Chapter 2
Elements of Differential Calculus

If I have seen further, it is by standing on the shoulders of giants.

—Isaac Newton, letter to Robert Hooke, February 5, 1675

Differential calculus is at the core of several sciences and techniques. Our world would not be the same without it: astronomy, electromagnetism, mechanics, optimization, thermodynamics, among others, use it as a fundamental tool.

The birth of differential calculus is usually attributed to Isaac Newton and Gottfried Wilhelm Leibniz in the latter part of the seventeenth century, with several other contributions. The pioneer work of Pierre de Fermat is seldom recognized, although he introduced the idea of approximation that is the backbone of differential calculus and that enabled him (and others) to treat many applications. During the eighteenth century, the topic reached maturity, and its achievements led to the principle of determinism in the beginning of the nineteenth century. But it is only with the appearance of functional analysis that it took its modern form.

Several notions of differentiability exist; they correspond to different needs or different situations. The most usual one is the notion of Fréchet differentiability, that is presented in Sect. 2.4. However, a weaker notion of directional differentiability due to Hadamard has some interest. We present it in Sect. 2.3 as a passage from the case of one-variable maps to the case of maps defined on open subsets of normed spaces. Its study has an interest of its own and forms a basis for a notion of subdifferential that will come to the fore in Chap. 4 along with a notion corresponding to the Fréchet derivative. For some results, differentiability does not suffice and one needs some continuity property of the derivative. Besides classical continuity, we consider a weaker continuity condition. The latter is seldom given attention. Still, it will serve as preparation for the limiting processes considered in Chap. 6.

The main questions we treat are the invertibility of nonlinear maps, its applications to geometrical notions, and its uses for optimization problems. The notions of normal cone and tangent cones appearing for optimality conditions in fact belong

J.-P. Penot, *Calculus Without Derivatives*, Graduate Texts in Mathematics 266,
DOI 10.1007/978-1-4614-4538-8_2, © Springer Science+Business Media New York 2013

to the realm of nonsmooth analysis. Many practitioners are unaware of this—rather like Molière's Monsieur Jourdain, who had been speaking prose all his life without knowing it. We end the chapter with an introduction to the calculus of variations, that has been a strong incentive for the development of differential calculus since the end of the seventeenth century. Differentiability questions for convex functions will be considered in the next chapter.

2.1 Derivatives of One-Variable Functions

The differentiation of one-variable vector-valued functions is not very different from the differentiation of one-variable real-valued functions. In both cases, the calculus relies on rules for limits. The aims are similar too. In both cases, the purpose consists in drawing some information about the behavior of the function from some knowledge concerning the derivative. In the vector-valued case, the direction of the derivative takes as great importance as its magnitude.

2.1.1 Differentiation of One-Variable Functions

In this section unless otherwise mentioned, T is an open interval of $\mathbb{R}$ and $f : T \to X$ is a map with values in a normed space X.

Definition 2.1. A map f is said to be *right-differentiable* (resp. *left-differentiable*) at $t \in T$ if the quotient $(f(t+s) - f(t))/s$ has a limit as $s \to 0_+$, i.e., $s \to 0$ with $s > 0$ (resp. as $s \to 0_-$, i.e., $s \to 0$ with $s < 0$). These limits, denoted by $f'_+(t)$ and $f'_-(t)$ respectively, are called the right and the left *derivatives* of f at t.

When these limits coincide, f is said to be *differentiable* at t, and their common value $f'(t)$ is called the *derivative* of f at t.

Thus f is differentiable at t if and only if the quotient $(f(t+s) - f(t))/s$ has a limit as $s \to 0$, with $s \neq 0$, or equivalently, if there exist some vector $v(= f'(t)) \in X$ and some function $r : T' := T - t \to X$ called a *remainder* such that $r(s)/s \to 0$ as $s \to 0$, for which one has the expansion

$$f(t') = f(t) + (t' - t)v + r(t' - t), \tag{2.1}$$

as can be seen by setting $s = t' - t$, $r(0) = 0$, $r(s) = s^{-1}(f(t+s) - f(t)) - v$ for $s \in T' \setminus \{0\}$.

The following rules are immediate consequences of the rules for limits.

Proposition 2.2. *If $f, g : T \to X$ are differentiable at $t \in T$ and $\lambda, \mu \in \mathbb{R}$, then $h := \lambda f + \mu g$ is differentiable at t and its derivative at t is $h'(t) = \lambda f'(t) + \mu g'(t)$.*

Proposition 2.3. *If $f : T \to X$ is differentiable at $t \in T$, if Y is another normed space and if $A : X \to Y$ is linear and continuous, then $g := A \circ f$ is differentiable at $t \in T$ and $g'(t) = A(f'(t))$.*

Similar rules hold for right derivatives and left derivatives. We will see later a more general composition rule (or chain rule). The following composition rule can be proved using quotients as for scalar functions. We prefer using expansions as in (2.1) because such expansions give the true flavor of differential calculus, i.e., approximations by continuous affine functions. Moreover, one does not need to take care of denominators taking the value 0.

Proposition 2.4. *If T, U are open intervals of $\mathbb{R}$, if $g : T \to U$ is differentiable at $\bar{t} \in T$, and if $h : U \to X$ is differentiable at $\bar{u} := g(\bar{t})$, then $f := h \circ g$ is differentiable at $\bar{t}$ and $f'(\bar{t}) = g'(\bar{t})h'(\bar{u})$.*

Proof. Let $v := h'(\bar{u})$ and let $\alpha : T \to \mathbb{R}$, $\beta : U \to X$ be such that $\alpha(t) \to 0$ as $t \to \bar{t}$, $\beta(u) \to 0$ as $u \to \bar{u}$ with $g(t) - g(\bar{t}) = (t - \bar{t})g'(\bar{t}) + (t - \bar{t})\alpha(t)$, $h(u) - h(\bar{u}) = (u - \bar{u})v + (u - \bar{u})\beta(u)$. Then one has

$$\begin{aligned} f(t) - f(\bar{t}) &= h(g(t)) - h(\bar{u}) = (g(t) - \bar{u})v + (g(t) - \bar{u})\beta(g(t)) \\ &= (t - \bar{t})g'(\bar{t})v + (t - \bar{t})\alpha(t)v + (t - \bar{t})(g'(\bar{t}) + \alpha(t))\beta(g(t)). \end{aligned}$$

Since $g(t) \to \bar{u}$ as $t \to \bar{t}$, one sees that $\alpha(t)v + (g'(\bar{t}) + \alpha(t))\beta(g(t)) \to 0$ as $t \to \bar{t}$, so that f is differentiable at $\bar{t}$ and $f'(\bar{t}) = g'(\bar{t})v = g'(\bar{t})h'(\bar{u})$. □

Now let us devise a rule for the derivative of a product. It can be generalized to a finite number of factors.

Proposition 2.5 (Leibniz rule). *Let X, Y, Z be normed spaces and let $b : X \times Y \to Z$ be a continuous bilinear map. Given functions $f : T \to X$, $g : T \to Y$ that are differentiable at t, the function $h : r \mapsto b(f(r), g(r))$ is differentiable at t and*

$$h'(t) = b(f'(t), g(t)) + b(f(t), g'(t)).$$

Proof. By assumption, there exist some $\alpha : (T - t) \to X$, $\beta : (T - t) \to Y$ satisfying $\alpha(s) \to 0$, $\beta(s) \to 0$ as $s \to 0$ such that

$$f(t + s) = f(t) + sf'(t) + s\alpha(s), \qquad g(t + s) = g(t) + sg'(t) + s\beta(s).$$

Plugging these expansions into b and setting $\gamma(s) := b(\alpha(s), g(t)) + b(f(t), \beta(s)) + sb(\alpha(s), \beta(s))$, so that $\gamma(s) \to 0$ as $s \to 0$, we get

$$h(t + s) - h(t) = sb(f'(t), g(t)) + sb(f(t), g'(t)) + s\gamma(s)$$

and $s^{-1}(h(t + s) - h(t)) \to b(f'(t), g(t)) + b(f(t), g'(t))$. □

2.1.2 The Mean Value Theorem

The mean value theorem is a precious tool for devising estimates. For this reason, it is a cornerstone of differential calculus. Let us note that the elementary version recalled in the following lemma is not valid when the function takes its values in a linear space of dimension greater than one.

Lemma 2.6. *Let $f : T \to \mathbb{R}$ be a continuous function on some interval $T := [a,b]$ of $\mathbb{R}$, with $a < b$. If f is differentiable on (a,b) then there exists some $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)(b-a).$$

Example. Let $f : [0,1] \to \mathbb{R}^2$ be given by $f(t) := (t^2, t^3)$ for $t \in T := [0,1]$. Then one cannot find every $c \in \operatorname{int} T$ satisfying the preceding relation, since the system $2c = 1$, $3c^2 = 1$ has no solution. □

Instead, a statement under the form of an inequality is valid.

Theorem 2.7. *Let X be a normed space, $T := [a,b]$ a compact interval of $\mathbb{R}$, and $f : T \to X$, $g : T \to \mathbb{R}$ continuous on T with right derivatives on (a,b) such that $\left\| f'_+(t) \right\| \leq g'_+(t)$ for every $t \in (a,b)$. Then*

$$\|f(b) - f(a)\| \leq g(b) - g(a). \tag{2.2}$$

Proof. It suffices to prove that for every given $\varepsilon > 0$, b belongs to the set

$$T_\varepsilon := \{t \in T : \|f(t) - f(a)\| \leq g(t) - g(a) + \varepsilon(t-a)\}.$$

This set is nonempty, since $a \in T_\varepsilon$, and closed, being defined by an inequality whose sides are continuous. Let $s := \sup T_\varepsilon \leq b$. Then $s \in T_\varepsilon$.

We first suppose f and g have right derivatives on $[a,b)$ and we show that assuming $s < b$ leads to a contradiction. The existence of the right derivatives of f and g at s yields some $\delta \in (0, b-s)$ such that

$$r \in (0,\delta] \Rightarrow \left\| \frac{f(s+r) - f(s)}{r} - f'_+(s) \right\| \leq \frac{\varepsilon}{2}, \quad \left| \frac{g(s+r) - g(s)}{r} - g'_+(s) \right| \leq \frac{\varepsilon}{2}.$$

It follows that for $r \in (0,\delta]$ one has

$$\|f(s+r) - f(s)\| \leq r \left\| f'_+(s) \right\| + r\varepsilon/2, \quad g(s+r) - g(s) \geq r g'_+(s) - r\varepsilon/2.$$

Therefore, since $s \in T_\varepsilon$ and $\left\| f'_+(s) \right\| \leq g'_+(s)$,

$$
\begin{aligned}
\|f(s+r)-f(a)\| &\leq \|f(s+r)-f(s)\| + \|f(s)-f(a)\| \\
&\leq rg'_+(s) + r\varepsilon/2 + g(s) - g(a) + \varepsilon(s-a) \\
&\leq g(s+r) - g(s) + r\varepsilon + g(s) - g(a) + \varepsilon(s-a) \\
&\leq g(s+r) - g(a) + \varepsilon(s+r-a).
\end{aligned}
$$

This string of inequalities shows that $s+r \in T_\varepsilon$, a contradiction to the definition of s. Thus $b \in T_\varepsilon$ and the result is established under the additional assumption that the right derivatives of f and g exist at a (note that we may have $s = a$ in what precedes).

When this additional assumption is not made, we take $a' \in (a,b]$ and we apply the preceding case to the interval $[a',b]$:

$$\left\|f(b) - f(a')\right\| \leq g(b) - g(a').$$

Then passing to the limit as $a' \to a_+$, we get the announced inequality. □

Remark. Since we allow the possibility that the right derivatives do not exist at the extremities of the interval, we may assume that the derivatives do not exist (or do not satisfy the assumed inequality) at a finite number of points of T. To prove this, it suffices to subdivide the interval into subintervals and to gather the obtained inequalities using the triangular inequality. In fact, one can exclude a countable set of points of T, but the proof is more involved; see [197], [294, p.153].

Theorem 2.8. *With the notation of Theorem 2.7, the estimate (2.2) holds when f and g are continuous on T and have right derivatives on $T \setminus D$, where D is countable, such that $\left\|f'_+(t)\right\| \leq g'_+(t)$ for every $t \in T \setminus D$.*

The most usual application is given in the following corollary, in which we take $g(t) = mt$ for some $m \in \mathbb{R}_+$ and $t \in T$. The Lipschitz property is obtained on substituting an arbitrary pair t, t' (with $t \leq t'$) for a, b.

Corollary 2.9. *Let $f : T \to X$ be continuous on $T := [a,b]$, let $m \in \mathbb{R}_+$, and let D be a countable subset of T. Suppose that for all $t \in (a,b) \setminus D$, f has a right derivative at t such that $\left\|f'_+(t)\right\| \leq m$. Then f is Lipschitzian with rate m on T, and in particular,*

$$\|f(b) - f(a)\| \leq m(b-a).$$

The case $m = 0$ yields the following noteworthy consequence.

Corollary 2.10. *Let $f : [a,b] \to X$ be continuous and such that f has a right derivative f'_+ on $(a,b) \setminus D$ that is null, D being countable. Then f is constant on $[a,b]$.*

The purpose of obtaining estimates often requires the introduction of auxiliary functions, as in the proof of the following useful corollary.

Corollary 2.11. *Let $f : T \to X$ be continuous on $T := [a,b]$, let $v \in X$, $r \in \mathbb{R}_+$, and let D be a countable subset of T. Suppose f has a right derivative on $(a,b) \setminus D$ such that $f'_+(t) \in v + rB_X$ for every $t \in (a,b) \setminus D$. Then*

$$f(b) \in f(a) + (b-a)v + (b-a)rB_X.$$

Proof. Define $h : T \to X$ by $h(t) := f(t) - tv$. Then h is continuous and for $t \in (a,b) \setminus D$ one has $\left\|h'_+(t)\right\| = \left\|f'_+(t) - v\right\| \leq r$. Then Corollary 2.10 entails that

$$\|f(b) - f(a) - (b-a)v\| = \|h(b) - h(a)\| \leq (b-a)r,$$

an estimate equivalent to the inclusion of the statement. □

Remark. The terminology for the theorem stems from the fact that the mean value $\overline{v} := (b-a)^{-1}(f(b) - f(a))$ is estimated by the approximate speed v, with an error r that is exactly the magnitude of the uncertainty of the estimate of the instantaneous speed $f'_+(t)$. Note that the shorter the lapse of time $(b-a)$, the more precise the localization of $f(b)$ by $f(a) + (b-a)v$. Thus, if you lose your dog, be sure to have a rather precise idea of his speed and direction and do not lose time in pursuing him.

2.2 Primitives and Integrals

The aim of this subsection is to present an inverse of the differentiation operator. In fact, as revealed by the Darboux property (Exercise 1), not all functions from some interval T of $\mathbb{R}$ to a real Banach space X are derivatives. Therefore, we will get a primitive g of a function f on T only if f is regular enough. Here we use the following terminology.

Definition 2.12. A function $g : T \to X$ is said to be a *primitive* of $f : T \to X$ if g is continuous and if there exists a countable subset D of T such that for all $t \in T \setminus D$, g is differentiable at t and $g'(t) = f(t)$.

Corollary 2.10 ensures uniqueness of g.

Proposition 2.13. *If g_1 and g_2 are two primitives of an arbitrary function $f : T \to X$, then $g_1 - g_2$ is constant.*

Proof. If g_1 and g_2 are two primitives of f, then there exist countable subsets D_1 and D_2 of T such that g_i is differentiable on $T \setminus D_i$ and $g'_i(t) = f(t)$ for all $t \in T \setminus D_i$ $(i = 1,2)$. Then for the countable set $D := D_1 \cup D_2$, the continuous function $g_1 - g_2$ is differentiable on $T \setminus D$ and its derivative is 0 there; thus $g_1 - g_2$ is constant. □

In order to construct g from f, we use an integration process. Such a process is useful for many other purposes and is well known when $X = \mathbb{R}$. Since we focus on vector-valued functions, we are not too exacting about regularity assumptions, so that we choose a construction that is simpler than the Lebesgue–Bochner integration theory. For most purposes, integrating continuous functions would suffice. However, admitting simple discontinuities may be useful. The class we select is described in the next definition.

Definition 2.14. A function $f : T \to X$ from a compact interval $T := [a,b]$ of $\mathbb{R}$ to a real Banach space X is said to be *regulated* if for all $t \in [a,b)$ (resp. $t \in (a,b]$), f has a limit on the right $f(t_+) := \lim_{r \to t, r > t} f(r)$ (resp. on the left $f(t_-) := \lim_{s \to t, s < t} f(s)$).

The function f is said to be a (right-) *normalized regulated function* if it is regulated, if $f(b) = f(b_-)$, and if for all $t \in [a,b)$ one has $f(t_+) = f(t)$.

Real-valued monotone functions, vector-valued continuous functions, and step functions are regulated functions. Recall that $f : T \to X$ is a *step function* if there is a finite sequence $\sigma := (s_0, s_1, \ldots, s_k)$ with $s_0 = a < s_1 < \cdots < s_k = b$, called a *subdivision* of T, such that f is constant on each open interval (s_{i-1}, s_i) for $i = 1, \ldots, k$. The step function f is said to be a (right-) normalized step function if f is constant on $[s_{i-1}, s_i)$ for $i = 1, \ldots, k-1$ and on $[s_{k-1}, b]$. We leave the proofs of the following results as exercises (see [294]).

Proposition 2.15. *Let X be a Banach space and let T be a compact interval of $\mathbb{R}$. A function $f : T \to X$ is regulated (resp. normalized regulated) if and only if it is the uniform limit of a sequence (f_n) of step functions (resp. normalized step functions).*

It follows that a regulated function on T is bounded. Moreover:

Proposition 2.16. *For every regulated function $f : T \to X$, the set $f(T)$ is relatively compact in X (i.e., $\mathrm{cl}(f(T))$ is compact). Moreover, the set of discontinuities of f is at most countable.*

The next statement can be either derived from Proposition 2.15 or proved directly (see [294]).

Proposition 2.17. *The space $R(T,X)$ (resp. $R_n(T,X)$) of regulated functions (resp. normalized regulated functions) from a compact interval T to a Banach space X endowed with the norm $\|\cdot\|_\infty$ given by $\|f\|_\infty := \sup_{t \in T} \|f(t)\|$ is a Banach space.*

The *integral* of a step function f can be defined unambiguously as follows: if $s_0 = a < s_1 < \cdots < s_k = b$ is such that $f(t) = c_i$ for $t \in (s_{i-1}, s_i)$, $i \in \mathbb{N}_k$, then

$$\int_T f := \int_a^b f(t)\mathrm{d}t := \sum_{i=1}^{k} (s_{i+1} - s_i) c_i.$$

It is easy to show that this element of X does not depend on the subdivision of T. Moreover, for every step function f from T to X, the triangle inequality ensures that

$$\left\| \int_T f \right\| \leq (b-a) \|f\|_\infty . \tag{2.3}$$

Since the space $S(T,X)$ of step functions is dense in the space $R(T,X)$, the map $f \mapsto \int_T f$ can be extended by continuity from $S(T,X)$ to $R(T,X)$:

$$\int_T f = \lim_n \int_T f_n \quad \text{if } f = \lim_n f_n, \quad f_n \in S(T,X).$$

This extension is linear, continuous, and with norm $b-a$, since (2.3) remains valid for $f \in R(T,X)$. Moreover, given $a \le b \le c$ in $\mathbb{R}$, for all $f \in R([a,c],X)$ one has *Chasles's relation*

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

It easily follows from the case of step functions by a passage to the limit.

The following composition property is crucial: using continuous linear forms x^* on X, it enables one to uniquely determine the integral of a regulated function $f \in R(T,X)$ with the help of the integrals of the real-valued functions $x^* \circ f$.

Proposition 2.18. *Given Banach spaces X, Y and $A \in L(X,Y)$, for every $f \in R(T,X)$ one has $A \circ f \in R(T,Y)$ and $\int_T A \circ f = A(\int_T f)$.*

Proof. The first assertion is a direct consequence of the definition. It can also be checked by taking a sequence (f_n) in $S(T,X)$ that converges uniformly to f. Since the relation $\int_T A \circ f_n = A(\int_T f_n)$ is immediate, the second assertion follows from the definition of the integral of $A \circ f$ as $\lim \int_T A \circ f_n$, since $(A(\int_T f_n)) \to A(\int_T f)$, A being continuous and $(\int_T f_n)$ converging to $\int_T f$. □

The next result gives a partial inverse of the differentiation operator.

Theorem 2.19. *For $f \in R(T,X)$, the map $g : t \mapsto \int_a^t f(s)\mathrm{d}s$ is a primitive of f on T.*

Proof. Given $t \in [a,b)$, $\varepsilon > 0$, let $\delta \in (0, b-t)$ be such that $\|f(t+r) - f(t_+)\| \le \varepsilon$ for every $r \in (0,\delta]$. Since for $c := f(t_+)$ one has $\int_t^{t+r} c = rc$, it follows from Chasles's relation and (2.3) that

$$\left\| \int_a^{t+r} f - \int_a^t f - rc \right\| = \left\| \int_t^{t+r} (f-c) \right\| \le r\varepsilon.$$

This relation shows that $g : t \mapsto \int_a^t f(s)\mathrm{d}s$ has a right derivative at t whose value is c. Similarly, if $t \in (a,b]$, then g has $f(t_-)$ as a left derivative at t. Therefore, if f is continuous at $t \in (a,b)$, then g is differentiable at t and $g'(t) = f(t)$. Since the set D of discontinuities of f is countable, we get that g is differentiable on $T \setminus D$ with derivative f. Moreover, g is continuous on T in view of Chasles's relation and (2.3). □

Corollary 2.20. *If $f : T \to X$ is continuous, then $g : t \mapsto \int_a^t f(s)\mathrm{d}s$ is of class C^1 (i.e., differentiable with a continuous derivative) and its derivative is f.*

Let us give two rules that are useful for the computation of primitives.

Proposition 2.21 (Change of variables). *Let $h : S = [\alpha,\beta] \to \mathbb{R}$ be the primitive of a regulated function h' such that $h(S) \subset T$ and let $f \in R(T,X)$. If either f is continuous or h is strictly monotone, then $s \mapsto h'(s)f(h(s))$ is regulated and for all $r \in [\alpha,\beta]$ one has*

$$\int_{\alpha}^{r} h'(s)f(h(s))\mathrm{d}s = \int_{h(\alpha)}^{h(r)} f(t)\mathrm{d}t. \tag{2.4}$$

Proof. When f is continuous, since h is continuous, $f \circ h$ is continuous and then $k : s \mapsto h'(s)f(h(s))$ is regulated; the same is true when h is either increasing or decreasing. Then the left-hand side of equality (2.4) is the value at r of the primitive j of k satisfying $j(\alpha) = 0$. The right-hand side is $g(h(r))$, where g is the primitive of f satisfying $g(h(\alpha)) = 0$. Under each of our assumptions, for a countable subset D of S, the derivative of $g \circ h$ at $r \in S \setminus D$ exists and is $h'(r)g'(h(r)) = h'(r)f(h(r))$. The uniqueness of the primitive of k null at α gives the equality. □

Proposition 2.22 (Integration by parts). *Let X, Y, and Z be Banach spaces, let $(x,y) \mapsto x * y$ be a continuous bilinear map from $X \times Y$ into Z, and let $f : T \to X$, $g : T \to Y$ be primitives of regulated functions, with $T := [a,b]$. Then*

$$\int_a^b f(t) * g'(t)\mathrm{d}t = f(b) * g(b) - f(a) * g(a) - \int_a^b f'(t) * g(t)\mathrm{d}t.$$

Proof. The functions $t \mapsto f(t) * g'(t)$ and $t \mapsto f'(t) * g(t)$ clearly have one-sided limits at all points of $T := [a,b]$. Moreover, their sum is the derivative of $h : t \mapsto f(t) * g(t)$ on $T \setminus D$, where D is the countable set of nondifferentiability of f or g. Thus the result amounts to the equality $\int_a^b h'(t)\mathrm{d}t = h(b) - h(a)$, which stems from the uniqueness of the primitive of h' that takes the value 0 at a. □

Exercises

1. (Darboux property) Show that the derivative f of a differentiable function $g : T \to \mathbb{R}$ satisfies the intermediate value property: given $a, b \in T$ with $f(a) < f(b)$ and $r \in (f(a), f(b))$, there exists some c between a and b such that $f(c) = r$.

2. Show that there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ and two continuous functions g_1, g_2 whose difference is not constant and are such that g_1 and g_2 are differentiable on $\mathbb{R} \setminus N$, where N is a set of measure zero, with $g_1'(t) = g_2'(t) = f(t)$ for all $t \in \mathbb{R} \setminus N$. [Hint: Take $f = 0$, $g_1 = 0$ and for g_2 take an increasing function whose derivative is 0 a.e.]

3. Prove Theorem 2.8. [See [197, 294].]

4. Prove Proposition 2.15. [See [294, 7.6.1].]

5. Show that every (right-) normalized step function on $T := [a,b]$ can be written as a linear combination of the functions $(e_t)_{t \in T}$ given by $e_b = 1$, and for $t \in [a,b)$, $e_t(r) = 1$ for $r \in [a,t)$, $e_t(r) = 0$ for $r \in [t,b]$. Give a generalization to the case of step functions taking their values in a normed space.

6. A function $v : T \to X$ from an interval $T := [a,b]$ of $\mathbb{R}$ to a normed space X is said to be of *bounded variation* if there exists some $c \in \mathbb{R}_+$ such that for every subdivision $\sigma := (s_0, s_1, \ldots, s_k)$ of T one has $\Sigma_{1\le i\le k}\|v(s_i) - v(s_{i-1})\| \le c$. The infimum of such constants c is denoted by $V_a^b(v)$ and called the variation of v on $[a,b]$.

(**a**) Prove that the space $\mathrm{BV}(T,X)$ of functions of bounded variation on T forms a normed space for the norm $v \mapsto \|v\|_{\mathrm{BV}(T,X)} := \|v(a)\| + V_a^b(v)$.
(**b**) Show that a function of bounded variation is regulated.
(**c**) Show that Lipschitzian functions with values in X and monotone functions with values in $\mathbb{R}$ are of bounded variation.
(**d**) Check that the function f defined by $f(0) := 0$, $f(x) := x^2 \sin(1/x^2)$ for $x \in \mathbb{R}\setminus\{0\}$ is not of bounded variation on $T := [0,1]$ although it has a derivative at each point of T.
(**e**) Given $a < b < c$ in $\mathbb{R}$ and $v \in \mathrm{BV}([a,c],X)$, show that $V_a^c(v) = V_a^b(v) + V_b^c(v)$ and that $s \mapsto V_a^s(v)$ is a nondecreasing function.
(**d**) Prove that for all $v \in \mathrm{BV}(T) := \mathrm{BV}(T,\mathbb{R})$ there exist nondecreasing functions v_1, v_2 such that $v = v_1 - v_2$. [Hint: Take $v_1 := (1/2)(w+v)$, $v_2 := (1/2)(w-v)$ with $w(t) := V_a^t(v)$ for $t \in T$.]

7. (Stieltjes integral) Given a function $v \in \mathrm{BV}(T) := \mathrm{BV}(T,\mathbb{R})$ for $T := [a,b]$ and a (right-) normalized step function f from T to a Banach space X, let $I_v(f) := \Sigma_{1\le i\le k} V_a^{t_i}(v)c_i$ if $f := \Sigma_{1\le i\le k} c_i e_{t_i}$, where $c_i \in X$ and e_{t_i} is defined as in Exercise 5.

(**a**) Show that $I_v(f)$ does not depend on the decomposition of f. Check that $\|I_v(f)\| \le V_a^b(v)\,\|f\|_\infty$.
(**b**) Deduce from the inequality above that the map $f \mapsto I_v(f)$ can be extended to a linear map from the space $R_n(T,X)$ of normalized regulated functions with values in X into X satisfying the same inequality. This map is called the *Stieltjes integral* of f relative to v.
(**c**) Conversely, given a continuous linear form f^* on the space $R_n(T) := R_n(T,\mathbb{R})$, let $v(t) := f^*(e_t)$, where e_t is defined in Exercise 5. Show that v is of bounded variation on T and that $V_a^b(v) \le \|f^*\|$.
(**d**) Deduce from what precedes a correspondence between the (topological) dual of the space $R_n(T)$ and the space $\mathrm{BV}(T)$. [See [692].]

2.3 Directional Differential Calculus

Now let us consider maps from an open subset W of a normed space X into another normed space Y. A natural means of reducing the study of differentiability to the one-variable case consists in taking restrictions to line segments or regular curves in W.

Definition 2.23. Let X, Y be normed spaces, let W be an open subset of X, let $\overline{x} \in W$, and let $f : W \to Y$. We say that f has a *radial derivative* at $\overline{x}$ in the direction $u \in X$ if $(1/t)(f(\overline{x}+tu) - f(\overline{x}))$ has a limit as $t \to 0_+$. We denote by $f'_r(\overline{x},u)$ or $d_r f(\overline{x},u)$ this limit. If f has a radial derivative at $\overline{x}$ in every direction u, we say that f is *radially differentiable* at $\overline{x}$. If, moreover, the map $D_r f(\overline{x}) : u \mapsto d_r f(\overline{x},u)$ is linear and continuous, we say that f is *Gâteaux differentiable* at $\overline{x}$ and call $D_r f(\overline{x})$ the Gâteaux derivative of f at $\overline{x}$.

One often says that f is directionally differentiable at $\overline{x}$, but we prefer to keep this terminology for a slightly more demanding notion that we consider now. In fact, although the notion of radial differentiability is simple and useful, it has several drawbacks; the main one is that this notion does not enjoy a chain rule. This variant does enjoy such a rule and reflects a smoother behavior of f when the direction u is submitted to small changes.

Definition 2.24. Let X, Y be normed spaces, let W be an open subset of X, let $\overline{x} \in W$, and let $f : W \to Y$. We say that f has a *directional derivative* at $\overline{x}$ in the direction $u \in X$, or that f is differentiable at $\overline{x}$ in the direction u, if $(1/t)(f(\overline{x}+tv) - f(\overline{x}))$ has a limit as $(t,v) \to (0_+,u)$. We denote by $f'(\overline{x},u)$ or $df(\overline{x},u)$ this limit. If f has a directional derivative at $\overline{x}$ in every direction u, we say that f is *directionally differentiable* at $\overline{x}$. If, moreover, the map $f'(x) := Df(x) : u \mapsto f'(\overline{x},u)$ is linear and continuous, we say that f is *Hadamard differentiable* at $\overline{x}$.

The concepts of directional derivative and radial derivative are different, as the next example shows. Thus, it is convenient to dispose of two notations.

Example–Exercise. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(r,s) = (r^4+s^2)^{-1}r^3 s$ for $(r,s) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $f(0,0) = 0$. It is Gâteaux differentiable at $(0,0)$ but not directionally differentiable at $(0,0)$. □

The (frequent) use of the same notation for the radial and directional derivatives is justified by the following observation showing the compatibility of the two notions.

Proposition 2.25. *If X and Y are normed spaces, if W is an open subset of X and if $f : W \to Y$ has a directional derivative at $\overline{x}$ in the direction u, then it has a radial derivative at $\overline{x}$ in the direction u and both derivatives coincide. In particular, if f is Hadamard differentiable at $\overline{x}$, then it is Gâteaux differentiable at $\overline{x}$.*

Conversely, if f is Lipschitzian on a neighborhood V of $\overline{x}$, then f is directionally differentiable at $\overline{x}$ in every direction u in which f is radially differentiable.

Proof. The first assertions stem from an application of the definition of a limit.

Let us prove the converse assertion. Let k be the Lipschitz rate of f on V and let $u \in X$ be such that f is radially differentiable at $\overline{x}$ in the direction u. Setting $r(t,v) := f(\overline{x}+tv) - f(\overline{x}) - tf'_r(\overline{x},u)$, we have $t^{-1}r(t,u) \to 0$ as $t \to 0$, and since $\left\|t^{-1}(r(t,v) - r(t,u))\right\| = \left\|t^{-1}(f(\overline{x}+tv) - f(\overline{x}+tu))\right\| \leq k\|v-u\| \to 0$ as $(t,v) \to (0_+,u)$, we get $t^{-1}r(t,v) \to 0$ as $(t,v) \to (0_+,u)$. □

While radial differentiability of f at $\bar{x}$ in the direction u is equivalent to differentiability of the function $f_{\bar{x},u} : t \mapsto f(\bar{x}+tu)$ at 0, directional differentiability of f at $\bar{x}$ amounts to differentiability of the composition of f with curves issued from $\bar{x}$ with the initial direction u, as the next proposition shows.

Proposition 2.26. *The map $f : W \to Y$ is differentiable at $\bar{x}$ in the direction $u \in X \setminus \{0\}$ if and only if f is radially differentiable at $\bar{x}$ in the direction u and for every $\tau > 0$ and every (continuous) $c : [0,\tau] \to W$ that is right differentiable at 0 with $c'_+(0) = u$, $c(0) = \bar{x}$, the map $f \circ c$ is right differentiable at 0 and $(f \circ c)'_+(0) = d_r f(\bar{x},u)$.*

Proof. Suppose f is differentiable at $\bar{x}$ in the direction $u \in X$. Given $\tau > 0$ and $c : [0,\tau] \to W$ that is right differentiable at 0 with $c'_+(0) = u$ and $c(0) = \bar{x}$, let us set $v_t := (1/t)(c(t) - c(0))$, so that $v_t \to u$ as $t \to 0_+$. Then

$$\frac{f(c(t)) - f(c(0))}{t} = \frac{f(\bar{x}+tv_t) - f(\bar{x})}{t} \to df(\bar{x},u) \text{ as } t \to 0_+.$$

Now let us prove the sufficient condition. Suppose f has a radial derivative at $\bar{x}$ in the direction u but is not differentiable at $\bar{x}$ in the direction $u \neq 0$. There exist $\varepsilon > 0$ and some sequence $(t_n, u_n) \to (0_+, u)$ such that $\bar{x} + t_n u_n \in W$ for all $n \in \mathbb{N}$ and

$$\left\| \frac{f(\bar{x}+t_n u_n) - f(\bar{x})}{t_n} - d_r f(\bar{x},u) \right\| \geq \varepsilon. \tag{2.5}$$

We may assume that $t_{n+1} \leq (1/2)t_n$. Then let us define $c : [0,t_0] \to X$ by $c(0) := \bar{x}$,

$$c(t) := \bar{x} + (t_n - t_{n+1})^{-1}[(t_n - t)t_{n+1}u_{n+1} + (t - t_{n+1})t_n u_n]$$

for $t \in [t_{n+1}, t_n)$. Then one sees that $(1/t)(c(t) - c(0)) \to u$ as t goes to 0, but since $c(t_n) = \bar{x} + t_n u_n$, in view of (2.5), $f \circ c$ is not differentiable at 0 with derivative $d_r f(\bar{x},u)$. □

Corollary 2.27. *Let X, Y be normed spaces, let T, W be open subsets of $\mathbb{R}$ and X respectively, let $c : T \to X$ be differentiable at $\bar{t} \in T$ and let $f : W \to Y$ be Hadamard differentiable at $\bar{x} \in W$ and such that $c(T) \subset W$, $\bar{x} = c(\bar{t})$. Then $f \circ c$ is differentiable at $\bar{t}$ and*

$$(f \circ c)'(\bar{t}) = Df(\bar{x})(c'(\bar{t})).$$

Thus, $Df(\bar{x})$ appears as the continuous linear map transforming velocities.

It is easy to show that every linear combination of maps having radial (resp. directional) derivatives at $\bar{x}$ in some direction u has a radial (resp. directional) derivative at $\bar{x}$ in the direction u. In particular, every linear combination of two Gâteaux (resp. Hadamard) differentiable maps is Gâteaux (resp. Hadamard) differentiable. One also deduces from Proposition 2.3 that if f has a directional (resp. radial) derivative at $\bar{x}$ in the direction u and if $A : Y \to Z$ is a continuous linear map, then $A \circ f$ has a directional (resp. radial) derivative at $\bar{x}$ in the direction u and $(A \circ f)'(\bar{x},u) = A(f'(\bar{x},u))$.

The preceding example–exercise shows that the composition of two radially differentiable maps is not necessarily radially differentiable. However, one does have a chain rule for directionally differentiable maps. These facts show that Hadamard differentiability is a more interesting property than Gâteaux differentiability.

Theorem 2.28. *Let X, Y, Z be normed spaces, let U and V be open subsets of X and Y respectively, and let $f : U \to Y$, $g : V \to Z$ be directionally differentiable maps at $\overline{x} \in W := f^{-1}(V)$ and $\overline{y} := f(\overline{x}) \in V$ respectively. Then $h := g \circ f$ is directionally differentiable at $\overline{x}$ and*

$$d\,(g \circ f)\,(\overline{x}, u) = dg(f(\overline{x}), df(\overline{x}, u)).$$

In particular, if f is Hadamard differentiable at $\overline{x}$ and g is Hadamard differentiable at $\overline{y} := f(\overline{x})$, then $h := g \circ f$ is Hadamard differentiable at $\overline{x}$ and

$$D(g \circ f)(\overline{x}) = Dg(\overline{y}) \circ Df(\overline{x}).$$

Proof. More generally, let us show that if f has a directional derivative at $\overline{x}$ in the direction $u \in X$ and if g has a directional derivative at $f(\overline{x})$ in the direction $v := df(\overline{x}, u)$, then $h := g \circ f$ has a directional derivative at $\overline{x}$ in the direction u. For (t, u') close enough to $(0, u)$ one has $\overline{x} + tu' \in W$. Let $q(t, u') := (1/t)(f(\overline{x} + tu') - f(\overline{x}))$. Then $q(t, u') \to v := df(\overline{x}, u)$ as $(t, u') \to (0_+, u)$. Therefore

$$\frac{h(\overline{x} + tu') - h(\overline{x})}{t} = \frac{g(\overline{y} + tq(t, u')) - g(\overline{y})}{t} \to dg(\overline{y}, v) \text{ as } (t, u') \to (0_+, u).$$

The statement can also be proved using Proposition 2.26. □

The notion of radial differentiability is sufficient to get a mean value theorem. Recall that the *segment* $[a, b]$ (respectively (a, b)) with endpoints a, b in a normed space is the set $\{(1 - t)a + tb : t \in [0, 1]\}$ (respectively $\{(1 - t)a + tb : t \in (0, 1)\}$).

Proposition 2.29. *If $f : W \to Y$ is radially differentiable at each point of a segment $[w, x]$ contained in W, then*

$$\|f(x) - f(w)\| \le \sup_{t \in (0,1)} \|d_r f(w + t(x - w), x - w)\|.$$

Proof. Let $h : [0, 1] \to Y$ be given by $h(t) := f((1-t)w + tx)$; it is right differentiable on $(0, 1)$, with right derivative $h'_+(t) = d_r f((1 - t)w + tx, x - w)$, and continuous on $[0, 1]$. Corollary 2.9 then yields the estimate. □

A variant can be deduced when f is Gâteaux differentiable at each point of $S := (a, b)$, since then one has $\|d_r f(z, x - w)\| \le \|D_r f(z)\| \,.\, \|x - w\|$ for all $z \in S$, $w, x \in X$.

Proposition 2.30. *Let X and Y be normed spaces, let W be an open subset of X containing the segment $[w, x]$, and let $f : W \to Y$ be continuous on $[w, x]$ and Gâteaux* *differentiable at each point of* $S := (w,x)$, *with* $m := \sup_{z\in S} \|D_r f(z)\| < +\infty$. *Then one has*

$$\|f(x) - f(w)\| \leq m\,\|x-w\|\,.$$

Corollary 2.31. *Let* X *and* Y *be normed spaces, let* W *be a convex open subset of* X, *and let* $f : W \to Y$ *be Gâteaux differentiable at each point of* W *and such that for some* $c \in \mathbb{R}$ *one has* $\|D_r f(w)\| \leq c$ *for every* $w \in W$. *Then* f *is Lipschitzian with rate* c: *for all* $x, x' \in W$ *one has*

$$\left\|f(x) - f(x')\right\| \leq c\left\|x - x'\right\|.$$

In particular, if $D_r f(w) = 0$ for every $w \in W$, then f is constant on W. Such a result is also valid if W is connected instead of convex. An extension of the estimate of Proposition 2.30 is also valid in the case that W is connected, provided one replaces the usual distance with the geodesic distance d_W in W defined as in Exercise 5.

In the usual case in which $X_0 = X$, the following corollary gives an approximation of f in the case that one has an approximate value of the derivative of f around $\overline{x}$.

Corollary 2.32. *Let* X *and* Y *be normed spaces, let* X_0 *be a linear subspace of* X, *let* W *be a convex open subset of* X, *and let* $f : W \to Y$ *be Gâteaux differentiable at each point of* W *and such that for some* $c \in \mathbb{R}$ *and some* $\ell \in L(X_0, Y)$ *one has* $\|D_r f(x)(u) - \ell(u)\| \leq c\,\|u\|$ *for every* $x \in W$, $u \in X_0$. *Then for every* $x, x' \in W$ *such that* $x - x' \in X_0$, *one has*

$$\left\|f(x) - f(x') - \ell(x - x')\right\| \leq c\left\|x - x'\right\|.$$

This result (obtained by changing f into $f - \ell$ in the preceding corollary) will serve to get Fréchet differentiability from Gâteaux differentiability. For the moment, let us point out another passage from Gâteaux differentiability to Hadamard differentiability.

Proposition 2.33. *Let* W *be an open subset of* X. *If* $f : W \to Y$ *is radially differentiable on a neighborhood* V *of* $\overline{x}$ *in* W *and if for some* $u \in X \setminus \{0\}$, *its radial derivative* $d_r f : V \times X \to Y$ *is continuous at* $(\overline{x}, u)$, *then* f *is directionally differentiable at* $\overline{x}$ *in the direction* u.

In particular, if f *is Gâteaux differentiable on* V *and if* $d_r f : V \times X \to Y$ *is continuous at each point of* $\{\overline{x}\} \times X$, *then* f *is Hadamard differentiable at* $\overline{x}$.

Proof. Without loss of generality, we may suppose u has norm 1. Given $\varepsilon > 0$, let $\delta \in (0,1)$ be such that $\|f'_r(x,v) - f'_r(\overline{x},u)\| \leq \varepsilon$ for all $(x,v) \in B(\overline{x}, 2\delta) \times B(u,\delta)$, with $B(\overline{x}, 2\delta) \subset V$. Setting $r(t,v) := f(\overline{x} + tv) - f(\overline{x}) - tf'_r(\overline{x},u)$, we observe that for every $v \in B(u,\delta)$ the map $r_v := r(\cdot, v)$ is differentiable on $[0,\delta]$ and $\|r'_v(t)\| = \|f'_r(\overline{x} + tv, v) - f'_r(\overline{x},u)\| \leq \varepsilon$. Since $r_v(0) = 0$, Corollary 2.9 yields $\|r(t,v)\| \leq \varepsilon t$. That shows that f has $f'_r(\overline{x},u)$ as a directional derivative at $\overline{x}$ in the direction u. The last assertion is an immediate consequence. □

The importance of this continuity condition leads us to introduce a definition.

Definition 2.34. Given normed spaces X,Y and an open subset W of X, a map $f : W \to Y$ is said to be of *class* D^1 at $\overline{w}$ (resp. on W) if it is Hadamard differentiable around $\overline{w}$ (resp. on W) and if $df : W \times X \to Y$ is continuous at $(\overline{w}, v)$ for all $v \in X$ (resp. on $W \times X$). We say that f is of class D^k with $k \in \mathbb{N}$, $k > 1$, if f is of class D^1 and if df is of class D^{k-1}.

We denote by $D^1(W,Y)$ the space of maps of class D^1 from W to Y and by $BD^1(W,Y)$ the space of maps $f \in D^1(W,Y)$ that are bounded and such that f' is bounded from W to $L(X,Y)$. Let us note the following two properties.

Proposition 2.35. *For every $f \in D^1(W,Y)$ the map $f' : w \mapsto Df(w) := df(w,\cdot)$ is locally bounded.*

Proof. Suppose, to the contrary, that there exist $w \in W$ and a sequence $(w_n) \to w$ such that $(r_n) := (\|Df(w_n)\|) \to +\infty$. For each $n \in \mathbb{N}$ one can pick some unit vector $u_n \in X$ such that $\|df(w_n,u_n)\| > r_n - 1$. Setting (for $n \in \mathbb{N}$ large) $x_n := r_n^{-1} u_n$, we see that $((w_n,x_n)) \to (w,0)$ but $(\|df(w_n,x_n)\|) \to 1$, a contradiction. □

Corollary 2.36. *Let $f : W \to Y$ be a Hadamard (or Gâteaux) differentiable function. Then f is of class D^1 if and only if f' is locally bounded and for all $u \in X$ the map $x \mapsto f'(x)u$ is continuous. In particular, if $Y = \mathbb{R}$ and if $f \in D^1(W,\mathbb{R})$, the derivative is continuous when X^* is provided with the topology of uniform convergence on compact sets (the bw* topology).*

Proof. The necessary condition stems from the preceding proposition. The sufficient condition follows from the inequalities

$$\left\|f'(w)v - f'(x)u\right\| \le \left\|f'(w)(v-u)\right\| + \left\|f'(w)u - f'(x)u\right\| \le m\varepsilon/(2m) + \varepsilon/2 = \varepsilon,$$

when for some $m > 0$ and a given $\varepsilon > 0$ one can find a neighborhood V of x in W such that $\|f'(w)\| \le m$ for $w \in V$ and $\|f'(w)u - f'(x)u\| \le \varepsilon/2$ for $w \in V$, $w \in B(u,\varepsilon/2m)$. □

Proposition 2.37. *If X,Y,Z are normed spaces, if U and V are open subsets of X and Y respectively, and if $f \in D^1(U,Y)$, $g \in D^1(V,Z)$, then $h := g \circ f \in D^1(W,Z)$ for $W := f^{-1}(V)$.*

Proof. This conclusion is an immediate consequence of the formula $dh(w,x) = dg(f(w),df(w,x))$ for all $(w,x) \in W \times X$. □

Under a differentiability assumption, convex functions, integral functionals, and Nemitskii operators are important examples of maps of class D^1.

Example (Nemitskii operators). Let $(S,\mathscr{F},\mu)$ be a measure space, let X, Y be Banach spaces, let $f : S \times X \to Y$ be a measurable map of class D^1 in its second variable and such that $g : (s,x,v) \mapsto df_s(x,v)$ is measurable, f_s being the map $x \mapsto f(s,x)$. Then, if for $p,q \in [1,+\infty)$, the Nemitskii operator $F : L_p(S,X) \to$

$L_q(S,Y)$ given by $F(u) := f(\cdot, u(\cdot))$ for $u \in L_p(S,X)$ is well defined and Gâteaux differentiable, with derivative given by $D_r F(u)(v) = df(\cdot, u(\cdot), v(\cdot))$, then F is of class D^1. This follows from the following result applied to $g := df(\cdot, u(\cdot), v(\cdot))$ (see [37]).

Lemma 2.38 (Krasnoselskii's theorem). *Let $(S, \mathscr{F}, \mu)$ be a measure space, let W, Z be Banach spaces, and let $g : S \times W \to Z$ be a measurable map such that for all $s \in S \setminus N$, where N has null measure, the map $g(s, \cdot)$ is continuous. If for some $p, q \in [1, +\infty)$ and all $u \in L_p(S,W)$ the map $g(\cdot, u(\cdot))$ belongs to $L_q(S,Z)$, then the Nemitskii operator $G : L_p(S,W) \to L_q(S,Z)$ given by $G(u) := g(\cdot, u(\cdot))$ for $u \in L_p(S,X)$ is continuous.*

Exercises

1. Let X, Y be normed spaces and let W be an open subset of X. Prove that $f : W \to Y$ is Hadamard differentiable at $\overline{x}$ if and only if there exists a continuous linear map $\ell : X \to Y$ such that the map q_t given by $q_t(v) := (1/t)(f(\overline{x} + tv) - f(\overline{x}))$ converges to ℓ as $t \to 0_+$, uniformly on compact subsets of X. Deduce another proof of Proposition 2.50 below from this characterization.

2. Prove that if $f : W \to Y$ is radially differentiable at $\overline{x}$ in the direction u and if f is *directionally steady* at $\overline{x}$ in the direction u in the sense that $(1/t)(f(\overline{x} + tv) - f(\overline{x} + tu)) \to 0$ as $(t, v) \to (0_+, u)$, then f is directionally differentiable at $\overline{x}$ in the direction u. Give an example showing that this criterion is more general than the Lipschitz condition of Proposition 2.25.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(r,s) := r^2 s (r^2 + s^2)^{-1}$ for $(r,s) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $f(0,0) = 0$. Show that f has a radial derivative (which is in fact a bilateral derivative) but is not Gâteaux differentiable at $(0,0)$.

4. Let E be a Hilbert space and let $X := D^1(T, E)$, where $T := [0,1]$. Endow X with the norm $\|x\| := \sup_{t \in T} \|x(t)\| + \sup_{t \in T} \|x'(t)\|$. Define the length of a curve $x : [0,1] \to E$ by

$$\ell(x) := \int_0^1 \|x'(t)\| \mathrm{d}t.$$

- **(a)** Show that ℓ is a continuous sublinear functional on X with Lipschitz rate 1.
- **(b)** Let W be the set of $x \in X$ such that $x'(t) \neq 0$ for all $t \in [0,1]$. Show that W is an open subset of X and that ℓ is Gâteaux differentiable on W.
- **(c)** Show that ℓ is of class D^1 on W [Hint: Use convergence results for integrals.] In order to prove that ℓ is of class C^1 one may use the following questions.
- **(d)** Let $E_0 := E \setminus \{0\}$ and let $D : E_0 \to E$ be given by $D(v) := \|v\|^{-1} v$. Given $u, v \in E_0$ show that $\|D(u) - D(v)\| \leq 2\|u\|^{-1} \|u - v\|$.
- **(e)** Deduce from the preceding inequality that ℓ' is continuous.

5. Prove the assertion following Corollary 2.31, defining the geodesic distance $d_W(x,x')$ between two points x,x' of W as the infimum of the lengths of curves joining x to x'.

6. Prove that if $f : W \to Y$ has a directional derivative at some point $\overline{x}$ of the open subset W of X, then its derivative $Df(\overline{x}) : u \mapsto df(\overline{x},u)$ is continuous if it is linear.

7. Prove Proposition 2.29 by deducing it from the classical mean value theorem (Lemma 2.6) for real-valued functions, using the Hahn–Banach theorem. [Hint: Take y^* with norm one such that $\langle y^*,y\rangle = \|y\|$ for $y := f(x) - f(w)$, set $g(t) := \langle y^*, f(x+t(w-x))\rangle$, and pick $\theta \in (0,1)$ such that $g(1) - g(0) = g'_+(\theta)$.]

8. Show that the norm $x \mapsto \|x\| := \sup_{t\in T} |x(t)|$ on the Banach space $X := C(T)$ of continuous functions on $T := [0,1]$ is Hadamard differentiable at $\overline{x} \in X$ if and only if the function $t \mapsto |x(t)|$ attains its maximum on T at a single point.

9. **(a)** Let a,b be two points of a normed space X. Show that the function g given by $g(t) := \|a+tb\|$ has a right derivative and a left derivative at all points of $\mathbb{R}$.

(b) Let $f : T \to X$, where T is an interval of $\mathbb{R}$. Show that if f has a right derivative $f'_+(t)$ at some $t \in T$, then $g \circ f$ has a right derivative at t and $(g\circ f)'_+(t) \leq \|f'_+(t)\|$.

10. Use the preceding exercise to deduce a mean value theorem from Lemma 2.6.

2.4 Fréchet Differential Calculus

Nonlinear maps are difficult to study. The main purpose of differential calculus consists in getting some information using an affine approximation to a given nonlinear map around a given point. Of course, the meaning of the word "approximation" has to be made precise. For that purpose, we define remainders. Fréchet differentiability consists in an approximation by a continuous affine map.

Definition 2.39. Given normed spaces X and Y, we denote by $o(X,Y)$ the set of maps $r : X \to Y$ such that $r(x)/\|x\| \to 0$ as $x \to 0$ in $X \setminus \{0\}$. The elements of $o(X,Y)$ will be called *remainders*.

Thus, $r : X \to Y$ is a remainder if and only if there exists some map $\alpha : X \to Y$ satisfying $\alpha(x) \to 0$ as $x \to 0$ and $r(x) = \|x\|\,\alpha(x)$. Moreover, $r \in o(X,Y)$ if and only if there exists a modulus $\mu : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ such that $\|r(x)\| \leq \mu(\|x\|)\,\|x\|$ (recall that $\mu : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is a *modulus* when μ is nondecreasing, $\mu(0) = 0$, and μ is continuous at 0). Such a case occurs when there exist $c > 0$ and $p > 1$ such that $\|r(x)\| \leq c\,\|x\|^p$. Following Landau, remainders are often denoted by $o(\cdot)$, and different remainders are often denoted by the same letters, since they are considered as inessential for the assigned purposes.

If $r,s : X \to Y$ are two maps that coincide on some neighborhood V of 0 in X, then s belongs to $o(X,Y)$ if and only if r belongs to $o(X,Y)$. Thus if $p : V \to Y$ is defined on some neighborhood V of 0 in X, we consider that p is a remainder if some extension r of p to all of X is a remainder. The preceding observation shows that this property does not depend on the choice of the extension.

The following result is a direct consequence of the rules for limits.

Lemma 2.40. *For every pair of normed spaces X,Y, the set $o(X,Y)$ of remainders is a linear space.*

The class of remainders is stable under composition by continuous linear maps.

Lemma 2.41. *For all normed spaces W, X, Y, Z, for every $r \in o(X,Y)$ and all continuous linear maps $A : W \to X$, $B : Y \to Z$ one has $r \circ A \in o(W,Y)$ and $B \circ r \in o(X,Z)$ (hence $B \circ r \circ A \in o(W,Z)$).*

Proof. Let $\alpha : X \to Y$ be such that $\alpha(x) \to 0$ as $x \to 0$ and $r(x) = \|x\| \alpha(x)$. Then if $A : W \to X$ is *stable* at 0, i.e., is such that there exists some $c > 0$ for which $\|A(w)\| \leq c\|w\|$ for w in a neighborhood of 0 in W, in particular if A is linear and continuous, then one has $\|r(A(w))\| = \|A(w)\| \|\alpha(A(w))\| \leq c\|w\| \|\alpha(A(w))\|$ and $\alpha(A(w)) \to 0$ as $w \to 0$, so that $r \circ A \in o(W,Y)$. Similarly, if $B : Y \to Z$ is stable at 0, then $B \circ r \in o(X,Z)$. The assertion about $B \circ r \circ A$ is a combination of the two other cases. □

The proof of the next lemma is an easy consequence of the rules for limits.

Lemma 2.42. *Given normed spaces X, $Y_1, \ldots, Y_k$, $Y := Y_1 \times \cdots \times Y_k$, a map $r : X \to Y$ is a remainder if and only if its components $r_1, \ldots, r_k$ are remainders.*

We are ready to define differentiability in the Fréchet sense; this notion is so usual that one often writes "differentiable" instead of "Fréchet differentiable."

Definition 2.43. Given normed spaces X, Y and an open subset W of X, a map $f : W \to Y$ is said to be *(Fréchet) differentiable* (or firmly differentiable, or just differentiable) at $\overline{x} \in W$ if there exist a continuous linear map $\ell : X \to Y$ and a remainder $r \in o(X,Y)$ such that for $x \in W$ one has

$$f(x) = f(\overline{x}) + \ell(x - \overline{x}) + r(x - \overline{x}). \tag{2.6}$$

It is often convenient to write the preceding relation in the form

$$f(\overline{x} + u) - f(\overline{x}) = \ell(u) + r(u)$$

for u close to 0. Here the continuous affine map $x \mapsto f(\overline{x}) + \ell(x - \overline{x})$ can be viewed as an approximation of f that essentially determines the behavior of f around $\overline{x}$. The continuous linear map ℓ is called the *derivative* of f at $\overline{x}$ and is denoted by $Df(\overline{x})$ or $f'(\overline{x})$. It is unique: given two approximations ℓ_1, ℓ_2 of $f(\overline{x} + \cdot)$ around 0 and two remainders r_1, r_2 such that $f(\overline{x} + u) - f(\overline{x}) = \ell_1(u) + r_1(u) = \ell_2(u) + r_2(u)$, one has

$\ell_1 = \ell_2$, since $\ell := \ell_1 - \ell_2$ is the remainder $r := r_2 - r_1$; in fact, for every $u \in X$ and every $t > 0$ small enough, one has

$$\ell(u) = \frac{1}{t} r(tu) = \frac{1}{t} \alpha(tu) \|tu\| = \alpha(tu) \|u\| \to 0 \text{ as } t \to 0,$$

so that $\ell(u) = 0$. Thus $L(X,Y) \cap o(X,Y) = \{0\}$. Uniqueness is also a consequence of Corollary 2.50 below and of the fact that the directional derivative is unique, since it is obtained as a limit.

When $Y := \mathbb{R}$, the derivative $Df(\overline{x})$ of f at $\overline{x}$ belongs to the dual X^* of X. When X is a Hilbert space with scalar product $(\cdot \mid \cdot)$ it may be convenient to use the *Riesz isometry* $R : X \to X^*$ given by $\langle R(x), y \rangle = (x \mid y)$ to get an element $\nabla f(\overline{x})$ of X, called the *gradient* of f at x, by setting $\nabla f(\overline{x}) := R^{-1}(Df(\overline{x}))$. It allows one to visualize the derivative, but in some respects, it is preferable to work with the derivative.

Proposition 2.44. *If $f : W \to Y$ is differentiable at $\overline{x} \in W$, then it is continuous at $\overline{x}$.*

Proof. This follows from the fact that every remainder is continuous at 0. □

Proposition 2.45. *If $f, g : W \to Y$ are differentiable at $\overline{x} \in W$, then for every $\lambda, \mu \in \mathbb{R}$ the map $h := \lambda f + \mu g$ is differentiable at $\overline{x}$ and $Dh(\overline{x}) = \lambda Df(\overline{x}) + \mu Dg(\overline{x})$.*

Proof. If $r(x) := f(\overline{x}+x) - f(\overline{x}) - f'(\overline{x})(x)$, $s(x) := g(\overline{x}+x) - g(\overline{x}) - g'(\overline{x})x$, one has $h(\overline{x}+x) = h(\overline{x}) + \lambda f'(\overline{x})(x) + \mu g'(\overline{x})(x) + t(x)$, where $t := \lambda r + \mu s \in o(X,Y)$. Thus h is differentiable at $\overline{x}$ and $h'(\overline{x}) = \lambda f'(\overline{x}) + \mu g'(\overline{x})$. □

Examples. (a) A constant map is everywhere differentiable and its derivative is 0.

(b) A continuous linear map $\ell \in L(X,Y)$ is differentiable at every point $\overline{x}$ and its derivative at $\overline{x}$ is ℓ since $\ell(\overline{x}+x) = \ell(\overline{x}) + \ell(x)$.

(c) A continuous affine map $f := \ell + c$, where $\ell \in L(X,Y)$ and $c \in Y$, is differentiable at every $\overline{x} \in X$ and $Df(\overline{x}) = \ell$.

(d) If $f : X := X_1 \times X_2 \to Y$ is a continuous bilinear map, then f is differentiable at every point $\overline{x} := (\overline{x}_1, \overline{x}_2) \in X$, and for $x = (x_1, x_2)$, one has $Df(\overline{x})(x) = f(x_1, \overline{x}_2) + f(\overline{x}_1, x_2)$, since $f(\overline{x}+x) - f(\overline{x}) = f(x_1, \overline{x}_2) + f(\overline{x}_1, x_2) + f(x_1, x_2)$. Here f is a remainder since $\|f(x)\| \le \|f\| \|x_1\| \|x_2\| \le \|f\| \|x\|^2$ whenever $\|x\| \ge \|x\|_\infty := \max(\|x_1\|, \|x_2\|)$.

(e) If $f : X \to Y$ is a continuous *quadratic map* in the sense that there exists a continuous bilinear map $b : X \times X \to Y$ such that $f(x) = b(x,x)$, then f is differentiable at every point $\overline{x} \in X$ and $Df(\overline{x})(x) = b(\overline{x}, x) + b(x, \overline{x})$ for $x \in X$. This follows from the chain rule below and the preceding example. Alternatively, one may observe that $r := f$ is a remainder, since for every $x \in X$ one has $\|f(x)\| \le \|b\| \|x\|^2$ and $f(\overline{x}+x) = f(\overline{x}) + b(\overline{x}, x) + b(x, \overline{x}) + f(x)$.

(f) If $f : T \to Y$ is defined on an open interval T of $\mathbb{R}$, then f is differentiable at $\overline{x} \in T$ if and only if f has a derivative at $\overline{x}$ and $Df(\overline{x})$ is the linear map $r \mapsto rf'(\overline{x})$, whence $f'(\overline{x}) = Df(\overline{x})(1)$. The key point in this example is illuminated in the following exercise. □

Exercise. Show that for every normed space Y the space $L(\mathbb{R},Y)$ is isomorphic (and even isometric) to Y via the evaluation map $\ell \mapsto \ell(1)$, whose inverse is the map $v \mapsto \ell_v$, where $\ell_v \in L(\mathbb{R},Y)$ is defined by $\ell_v(r) := rv$ for $r \in \mathbb{R}$. □

The following characterization will be helpful.

Lemma 2.46. *Given an open subset W of X, a map $f : W \to Y$ is differentiable at $\overline{x}$ if and only if there exists a map $F : W \to L(X,Y)$ that is continuous at $\overline{x}$ and such that $f(x) - f(\overline{x}) = F(x)(x - \overline{x})$ for all $x \in W$.*

Proof. Suppose there is a map $F : W \to L(X,Y)$ continuous at $\overline{x}$ such that $f(x) = f(\overline{x}) + F(x)(x - \overline{x})$ for all $x \in W$. Then $f(x) - f(\overline{x}) = F(\overline{x})(x - \overline{x}) + r(x)$, where r is the remainder defined by $r(x) := (F(\overline{x}+x) - F(\overline{x}))(x)$, so that f is differentiable at $\overline{x}$ and $Df(\overline{x}) = F(\overline{x})$. To prove the converse, using the Hahn–Banach theorem, for $x \in W$ we pick $\ell_x \in X^*$ such that $\|\ell_x\| = 1$ and $\ell_x(x) = \|x\|$. Then, setting $A := Df(\overline{x})$ and writing the remainder r appearing in (2.6) in the form $r(u) = \alpha(u)\|u\| = \alpha(u)\ell_u(u)$ with $\alpha(u) \to 0$ as $u \to 0$, we get

$$f(\overline{x}+u) - f(\overline{x}) = (A + \alpha(u)\ell_u)(u),$$

or $f(x) - f(\overline{x}) = F(x)(x - \overline{x})$ for $F(x) := A + \alpha(x - \overline{x})\ell_{x-\overline{x}} \to A = F(\overline{x})$ as $x \to \overline{x}$. □

Let us give a chain rule. It is a cornerstone of differential calculus.

Theorem 2.47 (Chain rule). *Let X, Y, Z be normed spaces, let U, V be open subsets of X and Y respectively, and let $f : U \to Y$, $g : V \to Z$ be differentiable at $\overline{x} \in U$ and $\overline{y} = f(\overline{x})$ respectively and be such that $f(U) \subset V$. Then $h := g \circ f$ is differentiable at $\overline{x}$ and*

$$Dh(\overline{x}) = Dg(\overline{y}) \circ Df(\overline{x}). \tag{2.7}$$

Proof. Let $\ell := Df(\overline{x})$, $m := Dg(\overline{y})$ and let $r \in o(X,Y)$, $s \in o(Y,Z)$ be defined by

$$r(x) := f(\overline{x}+x) - f(\overline{x}) - \ell(x), \qquad s(y) := g(\overline{y}+y) - g(\overline{y}) - m(y).$$

Then, setting $y := \ell(x) + r(x)$ for $x \in U - \overline{x}$, so that $f(\overline{x}+x) = \overline{y} + y$, we get

$$h(\overline{x}+x) - h(\overline{x}) - m(\ell(x)) = g(\overline{y}+y) - g(\overline{y}) - m(y - r(x)) = s(y) + m(r(x)). \tag{2.8}$$

Lemma 2.41 ensures that $m \circ r \in o(X,Z)$. Now, given $c > \|\ell\|$, there exists some $\rho > 0$ such that for $x \in B(0,\rho)$ one has $\|r(x)\| \le (c - \|\ell\|)\|x\|$ and hence $\|\ell(x) + r(x)\| \le c\|x\|$. Then the proof of Lemma 2.41 ensures that $s \circ (\ell + r) \in o(X,Z)$. Thus, the right-hand side $s \circ (\ell + r) + m \circ r$ of (2.8) is a remainder, and we conclude that h is differentiable at $\overline{x}$ with derivative the continuous linear map $m \circ \ell$. □

The following corollary is a consequence of the fact that the derivative of a continuous linear map ℓ at an arbitrary point is ℓ itself.

Corollary 2.48. *Let X,Y,Z be normed spaces, let U,V be open subsets of X and Y respectively, and let $f : U \to Y$, $g : V \to Z$ be such that $f(U) \subset V$ and let $h := g \circ f$.*

(a) If f is differentiable at $\overline{x}$ and $V := Y$, $g \in L(Y,Z)$, then h is differentiable at $\overline{x}$ and $Dh(\overline{x}) = g \circ Df(\overline{x})$.
(b) If g is differentiable at $\overline{y} := f(\overline{x})$ and $U := X$, $f \in L(X,Y)$, then h is differentiable at $\overline{x}$ and $Dh(\overline{x}) = Dg(\overline{y}) \circ f$.

Corollary 2.49. *The differentiability of $f : W \to Y$ (with W open in X) at $\overline{x}$ does not depend on the choices of the norms on X and Y within their equivalences classes.*

In fact, changing the norm amounts to composing with a continuous linear map.

Corollary 2.50. *Let X,Y be normed spaces, let W be an open subset of X, and let $f : W \to Y$. If f is Fréchet differentiable at $\overline{x} \in W$, then f is Hadamard differentiable at $\overline{x}$. If X is finite-dimensional, the converse holds.*

Thus, the mean value theorems of Sect. 2.1.2 are in force for Fréchet differentiability. Also, the interpretation of the derivative as a rule for the transformation of velocities remains valid for the Fréchet derivative.

Proof. The first assertion follows from the definitions or from Theorem 2.47 and Proposition 2.26.

Assuming that X is finite-dimensional, let us prove that if f is directionally differentiable at $\overline{x}$, and if its directional derivative $f'(\overline{x},\cdot)$ is continuous, then r given by

$$r(w) := f(\overline{x}+w) - f(\overline{x}) - f'(\overline{x},w)$$

is a remainder. Adding the assumption that $f'(\overline{x},\cdot)$ is linear will prove the converse assertion. Suppose, to the contrary, that there exist $\varepsilon > 0$ and a sequence $(w_n) \to 0$ such that for all $n \in \mathbb{N}$, $\|r(w_n)\| > \varepsilon \|w_n\|$. Then $t_n := \|w_n\|$ is positive; setting $u_n := t_n^{-1} w_n$, we may suppose the sequence (u_n) converges to some vector u of the unit sphere of X. Then, given $\varepsilon' \in (0,\varepsilon)$, we can find $k \in \mathbb{N}$ such that for $n \geq k$ we have $\|f'(\overline{x},u_n) - f'(\overline{x},u)\| \leq \varepsilon - \varepsilon'$, so that

$$\left\|f(\overline{x}+t_n u_n) - f(\overline{x}) - t_n f'(\overline{x},u)\right\| > \varepsilon t_n \|u_n\| - t_n \left\|f'(\overline{x},u_n) - f'(\overline{x},u)\right\| \geq \varepsilon' t_n,$$

a contradiction to the assumption that f is differentiable at $\overline{x}$ in the direction u. □

Another link between directional differentiability and firm differentiability is pointed out in the next statement. A direct proof using Corollary 2.32 is easy. We present a proof in the case that f' is continuous around $\overline{x}$.

Proposition 2.51. *If f is Gâteaux differentiable on W and if $f' : W \to L(X,Y)$ is continuous at $\overline{x} \in W$, then f is Fréchet differentiable at $\overline{x}$.*

Proof. Without loss of generality, replacing Y by its completion, we may suppose Y is complete; replacing W by a ball centered at $\overline{x}$, we may also suppose W is convex. Then for $x \in W$ one has $f(x) - f(\overline{x}) = F(x)(x - \overline{x})$ with

$$F(x) := \int_0^1 Df(\overline{x}+t(x-\overline{x}))\mathrm{d}t,$$

and F is continuous at $\overline{x}$, so that the criteria of Lemma 2.46 apply. □

This result shows that it may be a sensible strategy to start with radial differentiability in order to prove that a map is of *class* C^1, i.e., that it is differentiable with a continuous derivative. For instance, if one deals with an integral functional

$$f(x) := \int_S F(s,x(s))\mathrm{d}s,$$

where S is some measure space and x belongs to some space of measurable maps, it is advisable to use Lebesgue's theorem to differentiate inside the integral (under appropriate assumptions) by taking the limit in the quotient

$$\frac{1}{t}[f(\overline{x}+tu)-f(\overline{x})] = \int_S \frac{1}{t}[F(s,\overline{x}(s)+tu(s))-F(s,\overline{x}(s))]\mathrm{d}s.$$

Continuity arguments may be invoked later, for instance using Krasnoselskii's criterion.

Let us note other consequences of Theorem 2.47.

Proposition 2.52. *Let X, $Y_1,\ldots,Y_n$ be normed spaces, let W be an open subset of X, and let $f := (f_1,\ldots,f_n) : W \to Y := Y_1 \times \cdots \times Y_n$. Then f is differentiable at $\overline{x} \in W$ if and only if its components $f_i : W \to Y_i$ ($i = 1,\ldots,n$) are differentiable at $\overline{x}$ and for $v \in X$,*

$$Df(\overline{x})(v) = (Df_1(\overline{x})(v),\ldots,Df_n(\overline{x})(v))\,.$$

Proof. Let $p_i : Y \to Y_i$ denote the ith canonical projection. If f is differentiable at $\overline{x}$, then Corollary 2.48 ensures that $f_i := p_i \circ f$ is differentiable at $\overline{x}$ and $Df_i(\overline{x}) = p_i \circ Df(\overline{x})$. Conversely, suppose that $f_1,\ldots,f_n$ are differentiable at $\overline{x}$. Let $r_i \in o(X,Y_i)$ be given by $r_i(x) = f_i(\overline{x}+x) - f_i(\overline{x}) - Df_i(\overline{x})(x)$. Then by Lemma 2.42, we have that $r := (r_1,\ldots,r_n) \in o(X,Y)$ and $r(x) = f(\overline{x}+x) - f(\overline{x}) - \ell(x)$ for $\ell \in L(X,Y)$ given by $\ell(x) := (Df_1(\overline{x})(x),\ldots,Df_n(\overline{x})(x))$. Thus f is differentiable at $\overline{x}$, with derivative ℓ. □

Now, let us consider the case in which the source space X is a product $X_1 \times \cdots \times X_n$ and W is an open subset of X. One says that $f : W \to Y$ has a *partial derivative at* $\overline{x} \in W$ relative to X_i for some $i \in \mathbb{N}_n$ if the map $f_{i,\overline{x}} : x_i \mapsto f(\overline{x}_1,\ldots,\overline{x}_{i-1},x_i,\overline{x}_{i+1},\ldots,\overline{x}_n)$ is differentiable at $\overline{x}_i$. Then one denotes by $D_i f(\overline{x})$ or $\frac{\partial f}{\partial x_i}(\overline{x})$ the derivative of the map $f_{i,\overline{x}}$ at $\overline{x}_i$. Let $j_i \in L(X_i,X)$ be the insertion given by $j_i(x_i) := (0,\ldots,0,x_i,0,\ldots,0)$. Since the map $f_{i,\overline{x}}$ is just the composition of the affine map $x_i \mapsto j_i(x_i-\overline{x}_i)+\overline{x} = (\overline{x}_1,\ldots,\overline{x}_{i-1},x_i,\overline{x}_{i+1},\ldots,\overline{x}_n)$ with f, from Corollary 2.48 (b) and the fact that $v = j_1(v_1)+\cdots+j_n(v_n)$, while $D_i f(\overline{x}) = Df_{i,\overline{x}}(\overline{x}_i) = Df(\overline{x}) \circ j_i$, one gets the following proposition.

Proposition 2.53. *If $f : W \to Y$ is defined on an open subset W of a product space $X := X_1 \times \cdots \times X_k$ and if f is differentiable at $\overline{x}$, then for $i = 1, \ldots, k$, the map f has a partial derivative at $\overline{x}$ relative to X_i and*

$$\forall v := (v_1, \ldots, v_k), \qquad Df(\overline{x})(v) = D_1 f(\overline{x})v_1 + \cdots + D_k f(\overline{x})v_k.$$

When $X := \mathbb{R}^m$, $Y := \mathbb{R}^n$, the matrix $(D_i f_j(\overline{x}))$ of $Df(\overline{x})$ formed with the partial derivatives of the components $(f_j)_{1 \leq j \leq n}$ of f is called the *Jacobian matrix* of f at $\overline{x}$. It determines $Df(\overline{x})$.

Note that it may happen that f has partial derivatives at $\overline{x}$ with respect to all its variables but is not differentiable at $\overline{x}$.

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(r,s) := rs(r^2 + s^2)^{-1}$ for $(r,s) \neq (0,0)$ and $f(0,0) = 0$. Since $f(r,0) = 0 = f(0,s)$, f has partial derivatives with respect to its two variables at $(0,0)$. However, f is not continuous at $(0,0)$, hence is not differentiable at $(0,0)$. □

Now let us introduce a reinforced notion of differentiability that allows us to formulate several results with assumptions weaker than continuous differentiability.

Definition 2.54. Let X and Y be normed spaces, let W be an open subset of X, and let $\overline{x} \in W$. A map $f : W \to Y$ is said to be *circa-differentiable* (or peri-differentiable, or strictly differentiable) at $\overline{x}$ if there exists some continuous linear map $\ell \in L(X,Y)$ such that for every $x, x' \in W$ one has

$$\frac{\|f(x) - f(x') - \ell(x - x')\|}{\|x - x'\|} \to 0 \text{ as } x, x' \to \overline{x} \text{ with } x' \neq x. \tag{2.9}$$

If X_0 is a linear subspace of X, we say that f is *circa-differentiable* (or strictly differentiable) *at $\overline{x}$ with respect to X_0* if there exists some continuous linear map $\ell \in L(X_0, Y)$ such that (2.9) holds whenever $x, x' \in W$ satisfy $x - x' \in X_0$.

Let us relate the preceding notion to continuous differentiability. Taking $x' = \overline{x}$ in relation (2.9), one sees that if f is circa-differentiable at $\overline{x}$, then f is differentiable at $\overline{x}$ and $Df(\overline{x}) = \ell$.

Definition 2.55. The map $f : W \to Y$ will be said to be continuously differentiable at $\overline{x} \in W$, or of *class C^1* at $\overline{x}$, and we write $f \in C^1_{\overline{x}}(W,Y)$, if f is differentiable on some neighborhood V of $\overline{x}$ and if the derivative $f' : V \to L(X,Y)$ of f given by $f'(x) := Df(x)$ for $x \in V$ is continuous at $\overline{x}$. If f is of class C^1 at each point x of W, then f is said to be of class C^1 on W and one writes $f \in C^1(W,Y)$.

One says that f is of class C^k with $k \in \mathbb{N}$, $k > 1$, if f is of class C^1 and if f' is of class C^{k-1}. Then one writes $f \in C^k(W,Y)$.

Proposition 2.56. *Let X and Y be normed spaces, let W be an open subset of X and let $\overline{x} \in W$. A map $f : W \to Y$ that is differentiable on a neighborhood $U \subset W$ of $\overline{x}$ is circa-differentiable at $\overline{x} \in W$ if and only if $f \in C^1_{\overline{x}}(W,Y)$.*

Proof. Suppose $f \in C^1_{\overline{x}}(W,Y)$ and let $\ell := Df(\overline{x})$. Given $\varepsilon > 0$ one can find $\delta > 0$ such that $B(\overline{x},\delta) \subset W$ and for $x \in B(\overline{x},\delta)$ one has $\|Df(x) - \ell\| \leq \varepsilon$. Then using Corollary 2.32, for $x, x' \in B(\overline{x},\delta)$, one has

$$\left\|f(x') - f(x) - \ell(x' - x)\right\| \leq \varepsilon \left\|x' - x\right\|,$$

so that f is circa-differentiable at $\overline{x}$.

Conversely, suppose f is circa-differentiable at $\overline{x}$ and is differentiable on a neighborhood V of $\overline{x}$ contained in W. Given $u \in X$ and $\varepsilon > 0$, assuming that the preceding inequality holds whenever $x, x' \in B(\overline{x},\delta) \subset V$, one gets for all $x \in B(\overline{x},\delta)$, $u \in X$

$$\|Df(x)(u) - \ell(u)\| = \lim_{t \to 0_+} t^{-1} \|f(x+tu) - f(x) - \ell(tu)\| \leq \varepsilon \|u\|,$$

so that $\|Df(x) - \ell\| \leq \varepsilon$ and $f' : x \mapsto Df(x)$ is continuous at $\overline{x}$. □

We are now in a position to give a converse of Proposition 2.53.

Proposition 2.57. *If $f : W \to Y$ is defined on an open subset W of a product space $X := X_1 \times \cdots \times X_k$, if for $i = 1, \ldots, k$, f has a partial derivative at $\overline{x} \in W$ relative to X_i, and if f is circa-differentiable at $\overline{x}$ with respect to $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k$, then f is differentiable at $\overline{x}$. In particular, if f has partial derivatives on some neighborhood of $\overline{x}$ all of which but one are continuous at $\overline{x}$, then f is differentiable at $\overline{x}$.*

Proof. It suffices to give the proof for $k = 2$; an induction yields the general case.

Thus, let f be circa-differentiable at $\overline{x}$ with respect to X_1 and have a partial derivative at $\overline{x}$ relative to X_2. The first assumption means that there exists some $\ell_1 \in L(X_1,Y)$ such that for every $\varepsilon > 0$ one can find some $\delta > 0$ such that $B(\overline{x},2\delta) \subset W$ and for $x := (x_1,x_2) \in B(\overline{x},\delta)$, $u_1 \in X_1$, $\|u_1\| \leq \delta$ one has

$$\|f(x_1 + u_1, x_2) - f(x_1,x_2) - \ell_1(u_1)\| \leq \varepsilon \|u_1\|. \tag{2.10}$$

Setting $\ell_2 := D_2 f(\overline{x})$ and taking a smaller $\delta > 0$ if necessary, we may suppose that

$$\|f(\overline{x}_1, \overline{x}_2 + u_2) - f(\overline{x}_1,\overline{x}_2) - \ell_2(u_2)\| \leq \varepsilon \|u_2\|$$

for every $u_2 \in X_2$ satisfying $\|u_2\| \leq \delta$. Then, taking $(x_1,x_2) := (\overline{x}_1, \overline{x}_2 + u_2)$ in (2.10) with $u := (u_1,u_2) \in B(0,\delta)$, we get

$$\begin{aligned}
&\|f(\overline{x}+u) - f(\overline{x}) - \ell_1(u_1) - \ell_2(u_2)\| \\
&\leq \|f(\overline{x}+u) - f(\overline{x}_1, \overline{x}_2 + u_2) - \ell_1(u_1)\| + \|f(\overline{x}_1, \overline{x}_2 + u_2) - f(\overline{x}_1,\overline{x}_2) - \ell_2(u_2)\| \\
&\leq \varepsilon \|u_1\| + \varepsilon \|u_2\| = \varepsilon \|(u_1,u_2)\|
\end{aligned}$$

if one takes the norm on X given by $\|(u_1,u_2)\| := \|u_1\| + \|u_2\|$. □

Corollary 2.58. *A map $f : W \to Y$ defined on an open subset W of a product space $X := X_1 \times \cdots \times X_k$ is of class C^1 on W if and only if f has partial derivatives on W that are jointly continuous.*

Now let us give a result dealing with the interchange of limits and differentiation.

Theorem 2.59. *Let (f_n) be a sequence of Fréchet (resp. Hadamard) differentiable functions from a bounded, convex, open subset W of a normed space X to a Banach space Y. Suppose*

(a) There exists some $\overline{x} \in W$ such that $(f_n(\overline{x}))$ converges in Y
(b) The sequence (f_n') uniformly converges on W to some map $g : W \to L(X,Y)$

Then (f_n) uniformly converges on W to some map f that is Fréchet (resp. Hadamard) differentiable on W. Moreover, $f' = g$.

Proof. Let us prove the first assertion. Let $r > 0$ be such that W is contained in the ball $B(\overline{x}, r)$. Given n, p in $\mathbb{N}$, Corollary 2.31 yields, for every $x \in W$,

$$\left\|f_p(x) - f_p(\overline{x}) - (f_n(x) - f_n(\overline{x}))\right\| \le \sup_{w \in W} \left\|f_p'(w) - f_n'(w)\right\| . \|x - \overline{x}\| \le r \left\|f_p' - f_n'\right\|_\infty , \tag{2.11}$$

$$\left\|f_p(x) - f_n(x)\right\| \le \left\|f_p(\overline{x}) - f_n(\overline{x})\right\| + r \left\|f_p' - f_n'\right\|_\infty . \tag{2.12}$$

Since $\left\|f_p' - f_n'\right\|_\infty \to 0$ as $n, p \to \infty$ and since $(f_p(\overline{x}) - f_n(\overline{x})) \to 0$ as $n, p \to \infty$, we see that $(f_n(x))$ is a Cauchy sequence, hence has a limit in the complete space Y; we denote it by $f(x)$. Passing to the limit on p in (2.12) we see that the limit is uniform on W.

Now, given $x \in W$, let us prove that f is differentiable at x with derivative $g(x)$. Given $\varepsilon > 0$, we can find $k \in \mathbb{N}$ such that for $p > n \ge k$ one has $\left\|f_p' - f_n'\right\|_\infty \le \varepsilon/3$, hence $\|g' - f_n'\|_\infty \le \varepsilon/3$. Using again Corollary 2.31 with $x' := x + u \in W$, we get

$$\left\|(f_p(x+u) - f_p(x)) - (f_n(x+u) - f_n(x))\right\| \le (\varepsilon/3) \|u\| ,$$

and passing to the limit on p, we obtain

$$\|f(x+u) - f(x) - (f_n(x+u) - f_n(x))\| \le (\varepsilon/3) \|u\| . \tag{2.13}$$

In the Fréchet differentiable case, we can find $\delta > 0$ such that $B(x, \delta) \subset W$ and for all $u \in \delta B_X$,

$$\begin{aligned} \|f_k(x+u) - f_k(x) - g(x)(u)\| &\le \left\|f_k(x+u) - f_k(x) - f_k'(x)(u)\right\| \\ &+ \left\|f_k'(x)(u) - g(x)(u)\right\| \le (\varepsilon/3) \|u\| + (\varepsilon/3) \|u\| . \end{aligned}$$

Combining this estimate with relation (2.13), in which we take $n = k$, we get

$$\forall u \in \delta B_X, \qquad \|f(x+u) - f(x) - g(x)(u)\| \leq \varepsilon \|u\|,$$

so that f is Fréchet differentiable at x with derivative $g(x)$.

In the Hadamard differentiable case, given $\varepsilon > 0$ and a unit vector u, we take $\delta \in (0,1)$ such that $B(x, 2\delta) \subset W$ and for $t \in (0,\delta)$, $v \in B(u,\delta)$,

$$\begin{aligned}
&\|f_k(x+tv) - f_k(x) - g(x)(tu)\| \\
&\quad \leq \left\|f_k(x+tv) - f_k(x) - f_k'(x)(tu)\right\| + \left\|f_k'(x)(tu) - g(x)(tu)\right\| \leq (\varepsilon/3)t + (\varepsilon/3)t.
\end{aligned}$$

Gathering this estimate with relation (2.13), in which we take $n = k$, $u = tv$, we get

$$\forall (t,v) \in (0,\delta) \times B(u,\delta), \qquad \|f(x+tv) - f(x) - g(x)(tu)\| \leq \varepsilon t,$$

so that f is Hadamard differentiable at x and $f'(x) = g(x)$. □

Corollary 2.60. *Let X, Y be normed spaces, Y being complete, and let W be an open subset of X. The space $B^1(W,Y)$ (resp. $BC^1(W,Y)$) of bounded, Lipschitzian, differentiable (resp. of class C^1) maps from W to Y is complete for the norm $\|\cdot\|_{1,\infty}$ given by*

$$\|f\|_{1,\infty} := \sup_{x \in W} \|f(x)\| + \sup_{x \in W} \left\|f'(x)\right\|.$$

Here we use the fact that if f is Lipschitzian and differentiable, its derivative is bounded.

Proof. Let (f_n) be a Cauchy sequence of $\left(B^1(W,Y), \|\cdot\|_{1,\infty}\right)$. Then (f_n') is a Cauchy sequence of the space $B(W, L(X,Y))$ of bounded maps from W into $L(X,Y)$ for the uniform norm; thus it converges and its limit is continuous if $f_n \in BC^1(W,Y)$. Similarly, (f_n) converges in $B(W,Y)$. The theorem ensures that the limit f of (f_n) is Fréchet differentiable and its derivative is the limit of (f_n'), hence is bounded. Thus f belongs to $B^1(W,Y)$ and $(f_n) \to f$ for $\|\cdot\|_{1,\infty}$. If (f_n) is contained in $BC^1(W,Y)$, then f' is continuous, whence $f \in BC^1(W,Y)$. □

A directional version follows similarly from Theorem 2.59.

Corollary 2.61. *Let X, Y be normed spaces, Y being complete, and let W be an open subset of X. The space $BH^1(W,Y)$ of bounded, Lipschitzian, Hadamard differentiable maps from W to Y is complete for the norm $\|\cdot\|_{1,\infty}$. The same is true for its subspace $BD^1(W,Y)$ formed by bounded, Lipschitzian maps of class D^1.*

Now let us derive the important Borwein–Preiss smooth variational principle from the Deville–Godefroy–Zizler theorem (Theorem 1.152). When $Y := \mathbb{R}$, we simplify the notation $B^1(X,Y)$ into $B^1(X)$, and we adopt similar simplifications for the other spaces.

Theorem 2.62 (Borwein–Preiss variational principle). *Let X be a Banach space and let $F := B^1(X)$ (resp. $BH^1(X)$, $BC^1(X)$, $BD^1(X)$) with the norm $\|\cdot\|_{1,\infty}$ defined above. Suppose there exists some nonnull function $b \in F$ with bounded support.*

Then, given a lower semicontinuous function $f : X \to \mathbb{R}_\infty$ that is bounded below, the set G of $g \in F$ such that $f + g$ is well-posed is generic in F.

Moreover, there exists some $\kappa > 0$ depending only on X such that for every $\varepsilon > 0$ and every $u \in X$ satisfying $f(u) < \inf f(X) + \kappa\varepsilon^2$ one can find some $g \in F$ satisfying $\|g\|_{1,\infty} \leq \varepsilon$ and some minimizer v of $f + g$ belonging to $B(u, \varepsilon)$.

Note that one has $f(v) - \varepsilon \leq f(v) + g(v) \leq f(u) + g(u) \leq f(u) + \varepsilon$, hence $f(v) \leq f(u) + 2\varepsilon$.

Proof. Conditions (b) and (c) of Theorem 1.152 are obviously satisfied, whereas (a) is part of our assumptions (here we have changed W into F in order to avoid confusion with what precedes). Moreover, $(F, \|\cdot\|)$ is complete by the preceding corollary. The last assertion follows from the corresponding localization property in Theorem 1.152 and the relation $\|g(t\cdot)\| \leq t\,\|g\|$ for $t \geq 1$, $g \in F$. □

Exercises

1. (**a**) Show that $r : X \to Y$ is a remainder if and only if there exists a remainder ρ on $\mathbb{R}$ such that $\|r(x)\| \leq \rho(\|x\|)$ for all x close to 0.
(**b**) Prove the other two characterizations of remainders that follow the definition.

2. Define a notion of directional remainder that could be used for the study of Hadamard differentiability.

3. Show that when $f : W \to Y$ is Fréchet differentiable at $\bar{x}$, then it is *stable* at $\bar{x}$ in the sense that there exists $c > 0$ such that $\|f(\bar{x}+x) - f(\bar{x})\| \leq c\,\|x\|$ for $\|x\|$ small enough.

4. Give a direct proof that Fréchet differentiability implies Hadamard differentiability.

5. Show that if $f : X_1 \times X_2 \to Y$ is circa-differentiable at $\bar{x} := (\bar{x}_1, \bar{x}_2)$ with respect to X_1 and X_2, then it is circa-differentiable at $\bar{x}$.

6. In Theorem 2.59, when W is not bounded, assuming that (f_n') converges to g uniformly on bounded subsets of W, get a similar interchange result in which the convergence of (f_n) to f is uniform on bounded subsets of the open convex set W.

7. In Theorem 2.59, assuming that W is a connected open subset of X and that the convergence of (f_n') is locally uniform (in the sense that for every $x \in W$ there exists some ball with center x contained in W on which the convergence of (f_n') is uniform), prove that (f_n) is locally uniformly convergent and that its limit f is differentiable with derivative g.

8. Give a direct proof of Proposition 2.51.

9. With the hypothesis of Proposition 2.51, show that the map f is circa-differentiable at $\overline{x}$. Is it of class C^1 at $\overline{x}$?

10. Express the chain rule for differentiable maps between $\mathbb{R}^m$, $\mathbb{R}^n$, $\mathbb{R}^p$ in terms of a matrix product for the Jacobians of f and g.

11. Using the Hahn–Banach theorem, show that $f : W \to Y$ is circa-differentiable at $a \in W$ if and only if there exists a map $F : W \times W \to L(X,Y)$ continuous at (a,a) such that $f(u) - f(v) = F(u,v)(u-v)$. Then $f'(a) = F(a,a)$.

12. Show that if X is finite-dimensional, then $f : U \to Y$, with U open in X, is of class D^1 if and only if f is of class C^1. [Hint: For every element e of a basis of X the map $x \mapsto Df(x)(e)$ is continuous when f is of class D^1.]

13. Given normed spaces X, Y and a topology $\mathscr{T}$ (or a convergence) on the space of maps from B_X to Y, one can define a notion of $\mathscr{T}$-semiderivative at $\overline{x}$ of a map $f : B(\overline{x}, r) \to Y$: it consists in requiring that the family of maps $(f_t)_{0<t<r}$ from B_X to Y given by $f_t(v) := t^{-1}(f(\overline{x}+tv) - f(\overline{x}))$ have a limit as $t \to 0_+$. If the limit is the restriction to B_X of a continuous linear map, one speaks of a $\mathscr{T}$-derivative. Interpret Gâteaux, Hadamard, and Fréchet derivatives with the help of the topologies of uniform convergence on the families of finite subsets, compact subsets, and bounded subsets. Observe that such a process also applies to some other families of sets, such as the family of weakly compact subsets of B_X.

14. Show that the norm $x \mapsto \|x\| := \sup_{t\in T} |x(t)|$ on the Banach space $X := C(T)$ of continuous functions on $T := [0,1]$ is not Fréchet differentiable at any point. Compare with Exercise 8 of the preceding section.

15. Let X and Y be normed spaces, let $\overline{x} \in X$, $c,\ r > 0$, $W := B(\overline{x}, r)$, $f : W \to Y$ be of class C^1 and such that $\|f'(x) - f'(\overline{x})\| \le c\,\|x - \overline{x}\|$ for all $x \in W$.

(**a**) Show that $\|f(x) - f(\overline{x}) - f'(\overline{x})(x - \overline{x})\| \le (c/2)\,\|x - \overline{x}\|^2$ for all $x \in W$.
(**b**) Suppose that f' is Lipschitzian with rate c on W. Show that for all $w, x \in W$ one has $\|f(x) - f(w) - f'(w)(x-w)\| \le (c/2)\,\|x-w\|^2$.

2.5 Inversion of Differentiable Maps

In the present section, we show that simple methods linked with differentiability concepts lead to efficient ways of solving nonlinear systems or vectorial equations

$$f(x) = 0. \tag{2.14}$$

Here X and Y are Banach spaces, W is an open subset of X, and $f : W \to Y$ is a map. We start with a classical constructive algorithm.

2.5.1 Newton's Method

Newton's method is an iterative process that relies on a notion of approximation by a linear map. We formulate it as follows.

Definition 2.63. The map $f : W \to Y$ has a *Newton approximation* at $\overline{x} \in W$ if there exist $r > 0$, $\alpha > 0$ and a map $A : B(\overline{x}, r) \to L(X,Y)$ such that $B(\overline{x}, r) \subset W$ and

$$\forall x \in B(\overline{x}, r), \qquad \|f(x) - f(\overline{x}) - A(x)(x - \overline{x})\| \leq \alpha \|x - \overline{x}\|. \tag{2.15}$$

A map $A : V \to L(X,Y)$ is a *slant derivative* of f at $\overline{x}$ if V is a neighborhood of $\overline{x}$ contained in W and if for every $\alpha > 0$ there exists some $r > 0$ such that $B(\overline{x}, r) \subset V$ and relation (2.15) holds.

Thus f is differentiable at $\overline{x}$ if and only if f has a slant derivative at $\overline{x}$ that is constant on some neighborhood of $\overline{x}$. But condition (2.15) is much less demanding, as the next lemma shows.

Lemma 2.64. *The following assertions are equivalent:*

(a) f has a Newton approximation A that is bounded near $\overline{x}$
(b) f has a slant derivative A at $\overline{x}$ that is bounded on some neighborhood of $\overline{x}$
(c) f is stable at $\overline{x}$, i.e., there exist $c > 0$, $r > 0$ such that

$$\forall x \in B(\overline{x}, r), \qquad \|f(x) - f(\overline{x})\| \leq c \|x - \overline{x}\|. \tag{2.16}$$

Proof. (a)⇒(c) If for some α, $\beta > 0$ and some $r > 0$ a map $A : B(\overline{x}, r) \to L(X,Y)$ is such that (2.15) holds with $\|A(x)\| \leq \beta$ for all $x \in B(\overline{x}, r)$, then by the triangle inequality, relation (2.16) holds with $c := \alpha + \beta$.

(c)⇒(b) We use a corollary of the Hahn–Banach theorem asserting the existence of some map $s : X \to X^*$ such that $s(x)(x) = \|x\|$ and $\|s(x)\| = 1$ for all $x \in X$. Suppose (2.16) holds. Then setting $A(\overline{x}) = 0$ and for $w \in W \setminus \{\overline{x}\}$, $x \in X$,

$$A(w)(x) = \langle s(w - \overline{x}), x \rangle \frac{f(w) - f(\overline{x})}{\|w - \overline{x}\|},$$

we easily check that $\|A(w)\| \leq c$ for all $w \in W$ and that $A(x)(x - \overline{x}) = f(x) - f(\overline{x})$ for all $x \in W$, so that (2.15) holds with $\alpha = 0$ and A is a slant derivative of f at $\overline{x}$.

(b)⇒(a) is obvious, a slant derivative of f at $\overline{x}$ being a Newton approximation of f at $\overline{x}$. □

In the elementary Newton method that follows, we first assume that (2.14) has a solution $\overline{x}$.

Proposition 2.65. *Let $\overline{x}$ be a solution to (2.14), let α, $\beta, r > 0$ satisfy $\gamma := \alpha\beta < 1$, and let $A : B(\overline{x}, r) \to L(X,Y)$ be such that (2.15) holds, $A(x)$ being invertible with $\left\|A(x)^{-1}\right\| \leq \beta$ for all $x \in B(\overline{x}, r)$. Then the sequence (x_n) given by*

$$x_{n+1} := x_n - A(x_n)^{-1}(f(x_n)) \tag{2.17}$$

is well defined for every initial point $x_0 \in B(\overline{x}, r)$ *and converges linearly to* $\overline{x}$ *with rate* γ.

The last assertion means that $\|x_{n+1} - \overline{x}\| \leq \gamma \|x_n - \overline{x}\|$, hence $\|x_n - \overline{x}\| \leq c\gamma^n$ for some $c > 0$ (in fact $c := \|x_0 - \overline{x}\|$). Thus, if A is a slant derivative of f at $\overline{x}$, then (x_n) converges superlinearly to $\overline{x}$: for all $\varepsilon > 0$ there is some $k \in \mathbb{N}$ such that $\|x_{n+1} - \overline{x}\| \leq \varepsilon \|x_n - \overline{x}\|$ for all $n \geq k$.

Proof. Using the fact that $f(\overline{x}) = 0$, so that

$$x_{n+1} - \overline{x} = A(x_n)^{-1} \left(f(\overline{x}) - f(x_n) + A(x_n)(x_n - \overline{x}) \right),$$

we inductively obtain that

$$\|x_{n+1} - \overline{x}\| \leq \beta \|f(x_n) - f(\overline{x}) - A(x_n)(x_n - \overline{x})\| \leq \alpha\beta \|x_n - \overline{x}\|,$$

so that $x_{n+1} \in B(\overline{x}, r)$: the whole sequence (x_n) is well defined and converges to $\overline{x}$. □

Under reinforced assumptions, one can show the existence of a solution.

Theorem 2.66 (Kantorovich). *Let* $x_0 \in W$, α, $\beta > 0$, $r > 0$ *with* $\gamma := \alpha\beta < 1$, $B(x_0, r) \subset W$ *and let* $A : B(x_0, r) \to L(X, Y)$ *be such that for all* $x \in B(x_0, r)$ *the map* $A(x) : X \to Y$ *has a right inverse* $B(x) : Y \to X$ *satisfying* $\|B(x)(\cdot)\| \leq \beta \|\cdot\|$ *and*

$$\forall w, x \in B(x_0, r), \qquad \|f(w) - f(x) - A(x)(w - x)\| \leq \alpha \|w - x\|. \tag{2.18}$$

If $\|f(x_0)\| < \beta^{-1}(1 - \gamma)r$ *and if* f *is continuous, the sequence given by the Newton iteration*

$$x_{n+1} := x_n - B(x_n)(f(x_n)) \tag{2.19}$$

is well defined and converges to a solution $\overline{x}$ *of (2.14). Moreover, one has* $\|x_n - \overline{x}\| \leq r\gamma^n$ *for all* $n \in \mathbb{N}$ *and* $\|\overline{x} - x_0\| \leq \beta(1 - \gamma)^{-1} \|f(x_0)\| < r$.

Here $B(x)$ is a *right inverse* of $A(x)$ if $A(x) \circ B(x) = I_Y$; $B(x)$ is not assumed to be linear.

Proof. Let us prove by induction that $x_n \in B(x_0, r)$, $\|x_{n+1} - x_n\| \leq \beta\gamma^n \|f(x_0)\|$, and $\|f(x_n)\| \leq \gamma^n \|f(x_0)\|$. For $n = 0$ these relations are obvious. Assuming that they are valid for $n < k$, we get

$$\|x_k - x_0\| \leq \sum_{n=0}^{k-1} \|x_{n+1} - x_n\| \leq \beta \|f(x_0)\| \sum_{n=0}^{\infty} \gamma^n = \beta \|f(x_0)\| (1 - \gamma)^{-1} < r,$$

or $x_k \in B(x_0, r)$, and since $f(x_{k-1}) + A(x_{k-1})(x_k - x_{k-1}) = 0$, from (2.18), (2.19), we have

$$\|f(x_k)\| \le \|f(x_k) - f(x_{k-1}) - A(x_{k-1})(x_k - x_{k-1})\| \le \alpha \|x_k - x_{k-1}\| \le \gamma^k \|f(x_0)\|$$

and

$$\|x_{k+1} - x_k\| \le \beta \|f(x_k)\| \le \beta\gamma^k \|f(x_0)\|.$$

Since $\gamma < 1$, the sequence (x_n) is a Cauchy sequence, hence converges to some $\overline{x} \in X$ satisfying $\|\overline{x} - x_0\| \le \beta \|f(x_0)\| (1-\gamma)^{-1} < r$. Moreover, by the continuity of f, we get $f(\overline{x}) = \lim_n f(x_n) = 0$. Finally,

$$\|x_n - \overline{x}\| \le \lim_{p\to+\infty} \|x_n - x_p\| \le \lim_{p\to+\infty} \sum_{k=n}^{p-1} \|x_{k+1} - x_k\| \le r\gamma^n. \qquad \square$$

We deduce from Kantorovich's theorem a result that is the root of important estimates in nonlinear analysis.

Theorem 2.67 (Lyusternik–Graves theorem). *Let X and Y be Banach spaces, let W be an open subset of X, and let $g : W \to Y$ be circa-differentiable at some $\overline{x} \in W$ with a surjective derivative $Dg(\overline{x})$. Then g is open at $\overline{x}$. More precisely, there exist some $\rho, \sigma, \kappa > 0$ such that g has a right inverse $h : B(g(\overline{x}), \sigma) \to W$ satisfying $\|h(y) - \overline{x}\| \le \kappa \|g(\overline{x}) - y\|$ for all $y \in B(g(\overline{x}), \sigma)$ and*

$$\forall (w,y) \in B(\overline{x}, \rho) \times B(g(\overline{x}), \sigma) \quad \exists x \in W : \ g(x) = y, \ \|x - w\| \le \kappa \|g(w) - y\|. \tag{2.20}$$

Proof. Let $A : W \to L(X,Y)$ be the constant map with value $A := Dg(\overline{x})$ (we use a familiar abuse of notation). The open mapping theorem yields some $\beta > 0$ and some right inverse $B : Y \to X$ of A such that $\|B(\cdot)\| \le \beta \|\cdot\|$. Let $\alpha, r > 0$ be such that $\gamma := \alpha\beta < 1$, $B(\overline{x}, 2r) \subset W$ and

$$\forall w, x \in B(\overline{x}, 2r), \qquad \|g(w) - g(x) - Dg(\overline{x})(w - x)\| \le \alpha \|w - x\|. \tag{2.21}$$

Let $\sigma, \tau > 0$ be such that $\sigma + \tau < \beta^{-1}(1-\gamma)r$, and let $\rho \in (0, r]$ be such that $g(w) \in B(g(\overline{x}), \tau)$ for all $w \in B(\overline{x}, \rho)$. Given $w \in B(\overline{x}, \rho)$, $y \in B(g(\overline{x}), \sigma)$, let us set $f(x) := g(x) - y$ for $x \in B(\overline{x}, \rho)$, so that $\|f(w)\| \le \|g(w) - g(\overline{x})\| + \|g(\overline{x}) - y\| < \beta^{-1}(1-\gamma)r$, and by (2.21), we have that (2.18) holds in the ball $B(x_0, r)$, with $x_0 := w$. Using the estimate $\|x - x_0\| \le \beta \|f(x_0)\| (1-\gamma)^{-1} < r$ obtained in the proof of Kantorovich's theorem for a solution x of the equation $f(x) = 0$, we get some $x \in W$ such that $g(x) = y$, $\|x - w\| \le \kappa \|g(w) - y\|$ with $\kappa := \beta(1-\gamma)^{-1}$. The right inverse h is obtained by taking $w := \overline{x}$ in (2.20). $\square$

Exercises

1. Let X and Y be Banach spaces, let $\overline{x} \in X$, $b,\ c,\ r > 0$, $W := B(\overline{x}, r)$, $f : W \to Y$ be of class C^1 and such that f' is Lipschitzian with rate c on W and $\|f'(w)^{\mathsf{T}}(y^*)\| \ge$

$b\|y^*\|$ for all $w \in W$, $y^* \in Y^*$. Let $b > cr$. Using Kantorovich's theorem, prove that for all $y \in B(f(\overline{x}), (b-cr)r)$ there exists $x \in W$ satisfying $f(x) = y$ and $\|x - \overline{x}\| \le b^{-1}\|y - f(\overline{x})\|$. [Hint: Use the Banach–Schauder theorem to find a right inverse $B(w)$ of $A(w) := f'(w)$ for all $w \in W$ satisfying $\|B(w)(\cdot)\| \le b^{-1}\|\cdot\|$ and use Exercise 15 of Sect. 2.4 to check condition (2.18)]

2. Using Exercise 15 of Sect. 2.4 to establish a refined version of Kantorovich's theorem and prove that the conclusion of the preceding result can be extended to every $y \in B(f(\overline{x}), br)$.

3. (**Convexity of images of small balls** [842]). Let X be a Hilbert space, let Y be a normed space, let $a \in X$, c, ρ, $\sigma > 0$, $W := B(a, \rho)$, and let $f : W \to Y$ be differentiable and such that f' is Lipschitzian with rate c on W and $\|f'(a)^{\mathsf{T}}(y^*)\| \ge \sigma\|y^*\|$ for all $y^* \in Y^*$. Prove that for $r > 0$, $r < \min(\rho, \sigma/2c)$, the image $f(B)$ of $B := B(a, r)$ by the nonlinear map f is convex. [Hint: Given $x_0, x_1 \in B$, $y_0 := f(x_0)$, $y_1 := f(x_1)$, $y := (1/2)(y_0 + y_1)$, $\overline{x} := (1/2)(x_0 + x_1)$, show that $\|f'(w)^{\mathsf{T}}(y^*)\| \ge b\|y^*\|$ for all $w \in W$, $y^* \in Y^*$ for $b := \sigma - cr$ and apply the preceding exercise.]

4. Extend the (surprising!) result of the preceding exercise to the case that X is a Banach space with a uniformly convex norm.

2.5.2 The Inverse Mapping Theorem

The inverse mapping theorem is a milestone of differential calculus. It shows the interest and the power of derivatives. It has numerous applications in differential geometry, differential topology, and the study of dynamical systems.

When $f : T \to \mathbb{R}$ is a continuous function on some open interval T of $\mathbb{R}$, one can use the order of $\mathbb{R}$ and the intermediate value theorem to get results about invertibility of f. If, moreover, f is differentiable at some $r \in T$ and if $f'(r)$ is nonnull, one can conclude that $f(T)$ contains some neighborhood of $f(r)$. When f is a map of several variables, one would like to know whether such a conclusion is valid, and even more, whether f induces a bijection from some neighborhood of a given point $\overline{x}$ onto some neighborhood of $f(\overline{x})$. Of course, one cannot expect a global result without further assumptions, since the derivative is a local notion.

Following René Descartes's advice, we will reach our main results, concerning the possibility of inverting nonlinear maps, through several small steps; some of them have an independent interest.

First, given a bijection f between two metric spaces X, Y, we would like to know whether a map close enough to f is still a bijection. When X and Y are finite-dimensional normed spaces and f is a linear isomorphism, we know that every linear map g that is close enough to f for some norm on the space $L(X, Y)$ of linear continuous maps from X into Y is still an isomorphism: taking bases in X and Y, we see that if g is close enough to f, its determinant will remain different from 0. A similar result holds in infinite-dimensional spaces: the set of linear continuous maps that are isomorphisms onto their images is open in the space $L(X,Y)$. When X and Y are complete, a more precise result can be given.

Proposition 2.68. *Let f be a linear isomorphism between two Banach spaces X and Y. Then every $g \in L(X,Y)$ such that $\|f-g\| < \|f^{-1}\|^{-1}$ is an isomorphism.*

Proof. Let us first consider the case $X = Y$, $f = I_X$. Let $u := I_X - g$, so that $u \in L(X,X)$ satisfies $\|u\| < 1$. Since the map $(v,w) \mapsto w \circ v$ is continuous, since

$$I_X - u^{n+1} = (I_X - u) \circ \left(\sum_{k=0}^{n} u^k\right) = \left(\sum_{k=0}^{n} u^k\right) \circ (I_X - u),$$

and since the series $\sum_{k=0}^{\infty} u^k$ is absolutely convergent (since $\left\|u^k\right\| \leq \|u\|^k$), we get that its sum is a right and left inverse of $I_X - u$. Thus $I_X - u$ is invertible.

The general case can be deduced from this special case. Given $g \in L(X,Y)$ such that $\|f-g\| < r := \|f^{-1}\|^{-1}$, setting $u := I_X - f^{-1} \circ g$, we observe that $\|u\| \leq \|f^{-1} \circ (f-g)\| \leq \|f^{-1}\| \cdot \|f-g\| < 1$. Therefore, by what precedes, $f^{-1} \circ g = I_X - u$ is invertible. It follows that g is invertible, with inverse $(I_X - u)^{-1} \circ f^{-1}$. □

Now let us turn to a nonlinear situation. Let us first observe that if $f : U \to V$ is a bijection between two open subsets of normed spaces X and Y respectively, it may occur that f is differentiable at some $a \in U$ whereas its inverse g is not differentiable at $b = g(a)$: take $U = V = \mathbb{R}$, f given by $f(x) = x^3$, whose inverse $y \mapsto y^{1/3}$ is not differentiable at 0. However, if f is differentiable at some $a \in X$ and if its inverse g is differentiable at $b := g(a)$, then the derivative of g at b is the inverse $f'(a)^{-1}$ of the derivative $f'(a)$ of f at a. This fact simply follows from the chain rule: from $g \circ f = I_U$ and $f \circ g = I_V$ one deduces that $g'(b) \circ f'(a) = I_X$ and $f'(a) \circ g'(b) = I_Y$.

Our first step is not as obvious as the preceding observation, since one of its assumptions is now a conclusion.

Lemma 2.69. *Let U and V be two open subsets of normed spaces X and Y respectively. Assume that $f : U \to V$ is a homeomorphism that is differentiable at $a \in U$ and such that $f'(a)$ is an isomorphism. Then the inverse g of f is differentiable at $b = f(a)$ and $g'(b) = f'(a)^{-1}$.*

Proof. Using translations if necessary, we may suppose $a = 0$, $f(a) = 0$ without loss of generality. Changing f into $h^{-1} \circ f$, where $h := f'(a)$, we may also suppose $Y = X$ and $f'(a) = I_X$. Then setting $s(y) := g(y) - y$, we have to show that $s(y)/\|y\| \to 0$ as $y \to 0$, $y \neq 0$. Let us set $r(x) := f(x) - x$. Given $\varepsilon \in (0,1)$, we can find $\rho > 0$ such that $\|r(x)\| \leq (\varepsilon/2)\|x\|$ for $x \in \rho B_X$. Since g is continuous, we can find $\sigma > 0$ such that $\|g(y)\| \leq \rho$ for $y \in \sigma B_Y$. Then for $y \in \sigma B_Y$ and $x := g(y)$, we have $y = f(x) = x + r(x)$, and hence

$$\|y\| \geq \|x\| - \|r(x)\| \geq (1/2)\|x\|,$$
$$\|s(y)\| = \|g(y) - y\| = \|r(x)\| \leq (\varepsilon/2)\|x\| \leq \varepsilon \|y\|.$$ □

In order to get a stronger result in which the invertibility of f is part of the conclusion instead of being an assumption, we will use the reinforced differentiability property of Definition 2.54. Recall that a map $f : W \to Y$ from an open subset W of a normed space X into another normed space Y is *circa-differentiable* (or strictly differentiable) at $a \in W$ if there exists a continuous linear map $\ell : X \to Y$ such that the map $r = f - \ell$ is Lipschitzian with arbitrarily small Lipschitz rate on sufficiently small neighborhoods of a: for every $\varepsilon > 0$ there exists $\rho > 0$ such that $B(a,\rho) \subset W$ and

$$\forall w, w' \in B(a,\rho), \qquad \left\| f(w) - f(w') - \ell(w - w') \right\| \leq \varepsilon \left\| w - w' \right\|.$$

The criterion for circa-differentiability given in Proposition 2.56 uses continuous differentiability or slightly less. Thus, the reader who is not interested in refinements may suppose throughout that f is of class C^1.

Our next step is a perturbation result. We formulate it in a general framework.

Lemma 2.70. *Let (U,d) be a metric space, let Y be a normed space, let $j,h : U \to Y$ be such that*

(a) j is injective and its inverse $j^{-1} : j(U) \to U$ is Lipschitzian with rate γ;
(b) h is Lipschitzian with rate λ.

Then if $\gamma\lambda < 1$, the map $f := j + h$ is still injective and its inverse $f^{-1} : f(U) \to U$ is Lipschitzian with rate $\gamma(1-\gamma\lambda)^{-1}$.

Note that the Lipschitz rate of the inverse of the perturbed map f is close to the Lipschitz rate of j^{-1} when λ is small. It may be convenient to reformulate this lemma by saying that a map $e : X \to Y$ between two metric spaces is *expansive with rate* $c > 0$ if for all $x, x' \in X$ one has

$$d(e(x), e(x')) \geq c\,d(x,x').$$

This property amounts to

$$d(e^{-1}(y), e^{-1}(y')) \leq c^{-1} d(y,y')$$

for every $y, y' \in e(X)$, i.e., e is injective and its inverse is Lipschitzian on the image $e(X)$ of e. Thus the lemma can be rephrased as follows:

Lemma. *Let X be a metric space and let Y be a normed space. Let $e : X \to Y$ be expansive with rate $c > 0$ and let $h : X \to Y$ be Lipschitzian with rate $\ell < c$. Then $g := e + h$ is expansive with rate $c - \ell$.*

Proof. The lemma results from the following relations, valid for every $x, x' \in X$:

$$\left\| g(x) - g(x') \right\| \geq \left\| e(x) - e(x') \right\| - \left\| h(x) - h(x') \right\| \geq c\,d(x,x') - \ell d(x,x').$$

Note that for $c = \gamma^{-1}$, $\ell = \lambda$ one has $(c - \ell)^{-1} = \gamma(1 - \gamma\lambda)^{-1}$. □

Since we have defined differentiability only on open subsets, it will be important to ensure that $f(U)$ is open in order to apply Lemma 2.69. We reach this conclusion in two steps. The first one relies on the Banach–Picard contraction theorem.

Lemma 2.71. *Let W be an open subset of a Banach space Y and let $k : W \to Y$ be a Lipschitzian map with rate $c < 1$. Then the image of W by $f := I_W + k$ is open.*

Proof. We will prove that for every $a \in W$ and for every closed ball $B[a,r]$ contained in W, the closed ball $B[f(a),(1-c)r]$ is contained in the set $f(W)$, and in fact in the set $f(B[a,r])$. Without loss of generality, we may suppose $a = 0$, $k(a) = 0$, using translations if necessary. Given $y \in (1-c)rB_Y$ we want to find $x \in rB_Y$ such that $y = f(x)$. This equation can be written $y - k(x) = x$. We note that $x \mapsto y - k(x)$ is Lipschitzian with rate $c < 1$ and that it maps rB_Y into itself, since

$$\|y - k(x)\| \le \|y\| + \|k(x)\| \le (1-c)r + cr = r.$$

Since rB_Y is a complete metric space, the contraction theorem yields some fixed point x of this map. Thus $y = f(x) \in f(W)$. □

Lemma 2.72. *Let (U,d) be a metric space, let Y be a Banach space, let $\gamma > 0, \lambda > 0$ with $\gamma\lambda < 1$, and let $j,h : U \to Y$ be such that $W := j(U)$ is open and*

(a) j is injective and its inverse $j^{-1} : W \to U$ is Lipschitzian with rate γ;
(b) h is Lipschitzian with rate λ.

Then the map $f := j + h$ is injective, its inverse is Lipschitzian, and $f(U)$ is open.

Proof. Let $k := h \circ j^{-1}$, so that $f \circ j^{-1} = I_W + k$ and k is Lipschitzian with rate $\gamma\lambda < 1$. Then Lemma 2.71shows that $f(U) = f(j^{-1}(W)) = (I + k)(W)$ is open. □

We are ready to state the inverse mapping theorem.

Theorem 2.73 (Inverse mapping theorem). *Let X and Y be Banach spaces, let W be an open subset of X, and let $f : W \to Y$ be circa-differentiable at $a \in W$ and such that $f'(a)$ is an isomorphism from X onto Y. Then there exist neighborhoods U of a and V of $b := f(a)$ such that $U \subset W$ and such that f induces a homeomorphism from U onto V whose inverse is differentiable at b.*

Proof. In the preceding lemma, let us take $j := f'(a)$, $h = f - j$. Since j is an isomorphism, its inverse is Lipschitzian with rate $\|j^{-1}\|$. Let U be a neighborhood of a such that h is Lipschitzian with rate $\lambda < 1/\|j^{-1}\|$. Then by the preceding lemma, $V := f(U)$ is open and $f \mid U$ is a homeomorphism from U onto V, and by Lemma 2.69, its inverse is differentiable at b. □

Exercise. Show that the inverse of f is in fact circa-differentiable at b.

Exercise (Square root of an operator). Let E be a Banach space and let $X := L(E,E)$. Considering the map $f : X \to X$ given by $f(u) := u^2 := u \circ u$, show that there exist a neighborhood V of I_E in X and a differentiable map $g : V \to X$ such that $g(v)^2 := g(v) \circ g(v) = v$ for all $v \in V$.

The following classical terminology is helpful.

Definition 2.74. A C^k-*diffeomorphism* between two open subsets of normed spaces is a homeomorphism that is of class C^k, as is its inverse ($k \geq 1$).

The following example plays an important role in the sequel, so that we make it a lemma.

Lemma 2.75. *Let X and Y be Banach spaces. Then the set* $\mathrm{Iso}(X,Y)$ *of isomorphisms from X onto Y is open in $L(X,Y)$ and the map $i : \mathrm{Iso}(X,Y) \to \mathrm{Iso}(Y,X)$ given by $i(u) = u^{-1}$ is a C^∞-diffeomorphism, i.e., a C^k-diffeomorphism for all $k \geq 1$.*

Proof. The first assertion has been proved in Proposition 2.68. Let us prove the second assertion by first considering the case $X = Y$ and by showing that i is differentiable at the identity map I_X, with derivative $Di(I_X)$ given by $Di(I_X)(v) = -v$. Taking $\rho \in (0,1)$, this follows from the expansion

$$\forall v \in L(X,X),\ \ \|v\| \leq \rho, \qquad (I_X + v)^{-1} = I_X - v + s(v),$$

with $s(v) := v^2 \circ \sum_{k=0}^{\infty} (-1)^k v^k$: s defines a remainder, since $\left\|(-1)^k v^k\right\| \leq \rho^k$ and $\|s(v)\| \leq (1-\rho)^{-1} \|v\|^2$. Thus i is differentiable at I_X.

Now in the general case, for $u \in \mathrm{Iso}(X,Y)$, $w \in L(X,Y)$ satisfying $\|w\| < 1/\left\|u^{-1}\right\|$, $v := u^{-1} \circ w$, one has $u + w = u \circ (I_X + v) \in \mathrm{Iso}(X,Y)$,

$$\begin{aligned} i(u+w) &= \left[u \circ (I_X + u^{-1} \circ w)\right]^{-1} = (I_X + u^{-1} \circ w)^{-1} \circ u^{-1} \\ &= \left(I_X - u^{-1} \circ w + s(v)\right) \circ u^{-1}, \end{aligned}$$

and one sees that i is differentiable at u, with

$$Di(u)(w) = -u^{-1} \circ w \circ u^{-1}. \tag{2.22}$$

Thus the derivative $i' : \mathrm{Iso}(X,Y) \to L(L(X,Y), L(Y,X))$ is obtained by composing i with the map $k : L(Y,X) \to L(L(X,Y), L(Y,X))$ given by $k(z)(w) := -z \circ w \circ z$ for $z \in L(Y,X)$, $w \in L(X,Y)$, which is continuous and quadratic, hence is of class C^1. It follows that i' is continuous and i is of class C^1. Then i' is of class C^1. By induction, we obtain that i is of class C^k for all $k \geq 1$. Since i is a bijection with inverse $i^{-1} : \mathrm{Iso}(Y,X) \to \mathrm{Iso}(X,Y)$ given by $i^{-1}(z) = z^{-1}$, we get that i is a C^∞-diffeomorphism. □

Note that formula (2.22) generalizes the usual case $i(t) = t^{-1}$ on $\mathbb{R} \setminus \{0\}$ for which $i'(u) = -u^{-2}$ and $Di(u)(w) = -u^{-2} w$.

Corollary 2.76. *Let X and Y be Banach spaces, let W be an open subset of X, and let $f : W \to Y$ be of class C^k ($k \geq 1$) and such that $f'(a)$ is an isomorphism from X onto Y for some $a \in W$. Then there exist neighborhoods U of a and V of $b := f(a)$ such that $U \subset W$ and such that $f \mid U$ is a C^k-diffeomorphism between U and V.*

Proof. Let us first consider the case $k = 1$. The inverse mapping theorem ensures that f induces a homeomorphism from a neighborhood U of a onto a neighborhood V of b. Since f' is continuous at a and since the set $\mathrm{Iso}(X,Y)$ of isomorphisms from X onto Y is open in $L(X,Y)$, taking a smaller U if necessary, we may assume that $f'(x)$ is an isomorphism for all $x \in U$. Then Lemma 2.69 guarantees that $g := f^{-1}$ is differentiable at $f(x)$. Moreover, one has

$$g'(y) = (f'(g(y)))^{-1}.$$

Since the map $i : u \mapsto u^{-1}$ is of class C^1 on $\mathrm{Iso}(X,Y)$, $g' = i \circ f' \circ g$ is continuous. Thus g is of class C^1.

Now suppose by induction that g is of class C^k if f is of class C^k, and let us prove that when f is of class C^{k+1}, then g is of class C^{k+1}. That follows from the expression $g' = i \circ f' \circ g$, which shows that g' is of class C^k as a composite of maps of class C^k. □

Let us give a global version of the inverse mapping theorem.

Corollary 2.77. *Let X and Y be Banach spaces, let W be an open subset of X, and let $f : W \to Y$ be an injection of class C^k such that for every $x \in W$, the linear map $f'(x)$ is an isomorphism from X onto Y. Then $f(W)$ is open and f is a C^k-diffeomorphism between W and $f(W)$.*

Proof. The inverse mapping theorem ensures that $f(W)$ is open in Y. Thus f is a continuous bijection from W onto $f(W)$ and its inverse is locally of class C^k, hence is of class C^k. □

Exercise. Let $f : T \to \mathbb{R}$ be a continuous function on some open interval T of $\mathbb{R}$. Show that if f is differentiable at some $r \in T$ with $f'(r)$ nonnull, then $f(T)$ contains some neighborhood of $f(r)$. Show by an example that it may happen that there is no neighborhood of r on which f is injective.

Example–Exercise (Polar coordinates). Let $W := (0,+\infty) \times (-\pi,\pi) \subset \mathbb{R}^2$ and let $f : W \to \mathbb{R}^2$ be given by $f(r,\theta) = (r\cos\theta, r\sin\theta)$. Then f is a bijection from W onto $\mathbb{R}^2 \setminus D$, with $D := (-\infty,0] \times \{0\}$ and the Jacobian matrix of f at (r,θ) is

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Its determinant (called the *Jacobian* of f) is $r(\cos^2\theta + \sin^2\theta) = r > 0$; hence f is a diffeomorphism of class C^∞ from W onto $f(W)$. Using the relation $\tan(\theta/2) = 2\sin(\theta/2)\cos(\theta/2)/2\cos^2(\theta/2) = \sin\theta/(1+\cos\theta)$, show that its inverse is given by

$$(x,y) \mapsto \left(\sqrt{x^2+y^2}, 2\,\mathrm{Arctan}\,\frac{y}{x+\sqrt{x^2+y^2}}\right).$$

Example–Exercise (Spherical coordinates). Let $W := (0,+\infty)\times(-\pi,\pi)\times(\frac{-\pi}{2},\frac{\pi}{2})$ and let $f : W \to \mathbb{R}^3$ be given by $f(r,\theta,\omega) = (r\cos\theta\sin\omega, r\sin\theta\sin\omega, r\cos\omega)$. Show that f is a diffeomorphism from W onto its image. The angles θ,ω are known as *Euler angles*. On the globe, they can serve to measure latitude and longitude.

Example–Exercise. Is $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) := (x^2-y^2, 2xy)$ a diffeomorphism? Give an interpretation by considering $z \mapsto z^2$, with $z := x+\mathrm{i}y$, identifying $\mathbb{C}$ with $\mathbb{R}^2$.

2.5.3 The Implicit Function Theorem

Functions are sometimes defined in an implicit, indirect way. For example, in economics, the famous Phillips curve is defined through the equation

$$1.39u(w+0.9) = 9.64,$$

where u is the rate of unemployment and w is the annual rate of variation of nominal wages; in such a case one can express u in terms of w and vice versa. However, given Banach spaces X,Y,Z, an open subset W of $X\times Y$, and a map $f : W \to Z$, it is often impossible to determine an explicit map $h : X_0 \to Y$ from an open subset X_0 of X such that $(x,h(x)) \in W$ and $f(x,h(x)) = 0$ for all $x \in X_0$. When the existence of such a map is known (but not necessarily in an explicit form), one says that it is an *implicit function* determined by f. The following result guarantees the existence and regularity of such a map.

Theorem 2.78. *Let X,Y,Z be Banach spaces, let W be an open subset of $X\times Y$, and let $f : W \to Z$ be a map of class C^1 at $(a,b) \in W$ such that $f(a,b) = 0$ and the second partial derivative $D_Y f(a,b)$ is an isomorphism from Y onto Z. Then there exist open neighborhoods U of (a,b) and V of a in W and X respectively and a map $h : V \to Y$ of class C^1 at a such that $h(a) = b$ and*

$$((x,y) \in U,\ f(x,y) = 0) \Longleftrightarrow (x \in V,\ y = h(x)). \tag{2.23}$$

If f is of class C^k with $k \geq 1$ on W, then h is of class C^k on V. Moreover,

$$Dh(a) = -D_Y f(a,b)^{-1} \circ D_X f(a,b). \tag{2.24}$$

Proof. Let $F : W \to X\times Z$ be the map given by $F(x,y) := (x, f(x,y))$. Then F is of class C^1 at (a,b), as are its components, and

$$DF(a,b)(x,y) = (x, D_X f(a,b)x + D_Y f(a,b)y).$$

It is easy to check that $DF(a,b)$ is invertible and that its inverse is given by

$$(DF(a,b))^{-1}(x,z) = (x, -(D_Y f(a,b))^{-1} \circ D_X f(a,b)x + (D_Y f(a,b))^{-1} z).$$

Therefore, the inverse mapping theorem yields open neighborhoods U of (a,b) in W and U' of $(a,0)$ in $X \times Z$ such that F induces a homeomorphism from U onto U' of class C^1 at (a,b). Its inverse G is of class C^1 at $(a,0)$, satisfies $G(a,0) = (a,b)$, and has the form $(x,z) \mapsto (x, g(x,z))$. Let $V := \{x \in X : (x,0) \in U'\}$ and let $h : V \to Y$ be given by $h(x) = g(x,0)$. Then the equivalence

$$((x,y) \in U,\ (x,z) = (x, f(x,y))) \Leftrightarrow \big((x,z) \in U',\ (x,y) = (x, g(x,z))\big)$$

entails, by definition of V and h,

$$((x,y) \in U,\ f(x,y) = 0) \Leftrightarrow (x \in V,\ y = h(x)).$$

When f is of class C^k on W, with $k \geq 1$, F is of class C^k; hence G and h are of class C^k on U' and V respectively. Moreover, the computation of the inverse $DF(a,b)^{-1}$ we have done shows that

$$Dh(a) = D_X g(a,0) = -D_Y f(a,b)^{-1} \circ D_X f(a,b).$$

□

Example. Let X be a Hilbert space, and for $Y := \mathbb{R}$, let $f : X \times Y \to \mathbb{R}$ be given by $f(x,y) = \|x\|^2 + y^2 - 1$. Then f is of class C^∞ and for $(a,b) := (0,1)$ one has

$$Df(a,b)(u,v) = 2(a \mid u) + 2bv = 2v,$$

whence $D_Y f(a,b) = 2I_Y$ is invertible and $D_Y f(a,b)^{-1} = (1/2)I_Z$. Here we can take $U := B(a,1) \times (0,+\infty)$, $V := B(a,1)$, and the implicit function is given by $h(x) = (1 - \|x\|^2)^{1/2}$. As mentioned above, it is not always the case that U and h can be described explicitly as in this classical parameterization of the upper hemisphere.

When Z is finite-dimensional, the regularity assumption on f can be relaxed in two ways.

Theorem 2.79. *Let X, Y, Z be Banach spaces, Y and Z being finite-dimensional, let W be an open subset of $X \times Y$, and let $f : W \to Z$ be Fréchet differentiable at $(a,b) \in W$ such that $f(a,b) = 0$ and the partial derivative $D_Y f(a,b)$ is an isomorphism from Y onto Z. Then there exist open neighborhoods U of (a,b) and V of a in W and X respectively and a map $h : V \to Y$ Fréchet differentiable at a such that $h(a) = b$ and*

$$\forall x \in V, \qquad f(x, h(x)) = 0.$$

Differentiating this relation, we recover the value of $Dh(a)$:

$$Dh(a) = -D_Y f(a,b)^{-1} \circ D_X f(a,b).$$

The proof below is slightly simpler when $A := D_X f(a,b) = 0$; one can reduce it to that case by a linear change of variables.

Proof. Using translations and composing f with $D_Y f(a,b)^{-1}$, we may suppose $(a,b) = (0,0)$, $Z = Y$, and $D_Y f(a,b) = I_Y$. Let $r : W \to Y$ be a remainder such that

$$f(x,y) := Ax + y + r(x,y).$$

For $\varepsilon \in (0,1/2]$ let $\delta := \delta(\varepsilon) > 0$ be such that $\delta B_{X\times Y} \subset W$, $\|r(x,y)\| \leq \varepsilon(\|x\| + \|y\|)$ for all $(x,y) \in \delta B_{X\times Y}$. Let $\beta := \delta/2$, $\alpha := (2\|A\|+1)^{-1}\beta$, and for $x \in \alpha B_X$ let $k_x : \beta B_Y \to Y$ be given by

$$k_x(y) := -Ax - r(x,y).$$

Then k_x maps βB_Y into itself, since for $y \in \beta B_Y$ we have $\|k_x(y)\| \leq \|A\|\alpha + (1/2)(\alpha+\beta) \leq \beta$. The Brouwer fixed-point theorem ensures that k_x has a fixed point $y_x \in \beta B_Y$: $-Ax - r(x,y_x) = y_x$. Then setting $h(x) := y_x$, we have $f(x,h(x)) = Ax + h(x) + r(x,h(x)) = 0$. It remains to show that h is differentiable at 0. Since

$$\|h(x)\| = \|k_x(h(x))\| \leq \|A\|\,\|x\| + \varepsilon\,\|x\| + \varepsilon\,\|h(x)\|,$$

so that $\|h(x)\| \leq (1-\varepsilon)^{-1}(\|A\|+\varepsilon)\,\|x\|$, we get

$$\|h(x) + Ax\| = \|r(x,h(x))\| \leq \varepsilon\,\|x\| + \varepsilon\,\|h(x)\| \leq \varepsilon(1-\varepsilon)^{-1}(\|A\|+1)\,\|x\|.$$

This shows that h is differentiable at 0 with derivative $-A$. □

A similar (and simpler) proof yields the first assertion of the next statement.

Theorem 2.80 [785]. *Let X and Y be normed spaces, Y being finite-dimensional, and let $f : X \to Y$ be continuous on a neighborhood of $a \in X$ and differentiable at a, with $f'(a)(X) = Y$. Then there exist a neighborhood V of $b := f(a)$ in Y and a right inverse $g : V \to X$ that is differentiable at a and such that $g(b) = a$.*

If C is a convex subset of X, if $a \in C$, and if $f'(a)(\mathrm{cl}(\mathbb{R}_+(C-a))) = Y$, one can even get that $g(V) \subset C$ if one does not require that the directional derivative of g at b be linear.

The second weakening of the assumptions concerns the kind of differentiability.

Theorem 2.81. *Let X, Y, Z be Banach spaces, Y and Z being finite-dimensional, let W be an open subset of $X \times Y$, and let $f : W \to Z$ be a map of class D^1 at $(a,b) \in W$ such that $f(a,b) = 0$ and the partial derivative $D_Y f(a,b)$ is an isomorphism from Y onto Z. Then there exist open neighborhoods U of (a,b) and V of a in W and X respectively and a map $h : V \to Y$ of class D^1 such that $h(a) = b$ and*

$$((x,y) \in U,\ f(x,y) = 0) \Longleftrightarrow (x \in V,\ y = h(x)).$$

Proof. We may suppose W is a ball $B((a,b),\rho_0)$, $Y = Z$, $D_Y f(a,b) = I_Y$. With the notation of the preceding proof, using the compactness of the unit ball of Y, we may suppose the remainder r satisfies, for $\rho \in (0,\rho_0)$ and every $x \in \rho B_X,\ y,y' \in \rho B_Y$,

$$\left\|r(x,y) - r(x,y')\right\| = \left\|\int_0^1 (D_Y f(x,(1-t)y+ty') - I_Y)(y-y')\mathrm{d}t\right\| \le c(\rho)\left\|y-y'\right\|,$$

where $c(\rho) \to 0$ as $\rho \to 0_+$. Taking ρ_0 small enough, we see that the map k_x is a contraction with rate $c(\rho_0) \le 1/2$. Picking $\alpha \in (0,\rho_0)$ so that for $x \in \alpha B_X$, $\|k_x(0)\| = \|-f(x,0)\| \le \rho/2$, the Banach–Picard contraction theorem ensures that k_x has a unique fixed point y_x in the ball ρB_Y. Then setting $h(x) := y_x$, we have $f(x,h(x)) = 0$, and y_x is the unique solution of the equation $f(x,y) = 0$ in the ball ρB_Y. Moreover, h is continuous as a uniform limit of continuous maps given by iterations. Restricting f to $X_1 \times Y$, where X_1 is an arbitrary finite-dimensional subspace of X, we get that h is Gâteaux differentiable. Since $\mathrm{Iso}(Y)$ is an open subset of $L(Y,Y)$ and since $(x,y) \mapsto D_Y f(x,y)$ is continuous for the norm of $L(Y,Y)$ by the above argument, we obtain from the relation

$$Dh(x)v = -D_Y f(x,h(x))^{-1}(D_X f(x,h(x))v)$$

that $(x,v) \mapsto Dh(x)v$ is continuous. □

Exercises

1. Show that the inverse mapping theorem can be deduced from the implicit mapping theorem by considering the map $(x,y) \mapsto y - f(x)$.

2. Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be given by

$$f(w,x,y,z) = (w+x+y+z, w^2+x^2+y^2+z-2, w^3+x^3+y^3+z).$$

Show that there exist a neighborhood V of $a := 0$ in $\mathbb{R}$ and a map $h : V \to \mathbb{R}^3$ of class C^∞ such that $h(0) = (0,-1,1)$ and $f(h(z),z) = 0$ for every $z \in V$. Compute the derivative of h at 0.

3. Let X be the space of square $n \times n$ matrices and let $f : X \times \mathbb{R} \to \mathbb{R}$ be given by $f(A,r) = \det(A - rI)$. Let $r \in \mathbb{R}$ be such that $f(A,r) = 0$ and $D_2 f(A,r) \ne 0$. Show that there exist an open neighborhood U of A in X and a function $\lambda : U \to \mathbb{R}$ of class C^∞ such that for each B in U, $\lambda(B)$ is a simple eigenvalue of B.

4. Given Banach spaces $W, X, Z, Y := Z^*$, maps $f : W \times X \to \mathbb{R}$, $g : W \times X \to Z$ of class C^2, consider the parameterized mathematical programming problem

$$(\mathscr{P}_w)\ \text{ minimize } f(w,x) \text{ subject to } g(w,x) = 0$$

and let $p(w)$ be its value. Suppose that for some $\overline{w} \in W$ and a solution $\overline{x} \in X$ of $(\mathscr{P}_{\overline{w}})$ the derivative $B := D_X g(\overline{w}, \overline{x})$ is surjective and its kernel N has a topological supplement M. Let ℓ be the *Lagrangian* of $(\mathscr{P}_{\overline{w}})$:

$$\ell(w,x,y) := f(w,x) + \langle y, g(w,x) \rangle,$$

and let $\overline{y}$ be a multiplier at $\overline{x}$, i.e., an element of Y such that $D_X \ell(\overline{w}, \overline{x}, \overline{y}) = 0$. Suppose $D_X^2 \ell(\overline{w}, \overline{x}, \overline{y}) \mid N$ induces an isomorphism from N onto $N^* \simeq M^\perp$. Let $A := D_X^2 \ell(\overline{w}, \overline{x}, \overline{y})$.

(**a**) Show that for every $(x^*, z) \in X^* \times Z$ the system

$$Au + B^\mathsf{T} v = x^*,$$
$$Bu = z,$$

has a unique solution $(u, v) \in X \times Y$ continuously depending on (x^*, z).

(**b**) Show that the Karush–Kuhn–Tucker system

$$D_X f(w,x) + y \circ D_X g(w,x) = 0,$$
$$g(w,x) = 0,$$

determines $(x(w), y(w))$ as an implicit function of w in a neighborhood of $\overline{w}$ with $x(\overline{w}) = \overline{x}$, $y(\overline{w}) = \overline{y}$, the multiplier at $\overline{x}$.

(**c**) Suppose $x(w)$ is a solution to $(\mathscr{P}_w)$ for w close to $\overline{w}$. Show that p is of class C^1 near $\overline{w}$. Using the relations $p(w) = \ell(w, x(w), y(w))$, $D_X \ell(w, x(w), y(w)) = 0$, $D_Y \ell(w, x(w), y(w)) = 0$, show that $Dp(w) = D_W \ell(w, x(w), y(w))$.

(**d**) Deduce from what precedes that p is of class C^2 around $\overline{w}$ and give the expression of $D^2 p(\overline{w}) := (p'(\cdot))'(\overline{w})$.

2.5.4 The Legendre Transform

As an application of inversion results, let us give an account (and even a refinement) of the classical notion of Legendre function of class C^k. We will see that the Legendre transform enables one to pass from the Euler–Lagrange equations of the calculus of variations to the Hamilton equations, which are explicit (rather than implicit) differential equations of first order (instead of second order). Recall that a map $g : U \to V$ between two metric spaces is *stable* or is *Stepanovian* if for every $\overline{u} \in U$ there exist some $r > 0$, $c \in \mathbb{R}_+$ such that for every $u \in B(\overline{u}, r)$ one has

$$d(g(u), g(\overline{u})) \leq c d(u, \overline{u}).$$

Such an assumption is clearly a weakening of the Lipschitz condition.

Definition 2.82. A function $f : U \to \mathbb{R}$ on an open subset U of a normed space X is a (classical) *Legendre function* if it is differentiable, and its derivative $f' : U \to Y := X^*$ is a Stepanovian bijection onto an open subset V of Y whose inverse h is Stepanovian.

Then one defines the *Legendre transform* of f as the function $f^L : V \to \mathbb{R}$ given by

$$f^L(y) := \langle h(y), y\rangle - f(h(y)), \qquad y \in V.$$

Since h is just a Stepanovian function, it is surprising that f^L is in fact of class C^1.

Lemma 2.83. *If f is a Legendre function on U, then its Legendre transform f^L is of class C^1 on $V := f'(U)$ and of class C^k ($k \geq 1$) if f is of class C^k. Moreover, f^L is a Legendre function, $\left(f^L\right)^L = f$ and for all $(u,v) \in U \times V$ one has*

$$v = Df(u) \Leftrightarrow u = Df^L(v).$$

Proof. Given $v := Df(u) \in V$, let $y \in V - v$, let $x := h(v+y) - h(v) \in U - u$, and let $r(x) = f(u+x) - f(u) - Df(u)x$. Then since $h(v) = u$, $h(v+y) = u + x$, one has

$$\begin{aligned} f^L(v+y) - f^L(v) - \langle u,y\rangle &= \langle u+x, v+y\rangle - f(u+x) - \langle u,v\rangle + f(u) - \langle u,y\rangle \\ &= \langle x, v+y\rangle - Df(u)(x) - r(x) = \langle x,y\rangle - r(x). \end{aligned}$$

Since there exists $c \in \mathbb{R}_+$ such that $\|x\| \leq c\|y\|$ for $\|y\|$ small enough, the last right-hand side is a remainder as a function of y. Thus f^L is differentiable at v and $Df^L(v) = u = h(v)$. Therefore $(f^L)' = h$ is a bijection with inverse f' and f^L is a Legendre function. Now

$$\left(f^L\right)^L(u) = \langle Df^L(v), v\rangle - f^L(v) = \langle u,v\rangle - (\langle u,v\rangle - f(u)) = f(u).$$

When f is of class C^k, $(f^L)' = h$ is of class C^{k-1}, as an induction shows, thanks to the Stepanov property of f' and h. □

Exercise. Let X be a normed space, let $A : X \to X^*$ be a linear isomorphism, let $b \in X^*$, and let f be given by $f(x) := (1/2)\langle Ax, x\rangle + \langle b, x\rangle$ for $x \in X$. Show that f is a Legendre function and compute f^L.

2.5.5 Geometric Applications

When looking at familiar objects such as forks, knives, funnels, roofs, spires, one sees that some points are smooth, while some other points of the objects present ridges or peaks or cracks. Mathematicians have found concepts that enable them to deal with such cases.

The notions of (regular) curve, surface, hypersurface, and so on can be embodied in a general framework in which some differential calculus can be done. The underlying idea is the possibility of straightening a piece of the set; for this purpose, some forms of the inverse mapping theorem will be appropriate.

We first define a notion of smoothness for a subset S of a normed space X around some point a.

Definition 2.84. A subset S of a normed space X is said to be C^k*-smooth* around a point $a \in S$ if there exist normed spaces Y, Z, an open neighborhood U of a in X, an open neighborhood V of 0 in $Y \times Z$, and a C^k-diffeomorphism $\varphi : U \to V$ such that $\varphi(a) = 0$ and

$$\varphi(U \cap S) = (Y \times \{0\}) \cap V. \tag{2.25}$$

A subset S of a normed space X is said to be a *submanifold of class* C^k if it is C^k-smooth around each of its points.

Thus, φ straightens $U \cap S$ onto the piece $(Y \times \{0\}) \cap V$ of the linear space $Y \times \{0\}$, which can be identified with a neighborhood of 0 in Y. The map φ is called a *chart*, and a collection $\{\varphi_i\}$ of charts whose domains form a covering of S is called an *atlas*. When Y is of dimension d, one says that S is of dimension d around a. When Z is of dimension c, one says that S is of codimension c around a.

The following example can be seen as a general model.

Example. Let $X := Y \times Z$, where Y, Z are normed spaces, let W be an open subset of Y, and let $f : W \to Z$ be a map of class C^k. Then its graph $S := \{(w, f(w)) : w \in W\}$ is a C^k-submanifold of X: taking $U := V := W \times Z$, and setting $\varphi(w,z) := (w, z - f(w))$, we define a C^k-diffeomorphism from U onto V with inverse given by $\varphi^{-1}(w,z) = (w, z + f(w))$ for which (2.25) is satisfied. □

When in the preceding example we take $Z := \mathbb{R}$ and the epigraph $E := \{(w,y) \in W \times \mathbb{R} : y \geq f(w)\}$ of f, we get a model for the notion of submanifold with boundary. We just give a formal definition in which a subset Z_+ of a normed space Z is said to be a *half-space* of Z if there exists some $h \in Z^* \setminus \{0\}$ such that $Z_+ := h^{-1}(\mathbb{R}_+)$.

Definition 2.85. A subset S of a normed space X is said to be a C^k*-submanifold with boundary* if for every point a of S, either S is C^k-smooth around a or there exist normed spaces Y, Z, a half-space Z_+ of Z, an open neighborhood U of a in X, an open neighborhood V of 0 in $Y \times Z$, and a C^k-diffeomorphism $\varphi : U \to V$ such that $\varphi(a) = 0$ and

$$\varphi(U \cap S) = (Y \times Z_+) \cap V.$$

Such a notion is useful for giving a precise meaning to the expression "S is a regular open subset of $\mathbb{R}^d$" (an improper expression, since usually one considers the closure of such a set).

There are two usual ways of obtaining submanifolds: through equations and through parameterizations. For instance, the graph S of the preceding example can be defined either as the image under $(I_W, f) : w \mapsto (w, f(w))$ of the parameter space

W or as the set of points $(y,z) \in Y \times Z$ satisfying $y \in W$ and the equation $z - f(y) = 0$. As a more concrete example, we observe that for given $a, b \in \mathbb{P}$, the ellipse

$$E := \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$$

can be seen as the image of the parameterization $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(t) := (a\cos t, b\sin t)$.

Exercise. Give parameterizations for the ellipsoid $\left\{ (x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$ and do the same for the other surfaces of $\mathbb{R}^3$ defined by quadratic forms.

Even if S is not smooth around $a \in S$, one can get an idea of its shape around a using an approximation. The concept of tangent cone offers such an approximation; it can be seen as a geometric counterpart to the directional derivative.

Definition 2.86. The *tangent cone* (or *contingent cone*) to a subset S of a normed space X at some point a in the closure of S is the set $T(S,a)$ of vectors $v \in X$ such that there exist sequences $(v_n) \to v$, $(t_n) \to 0_+$ for which $a + t_n v_n \in S$ for all $n \in \mathbb{N}$.

Equivalently, one has $v \in T(S,a)$ if and only if there exist sequences (a_n) in S, $(t_n) \to 0_+$ such that $(v_n) := (t_n^{-1}(a_n - a)) \to v$: v is the limit of a sequence of secants to S issued from a.

Some rules for dealing with tangent cones are given in the next lemma, whose elementary proof is left as an exercise.

Lemma 2.87. *Let X be a normed space, let S, S' be subsets of X such that $S \subset S'$. Then for every $a \in S$ one has $T(S,a) \subset T(S',a)$.*

If U is an open subset of X, then for every $a \in S \cap U$ one has $T(S,a) = T(S \cap U, a)$.

If X' is another normed space, if $g : U \to X'$ is Hadamard differentiable at a, and if $S' \subset X'$ contains $g(S \cap U)$, then one has $Dg(a)(T(S,a)) \subset T(S', g(a))$.

If $\varphi : U \to V$ is a C^k-diffeomorphism between two open subsets of normed spaces X, X' and if S is a subset of X containing a, then for $S' := \varphi(S \cap U)$ and $a' := \varphi(a)$, one has $T(S', a') = D\varphi(a)(T(S,a))$.

Exercise. Deduce from the second assertion of the lemma that for $g : U \to X'$ Hadamard differentiable at a, $b := g(a)$, $S := g^{-1}(b)$ one has $T(S,a) \subset \ker Dg(a)$. Moreover, if for some $c > 0$, $\rho > 0$ one has $d(x, g^{-1}(b)) \leq cd(g(x), b)$ for all $x \in B(a, \rho)$, then one has $T(S,a) = \ker Dg(a)$.

Exercise. Let $S := \{(x,y) \in \mathbb{R}^2 : x^3 = y^2\}$. Check that $T(S,(0,0)) = \mathbb{R}_+ \times \{0\}$.

When S is smooth around $a \in S$ in the sense of Definition 2.84, one can give an alternative characterization of $T(S,a)$ in terms of velocities.

Proposition 2.88. *If S is C^1-smooth around $a \in S$, then the tangent cone $T(S,a)$ to S at a coincides with the set $T^I(S,a)$ of $v \in X$ such that there exist $\tau > 0$ and $c : [0, \tau] \to X$ right differentiable at 0 with $c'_+(0) = v$ and satisfying $c(0) = a$, $c(t) \in S$ for all $t \in [0, \tau]$. Moreover, if $\varphi : U \to V$ is a C^1-diffeomorphism such that $\varphi(a) = 0$*

and $\varphi(S\cap U)=(Y\times\{0\})\cap V$*, then one has* $T(S,a)=(D\varphi(a))^{-1}(Y\times\{0\})$*, and* $T(S,a)$ *is a closed linear subspace of* X*.*

Proof. The result follows from Lemma 2.87 and the observation that if S is an open subset of some closed linear subspace L of X then $T(S,a)=L=T^I(S,a)$. □

Now let us turn to sets defined by equations. We need the following result.

Theorem 2.89 (Submersion theorem). *Let* X *and* Z *be Banach spaces, let* W *be an open subset of* X*, and let* $g:W\to Z$ *be a map of class* C^k *with* $k\geq 1$ *such that for some* $a\in W$ *the map* $Dg(a)$ *is surjective and its kernel* N *has a topological supplement* M *in* X*. Then there exist an open neighborhood* U *of* a *in* W *and a diffeomorphism* φ *of class* C^k *from* U *onto a neighborhood* V *of* $(0,g(a))$ *in* $N\times Z$ *such that* $\varphi(a)=(0,g(a))$*,*

$$g\mid U=p\circ\varphi,$$

where p *is the canonical projection from* $N\times Z$ *onto* Z*. In particular,* g *is open around* a *in the sense that for every open subset* U' *of* U*, the image* $g(U')$ *is open.*

This result shows that the nonlinear map g has been straightened into a simple continuous linear map, a projection, using the diffeomorphism φ.

Proof. Let $F:W\to N\times Z$ be given by $F(x)=(p_N(x)-p_N(a),g(x))$, where $p_N:X\to N$ is the projection on N associated with the isomorphism between X and $M\times N$. Then F is of class C^k and $DF(a)(x)=(p_N(x),Dg(a)(x))$. Clearly $DF(a)$ is injective: when $p_N(x)=0$, $Dg(a)(x)=0$, one has $x\in M\cap N$, hence $x=0$. Let us show that $DF(a)$ is surjective: given $(y,z)\in N\times Z$, there exists $v\in X$ such that $Dg(a)(v)=z$, and since $y-p_N(v)\in N$, for $x:=v+y-p_N(v)$, we have that $Dg(a)(x)=Dg(a)(v)=z$ and $p_N(x)=p_N(y)=y$. Thus, by the Banach isomorphism theorem, we have that $DF(a)$ is an isomorphism of X onto $N\times Z$. The inverse mapping theorem ensures that the restriction φ of F to some open neighborhood U of a is a C^k-diffeomorphism onto some neighborhood V of $(0,g(a))$. □

Note that for $Z:=\mathbb{R}$, the condition on g reduces to the following: g is of class C^k and $g'(a)\neq 0$. Note also that when $N:=\{0\}$, we recover the inverse function theorem.

The application we have in view follows readily.

Corollary 2.90. *Let* X *and* Z *be Banach spaces, let* W *be an open subset of* X*, and let* $g:W\to Z$ *be a map of class* C^k *with* $k\geq 1$*. Let*

$$S:=\{x\in W:g(x)=0\}.$$

Suppose that for some $a\in S$ *the map* $g'(a):=Dg(a)$ *is surjective and its kernel* N *has a topological supplement in* X*. Then* S *is* C^k*-smooth around* a*. Moreover,* $T(S,a)=\ker g'(a)$*.*

Proof. Using the notation of the submersion theorem, setting $Y:=N$, we see that Definition 2.84 is satisfied, noting that for $x\in U$ we have $x\in S\cap U$ iff $p(\varphi(x))=$

$g(x) = 0$, iff $\varphi(x) \in (Y \times \{0\}) \cap V$. Now, the preceding proposition asserts that $T(S,a) = (\varphi'(a))^{-1}(Y \times \{0\})$. But since $g \mid U = p \circ \varphi$, we have $g'(a) = p \circ \varphi'(a)$, $\ker g'(a) = (\varphi'(a))^{-1}(\ker p) = (\varphi'(a))^{-1}(Y \times \{0\})$. Hence $T(S,a) = \ker g'(a)$. □

The regularity condition on g can be relaxed thanks to the Lyusternik–Graves theorem.

Proposition 2.91 (Lyusternik). *Let X and Y be Banach spaces, let W be an open subset of X, and let $g : W \to Y$ be circa-differentiable at $a \in S := \{x \in W : g(x) = 0\}$, with $g'(a)(X) = Y$. Then $T(S,a) = \ker g'(a)$.*

Proof. The inclusion $T(S,a) \subset \ker g'(a)$ follows from Lemma 2.87. Conversely, let $v \in \ker g'(a)$. Theorem 2.67 yields some κ, $\rho > 0$ such that for all $w \in B(a,\rho)$ there exists some $x \in W$ such that $g(x) = y := 0$, $\|x - w\| \le \kappa \|g(w)\|$. Taking $w := a + tv$ with $t > 0$ so small that $w \in B(a,\rho)$, we get some $x_t \in S$ satisfying $\|x_t - (a + tv)\| \le o(t) := \kappa \|g(\overline{x} + tv)\|$. Thus $v \in T(S,a)$ and even $v \in T^I(S,a)$. □

In the following example, we use the fact that when $Y = \mathbb{R}$, the surjectivity condition on $g'(a)$ reduces to $g'(a) \neq 0$ (or $\nabla g(a) \neq 0$ if X is a Hilbert space).

Example–Exercise. Let X be a Hilbert space and let $g : X \to \mathbb{R}$ be given by $g(x) := \frac{1}{2}(A(x) \mid x) - \frac{1}{2}$, where A is a linear isomorphism from X onto X that is symmetric, i.e., such that $(Ax \mid y) = (Ay \mid x)$ for every $x, y \in X$. Let $S := g^{-1}(\{0\})$. For all $a \in S$ one has $\nabla g(a) = A(a) \neq 0$, since $(A(a) \mid a) = 1$. Thus S is a C^∞-submanifold of X. Taking $X = \mathbb{R}^2$ and appropriate isomorphisms A, find the classical conic curves; then take $X = \mathbb{R}^3$ and find the classical conic surfaces, including the sphere, the ellipsoid, the paraboloid, and the hyperboloid.

A variant of the submersion theorem can be given with differentiability instead of circa-differentiability when the spaces are finite-dimensional. Its proof (we skip) relies on the Brouwer fixed-point theorem rather than on the contraction theorem.

Proposition 2.92. *Let X and Z be Banach spaces, Z being finite-dimensional, let W be an open subset of X, and let $g : W \to Z$ be Hadamard differentiable at $a \in W$, with $Dg(a)(X) = Z$. Then there exist open neighborhoods U of a in W, V of $g(a)$ in Z and a map $h : V \to U$ that is differentiable at $g(a)$ and such that $h(g(a)) = a$, $g \circ h = I_V$. In particular, g is open at a.*

Now let us turn to representations via parameterizations. We need the following result.

Theorem 2.93 (Immersion theorem). *Let P and X be Banach spaces, let O be an open subset of P, and let $f : O \to X$ be a map of class C^k with $k \ge 1$ such that for some $\overline{p} \in O$ the map $Df(\overline{p})$ is injective and its image Y has a topological supplement Z in X. Then there exist open neighborhoods U of $a := f(\overline{p})$ in X, Q of $\overline{p}$ in O, W of 0 in Z and a C^k-diffeomorphism $\psi : V := Q \times W \to U$ such that $\psi(q,0) = f(q)$ for all $q \in Q$.*

Again the conclusion can be written in the form of a commutative diagram, since $f \,|\, Q = \psi \circ j$, where $j : Q \to Q \times W$ is the canonical injection $y \mapsto (y,0)$. Again the nonlinear map f has been straightened by ψ into a linear map $j = \psi^{-1} \circ (f \,|\, Q)$.

Proof. Let $F : O \times Z \to X$ be given by $F(p,z) = f(p) + z$. Then F is of class C^k and $F'(\overline{p},0)(p,z) = f'(\overline{p})(p) + z$ for $(p,z) \in P \times Z$, so that $F'(\overline{p},0)$ is an isomorphism from $P \times Z$ onto $Y + Z = X$. The inverse mapping theorem asserts that F induces a C^k-diffeomorphism ψ from some open neighborhood of $(\overline{p},0)$ onto some open neighborhood U of $f(\overline{p})$. Taking a smaller neighborhood of $(\overline{p},0)$ if necessary, we may suppose it has the form of a product $Q \times W$. Clearly, $\psi(q,0) = f(q)$ for $q \in Q$. □

Example–Exercise. Let $P := \mathbb{R}^2$, $O := (-\pi,\pi) \times (-\pi/2, \pi/2)$, $X := \mathbb{R}^3$, and let f be given by $f(\varphi,\theta) := (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\varphi)$. Identify the image of f.

Exercise. Let us note that the image $f(O)$ of f is not necessarily a C^k-submanifold of X. Find a counterexample with $P := \mathbb{R}$, $X := \mathbb{R}^2$.

A topological assumption ensures that the image $f(O)$ is a C^k-submanifold of X.

Corollary 2.94 (Embedding theorem). *Let P and X be Banach spaces, let O be an open subset of P, and let $f : O \to X$ be a map of class C^k with $k \geq 1$ such that for every $p \in O$ the map $f'(p)$ is injective and its image has a topological supplement in X. Then if f is a homeomorphism from O onto $f(O)$, its image $S := f(O)$ is a C^k-submanifold of X.*

Moreover, for every $p \in O$ one has $T(S, f(p)) = f'(p)(P)$.

One says that f is an embedding of O into X and that S is parameterized by O.

Proof. Given $a := f(p)$ in S, with $p \in O$, we take $Q_a \subset O$, $U_a \subset X$, $W_a \subset Z$ and a C^k-diffeomorphism $\psi_a : V_a := Q_a \times W_a \to U_a$ such that $\psi_a(q,0) = f(q)$ for all $q \in Q_a$ as in the preceding theorem. Performing a translation in P, we may suppose $p = 0$. Using the assumption that f is a homeomorphism from O onto $S = f(O)$, we can find an open subset U'_a of X such that $f(Q_a) = S \cap U'_a$. Let $U := U_a \cap U'_a$, $V := \psi_a^{-1}(U)$, $\varphi := \psi_a^{-1} \,|\, U$, $Y := P$, so that $\varphi(a) = (0,0)$. Let us check relation (2.25), i.e., $\varphi(S \cap U) = (Y \times \{0\}) \cap V$. For all $(y,0) \in (Y \times \{0\}) \cap V$, we have $x := \varphi^{-1}(y,0) = \psi_a(y,0) = f(y) \in S$, hence $x \in S \cap U$; conversely, when $x \in S \cap U = f(Q_a)$ there is a unique $q \in Q_a$ such that $x = f(q)$, so that $x = \psi_a(q,0) = \varphi^{-1}(q,0)$ and $\varphi(x) = (q,0) \in (Y \times \{0\}) \cap V$.

Then $T(S,a) = T(S \cap U, a) = (\varphi'(a))^{-1}(T((Y \times \{0\}) \cap V, 0))$, and, since $T((Y \times \{0\}) \cap V, 0) = \psi'_a(0)(P \times \{0\}) = Y \times \{0\}$, we get $T(S,a) = Y = f'(p)(P)$. □

Exercises

1. (**Conic section**) Let $S \subset \mathbb{R}^3$ be defined by the equations $x^2 + y^2 - 1 = 0$, $x - z = 0$. Show that S is a submanifold of $\mathbb{R}^3$ of class C^∞ (it has been known since Apollonius that S is an ellipse). Find an explicit diffeomorphism (in fact linear isomorphism) sending S onto an ellipse of the plane $\mathbb{R}^2 \times \{0\}$.

2. (**Viviani's window**) Let S be the subset of $\mathbb{R}^3$ defined by the system $x^2+y^2=x$, $x^2+y^2+z^2-1=0$. Show that S is a submanifold of $\mathbb{R}^3$ of class C^∞.

3. (**The torus**) Let $r>s>0$, let $O:=(0,2\pi)\times(0,2\pi)$, and let $f:O\to\mathbb{R}^3$ be given by $f(\alpha,\beta)=((r+s\cos\beta)\cos\alpha,(r+s\cos\beta)\sin\alpha,s\sin\beta)$. Show that f is an embedding onto the torus $\mathbb{T}$ deprived from its greatest circle and from the set $\mathbb{T}\cap(\mathbb{R}_+\times\{0\}\times\mathbb{R})$, where

$$\mathbb{T}:=\left\{(x,y,z)\in\mathbb{R}^3:\left(\sqrt{x^2+y^2}-r\right)^2+z^2=s^2\right\}.$$

4. Using the submersion theorem, show that $\mathbb{T}$ is a C^∞-submanifold of $\mathbb{R}^3$.

5. (**a**) (**Beltrami's tractricoid**) Let $f:\mathbb{R}\to\mathbb{R}^2$ be given by $f(t):=(1/\cosh t, t-\tanh t)$. Determine the points of $T:=f(\mathbb{R})$ that are smooth.
(**b**) (**Beltrami's pseudosphere**) Let $g(s,t):=(\cos s/\cosh t,\sin s/\cosh t,t-\tanh t)$. Determine the points of $S:=g(\mathbb{R}^2)$ that are smooth. They form a surface of (negative) constant Gaussian curvature. It can serve as a model for hyperbolic geometry.

6. Study the **Roman surface** of equation $x^2y^2+y^2z^2+z^2x^2-xyz=0$. Consider its parameterization $(\theta,\varphi)\mapsto(\cos\theta\cos\varphi\sin\varphi,\sin\theta\cos\varphi\sin\varphi,\cos\theta\sin\theta\cos^2\varphi)$.

7. Study the **cross-cap surface** $\{((1+\cos v)\cos u,(1+\cos v)\sin u,\tanh(u-\pi)\sin v):(u,v)\in[0,2\pi]\times[0,1]\}$ and compare it with the *self-intersecting disk*, the image of $[0,2\pi]\times[0,1]$ by the parameterization $(u,v)\mapsto(v\cos 2u, v\sin 2u, v\cos u)$.

8. Study **Whitney's umbrella** $\{(uv,u,v^2):(u,v)\in\mathbb{R}^2\}$. Check that it is determined by the equation $x^2-y^2z=0$. Such a surface is of interest in the theory of singularities. For this surface or the preceding one, make some drawings if you can or find some on the Internet.

9. Let $O:=(0,1)\cup(1,\infty)\subset\mathbb{R}$, $f:O\to\mathbb{R}^2$ being given by $f(t)=(t+t^{-1},2t+t^{-2})$. Show that f is an embedding, but that its continuous extension to $(0,+\infty)$ given by $f(1)=(2,3)$ is of class C^k but is not an immersion.

10. Let X be a normed space and let $f:X\to\mathbb{R}$ be Lipschitzian around $x\in X$. Show that f is Hadamard differentiable at $x\in X$ iff the tangent cone to the graph G of f at $(x,f(x))$ is a hyperplane.

11. Show that the fact that the tangent cone at $(x,f(x))$ to the epigraph E of f is a half-space does not imply that f is Hadamard differentiable at x.

2.5.6 The Method of Characteristics

Let us consider the partial differential equation

$$F(w, Du(w), u(w)) = 0, \qquad w \in W_0, \tag{2.26}$$

where W is a reflexive Banach space, W_0 is an open subset of W whose boundary ∂W_0 is a submanifold of class C^2, and $F : (w, p, z) \mapsto F(w, p, z)$ is a function of class C^2 on $W_0 \times W^* \times \mathbb{R}$. We look for a solution u of class C^2 satisfying the boundary condition

$$u \mid \partial W_0 = g, \tag{2.27}$$

where $g : \partial W_0 \to \mathbb{R}$ is a given function of class C^2. We leave aside the question of compatibility conditions for the data (F, g). The method of characteristics consists in associating to (2.26) a system of ordinary differential equations (in which W^{**} is identified with W) called the *system of characteristics*:

$$w'(s) = D_p F(w(s), p(s), z(s)), \tag{2.28}$$

$$p'(s) = -D_w F(w(s), p(s), z(s)) - D_z F(w(s), p(s), z(s)) p(z), \tag{2.29}$$

$$z'(s) = \langle D_p F(w(s), p(s), z(s)), p(s) \rangle. \tag{2.30}$$

Suppose a smooth solution u of (2.26) is known. Let us relate it to a solution $s \mapsto (w(s), p(s), z(s))$ of the system (2.28)–(2.30). Let

$$q(s) := Du(y(s)), \qquad r(s) := u(y(s)),$$

where $y(\cdot)$ is the solution of the differential equation

$$y'(s) := D_p F(y(s), Du(y(s)), u(y(s))), \qquad y(0) = w_0.$$

Then

$$r'(s) = Du(y(s)) \cdot y'(s) = \langle q(s), D_p F(y(s), p(s), z(s)) \rangle.$$

For all $e \in W$, identifying W^{**} and W, we have

$$q'(s) \cdot e = D^2 u(y(s)) \cdot y'(s) \cdot e = \langle D_p F(y(s), p(s), z(s)), D^2 u(y(s)) \cdot e \rangle.$$

Now, taking the derivative of the function $F(\cdot, Du(\cdot), u(\cdot))$ and writing u, Du instead of $u(w), Du(w)$, we have

$$D_w F(w, Du, u)e + D_p F(w, Du, u) D^2 u(w) \cdot e + D_z F(w, Du, u) Du(w) e = 0.$$

Thus, replacing (w, Du, u) by $(y(s), q(s), r(s))$ and noting that e is arbitrary in W, we get

$$q'(s) = -D_w F(w(s), q(s), r(s)) - D_z F(w(s), q(s), r(s)) q(s).$$

It follows that $s \mapsto (y(s), q(s), r(s))$ is a solution of the characteristic system. Taking the same initial data $(w_0, p_0, g(w_0))$, by uniqueness of the solution of the characteristic system, we get $y(s) = w(s)$, $p(s) = q(s)$, and $z(s) = r(s) := u(w(s))$. This means that knowing the solution of the characteristic system, we get the value of u at $w(s)$. If around some point $\overline{w} \in W_0$ we can represent every point w of a neighborhood of $\overline{w}$ as the value $w(s)$ for the solution of (2.28)–(2.30) issued from some initial data, then we get u around $\overline{w}$. In the following classical example, the search for the initial data is particularly simple.

Example. Let $W := \mathbb{R}^n$, $W_0 := \mathbb{R}^{n-1} \times \mathbb{P}$, F being given by $F(w, p, z) := p \cdot b(w, z) - c(w, z)$, where $b : W_0 \times \mathbb{R} \to W$, $c : W_0 \times \mathbb{R} \to \mathbb{R}$. Then, taking into account the relation $D_p F(w(s), p(s), z(s)) \cdot p(s) = p(s) \cdot b(w(s), z(s)) = c(w(s), z(s))$, (2.28), (2.30) of the characteristic system read as a system in (w, z):

$$w'(s) = b(w(s), z(s)),$$
$$z'(s) = c(w(s), z(s)).$$

In the case that $b := (b_1, \ldots, b_n)$ is constant with $b_n \neq 0$ and $c(w, z) := z^{k+1}/k$, with $k > 0$, the solution of this system with initial data $((v, 0), g(v)) \in \mathbb{R}^n \times \mathbb{P}$ is given by

$$w_i(s) = b_i s + v_i \ (i = 1, \ldots, n-1), \quad w_n(s) = b_n s, \qquad z(s) = \frac{g(v)}{(1 - g(v)^k s)^{1/k}}.$$

It is defined for s in the interval $S := [0, g(v)^{-k})$. Given $x := (x_1, \ldots, x_n) \in W_0$ near $\overline{x} \in W_0$, the initial data v is found by solving the equations $b_i s + v_i = x_i \ (i \in \mathbb{N}_{n-1})$, $x_n = b_n s$: $v_i = x_i - a_i x_n$ with $a_i := b_i / b_n$. What precedes shows that u is given by

$$u(x) = \frac{g(x_1 - a_1 x_n, \ldots, x_{n-1} - a_{n-1} x_n)}{(1 - g(x_1 - a_1 x_n, \ldots, x_{n-1} - a_{n-1} x_n)^k x_n / b_n)^{1/k}}$$

and is defined in the set $\{(x_1, \ldots, x_n) : x_n g(x_1 - a_1 x_n, \ldots, x_{n-1} - a_{n-1} x_n)^k < b_n\}$. □

A special case of (2.26) is of great importance. It corresponds to the case $w := (x, t) \in W_0 := U \times (0, \tau)$ for some $\tau \in (0, +\infty]$ and some open subset U of a hyperplane X of W and $F((x, t), (y, v), z) := v + H(x, t, y, z)$, so that (2.26) and the boundary condition (2.27) take the form

$$D_t u(x, t) + H(x, t, D_x u(x, t), u(x, t)) = 0, \qquad (x, t) \in W_0 \times (0, \tau), \tag{2.31}$$

$$u(x, 0) = g(x), \qquad x \in W_0. \tag{2.32}$$

Such a system is called a *Hamilton–Jacobi equation*.

Let us note that as in the example of quasilinear equations, the general case can be reduced to this form under a mild condition. First, since W_0 is the interior of a smooth manifold with boundary, taking a chart, we may assume for a local study that $W_0 = U \times (0,\tau)$ for some $\tau > 0$ and some open subset U of a hyperplane X of W. Now, using the implicit function theorem around $\overline{w} \in \partial W_0$, F can be reduced to the form $F((x,t),(y,v),z) := v + H(x,t,y,z)$, provided $D_v F(\overline{w},\overline{p},\overline{z}) \neq 0$. Such a condition can be expressed intrinsically (i.e., without using the chart) by finding a vector $\overline{v}$ transverse to ∂W_0 at $\overline{w}$ such that $D_w F(\overline{w},\overline{y},\overline{z}) \cdot \overline{v} \neq 0$.

The characteristic system associated with (2.31) can be reduced to

$$x'(s) = D_y H(x(s),s,y(s),z(s)), \tag{2.33}$$

$$y'(s) = -D_x H(x(s),s,y(s),z(s)) - D_z H(x(s),s,y(s),z(s))y(s), \tag{2.34}$$

$$z'(s) = D_y H(x(s),s,y(s),z(s)) \cdot y(s) - H(x(s),s,y(s),z(s)), \tag{2.35}$$

by dropping the equation $t'(s) = 1$ and observing that we do not need an equation for $D_t u(x(s),t(s))$, since this derivative is known to be $-H(x(s),s,y(s),z(s))$. In order to take into account the dependence on the initial condition $(v, Dg(v), g(v))$, the one-jet of g at $v \in U \subset X$, let us denote by $s \mapsto (\widehat{x}(s,v),\widehat{y}(s,v),\widehat{z}(s,v))$ the solution to the system (2.33)–(2.35). Since the right-hand side of this system is of class C^1, the theory of differential equations ensures that the solution is a mapping of class C^1 in (s,v). In view of the initial data, we have

$$\forall v \in U,\ v' \in X, \qquad D_v \widehat{x}(0,v)v' = v'.$$

It follows that for all $\overline{v} \in U$ there exist a neighborhood V of $\overline{v}$ in U and some $\sigma \in (0,\tau)$ such that for $s \in (0,\sigma)$, the map $\widehat{x}_s : v \mapsto \widehat{x}(s,v)$ is a diffeomorphism from V onto $V_s := \widehat{x}(s,V)$. From the analysis that precedes, we get that for $x \in V_s$ one has $u(x,s) = \widehat{z}(s,v)$ with $v := (\widehat{x}_s)^{-1}(x)$. Thus we get a local solution to the system (2.31)–(2.32). In general, one cannot get a global solution with such a method: it may happen that for two values v_1, v_2 of v the characteristic curves issued from v_1 and v_2 take the same value for some $t > 0$.

Exercises

1. Write down the characteristic system for the **conservation law**

$$D_t u(x,t) + D_x u(x,t) \cdot b(u(x,t)) = 0, \qquad u(v,0) = g(v),$$

where $b : \mathbb{R} \to X$, $g : X \to \mathbb{R}$ are of class C^1. Check that its solution satisfies $\widehat{x}(s,v) = v + sb(g(v))$, $\hat{z}(s,v) = g(v)$. Compute $D_v \widehat{x}(s,v)$ and check that for all $\overline{v} \in X$, this element of $L(X,X)$ is invertible for (s,v) close enough to $(0,\overline{v})$. Deduce a local solution of the equation of conservation law from this property.

2. (**Haar's uniqueness theorem**) Suppose $X = \mathbb{R}$ and $H : X \times \mathbb{R} \times X^* \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition with constants k, ℓ:

$$\forall (t,x,y,y',z,z') \in T \times \mathbb{R}^4, \qquad \left|H(x,t,y,z) - H(x,t,y',z')\right| \leq k\left|y-y'\right| + \ell\left|z-z'\right|,$$

where T is the triangle $T := \{(x,t) \in X \times [0,a] : x \in [b+\ell t, c-\ell t]\}$, for some constants a,b,c. Show that if u_1, u_2 are two solutions of class C^1 in T of the system (2.31)–(2.32), then $u_1 = u_2$. [For a generalization to $X := \mathbb{R}^n$ see [925, Theorem 1.6], [960].]

3. Suppose $X = \mathbb{R}$, $g = I_X$, and $H : X \times \mathbb{R} \times X^* \times \mathbb{R} \to \mathbb{R}_\infty$ is given by $H(x,t,y,z) := |t-1|^{-1/2} y$ for $t \in [0,1)$, $+\infty$ otherwise. Using the method of characteristics, show that a solution to the system (2.31)–(2.32) is given by $u(x,t) = x - 2 + 2\sqrt{1-t}$ for $(x,t) \in X \times (0,1)$.

4. Suppose $X = \mathbb{R}$, and that g and H are given by $H(x,t,y,z) := -y^2/2$, $g(x) := x^2/2$. Using the method of characteristics, show that a solution to the system (2.31)–(2.32) is given by $u(x,t) = x^2/2(1-t)$ for $(x,t) \in X \times (0,1)$.

5. Suppose $X = \mathbb{R}$, and that g and H are given by $H(x,t,y,z) := e^{-3t} yz(a'(t)e^{2t} + b'(t)z^2) - z$, $g(x) := x$, where a and b are nonnegative functions of class C^1 satisfying $a(0) = 1$, $b(0) = 0$, $a + b > 0$. Show that the characteristics associated with the system (2.31)–(2.32) satisfy $\widehat{x}(t,v) = a(t)v + b(t)v^3$, $\widehat{z}(t) = e^t v$, so that $v \mapsto \widehat{x}(t,v)$ is a bijection. Assuming that there exists some $\tau > 0$ such that $a(t) = 0$ for $t \geq \tau$, show that $u(x,t) = e^t b(t)^{-1/3} x^{1/3}$ for $(x,t) \in X \times [\tau, \infty)$, so that u is not differentiable at $(0,t)$.

6. Suppose $X = \mathbb{R}$, and that g and H are given by $g(x) := x^2/2$, $H(x,t,y,z) := a'(t)e^{-t}y^2/2 + b'(t)e^{-3t}y^4 - z$, where a and b are as in the preceding exercise. Show that the characteristics associated with the system (2.31)–(2.32) satisfy $\widehat{x}(t,v) = a(t)v + 4b(t)v^3$, $\widehat{z}(t) = e^t(a(t)v^2/2 + 3b(t)v^4)$, so that for $t \geq \tau$, $v \mapsto \widehat{x}(t,v)$ is a bijection on a neighborhood of 0, in spite of the fact that $D_v\widehat{x}(t,0) = 0$ and $u(x,t) = 3.4^{-4/3} b(t) x^{4/3}$, so that u is of class C^1 but not C^2 around $(0,t)$.

2.6 Applications to Optimization

We will formulate necessary optimality conditions for the problem with constraint

$$(\mathscr{P}) \qquad \text{minimize } f(x) \text{ under the constraint } x \in F,$$

where F is a nonempty subset of the normed space X called the *feasible set* or the *admissible set*. These conditions will involve the concept of normal cone.

2.6.1 Normal Cones, Tangent Cones, and Constraints

In fact, we will use some variants of the concept of normal cone that fit different differentiability assumptions on the function f. When the feasible set is a convex set these variants coincide (Exercise 6) and the concept is very simple.

Definition 2.95. The *normal cone* $N(C,\overline{x})$ to a convex subset C of X at $\overline{x} \in C$ is the set of $\overline{x}^* \in X^*$ that attain their maximum on C at $\overline{x}$:

$$N(C,\overline{x}) := \{\overline{x}^* \in X^* : \forall x \in C,\ \langle \overline{x}^*, x-\overline{x}\rangle \leq 0\}.$$

Thus, when C is a linear subspace, $N(C,\overline{x}) = C^{\perp}$, where $C^{\perp}$ is the orthogonal of C (or annihilator of C) in X^*:

$$C^{\perp} := \{\overline{x}^* \in X^* : \forall x \in C,\ \langle \overline{x}^*, x\rangle = 0\}.$$

When C is a cone, one has $N(C,0) = C^0$, where C^0 is the polar cone of C.

In the nonconvex case the preceding definition has to be modified by introducing a remainder in the inequality in order to allow a certain curvature or inaccuracy.

Definition 2.96. The *firm or Fréchet normal cone* $N_F(F,\overline{x})$ to a subset F of X at $\overline{x} \in F$ is the set of $\overline{x}^* \in X^*$ for which there exists a remainder $r(\cdot)$ such that $\overline{x}^*(\cdot) + r(\cdot - \overline{x})$ attains its maximum on F at $\overline{x}$:

$$\overline{x}^* \in N_F(F,\overline{x}) \Longleftrightarrow \exists r \in o(X,\mathbb{R}) \quad \forall x \in F,\ \langle \overline{x}^*, x-\overline{x}\rangle + r(x-\overline{x}) \leq 0.$$

In other words, $\overline{x}^* \in X^*$ is a firm normal to F at $\overline{x}$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in F \cap B(\overline{x},\delta)$ one has $\langle \overline{x}^*, x-\overline{x}\rangle \leq \varepsilon \|x-\overline{x}\|$.

Equivalently,

$$\overline{x}^* \in N_F(F,\overline{x}) \quad \Longleftrightarrow \quad \limsup_{x\to\overline{x},\ x\neq\overline{x}} \frac{1}{\|x-\overline{x}\|}\langle \overline{x}^*, x-\overline{x}\rangle \leq 0.$$

We will give some properties and calculus rules in the next subsection. For the moment it is important to convince oneself that this notion corresponds to the intuitive idea of an "exterior normal" to a set, for instance by making drawings in simple cases. We shall present a necessary condition using this concept without delay. In it we say that f attains a *local maximum* (resp. *local minimum*) on F at $\overline{x}$ if $f(x) \leq f(\overline{x})$ (resp. $f(x) \geq f(\overline{x})$) for all x in some neighborhood of $\overline{x}$ in F. It is convenient to say that $\overline{x}$ is a *local maximizer* (resp. *local minimizer*) of f on F.

Theorem 2.97 (Fermat's rule). *Suppose f attains a local maximum on F at $\overline{x}$ and is Fréchet differentiable at $\overline{x}$. Then*

$$f'(\overline{x}) \in N_F(F,\overline{x}).$$

If f attains a local minimum on F at $\overline{x}$ and is Fréchet differentiable at $\overline{x}$ then

$$0 \in f'(\overline{x}) + N_F(F,\overline{x}).$$

Proof. Suppose f attains a local maximum on F at $\overline{x}$ and is differentiable at $\overline{x}$. Set

$$f(x) = f(\overline{x}) + \langle \overline{x}^*, x - \overline{x} \rangle + r(x - \overline{x})$$

with r a remainder, $\overline{x}^* := f'(\overline{x})$, so that for $x \in F$ close enough to $\overline{x}$ one has

$$\langle \overline{x}^*, x - \overline{x} \rangle + r(x - \overline{x}) = f(x) - f(\overline{x}) \leq 0.$$

Hence $\overline{x}^* \in N_F(F,\overline{x})$. Changing f into $-f$, one obtains the second assertion. □

The second formula shows how the familiar rule $f'(\overline{x}) = 0$ of unconstrained minimization has to be changed by introducing an additional term involving the normal cone. Without such an additional term the condition would be utterly invalid.

Example. The identity map $f = I_{\mathbb{R}}$ on $\mathbb{R}$ attains its minimum on $F := [0,1]$ at 0 but $f'(0) = 1$.

Example. Suppose F is the unit sphere of the Euclidean space $\mathbb{R}^3$ representing the surface of the earth and suppose f is a smooth function representing the temperature. If f attains a local minimum on F at $\overline{x}$, in general $\nabla f(\overline{x})$ is not 0; however, $\nabla f(\overline{x})$ is on the downward vertical at $\overline{x}$, and if one can increase one's altitude at that point, one usually experiences a decrease of the temperature. □

When the objective function f is not Fréchet differentiable but just Hadamard differentiable, an analogue of Fermat's rule can still be given by introducing a variant of the notion of firm normal cone. It goes as follows; although this variant appears to be more technical than the concept of Fréchet normal cone, it is a general and important notion. It can be formulated with the help of the notion of directional remainder: $r : X \to Y$ is a *directional remainder* if for all $u \in X \setminus \{0\}$ one has $r(tv)/t \to 0$ as $t \to 0_+$, $v \to u$; we write $r \in o_D(X,Y)$.

Definition 2.98. The *normal cone* (or *directional normal cone*) to the subset F at $\overline{x} \in \mathrm{cl}(F)$ is the set $N(F,\overline{x}) := N_D(F,\overline{x})$ of $\overline{x}^* \in X^*$ for which there exists a directional remainder $r(\cdot)$ such that $\overline{x}^*(\cdot) + r(\cdot - \overline{x})$ attains its maximum on F at $\overline{x}$:

$$\overline{x}^* \in N(F,\overline{x}) := N_D(F,\overline{x}) \Longleftrightarrow \exists r \in o_D(X,\mathbb{R}) \quad \forall x \in F, \quad \langle \overline{x}^*, x - \overline{x} \rangle + r(x - \overline{x}) \leq 0.$$

In other words, $\overline{x}^* \in X^*$ is a normal to F at $\overline{x}$ iff for all $u \in X \setminus \{0\}$, $\varepsilon > 0$ there exists $\delta > 0$ such that $\langle \overline{x}^*, v \rangle \leq \varepsilon$ for every $(t,v) \in (0,\delta] \times B(u,\delta)$ satisfying $\overline{x} + tv \in F$:

$$\overline{x}^* \in N(F,\overline{x}) := N_D(F,\overline{x}) \Longleftrightarrow \forall u \in X, \quad \limsup_{(t,v) \to (0_+,u),\ \overline{x}+tv \in F} \frac{1}{t} \langle \overline{x}^*, (\overline{x} + tv) - \overline{x} \rangle \leq 0.$$

Let us note that the case $u=0$ can be discarded in the preceding reformulation because the condition is automatically satisfied in this case with $\delta = \varepsilon \min(1, \|\overline{x}^*\|^{-1})$. This cone often coincides with the Fréchet normal cone and it always contains it, as the preceding reformulations show.

Lemma 2.99. *For every subset F and every $\overline{x} \in \mathrm{cl}(F)$ one has $N_F(F,\overline{x}) \subset N(F,\overline{x})$.*

The duality property we prove now compensates the complexity of the definition of the (directional) normal cone compared to the definition of the firm normal cone.

Proposition 2.100. *The normal cone to F at $\overline{x}$ is the polar cone to the tangent cone to F at $\overline{x}$:*

$$(\overline{x}^* \in N(F,\overline{x})) \Leftrightarrow (\forall u \in T(F,\overline{x}), \quad \langle \overline{x}^*, u \rangle \leq 0).$$

Proof. Given $\overline{x}^* \in N(F,\overline{x})$ and $u \in T(F,\overline{x}) \setminus \{0\}$, for every $\varepsilon > 0$, taking $\delta \in (0,\varepsilon)$ such that $\langle \overline{x}^*, v \rangle \leq \varepsilon$ for every $(t,v) \in (0,\delta] \times B(u,\delta)$ satisfying $\overline{x} + tv \in F$ and observing that such a pair (t,v) exists since $u \in T(F,\overline{x})$, we get $\langle \overline{x}^*, u \rangle \leq \langle \overline{x}^*, v \rangle + \|\overline{x}^*\| \, \|u - v\| \leq \varepsilon + \varepsilon \|\overline{x}^*\|$. Since ε is arbitrarily small, we get $\langle \overline{x}^*, u \rangle \leq 0$.

Conversely, given $\overline{x}^*$ in the polar cone of $T(F,\overline{x})$, given $u \in T(F,\overline{x})$, and given $\varepsilon > 0$, taking $\delta > 0$ such that $\delta \|\overline{x}^*\| \leq \varepsilon$, the inequality $\langle \overline{x}^*, v \rangle \leq \varepsilon$ holds whenever $t \in (0,\delta)$, $v \in t^{-1}(F - \overline{x}) \cap B(u,\delta)$, since

$$\langle \overline{x}^*, v \rangle \leq \langle \overline{x}^*, u \rangle + \langle \overline{x}^*, v - u \rangle \leq \|\overline{x}^*\| \, \|u - v\| \leq \delta \|\overline{x}^*\| \leq \varepsilon.$$

If $u \in X \setminus T(F,\overline{x})$ we can find $\delta > 0$ such that no such pair (t,v) exists. Thus, we have $\langle \overline{x}^*, v \rangle \leq \varepsilon$ for every $(t,v) \in (0,\delta] \times B(u,\delta)$ satisfying $\overline{x} + tv \in F$: $\overline{x}^* \in N(F,\overline{x})$. □

Theorem 2.101 (Fermat's rule). *Suppose f attains a local maximum on F at $\overline{x} \in F$ and is Hadamard differentiable at $\overline{x}$. Then for all $v \in T(F,\overline{x})$ one has $f'(\overline{x})v \leq 0$:*

$$f'(\overline{x}) \in N(F,\overline{x}).$$

If f attains a local minimum on F at $\overline{x}$, then for all $v \in T(F,\overline{x})$ one has $f'(\overline{x})v \geq 0$:

$$0 \in f'(\overline{x}) + N(F,\overline{x}).$$

Proof. Let V be an open neighborhood of $\overline{x}$ in X such that $f(x) \leq f(\overline{x})$ for all $x \in F \cap V$. Given $v \in T(F,\overline{x})$, let $(v_n) \to v$, $(t_n) \to 0_+$ be sequences such that $\overline{x} + t_n v_n \in F$ for all $n \in \mathbb{N}$. For n large enough, we have $\overline{x} + t_n v_n \in F \cap V$, hence $f(\overline{x} + t_n v_n) - f(\overline{x}) \leq 0$. Dividing by t_n and passing to the limit, the (Hadamard) differentiability of f at $\overline{x}$ yields $f'(\overline{x})(v) \leq 0$. □

It is possible to give a third version of Fermat's rule that does not suppose that f is differentiable; it is set in the space X instead of its dual X^*. In it, we use the *directional (lower) derivative* (or *contingent derivative*) of f given by

$$f^D(\overline{x},u) := \liminf_{(t,v)\to(0_+,u)} \frac{1}{t}(f(\overline{x}+tv) - f(\overline{x}))$$

and the tangent cone to F at $\overline{x}$ as introduced in Definition 2.86.

In view of their fundamental character, we will return to these notions of tangent and normal cones. For the moment, the definition itself suffices to give the primal version of Fermat's rule we announced. Note that this version entails the preceding theorem, since $f^D(\overline{x},\cdot) = f'(\overline{x})$ when f is Hadamard differentiable at $\overline{x}$.

Theorem 2.102. *Suppose f attains a local maximum on F at $\overline{x}$. Then*

$$f^D(\overline{x},u) \le 0 \textit{ for all } u \in T(F,\overline{x}).$$

Proof. Let $u \in T(F,\overline{x})$. There exist $(t_n) \to 0_+$, $(u_n) \to u$ such that $\overline{x}+t_n u_n \in F$ for all $n \in \mathbb{N}$. For n large enough we have $f(\overline{x}+t_n u_n) \le f(\overline{x})$, so that

$$f^D(\overline{x},u) \le \liminf_n \frac{1}{t_n}(f(\overline{x}+t_n u_n) - f(\overline{x})) \le 0. \qquad \square$$

For minimization problems, a variant of the tangent cone is required, since the rule $f^D(\overline{x},u) \ge 0$ for $u \in T(F,\overline{x})$ is not valid in general.

Example. Let $F := \{0\} \cup \{2^{-2n} : n \in \mathbb{N}\} \subset \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be even and given by $f(x) = 0$ for every $x \in F$, $f(2^{-2k+1}) = -2^{-2k+1}$, f being affine on each interval $[2^{-j}, 2^{-j+1}]$. Show that $f^D(\overline{x},1) = -1$ for $\overline{x} := 0$, although $f(\overline{x}) = \min f(F)$.

Definition 2.103. The *incident cone* (or *adjacent cone*) to F at $\overline{x} \in \mathrm{cl}(F)$ is the set

$$\begin{aligned} T^I(F,\overline{x}) &:= \{u \in X : \forall (t_n) \to 0_+, \exists (u_n) \to u, \quad \overline{x}+t_n u_n \in F \quad \forall n\} \\ &= \left\{u \in X : \forall (t_n) \to 0_+, \exists (x_n) \to \overline{x},\ (t_n^{-1}(x_n - \overline{x})) \to u, \quad x_n \in F \quad \forall n\right\}. \end{aligned}$$

It is easy to show that

$$u \in T^I(F,\overline{x}) \Leftrightarrow \lim_{t\to 0_+} \frac{1}{t} d(\overline{x}+tu, F) = 0.$$

Let us also introduce the *incident derivative* of a function f at $\overline{x}$ by

$$f^I(\overline{x},u) := \inf\{r \in \mathbb{R} : (u,r) \in T^I(E_f, \overline{x}_f)\},$$

where E_f is the epigraph of f and $\overline{x}_f := (\overline{x}, f(\overline{x}))$.

Proposition 2.104. *Suppose f is directionally stable at $\overline{x}$ in the sense that for all $u \in X \setminus \{0\}$ one has $(1/t)(f(\overline{x}+tv) - f(\overline{x}+tu)) \to 0$ as $(t,v) \to (0,u)$. If f attains a local minimum on F at $\overline{x}$, then*

$$f^I(\overline{x},u) \geq 0 \textit{ for all } u \in T(F,\overline{x}),$$
$$f^D(\overline{x},u) \geq 0 \textit{ for all } u \in T^I(F,\overline{x}).$$

Proof. Suppose, to the contrary, that there exists some $u \in T(F,\overline{x})$ such that $f^I(\overline{x},u) < 0$. Then there exists some $r < 0$ such that $(u,r) \in T^I(E_f,\overline{x}_f)$; thus, if $(t_n) \to 0_+$ and $(u_n) \to u$ are such that $\overline{x}+t_n u_n \in F$ for all $n \in \mathbb{N}$, one can find a sequence $((v_n,r_n)) \to (u,r)$ such that $\overline{x}_f + t_n(v_n,r_n) \in E_f$ for all $n \in \mathbb{N}$. Then $f(\overline{x})+t_n r_n \geq f(\overline{x}+t_n v_n)$ for all $n \in \mathbb{N}$ and

$$0 > r \geq \limsup_n (1/t_n)(f(\overline{x}+t_n v_n) - f(\overline{x})) = \limsup_n (1/t_n)(f(\overline{x}+t_n u_n) - f(\overline{x}) \geq 0,$$

a contradiction. The proof of the second assertion is similar. □

Exercises

1. Given an element $\overline{x}$ of the closure of a subset F of a normed space X, show that the tangent cone and the incident cone can be expressed in terms of limits of sets:

$$T(F,\overline{x}) = \limsup_{t\to 0_+}(1/t)(F-\overline{x}), \qquad T^I(F,\overline{x}) = \liminf_{t\to 0_+}(1/t)(F-\overline{x}).$$

2. Deduce from Exercise 1 that $v \in T(F,\overline{x})$ iff $\liminf_{t\to 0_+}(1/t)d(\overline{x}+tv,F) = 0$ and that $v \in T^I(F,\overline{x})$ if and only if $\lim_{t\to 0_+}(1/t)d(\overline{x}+tv,F) = 0$.

3. Find a subset F of $\mathbb{R}$ such that $1 \in T(F,0)$ but $T^I(F,0) = \{0\}$.

4. Show that if X is a finite-dimensional normed space, then for every subset F of X and every $\overline{x} \in \mathrm{cl}(F)$, one has $N(F,\overline{x}) = N_F(F,\overline{x})$.

5. Show that for every subset F of a normed space and every $\overline{x} \in \mathrm{cl}(F)$, the cones $N(F,\overline{x})$ and $N_F(F,\overline{x})$ are convex and closed.

6. Show that for every convex subset C of a normed space X and every $\overline{x} \in \mathrm{cl}(C)$ the cones $N(C,\overline{x})$ and $N_F(C,\overline{x})$ coincide with the normal cone in the sense of convex analysis described in Definition 2.95.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $a \in \mathbb{R}$ and such that a is a minimizer of f on some interval $[a,b]$ with $b > a$. Check that $f'(a) \geq 0$.

8. Show that the incident cone $T^I(F,\overline{x})$ can be called the velocity cone of F at $\overline{x}$ since $v \in T^I(F,\overline{x})$ iff there exists some $c : [0,1] \to F$ such that $c(0) = \overline{x}$, c is right differentiable at 0, and $c'_+(0) = v$.

2.6.2 Calculus of Tangent and Normal Cones

We devote this subsection to some calculus rules for normal cones. These rules will enable us to compute the normal cones to sets defined by equalities and inequalities, an important topic for the application to concrete optimization problems.

In order to show that the two notions of normal cone we introduced correspond to the classical notion in the smooth case, let us make some easy but useful observations.

Proposition 2.105. *The notions of normal cone and of Fréchet normal cone are local notions: if F and G are two subsets such that $F \cap V = G \cap V$ for some neighborhood V of $\overline{x}$, then $N(F,\overline{x}) = N(G,\overline{x})$ and $N_F(F,\overline{x}) = N_F(G,\overline{x})$.*

Proposition 2.106. *Given normed spaces X,Y and $\overline{x} \in F \subset X$, $\overline{y} \in G \subset Y$, one has*

$$N(F \times G, (\overline{x},\overline{y})) = N(F,\overline{x}) \times N(G,\overline{y}),$$
$$N_F(F \times G, (\overline{x},\overline{y})) = N_F(F,\overline{x}) \times N_F(G,\overline{y}).$$

Proposition 2.107. *The normal cone and the firm normal cone are antitone: for $F \subset G$ and every $\overline{x} \in \mathrm{cl}F$ one has $N(G,\overline{x}) \subset N(F,\overline{x})$ and $N_F(G,\overline{x}) \subset N_F(F,\overline{x})$. Moreover, if F is a finite union, $F = \bigcup_{i\in I} F_i$, then*

$$N(F,\overline{x}) = \bigcap_{i\in I} N(F_i,\overline{x}), \qquad N_F(F,\overline{x}) = \bigcap_{i\in I} N_F(F_i,\overline{x}).$$

This fact helps in the computation of normal cones, as the next example shows.

Example. Let $F := \{(r,s) \in \mathbb{R}^2 : rs = 0\}$, so that $F = F_1 \cup F_2$ with $F_1 := \mathbb{R} \times \{0\}$, $F_2 := \{0\} \times \mathbb{R}$. Then since F_i is a linear subspace, one has $N(F_i,0) = F_i^{\perp}$; hence $N(F,0) = F_1^{\perp} \cap F_2^{\perp} = \{0\}$.

However, the computations of normal cones to intersections are not obvious. One may just have the inclusions

$$N(F \cap G,\overline{x}) \supset N(F,\overline{x}) \cup N(G,\overline{x}), \qquad N_F(F \cap G,\overline{x}) \supset N_F(F,\overline{x}) \cup N_F(G,\overline{x}).$$

Example. Let $X := \mathbb{R}^2$ with its usual Euclidean norm and let $F := B_X + e$, $G := B_X - e$, where $e = (0,1)$. Then $N(F \cap G, 0) = \mathbb{R}^2$, whereas $N(F,0) \cup N(G,0) = \{0\} \times \mathbb{R}$.

Now let us show that the notions of normals and firm normals are invariant under differentiable transformations (diffeomorphisms).

Proposition 2.108. *Let $g : U \to V$ be a map between two open subsets of the normed spaces X and Y respectively and let $B \subset U$, $C \subset V$ be such that $g(B) \subset C$. Then if g is F-differentiable, respectively H-differentiable, at $\overline{x} \in B$, then for $\overline{y} := g(\overline{x})$, one has respectively*

$$N_F(C,\overline{y}) \subset (g'(\overline{x})^\intercal)^{-1}(N_F(B,\overline{x})), \tag{2.36}$$
$$N(C,\overline{y}) \subset (g'(\overline{x})^\intercal)^{-1}(N(B,\overline{x})).$$

Relation (2.36) is an equality when $C = g(B)$ and there exist $\rho > 0$, $c > 0$ such that

$$\forall y \in C \cap B(\overline{y},\rho), \qquad d(\overline{x}, g^{-1}(y) \cap B) \leq cd(y,\overline{y}). \tag{2.37}$$

Proof. Let $\overline{y}^*$ be an element of $N_F(C,\overline{y})$: for some remainder $r(\cdot)$ and for all $y \in C$ we have $\langle \overline{y}^*, y - \overline{y}\rangle \leq r(\|y-\overline{y}\|)$. The differentiability of g at $\overline{x}$ can be written for some remainder s:

$$g(x) - g(\overline{x}) = A(x - \overline{x}) + s(\|x - \overline{x}\|), \tag{2.38}$$

where $A := g'(\overline{x})$. Taking $x \in B$, since $y := g(x) \in C$, we get

$$\begin{aligned}\langle A^\intercal(\overline{y}^*), x - \overline{x}\rangle &= \langle \overline{y}^*, g(x) - g(\overline{x}) - s(\|x-\overline{x}\|)\rangle \\ &\leq r(\|g(x) - g(\overline{x})\|) - \langle \overline{y}^*, s(\|x-\overline{x}\|)\rangle := t(\|x - \overline{x}\|),\end{aligned}$$

where t is a remainder, since $\|g(x) - g(\overline{x})\| \leq (\|A\|+1)\|x-\overline{x}\|$ for x close enough to $\overline{x}$. The proof for the normal cone is similar. It can also be deduced from the inclusion $g'(\overline{x})(T(B,\overline{x})) \subset T(C,\overline{y})$.

Now suppose $C = g(B)$ and relation (2.37) holds for some $\rho > 0$, $c > 0$. Then for all $y \in C \cap B(\overline{y},\rho)$, there exists some $x_y \in g^{-1}(y) \cap B$ satisfying $\|x_y - \overline{x}\| \leq 2c\|y - \overline{y}\|$. Let $\overline{y}^* \in Y^*$ be such that $\overline{x}^* := g'(\overline{x})^\intercal(\overline{y}^*) \in N_F(B,\overline{x})$. Then there exists a remainder $r(\cdot)$ such that

$$\forall x \in B, \qquad \langle \overline{y}^*, g'(\overline{x})(x - \overline{x})\rangle = \langle \overline{x}^*, x - \overline{x}\rangle \leq r(x - \overline{x}).$$

Taking into account (2.38), we get for all $y \in C \cap B(\overline{y},\rho)$,

$$\langle \overline{y}^*, y - \overline{y}\rangle = \langle \overline{y}^*, g(x_y) - g(\overline{x})\rangle \leq r(\|x_y - \overline{x}\|) + \|\overline{y}^*\| s(\|x_y - \overline{x}\|),$$

and since $\|x_y - \overline{x}\| \leq 2c\|y-\overline{y}\|$, we conclude that $\overline{y}^* \in N_F(C,\overline{y})$. □

Corollary 2.109. *Let $g : U \to V$ be a bijection between two open subsets of the normed spaces X and Y respectively such that g and $h := g^{-1}$ are H-differentiable, respectively F-differentiable, at $\overline{x}$ and $\overline{y} := g(\overline{x})$ respectively, and let $B \subset U$, $C = g(B)$. Then we have respectively*

$$N(B,\overline{x}) = g'(\overline{x})^\intercal(N(C,\overline{y}))$$

and

$$N_F(B,\overline{x}) = g'(\overline{x})^\intercal(N_F(C,\overline{y})).$$

Proof. Since $h'(\overline{y})^\intercal$ is the inverse of $g'(\overline{x})^\intercal$, one has the inclusions of Proposition 2.108 and their analogues in which $h, \overline{y}, C$ take the roles of $g, \overline{x}, B$, respectively. □

For an inverse image, it is possible to ensure equality in the inclusions of Proposition 2.108. However, a technical assumption called a qualification condition should be added, for otherwise, the result may be invalid, as the following example shows.

Example. Let $X = Y = \mathbb{R}$, $g(x) = x^2$, $C = \{0\}$, $B = g^{-1}(C)$. Then $N(B,0) = \mathbb{R} \neq g'(\overline{x})^\intercal(N(C,0)) = \{0\}$.

The factorization of Lemma 1.108 will be helpful for handling inverse images.

Proposition 2.110 (Lyusternik). *Let X, Y be Banach spaces, let U be an open subset of X, and let $g : U \to Y$ be circa-differentiable at $\overline{x} \in U$ with $g'(\overline{x})(X) = Y$. Then for $S := g^{-1}(\overline{y})$ with $\overline{y} := g(\overline{x})$ one has $N(S,\overline{x}) = N_F(S,\overline{x}) = g'(\overline{x})^\intercal(Y^*)$.*

Proof. Proposition 2.108 ensures that $g'(\overline{x})^\intercal(Y^*) \subset N_F(S,\overline{x}) \subset N(S,\overline{x})$. Now, given $x^* \in N(S,\overline{x})$, for all $v \in T(S,\overline{x}) = \ker g'(\overline{x}) = -T(S,\overline{x})$ we have $g'(\overline{x})v = 0$, so that Lemma 1.108 yields some $y^* \in Y^*$ such that $x^* = y^* \circ g'(\overline{x}) = g'(\overline{x})^\intercal(y^*)$. □

A more general case is treated in the next theorem.

Theorem 2.111. *Let X, Y be Banach spaces, let U be an open subset of X, and let $g : U \to Y$ be a map that is circa-differentiable at $\overline{x} \in U$ with $A := g'(\overline{x})$ surjective. Then if C is a subset of Y and if $\overline{x} \in B := g^{-1}(C)$, $\overline{y} := g(\overline{x}) \in C$, one has*

$$N(B,\overline{x}) = g'(\overline{x})^\intercal(N(C,\overline{y})),$$
$$N_F(B,\overline{x}) = g'(\overline{x})^\intercal(N_F(C,\overline{y})).$$

Proof. We prove the Fréchet case only, leaving the directional case to the reader. The Lyusternik–Graves theorem (Theorem 2.67) asserts the existence of $\sigma > 0$, $c > 0$ such that for all $y \in B(\overline{y}, \sigma)$ there exists $x_y \in g^{-1}(y)$ satisfying $\|x_y - \overline{x}\| \leq c\,\|y - \overline{y}\|$. When $y \in C \cap B(\overline{y}, \sigma)$ we have $x_y \in g^{-1}(C) = B$; hence $d(\overline{x}, g^{-1}(y) \cap B) \leq d(\overline{x}, x_y) \leq cd(\overline{y}, y)$. Moreover, setting $V := B(\overline{y}, \sigma)$, $U := g^{-1}(V)$, $B' := B \cap U$, $C' := C \cap V$, we have $g(B') = C'$ and $N_F(B,\overline{x}) = N_F(B',\overline{x})$ and $N_F(C,\overline{y}) = N_F(C',\overline{y})$. Thus, we can replace B with B' and C with C'. Then Proposition 2.108 ensures that $N_F(B,\overline{x}) = g'(\overline{x})^\intercal(N_F(C,\overline{y}))$. □

2.6.3 Lagrange Multiplier Rule

As observed above, the usual necessary condition $f'(a) = 0$ in order that a function $f : X \to \mathbb{R}$ attain at a its minimum when it is differentiable there has to be modified when some restrictions are imposed. In the present section we consider the frequent case of constraints defined by equalities and we present a practical rule. The case of inequalities will be dealt with later on. The famous Lagrange multiplier rule is a direct consequence of Fermat's rule and Proposition 2.110.

Theorem 2.112 (Lagrange multiplier rule). *Let X, Y be Banach spaces, let W be an open subset of X, let $f : W \to \mathbb{R}$ be differentiable at a, and let $g : W \to Y$ be circa-differentiable at a with $g'(a)(X) = Y$. Let $b := g(a)$. Suppose that f attains on $S := g^{-1}(b)$ a local minimum at a. Then there exists some $y^* \in Y^*$ (called the Lagrange multiplier) such that*

$$f'(a) = y^* \circ g'(a).$$

Example. Let us find the shape of a box having a given volume $v > 0$ and minimal area. Denoting by x, y, z the lengths of the sides of the box, we are led to minimize

$$f(x,y,z) := 2(xy + yz + zx) \quad \text{subject to } g(x,y,z) := xyz - v = 0,\ x,y,z > 0.$$

First, we secure the existence of a solution by showing that f is coercive on $S := g^{-1}(0)$. In fact, if $w_n := (x_n, y_n, z_n) \in S$ and $(\|w_n\|) \to +\infty$, one of the components of w_n, say x_n, converges to $+\infty$; then, since $y_n + z_n \geq 2\sqrt{y_n z_n} = 2\sqrt{v/x_n}$, we get

$$f(w_n) \geq 2x_n(y_n + z_n) \geq 4\sqrt{v x_n} \to +\infty.$$

Now let (x, y, z) be a minimizer of f on S. Since the derivative of g is nonzero at (x, y, z), the Lagrange multiplier rule yields some $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} 2\,(y+z) &= \lambda yz, \\ 2\,(z+x) &= \lambda zx, \\ 2\,(x+y) &= \lambda xy. \end{aligned}$$

Then multiplying each side of the first equation by x, and doing similar operations with the other two equations, we get

$$\lambda v = \lambda xyz = 2x(y+z) = 2y(z+x) = 2z(x+y),$$

whence by summation, $3\lambda v = 4(xy + yz + zx) > 0$. Subtracting the above equations one from another, we get

$$2\,(y-x) = \lambda z(y-x), \quad 2(z-y) = \lambda x(z-y), \quad 2(x-z) = \lambda y(x-z).$$

Since λ, x, y, z are positive, considering the various cases, we get $x = y = z$. Since the unique solution of the necessary condition is $w := (v^{1/3}, v^{1/3}, v^{1/3})$, we conclude that w is the solution of the problem and the optimal box is a cube. We also note that the least area is $a(v) := f(w) = 6v^{2/3}$ and that $\lambda = 4v^{-1/3}$ is exactly the derivative of the function $v \mapsto a(v)$, a general fact we will explain later on that shows that the artificial multiplier λ has in fact an important interpretation as the measure for the change of the optimal value when the parameter v varies.

Example–Exercise. Let X be some Euclidean space and let $A \in L(X,X)$ that is symmetric. Let f and g be given by $f(x) = (Ax \mid x)$, $g(x) = \|x\|^2 - 1$. Take $v \in S_X$ such that f attains its minimum on the unit sphere S_X at v. Then show that there exists some $\lambda \in \mathbb{R}$ such that $Av = \lambda v$. Deduce from this result that every symmetric square matrix is diagonalizable.

Exercises

1. (Simplified **Karush–Kuhn–Tucker theorem**) Let X,Y be Banach spaces, let $g : X \to Y$ be circa-differentiable at $\overline{x}$ with $g'(\overline{x})(X) = Y$, and let $C \subset Z$ be a closed convex cone of Y. Suppose $\overline{x} \in F := g^{-1}(C)$ is a minimizer on F of a function $f : X \to \mathbb{R}$ that is differentiable at $\overline{x}$. Use Theorem 2.111 and Fermat's rule in order to get the existence of some $\overline{y}^* \in C^0$ such that $\langle \overline{y}^*, g(\overline{x})\rangle = 0$, $f'(\overline{x}) + \overline{y}^* \circ g'(\overline{x}) = 0$.

2. (**a**) Compute the tangent cone at $(0,0)$ to the set

$$F := \left\{(r,s) \in \mathbb{R}^2 : s \geq \mid r \mid (1+r^2)^{-1}\right\}.$$

(**b**) Use Fermat's rule to give a necessary condition in order that $(0,0)$ be a local minimizer of a function f on F, assuming that f is differentiable at $(0,0)$.

(**c**) Rewrite F as $F = \left\{(r,s) \in \mathbb{R}^2 : g_1(r,s) \leq 0, g_2(r,s) \leq 0\right\}$ with g_1, g_2 given by $g_1(r,s) = r(1+r^2)^{-1} - s$, $g_2(r,s) = -r(1+r^2)^{-1} - s$ and apply the Karush–Kuhn–Tucker theorem to get the condition obtained in (**b**).

3. (**a**) Compute the tangent cone to the set $F = F' \cup F''$, where

$$F' := \left\{(r,s) \in \mathbb{R}^2 : r^4 + s^4 - 2rs = 0\right\},$$

$$F'' := \left\{(r,s) \in \mathbb{R}^2 : r^4 + s^4 + 2rs = 0\right\},$$

first for some point $a \neq (0,0)$, then for $a = (0,0)$. [Hint: First study the symmetry properties of F and set $s = tr$.]

(**b**) Write a necessary condition in order that a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ attains on F a local minimizer at $(0,0)$. Assuming that f is twice differentiable at $(0,0)$, write a second-order necessary condition.

4. Give the dimensions of a cylindrical can that has a given volume v and minimal area $a(v)$. Give an interpretation of the multiplier in terms of the derivative of $a(\cdot)$.

5. Give the dimensions of a cylindrical can that has a given area a and maximal volume $v(a)$. Give an interpretation of the multiplier in terms of the derivative of $v(\cdot)$.

6. Give the dimensions of a box without lid that has a given volume v and minimal area $a(v)$. Give an interpretation of the multiplier in terms of the derivative of $a(\cdot)$.

7. Give the dimensions of a box without lid that has a given area a and maximal volume $v(a)$. Give an interpretation of the multiplier in terms of the derivative of $v(\cdot)$.

2.7 Introduction to the Calculus of Variations

The importance of the calculus of variations stems from its role in the history of the development of analysis and from its efficacy in presenting general principles that govern a number of physical phenomena. Among these are Fermat's principle governing the path of a ray of light and the Euler–Maupertuis principle of least action governing mechanics. Historically, the calculus of variations appeared at the end of the seventeenth century with the *brachistochrone problem*, solved in 1696 by Johann Bernoulli. This problem consists in determining a curve joining two given points along which a frictionless bead slides under the action of gravity in minimal time. The novelty of such a problem lies in the fact that the unknown is a geometrical object, a curve or a function, not a real number or a finite sequence of real numbers. Thus, such a topic brings to the fore the use of functional spaces, even if one limits one's attention to one-dimensional problems.

In fact, the choice of an appropriate space of functions is part of the problem. Several choices are possible. The most general one involves absolutely continuous maps and Lebesgue null sets and is a bit technical; for many problems piecewise C^1 curves would suffice. We adopt an intermediate choice.

Let E be a Banach space and let T be a compact interval of $\mathbb{R}$ (we will not consider higher-dimensional problems, in spite of the fact that problems such as the problem of minimal surfaces are important and although many partial differential equations are derived from problems in the calculus of variations). Without loss of generality, we may suppose $T := [0,1]$. We will use the space $X := R^1(T,E)$ of functions $x : T \to E$ that are primitives of (normalized) regulated functions from T to E; this means that there exists a function $x' : T \to E$ that is right continuous on $[0,1)$ and has a left limit $x'(t_-)$ for all $t \in (0,1]$ with $x'_-(1) = x'(1_-)$ such that

$$x(t) = x(0) + \int_0^t x'(s)\mathrm{d}s, \qquad t \in T.$$

Then x' is determined by x, since for each $t \in [0,1)$, $x'(t)$ is the right derivative of x at t and $x'(1)$ is the left derivative of x at 1. We endow X with the norm

$$\|x\| = \sup_{t\in T} \|x(t)\| + \sup_{t\in T} \left\|x'(t)\right\|.$$

It is equivalent to the norm $x \mapsto \|x(0)\| + \sup_{t\in T} \|x'(t)\|$, as is easily seen. Then X is a Banach space (use Theorem 2.59).

Given $(e_0,e_1) \in E \times E$, an open subset U of $E \times E \times T$, and a continuous function $L : U \to \mathbb{R}$, the problem consists in minimizing the function j given by

$$j(x) = \int_0^1 L(x(t),x'(t),t)\mathrm{d}t$$

over the set $W(e_0,e_1)$ of elements x of X such that $x(0) = e_0$, $x(1) = e_1$, and $(x(t),x'(t),t) \in U$ for each $t \in T$. We note that since L is continuous, the function $t \mapsto L(x(t),x'(t),t)$ is regulated, so that the integral is well defined. We have more.

Lemma 2.113. *Given U, L, and j as above, the set $W := \{x \in X : \mathrm{cl}(J^1x(T)) \subset U\}$, where $J^1x(T) := \{(x(t),x'(t),t) : t \in T\}$, is open in X and j is continuous on W.*

Proof. By Proposition 2.16, for all $x \in W$, the set $\mathrm{cl}(J^1x(T))$ is a compact subset of $E \times E \times T$. Thus, there exists some $r > 0$ such that $B(J^1x(T),r) \subset U$. Then for all $w \in X$ satisfying $\|w - x\| < r$ one has $w \in W$. Thus W is open in X.

Moreover, L being continuous is uniformly continuous around $\mathrm{cl}(J^1x(T))$ in the sense that for every $\varepsilon > 0$ one can find $\delta > 0$ such that for all $(e,v,t) \in \mathrm{cl}(J^1x(T))$ and all $(e',v',t') \in B((e,v,t),\delta)$ one has $|L(e',v',t') - L(e,v,t)| \le \varepsilon$. Therefore, for all $w \in X$ satisfying $\|w - x\| \le \delta$, one has $|L(w(t),w'(t),t) - L(x(t),x'(t),t)| \le \varepsilon$, hence $|j(w) - j(x)| \le \varepsilon$. □

Proposition 2.114. *Suppose L is continuous on U and has partial derivatives with respect to its first and second variables that are continuous on U. Then j is Hadamard differentiable on W and for $\overline{x} \in W$, $x \in X$ one has*

$$j'(\overline{x})x = \int_0^1 [D_1L(\overline{x}(t),\overline{x}'(t),t)x(t) + D_2L(\overline{x}(t),\overline{x}'(t),t)x'(t)]\mathrm{d}t.$$

Proof. Let us set $L_t(e,v) = L(e,v,t)$ for $(e,v,t) \in U$ and

$$Y := \{(e_1,e_2,v_1,v_2,t) : \forall s \in [0,1],\ ((1-s)e_1 + se_2,(1-s)v_1 + sv_2,t) \in U\},$$
$$Z := \{(w_1,w_2) \in W^2 : \forall t \in T,\ (w_1(t),w_2(t),w_1'(t),w_2'(t),t) \in Y\},$$

and

$$K(e_1,e_2,v_1,v_2,t) := DL_t((1-s)e_1 + se_2,(1-s)v_1 + sv_2).$$

The compactness of $[0,1]$ easily yields that Y is open in $E^2 \times E^2 \times T$. Then a proof similar to that of Lemma 2.113 shows that Z is open in $X \times X$ and that setting

$$J(w_1,w_2,x) := \int_0^1 K(w_1(t),w_2(t),w_1'(t),w_2'(t),t).(x(t),x'(t))\mathrm{d}t$$

for $(w_1,w_2,x) \in Z \times X$, the map J is continuous from $Z \times X$ into $\mathbb{R}$. Now, since

$$L(e_1,v_1,t)-L(e_2,v_2,t)=K(e_1,e_2,v_1,v_2,t),$$

substituting w_1 and w_2 and integrating over T, we get

$$j(w_1)-j(w_2)=J(w_1,w_2,w_1-w_2).$$

Since J is continuous, the function j is of class D^1 on W. In particular, it is Hadamard differentiable on W and for $\overline{x}\in W$, $x\in X$ one has $j'(\overline{x})(x)=J(\overline{x},\overline{x},x)$. □

Exercise. Prove that j is Gâteaux differentiable using the definition and an interchange theorem between integration and derivation.

Exercise. Prove that in fact j is Fréchet differentiable.

Proposition 2.115. *Suppose L satisfies the assumptions of the preceding proposition and $\overline{x}$ is a local minimizer of j on $W(e_0,e_1)$. Then $\overline{x}$ is a critical point of j on $W(e_0,e_1)$ in the following sense:*

$$j'(\overline{x})v=0 \qquad \forall v\in X_0:=W(0,0):=\{x\in X : x(0)=0=x(1)\}.$$

Proof. Let N be a neighborhood of $\overline{x}$ in X such that $j(w)\geq j(\overline{x})$ for every $w\in N\cap W(e_0,e_1)$. Given $v\in X_0$, for $r\in\mathbb{R}$ with $|r|$ small enough, we have $w:=\overline{x}+rv\in W$ by Lemma 2.113 and $w(0)=e_0$, $w(1)=e_1$. Thus $w\in N\cap W(e_0,e_1)$, hence $j(\overline{x}+rv)\geq j(\overline{x})$ for $|r|$ small enough. It follows that $j'(\overline{x})v=0$. □

Theorem 2.116 (Euler–Lagrange condition). *Suppose L satisfies the assumptions of Proposition 2.114 and $\overline{x}\in W$ is a critical point of j on $W(e_0,e_1)$. Then the function $D_1L(\overline{x}(\cdot),\overline{x}'(\cdot),\cdot)$ is a primitive of $D_2L(\overline{x}(\cdot),\overline{x}'(\cdot),\cdot)$: for every $t\in[0,1)$ the right derivative of $D_2L(\overline{x}(\cdot),\overline{x}'(\cdot),\cdot)$ exists and is such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(D_2L(\overline{x}(t),\overline{x}'(t)),t\big)=D_1L(\overline{x}(t),\overline{x}'(t),t). \tag{2.39}$$

The solutions of this equation are called *extremals*.

We break the proof into three steps. Taking $A(t):=D_2L(t,\overline{x}(t),\overline{x}'(t))$, $B(t):=D_1L(t,\overline{x}(t),\overline{x}'(t))$ in the last one, we shall get the result. The first step is as follows.

Lemma 2.117. *Let f be a nonnegative element of the space $R_n(T,\mathbb{R})$ of normalized regulated functions on T such that $\int_0^1 f(t)\mathrm{d}t=0$. Then $f=0$.*

Proof. Suppose, to the contrary, that there exists some $r\in T$ such that $f(r)>0$. When $r<1$, using the right continuity of f at r we can find some $\alpha,\delta>0$ such that $r+\delta<1$ and $f(s)\geq\alpha$ for $s\in[r,r+\delta]$. Then we get $\int_0^1 f(t)\mathrm{d}t\geq\int_r^{r+\delta}f(t)\mathrm{d}t\geq\alpha\delta>0$, a contradiction. If $r=1$, a similar argument using the left continuity of f at 1 also leads to a contradiction. □

Lemma 2.118. *Let $F\in R_n(T,E^*)$ be such that for all $x\in X_0:=\{x\in X : x(0)=0=x(1)\}$ one has $\int_0^1 F(t)\cdot x'(t)\mathrm{d}t=0$. Then $F(\cdot)$ is constant.*

More precisely, for $e^* := \int_0^1 F(t)\mathrm{d}t$ one has $F(t) = e^*$ for all $t \in T$.

Proof. Since $\int_0^1 e^* \cdot x'(t)\mathrm{d}t = 0$ for all $x \in X_0$, subtracting from F its means e^*, we are reduced to showing that $F(\cdot) = 0$ when $\int_0^1 F(t) \cdot x'(t)\mathrm{d}t = 0$ for every $x \in X_0$. Given $e \in E$, let us introduce $f, g : T \to \mathbb{R}$, $v, x : T \to E$ given by $g(t) = F(t)(e) := \langle F(t), e \rangle$, $f(t) = (g(t))^2$, $v(s) := F(s)(e)e := \langle F(s), e \rangle e$, $x(t) = \int_0^t v(s)\mathrm{d}s$. We see that $x(0) = 0$, $x'_+(t) = v(t)$ for $t \in [0,1)$, $x(1) = \int_0^1 v(t)\mathrm{d}t = \langle \int_0^1 F(t)\mathrm{d}t, e \rangle e = 0$, since the means of F is 0, so that $x \in X_0$. Our assumption yields

$$\int_0^1 f(t)\mathrm{d}t = \int_0^1 \langle F(t), e \rangle F(t)(e)\,\mathrm{d}t = \int_0^1 F(t)(\langle F(t), e \rangle e)\,\mathrm{d}t = \int_0^1 F(t) \cdot x'(t)\mathrm{d}t = 0.$$

The preceding lemma ensures that $f(t) = 0$ for every $t \in T$. Since e is arbitrary in E, we get $F(t) = 0$ for every $t \in T$. $\square$

Lemma 2.119 (Dubois–Reymond lemma). *Let $A, B \in R_n(T, E^*)$ be such that*

$$\forall x \in X_0, \qquad \int_0^1 \left[A(t)x(t) + B(t)x'(t)\right] \mathrm{d}t = 0.$$

Then B is a primitive of A: for every $t \in T$ one has $B(t) = B(0) + \int_0^t A(s)\mathrm{d}s$.

Proof. Let us set $C(t) := B(0) + \int_0^t A(s)\mathrm{d}s$. Then for each $x \in X_0$ the function $t \mapsto C(t)x(t)$ has a right derivative $t \mapsto A(t)x(t) + C(t)x'(t)$, and by assumption,

$$\begin{aligned} 0 &= \int_0^1 \left[A(t)x(t) + B(t)x'(t)\right] \mathrm{d}t = \int_0^1 \left[\frac{\mathrm{d}}{\mathrm{d}t}(C(t)x(t)) + (B(t) - C(t))x'(t)\right] \mathrm{d}t \\ &= C(1)x(1) - C(0)x(0) + \int_0^1 (B(t) - C(t))x'(t)\mathrm{d}t = \int_0^1 (B(t) - C(t))x'(t)\mathrm{d}t. \end{aligned}$$

Lemma 2.118 ensures that $B - C$ is constant. Since $B(0) - C(0) = 0$, $B = C$. $\square$

Corollary 2.120. *Suppose the Lagrangian L is independent of e: $L(e, v, t) = \widehat{L}_t(v)$. Then for every extremal $\overline{x}(\cdot)$, the function $t \mapsto D\widehat{L}_t(\overline{x}'(t))$ is a constant.*

Proof. Since $D_1 L = 0$, (2.39) is reduced to $\frac{\mathrm{d}}{\mathrm{d}t} D_2 L(t, \overline{x}(t), \overline{x}'(t)) = 0$, and hence $\widehat{L}(\cdot, \overline{x}'(\cdot))$ is constant. $\square$

When L is of class C^2, the Euler–Lagrange equation (2.39) is an implicit ordinary differential equation of order two. Let us show how it can be reduced to an explicit first-order differential system under the assumption that for $(e, t) \in T \times E$ the function $L_{e,t} : v \mapsto L(e, v, t)$ is a Legendre function on $U_{e,t} := \{v \in E : (e, v, t) \in U\}$. We set $V_{e,t} := DL_{e,t}(U_{e,t})$ and we denote by V the union of the sets $\{e\} \times V_{e,t} \times \{t\}$ and by $H : V \to \mathbb{R}$ the *Hamiltonian* given by

$$H(e, p, t) = \langle p, v \rangle - L(e, v, t) \qquad \text{for } p := D_2 L(e, v, t),$$

so that $H_{e,t} := H(e,\cdot,t)$ is the Legendre transform of $L_{e,t}$. We have seen that

$$D_2H(e,p,t) = v \Longleftrightarrow p = D_2L(e,v,t).$$

Assuming that D_2L is of class C^1, with $D_2^2L(e,v,t)$ invertible, we get that the function $v(e,p,t)$ determined by the implicit equation

$$p - D_2L(e,v(e,p,t),t) = 0$$

is differentiable with respect to e. Then in view of the expression of H and of the preceding relation, abbreviating $v(e,p,t)$ into v, for all $e' \in E$, one has

$$\begin{aligned} D_1H(e,p,t)e' &= \langle p, D_1v(e,p,t).e'\rangle - D_1L(e,v,t)e' - D_2L(e,v,t)(D_1v(e,p,t)e') \\ &= -D_1L(e,v(e,p,t),t)e', \end{aligned}$$

or

$$D_1H(e,p,t) = -D_1L(e,v(e,p,t),t). \tag{2.40}$$

Theorem 2.121 (Hamilton). *Suppose that for all $(e,t) \in T \times E$, the map $D_2L(e,\cdot,t)$ is a diffeomorphism from $U_{e,t}$ onto its image $V_{e,t}$. Let $\overline{x}$ be an extremal and let $\overline{y}(t) := D_2L(\overline{x}(t),\overline{x}'(t),t)$. Then the pair $(\overline{x},\overline{y})$ satisfies the Hamilton differential system*

$$\begin{aligned} \overline{x}'(t) &= D_2H(\overline{x}(t),\overline{y}(t),t), \\ \overline{y}'(t) &= -D_1H(\overline{x}(t),\overline{y}(t),t). \end{aligned}$$

Proof. Plugging $e = \overline{x}(t)$, $v = \overline{x}'(t)$, $p := \overline{y}(t)$ into the relation $v = D_2H(e,p,t)$, we get the first equation. By the Euler–Lagrange equation (2.39) and relation (2.40), we have

$$\overline{y}'(t) := \frac{\mathrm{d}}{\mathrm{d}t}\left(D_2L(\overline{x}(t),\overline{x}'(t),t)\right) = D_1L(\overline{x}(t),\overline{x}'(t),t) = -D_1H(\overline{x}(t),\overline{y}(t),t).$$

Exercises

1. (**Geodesics in a Hilbert space**) Let E be a Hilbert space, $U := T \times E \times (E \setminus \{0\})$, with $T := [0,1]$, and let L be the Lagrangian given by $L(e,v,t) := \|v\|$. Given e_0, $e_1 \in E$, show that if $\overline{x} : T \to E$ is an extremal over the set $W(e_0,e_1) := \{x \in R^1(T,E) : x'(T) \subset E \setminus \{0\},\ x(0) = e_0,\ x(1) = e_1\}$, then $t \mapsto \overline{x}'(t)/\|\overline{x}'(t)\|$ is a constant vector u. Setting $s(t) := \int_0^t \|x'(r)\|\,\mathrm{d}r$, show that $\overline{x}(t) = e_0 + s(t)u$ with $u = (e_1 - e_0)/s(1)$, so that $\overline{x}$ runs along the segment $[e_0,e_1]$.

2. (**Classical mechanics**) Let us consider a solid with mass m whose position is determined by parameters $(q_1,\dots,q_n) \in \mathbb{R}^n$ subject to a force $F(q_1,\dots,q_n)$ deriving from a potential $U(q_1,\dots,q_n)$ in the sense that $F(q_1,\dots,q_n) = \nabla U(q_1,\dots,q_n)$. Its kinetic energy is given by $T(v_1,\dots,v_n) = (1/2)m(v_1^2 + \cdots + v_n^2)$. Setting

$$L(q_1,\dots,q_n,v_1,\dots,v_n) = T(v_1,\dots,v_n) + U(q_1,\dots,q_n),$$

show that the Euler–Lagrange equations turn out to be the **Newton equation**

$$mq''(t) = F(q(t)),$$

in which $q''(t) := (q_1''(t),\dots,q_n''(t))$ is the acceleration.

3. Suppose that the Lagrangian L is independent of t. Show that if $x(\cdot)$ is an extremal, then the function $t \mapsto L(x(t),x'(t)) - D_1L(x(t),x'(t)) \cdot x'(t)$ is constant on T.

4. Let $(e,v) \mapsto L(e,v)$ be a nonnegative Lagrangian on some open subset of $E \times E$. Using Exercise **3**, show that every extremal of L is also an extremal of $\sqrt{L}$.

Conversely, show that if $x(\cdot)$ is an extremal of $\sqrt{L}$ such that for some reparameterization $s \mapsto \theta(s)$, the Lagrangian $L(y(s),y'(s))$ is constant, where $y(s) := x(\theta(s))$, then y is an extremal of L.

5. Let E be a Hilbert space and let L be the Lagrangian given by $L(e,v) := \|v\|^2$. Show that if $\bar{x} : T := [0,1] \to E$ minimizes $j : x \mapsto \int_0^1 \|x'(t)\|^2 \,\mathrm{d}t$ over the set $W(e_0,e_1) := \{x \in R^1(T,E) : x(0) = e_0,\ x(1) = e_1,\ x'(T) \subset E \setminus \{0\}\}$, then $t \mapsto \bar{x}'(t)$ is constant on T and $\bar{x}$ is also an extremal of the length functional $\ell : x \mapsto \int_0^1 \|x'(t)\| \,\mathrm{d}t$ over the set $W(e_0,e_1)$. Use the preceding exercise to show that conversely, if $\bar{x}$ is an extremal of the length functional ℓ and if for some reparameterization θ the function $s \mapsto \|\bar{x}'(\theta(s))\|$ is constant, then $\bar{x} \circ \theta$ is an extremal of j.

6. **Fermat's principle** states that the trajectory of light is an extremal of the travel time functional T, associated with the Lagrangian L given by $L(e,v,t) := 1/\|v\|$. Derive the **Descartes–Snell law** of light refraction on the boundary of two media of constant indices c_i $(i = 1,2)$ separated by a hyperplane.

7. (**Lobachevskian geometry**) Find the extremals of the length function

$$\ell(x) := \int_0^1 \frac{\sqrt{x_1'(t)^2 + x_2'(t)^2}}{x_2(t)} \mathrm{d}t,$$

i.e., the *geodesics*, on the Poincaré half-plane $P := \mathbb{R} \times (0,+\infty)$ endowed with the Riemannian metric $L(e,v) = \|v\|/e_2$, where e_2 is the second component of $e \in P$. [Hint: Show that the half-circles with centers in $\mathbb{R} \times \{0\}$ are geodesics, as well as the half-lines issued from $(0,0)$.]

8. (**Brachistochrone problem**) Show that for all $a,b > 0$, the *cycloids* given by $x(t) := (a(t - \sin t) + b, a(1 - \cos t))$ are extremals of the integral functional whose Lagrangian $L : P \times \mathbb{R}^2 \to \mathbb{R}$ is given by $L(e,v) := (e_2)^{-1/2} \|v\|$, $P := \mathbb{R} \times \mathbb{P}$.

9. (Minimal surfaces of revolution) Show that the *catenaries* $x(t) = c\cosh(t/c)$ $(c > 0)$ are extremals of the integral functional whose Lagrangian $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by $L(e,v) := e\sqrt{v^2+1}$. They can be seen as sections of minimal surfaces of revolution used in power stations.

2.8 Notes and Remarks

Differential calculus is part of every course in analysis, so that numerous textbooks are devoted to it. Here we have been inspired by the books of Cartan [197], Dieudonné [294], Lang [611, 612], which were among the first to give modern presentations of the theory. A detailed study of the theory in topological linear spaces are the papers by Averbukh and Smolyanov [49, 50]; see also [946, 947], which contain interesting historical views. These works show that the notion of differentiability has many variants. Mappings of class D^1 were introduced in [779, 805]. Richard Hamilton showed the importance of such a class for implicit function theorems in Fréchet spaces [466]. Theorems 2.79–2.81 are in the line of results in [461,462] and [785, Theorem 4.1] but have new features. The terminology "circa-differentiable" is not traditional but it reflects the nature of the concept and it fits the notion of circa subdifferential (or subdifferential in the sense of Clarke). The initial terminology was "strongly differentiable" [755] and was turned into "strictly differentiable," despite the fact that there is no strict inequality in the definition.

The paper of Dolecki and Greco [307] shows the difficulties in giving due credit with the example of the contribution of Peano [778], that remained in shadow for a long time. Another example is the credit given to Hadamard here that should be confirmed [459].

The version of the Borwein–Preiss variational principle we present slightly differs from the original one in [128]; it covers other cases, but the perturbation is not given a precise form as in [128].

The name of Kantorovich is associated with Newton's method in view of the improvements made by this author (see [584]). The last exercise of Sect. 2.5.3 is inspired by [404], which contains several applications of the result. A proof of the submersion theorem in the case that the image space is finite-dimensional can be found in [462, 785].

A breakthrough in differential geometry was the book [611], by Serge Lang, that introduced in a neat manner differentiable manifolds modeled on Banach spaces. Lyusternik is considered a pioneer in the computation of the tangent cone to an inverse image using metric estimates (see [693, 694]). The subject was greatly extended with the works of Ioffe [511, 531, 538, and others].

As mentioned above, the calculus of variations was a strong incentive for the development of differential calculus and analysis. Books on the topic abound. In particular, [192, 197, 418, 549, 988] can be recommended as introductions. A historical account is given in [450].

http://www.springer.com/978-1-4614-4537-1

Calculus Without Derivatives
Penot, J.-P.
2013, XX, 524 p., Hardcover
ISBN: 978-1-4614-4537-1