, whereas most of the books on that topic focus on a particular theory. Indeed, we believe that it is appropriate to deal with nonsmooth problems with an open mind. It is often the nature of the problem that suggests the choice of the spaces. In turn, the choice of the nonsmooth concepts (normal cones, subdifferentials, etc.) depends on the properties of the chosen spaces and on the objectives of the study. Some concepts are accurate, but are lacking good calculus rules; some enjoy nice convexity or duality properties but are not so precise. We would like to convince the reader that such a variety is a source of richness rather than disorder.

The quotation below would be appropriate if in the present case it corresponded to what actually occurred. But the truth is that the book would never had been written if Alexander Ioffe had not suggested the idea to the author and contributed to many aspects of it. The author expresses his deepest gratitude to him. He also wants to thank the many colleagues and friends, in particular, D. Azé, A. Dontchev, E. Giner, A. Ioffe, M. Lassonde, K. Nachi, L. Thibault, who made useful criticisms or suggestions, and he apologizes to those who are not given credit or given not enough credit.

N'écrire jamais rien qui de soi ne sortit,
Et modeste d'ailleurs, se dire mon petit,
Soit satisfait des fleurs, des fruits, même des feuilles,
Si c'est dans ton jardin à toi que tu les cueilles!
...Ne pas monter bien haut, peut-être, mais tout seul!

Edmond Rostand, *Cyrano de Bergerac*, Acte II, Scène 8

Never to write anything that does not proceed from the heart,
and, moreover, to say modestly to myself, “My dear,
be content with flowers, with fruits, even with leaves,
if you gather them in your own garden!”
. . . Not to climb very high perhaps, but to climb all alone!

Pau and Paris, France Jean-Paul Penot

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