Chapter 2
Scalar Delay Differential Equations on Semiaxes

2.1 Introduction

This chapter deals with nonoscillation properties of scalar differential equations with a finite number of delays. There are a lot of papers devoted to oscillation conditions for this class of equations. In comparison with oscillation, there are not so many results on nonoscillation of these equations, especially in monographs on oscillation theory. One of the aims of this chapter is to consider nonoscillation together with relevant problems: differential inequalities, comparison results, solution estimations, stability and so on. The second purpose is to derive some nonoscillation methods that will be used for other classes of functional differential equations. In particular, we apply a solution representation formula, so the most important nonoscillation property is positivity of the fundamental function of the considered equation.

The chapter is organized as follows. Section 2.2 contains relevant definitions and the solution representation formula. In Sect. 2.3, we prove that the following four assertions are equivalent: nonoscillation of the equation and the corresponding differential inequality, positivity of the fundamental function and existence of a nonnegative solution for a certain nonlinear integral inequality that is constructed explicitly from the differential equation. Section 2.4 involves comparison theorems that compare oscillation properties of various equations and also solutions of these equations. Next, in Sects. 2.5 and 2.6, explicit nonoscillation conditions for several classes of equations are considered. Section 2.7 includes several oscillation conditions that will be used in the following chapters. In Sect. 2.8, we obtain estimations for solutions and for the fundamental function of nonoscillatory equations. Section 2.9 presents conditions on initial functions and initial values that imply positivity of solutions. Section 2.10 considers slowly oscillating solutions. In Sect. 2.11, connection between nonoscillation and stability is established. Finally, Sect. 2.12 involves some discussion and open problems.

R.P. Agarwal et al., *Nonoscillation Theory of Functional Differential Equations with Applications*, DOI 10.1007/978-1-4614-3455-9_2,

2.2 Preliminaries

We consider the scalar delay differential equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) = 0, \ t \geq 0, \tag{2.2.1}$$

under the following conditions:

(a1) a_k, $k = 1, \cdots, m$, are Lebesgue measurable functions essentially bounded on each finite interval $[0, b]$.
(a2) $h_k : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$, $k = 1, \cdots, m$.

Together with (2.2.1), we consider for each $t_0 \geq 0$ the initial value problem

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) = f(t), \ t \geq t_0, \tag{2.2.2}$$

$$x(t) = \varphi(t), \ t < t_0, \ x(t_0) = x_0. \tag{2.2.3}$$

We also assume that the following hypothesis holds:

(a3) $f : [t_0, \infty) \to \mathbb{R}$ is a Lebesgue measurable function essentially bounded in each finite interval $[t_0, b]$, and $\varphi : (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

Definition 2.1 A function $x : \mathbb{R} \to \mathbb{R}$ absolutely continuous on each interval $[t_0, b]$ is called *a solution* of problem (2.2.2), (2.2.3) if it satisfies (2.2.2) for almost all $t \in [t_0, \infty)$ and equalities (2.2.3) for $t \leq t_0$.

Definition 2.2 For each $s \geq 0$, the solution $X(t, s)$ of the problem

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) = 0, \ x(t) = 0, \ t < s, \ x(s) = 1, \tag{2.2.4}$$

is called *a fundamental function* of (2.2.1).

We assume $X(t, s) = 0, 0 \leq t < s$.

Theorem B.1 implies the following result.

Lemma 2.1 *Let* (a1)–(a3) *hold. Then there exists a unique solution of problem* (2.2.2), (2.2.3) *that has the form*

$$x(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)f(s)ds - \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s)a_k(s)\varphi\big(h_k(s)\big)ds, \tag{2.2.5}$$

where $\varphi(h_k(s)) = 0$, *if* $h_k(s) > t_0$.

2.3 Nonoscillation Criteria

Definition 2.3 We will say that (2.2.1) *has a positive solution* for $t_0 \geq 0$ if there exist an initial function φ and a number x_0 such that the solution of initial value problem (2.2.2), (2.2.3) ($f \equiv 0$) is positive.

Consider together with (2.2.1) the delay differential inequality

$$\dot{y}(t) + \sum_{k=1}^{m} a_k(t) y\big(h_k(t)\big) \leq 0. \tag{2.3.1}$$

The following theorem establishes nonoscillation criteria.

Theorem 2.1 *Suppose $a_k(t) \geq 0$, $k = 1, \cdots, m$. Then the following hypotheses are equivalent*:

1) *There exists $t_0 \geq 0$ such that (2.3.1) has a positive solution for $t_0 \geq 0$.*
2) *There exist a point $t_1 \geq 0$ and a locally essentially bounded function $u(t)$ nonnegative for $t \geq t_1$ and satisfying the inequality*

$$u(t) \geq \sum_{k=1}^{m} a_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\},\ t \geq t_1, \tag{2.3.2}$$

where we assume $u(t) = 0$, $t < t_1$.
3) *There exists $t_1 \geq 0$ such that $X(t,s) > 0$, $t \geq s \geq t_1$.*
4) *There exists $t_1 \geq 0$ such that (2.2.1) has a positive solution for $t \geq t_1$.*

Proof 1) $\Rightarrow$ 2) Let $y(t)$ be a positive solution of (2.3.1) for $t \geq t_0$. Without loss of generality, we can assume that $y(h_k(t)) > 0$, $t \geq t_0$. By (a2), there exists a point t_1 such that $h_k(t) \geq t_0$ if $t \geq t_1$, $k = 1, \cdots, m$.

Denote

$$u_1(t) = -\frac{d}{dt} \ln \frac{y(t)}{y(t_1)},\ t \geq t_0.$$

Then

$$\begin{aligned} y(t) &= y(t_1) \exp\left\{-\int_{t_1}^{t} u_1(s)ds\right\}, \\ y\big(h_k(t)\big) &= y(t_1) \exp\left\{-\int_{t_1}^{h_k(t)} u_1(s)ds\right\}, \\ \dot{y}(t) &= -u_1(t) y(t_1) \exp\left\{-\int_{t_1}^{t} u_1(s)ds\right\},\ t \geq t_1. \end{aligned} \tag{2.3.3}$$

We substitute (2.3.3) into (2.3.1) and obtain

$$-u_1(t) y(t_1) \exp\left\{-\int_{t_1}^{t} u_1(s)ds\right\} + \sum_{k=1}^{m} y(t_1) a_k(t) \exp\left\{-\int_{t_1}^{h_k(t)} u_1(s)ds\right\} \leq 0.$$

Hence

$$-\exp\left\{-\int_{t_1}^{t} u_1(s)ds\right\} y(t_1)\left[u_1(t)-\sum_{k=1}^{m} a_k(t)\exp\left\{\int_{h_k(t)}^{t} u_1(s)ds\right\}\right] \le 0. \quad (2.3.4)$$

Since $y(t) > 0$ for $t \ge t_0$ and $a_k(t) \ge 0$, we have $y(t_1) > 0$ and

$$u_1(t) \ge \sum_{k=1}^{m} a_k(t)\exp\left\{\int_{h_k(t)}^{t} u_1(s)ds\right\}, \ t \ge t_1. \quad (2.3.5)$$

After denoting

$$u(t) = \begin{cases} u_1(t), \ t \ge t_1 \\ 0, \qquad t < t_1, \end{cases}$$

(2.3.5) implies (2.3.2).

2) $\Rightarrow$ 3) **Step 1**. Consider the initial value problem

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) = f(t), \ t \ge t_1, \ x(t) = 0, \ t \le t_1. \quad (2.3.6)$$

Denote

$$z(t) = \dot{x}(t) + u(t)x(t), \quad (2.3.7)$$

where x is the solution of (2.3.6) and u is a nonnegative solution of (2.3.2). The assumption $x(t) = 0$, $t \le t_1$ implies $z(t) = 0$ for $t \le t_1$.

The solution $x(t)$ of (2.3.7) satisfies

$$x(t) = \int_{t_1}^{t} \exp\left\{-\int_{s}^{t} u(\tau)d\tau\right\} z(s)ds, \quad (2.3.8)$$

$$x\big(h_k(t)\big) = \int_{t_1}^{h_k(t)} \exp\left\{-\int_{s}^{h_k(t)} u(\tau)d\tau\right\} z(s)ds, \quad (2.3.9)$$

$$\dot{x}(t) = z(t) - u(t)\int_{t_1}^{t} \exp\left\{-\int_{s}^{t} u(\tau)d\tau\right\} z(s)ds. \quad (2.3.10)$$

After substituting (2.3.9) and (2.3.10) into the left-hand side of (2.3.6), we have

$$\begin{aligned} & z(t) - u(t)\int_{t_1}^{t} \exp\left\{-\int_{s}^{t} u(\tau)d\tau\right\} z(s)ds \\ & \quad + \sum_{k=1}^{m} \int_{t_1}^{h_k(t)} \exp\left\{-\int_{s}^{h_k(t)} u(\tau)d\tau\right\} z(s)ds \\ & = z(t) - \int_{t_1}^{t} \exp\left\{-\int_{s}^{t} u(\tau)d\tau\right\} z(s)ds\left[u(t) - \sum_{k=1}^{m} a_k(t)\exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}\right]. \end{aligned}$$

Hence (2.3.6) can be rewritten in the form

$$z - Hz = f, \quad (2.3.11)$$

where

$$(Hz)(t) = \int_{t_1}^{t} \exp\left\{-\int_s^t u(\tau)d\tau\right\} z(s)ds \left[u(t) - \sum_{k=1}^{m} a_k(t)\exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}\right].$$

Inequality (2.3.2) yields that if $z(t) \geq 0$ then $(Hz)(t) \geq 0$ (i.e., operator H is positive). Besides, the operator $H : L_\infty[t_1, b] \to L_\infty[t_1, b]$ is an integral Volterra operator with the kernel essentially bounded on $[t_1, b] \times [t_1, b]$. By Theorem A.4, operator H is weakly compact in the space $L_\infty[t_1, b]$; Theorem A.7 implies that the spectral radius is $r(H) = 0 < 1$.

Thus, if in (2.3.11) $f(t) \geq 0$, then

$$z(t) = f(t) + (Hf)(t) + \left(H^2 f\right)(t) + \cdots \geq 0.$$

The solution of (2.3.6) has the form (2.3.8), with z being a solution of (2.3.11). Hence, if in (2.3.6) $f(t) \geq 0$, then for the solution of this equation we have $x(t) \geq 0$. On the other hand, the solution of (2.3.6) can be presented in the form (2.2.5)

$$x(t) = \int_{t_1}^{t} X(t, s) f(s)ds.$$

As was shown above, $f(t) \geq 0$ implies $x(t) \geq 0$, and consequently the kernel of the integral operator is nonnegative; i.e., $X(t, s) \geq 0$ for $t \geq s > t_1$.

Step 2. Let us prove that in fact the strict inequality $X(t, s) > 0$ holds. Denote

$$x(t) = X(t, t_1) - \exp\left\{-\int_{t_1}^{t} u(s)ds\right\}, \quad x(t) = 0, \; t < t_1.$$

The function $X(t, t_1)$ is a solution of homogeneous equation (2.3.6). After substituting $x(t)$ into the left-hand side of (2.3.6), we have

$$u(t)\exp\left\{-\int_{t_1}^{t} u(s)ds\right\} - \sum_{k=1}^{m} a_k(t)\exp\left\{-\int_{t_1}^{h_k(t)} u(s)ds\right\}$$
$$= \exp\left\{-\int_{t_1}^{t} u(s)ds\right\}\left[u(t) - \sum_{k=1}^{m} a_k(t)\exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}\right] \geq 0.$$

Hence $x(t)$ is a solution of (2.3.6) with $f(t) \geq 0$; as demonstrated above, $x(t) \geq 0$. Consequently,

$$X(t, t_1) \geq \exp\left\{-\int_{t_1}^{t} u(s)ds\right\} > 0.$$

For $s > t_1$, the inequality $X(t, s) > 0$ can be justified similarly.

3) ⇒ 4) A function $x(t) = X(t, t_1)$ is a positive solution of (2.2.1) for $t \geq t_1$.

Implication 4) ⇒ 1) is evident. □

Remark 2.1 If there exists a nonnegative solution of inequality (2.3.2) for $t \geq t_1$, then assertions 1), 3) and 4) of Theorem 2.1 are also fulfilled for $t \geq t_1$.

We will end this section with the result on the asymptotic behavior of nonoscillatory solutions.

Theorem 2.2 *Suppose* $a_k(t) \geq 0$, $k = 1, \cdots, m$, $\int_{t_0}^{\infty} \sum_{k=1}^{m} a_k(s)ds = \infty$. *Then, for any nonoscillatory solution of* (2.2.1), *we have* $\lim_{t\to\infty} x(t) = 0$.

Proof Suppose x is an eventually positive solution of (2.2.1). Then x is an eventually monotonically decreasing function, and hence there exists a nonnegative limit $\lim_{t\to\infty} x(t) = d < \infty$. If $d > 0$, then for some t_1 we have $x(t) > d - \varepsilon > 0$, $t \geq t_1$. Hence

$$x(t) = x(t_1) - \int_{t_1}^{t} \sum_{k=1}^{m} a_k(s)x\big(h_k(s)\big)ds \leq x(t_0) - (d - \varepsilon) \int_{t_1}^{t} \sum_{k=1}^{m} a_k(s)ds.$$

Thus $\lim_{t\to\infty} x(t) = -\infty$, and we have a contradiction, so $d = 0$, which completes the proof. □

2.4 Comparison Theorems

Theorem 2.1 can be employed to obtain comparison results in oscillation theory. To this end, consider together with (2.2.1) the equation

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x\big(h_k(t)\big) = 0, \;\; t \geq 0. \tag{2.4.1}$$

Suppose (a1) and (a2) hold for (2.4.1). Denote by $Y(t,s)$ the fundamental function of (2.4.1).

Theorem 2.3 *Suppose* $a_k(t) \geq 0$, $a_k(t) \geq b_k(t)$, $t \geq t_0$, *and condition* 2) *of Theorem* 2.1 *holds for* (2.2.1). *Then* (2.4.1) *has a positive solution for* $t \geq t_1$ *and* $Y(t,s) > 0$ *for* $t \geq s \geq t_1$.

Proof By Theorem 2.1 and Remark 2.1, the fundamental function $X(t,s)$ of (2.2.1) is positive for $t \geq t_1$.

Consider the equation with the zero initial conditions

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x\big(h_k(t)\big) = f(t), \;\; t \geq t_1, \;\; x(t) = 0, \;\; t \leq t_1. \tag{2.4.2}$$

We will show that if $f(t) \geq 0$, then the solution of (2.4.2) is nonnegative. To this end, let us rewrite (2.4.2) in the form

$$\begin{aligned}\dot{x}(t) &+ \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) + \sum_{k=1}^{m} \big[b_k(t) - a_k(t)\big]x\big(h_k(t)\big) \\ &= f(t), \;\; t \geq t_1, \;\; x(t) = 0, t \leq t_1.\end{aligned}$$

Denote

$$z(t) = \dot{x}(t) + \sum_{k=1}^{m} a_k(t) x\bigl(h_k(t)\bigr).$$

By solution representation formula (2.2.5),

$$x(t) = \int_{t_1}^{t} X(t,s) z(s) ds,\ \ x\bigl(h_k(t)\bigr) = \chi_{[t_1,\infty)}\bigl(h_k(t)\bigr) \int_{t_1}^{h_k(t)} X\bigl(h_k(t), s\bigr) z(s) ds,$$

where χ_I is the characteristic set of the interval I,

$$\chi_{[t_1,\infty)}(t) = \begin{cases} 1, \ t \ge t_1, \\ 0, \ t < t_1. \end{cases}$$

Thus (2.4.2) is equivalent to the equation

$$z - Tz = f, \tag{2.4.3}$$

where

$$(Tz)(t) = \sum_{k=1}^{m} \bigl[a_k(t) - b_k(t)\bigr] \chi_{[t_1,\infty)}\bigl(h_k(t)\bigr) \int_{t_1}^{h_k(t)} X\bigl(h_k(t), s\bigr) z(s) ds.$$

By Corollary B.1, we have the estimate

$$\bigl|X(t,s)\bigr| \le \exp \sum_{k=1}^{m} \int_{t_1}^{b} \bigl|a_k(\tau)\bigr| d\tau,\ \ t_1 \le s \le t \le b,$$

so the kernel of the integral operator T is essentially bounded on $[t_1, b] \times [t_1, b]$. By Theorem A.4, operator T is a weakly compact operator in the space $L_\infty[t_1, b]$. Theorem A.7 implies that the spectral radius $r(T) = 0 < 1$.

Theorem 2.1 implies $X(t,s) > 0$, $t \ge s \ge t_1$, and hence operator T is positive. Therefore, for the solution of (2.4.3), we have

$$z(t) = f(t) + (Tf)(t) + \bigl(T^2 f\bigr)(t) + \cdots \ge 0 \text{ if } f(t) \ge 0.$$

Then, as in the proof of Theorem 2.1, we conclude that $Y(t,s) > 0$, $t \ge s \ge t_1$, and therefore $x(t) = Y(t, t_1)$ is a positive solution of (2.4.1).

Positivity of $Y(t,s)$ for an arbitrary $s > t_1$ is demonstrated similarly. □

Corollary 2.1 *Suppose that $a_k(t) \ge 0$, $a_k(t) \ge b_k(t)$ for $t \ge t_0$ and (2.2.1) has a positive solution for $t \ge t_0$. Then there exists $t_1 \ge t_0$ such that (2.4.1) has a positive solution for $t \ge t_1$.*

Denote

$$a^+ = \max\{a, 0\}.$$

Corollary 2.2

1) *If the inequality*

$$\dot{x}(t) + \sum_{k=1}^{m} a_k^+(t) x\big(h_k(t)\big) \le 0 \tag{2.4.4}$$

has an eventually positive solution, then (2.2.1) *also has an eventually positive solution.*

2) *If condition* 2) *of Theorem* 2.1 *holds for* (2.2.1), *where inequality* (2.3.2) *is replaced by*

$$u(t) \ge \sum_{k=1}^{m} a_k^+(t) \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}, \ t \ge t_1, \tag{2.4.5}$$

then (2.2.1) *has a positive solution for $t \ge t_1$ and $X(t,s) > 0$ for $t \ge s \ge t_1$.*

Proof Consider the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k^+(t) x\big(h_k(t)\big) = 0.$$

Either of the two assumptions of the corollary imply that all hypotheses of Theorem 2.1 hold. Since $a_k(t) \le a_k^+(t)$ and $a_k^+(t) \ge 0$, Theorem 2.3 implies this corollary. □

Inequality (2.4.5) can be employed to obtain a comparison result that improves the statement of Theorem 2.3.

Consider the equation

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t) x\big(g_k(t)\big) = 0, \tag{2.4.6}$$

and suppose that the hypotheses (a1) and (a2) hold for (2.4.6); denote by $Y(t,s)$ the fundamental function of this equation.

Theorem 2.4 *Suppose that $a_k(t) \ge 0$ and there exists $t_0 \ge 0$ such that for* (2.2.1) *anyone of assertions* 1)–4) *of Theorem* 2.1 *holds for $t \ge t_0$. If*

$$b_k(t) \le a_k(t), \ h_k(t) \le g_k(t), \ k = 1, \cdots, m, \tag{2.4.7}$$

then there exists $t_1 \ge t_0$ such that (2.4.6) *has a positive solution for $t \ge t_1$ and $Y(t,s) > 0$, $t \ge s \ge t_1$.*

Proof Theorem 2.1 implies that for some $t_1 \ge t_0$ there exists a nonnegative solution u of inequality (2.3.2) for $t \ge t_1$. Inequalities (2.4.7) yield that this function is also a solution of the inequality

$$u(t) \ge \sum_{k=1}^{m} b_k^+(t) \exp\left\{\int_{g_k(t)}^{t} u(s)ds\right\}, \ t \ge t_1.$$

Hence, by Corollary 2.2, (2.4.6) has a positive solution for $t \geq t_1$ and the fundamental function of (2.4.6) is positive, which completes the proof. □

The inequality $X(t,s) > 0$ can be employed to compare solutions of two distinct differential equations. To this end, consider together with (2.2.2), (2.2.3) the initial value problem with the same delays:

$$\dot{y}(t) + \sum_{k=1}^{m} b_k(t) y\big(h_k(t)\big) = g(t), \ t \geq t_1, \tag{2.4.8}$$

$$y(t) = \psi(t), \ t < t_1, \ y(t_1) = y_0. \tag{2.4.9}$$

Suppose (a1)–(a3) hold for (2.4.8), (2.4.9). Denote by $x(t)$, $X(t,s)$ the solution and the fundamental function of problem (2.2.2), (2.2.3), where the initial point t_0 is replaced by t_1 and by $y(t)$, $Y(t,s)$ the solution and the fundamental function of problem (2.4.8), (2.4.9).

Theorem 2.5 *Suppose that condition* 2) *of Theorem* 2.1 *holds for* (2.2.1), $x(t) > 0$ *and*

$$a_k(t) \geq b_k(t) \geq 0, \ g(t) \geq f(t), \ \varphi(t) \geq \psi(t), \ t < t_1, \ y_0 \geq x_0.$$

Then $y(t) \geq x(t) > 0$.

Proof Denote by $u(t)$ a nonnegative solution of (2.3.2). Inequality $a_k(t) \geq b_k(t)$ yields that the function $u(t)$ is also a solution of the inequality corresponding to (2.3.2) for (2.4.8). Hence, by Theorem 2.1 we have $X(t,s) > 0$ and $Y(t,s) > 0$ for $t_0 \leq s < t$.

Equation (2.2.2) can be rewritten in the form

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t) x\big(h_k(t)\big) = \sum_{k=1}^{m} \big[b_k(t) - a_k(t)\big] x\big(h_k(t)\big) + f(t), \ t \geq t_1,$$

which implies

$$\begin{aligned} x(t) &= Y(t,t_1)x_0 - \sum_{k=1}^{m} \int_{t_1}^{t} Y(t,s) b_k(s) \varphi\big(h_k(s)\big) ds \\ &\quad + \int_{t_1}^{t} Y(t,s) f(s) ds - \sum_{k=1}^{m} \int_{t_1}^{t} Y(t,s) \big[a_k(s) - b_k(s)\big] x\big(h_k(s)\big) ds \\ &\leq Y(t,t_1) y_0 - \sum_{k=1}^{m} \int_{t_1}^{t} Y(t,s) b_k(s) \psi\big(h_k(s)\big) ds + \int_{t_1}^{t} Y(t,s) g(s) ds = y(t), \end{aligned}$$

where $\varphi(h_k(s)) = \psi(h_k(s)) = 0$ if $h_k(s) \geq t_1$ and $x(h_k(s)) = 0$ if $h_k(s) < t_1$. Therefore $y(t) \geq x(t) > 0$. □

Corollary 2.3 *Suppose that* $a_k(t) \geq 0$, *condition* 2) *of Theorem* 2.1 *holds for* (2.2.1) *and* x *and* y *are positive solutions of* (2.2.1) *and* (2.3.1) *for* $t \geq t_1$, *respectively. If* $x(t) \leq y(t)$ *for* $t < t_1$ *and* $x(t_1) = y(t_1)$, *then* $x(t) \geq y(t)$ *for* $t \geq t_1$.

Since the fundamental function of any ordinary differential equation $\dot{x}(t) + a(t)x(t) = 0$ is positive, we immediately obtain the following result.

Corollary 2.4 *If $a_k(t) \geq 0$, $k = 1, \cdots, m$, then the fundamental function $X(t,s)$ of the equation*

$$\dot{x}(t) + a(t)x(t) - \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) = 0, \ t \geq 0,$$

is positive for $0 \leq s \leq t$. In addition, for the solutions y and z of the inequalities

$$\dot{y}(t) + a(t)y(t) - \sum_{k=1}^{m} a_k(t)y\big(h_k(t)\big) \leq 0, \ t \geq 0,$$

$$\dot{z}(t) + a(t)z(t) - \sum_{k=1}^{m} a_k(t)z\big(h_k(t)\big) \geq 0, \ t \geq 0,$$

satisfying for any t_0 the equality $x(t) = y(t) = z(t)$, $t \leq t_0$, we have $y(t) \leq x(t) \leq z(t)$ for $t > t_0$.

2.5 Nonoscillation Conditions, Part 1

Inequality (2.4.5) can be applied to obtain explicit nonoscillation conditions. Corollary 2.2, Part 2, immediately implies the following result if we assume $u(t) \equiv \lambda$.

Theorem 2.6 *Suppose that there exist a point $t_1 \geq 0$ and a constant $\lambda > 0$ such that*

$$\sum_{j=1}^{m} a_j^+(t)e^{\lambda(t-h_k(t))} \leq \lambda, \ t \geq t_1.$$

Then the fundamental function $X(t,s)$ of (2.2.1) is positive for $t \geq s \geq t_1$.

Theorem 2.7 *Suppose that there exists a point $t_1 \geq 0$ such that*

$$\int_{\min_k\{h_k(t)\}}^{t} \sum_{j=1}^{m} a_j^+(s)ds \leq \frac{1}{e}, \ t \geq t_1. \tag{2.5.1}$$

Then the fundamental function $X(t,s)$ of (2.2.1) is positive for $t \geq s \geq t_1$.

Proof Let us demonstrate that the function

$$u(t) = e\sum_{k=1}^{m} a_k^+(t)$$

is a nonnegative solution of (2.4.5). By (2.5.1), we have for $t \geq t_1$

$$
\begin{aligned}
&\sum_{k=1}^{m} a_k^+(t) \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\} \\
&\quad = \sum_{k=1}^{m} a_k^+(t) \exp\left\{\int_{h_k(t)}^{t} e \sum_{i=1}^{m} a_i^+(s)ds\right\} \\
&\quad \le \sum_{k=1}^{m} a_k^+(t) \exp\left\{e \int_{\min_k h_k(t)}^{t} \sum_{i=1}^{m} a_i^+(s)ds\right\} \\
&\quad \le \sum_{k=1}^{m} a_k^+(t) e = u(t),
\end{aligned}
$$

so $u(t)$ satisfies (2.4.5). By Corollary 2.2, the fundamental function of (2.2.1) is positive for $t \ge t_1$. □

Let us note that the constant $1/e$ is the best possible since the equation

$$\dot{x}(t) + x(t-\tau) = 0$$

is oscillatory for $\tau > 1/e$.

Corollary 2.5 *Suppose*

$$\limsup_{t\to\infty} \int_{\min_k\{h_k(t)\}}^{t} \sum_{j=1}^{m} a_j^+(s)ds < \frac{1}{e}. \tag{2.5.2}$$

Then there exists an eventually positive solution of (2.2.1).

Corollary 2.6 *Suppose that there exists* $\tau > 0$ *such that* $t - h_k(t) \le \tau$, $k = 1, \cdots, m$ *and*

$$\int_{t_0}^{\infty} \sum_{k=1}^{m} a_k^+(s)ds < \infty.$$

Then there exists an eventually positive solution of (2.2.1).

In the monograph [192], the authors construct a counterexample that demonstrates that condition (2.5.2) is not necessary for nonoscillation of (2.2.1).

By [192, Theorem 3.4.3], the inequality

$$\limsup_{t\to\infty} \int_{\max_k\{h_k(t)\}}^{t} \sum_{j=1}^{m} a_j(s)ds \le 1 \tag{2.5.3}$$

is necessary for nonoscillation of (2.2.1) with nonnegative coefficients $a_k(t) \ge 0$ and monotonically nondecreasing deviations of arguments $h_k(t)$.

Let us find sufficient nonoscillation conditions when the number

$$\limsup_{t\to\infty} \int_{\max_k\{h_k(t)\}}^{t} \sum_{j=1}^{m} a_j^+(s)ds$$

is between $1/e$ and 1.

First, consider (2.2.1) with constant delays $\tau_k(t) = \text{const}$:

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(t-\tau_k) = 0, \ \tau_k > 0, \ k = 1, \cdots, m. \tag{2.5.4}$$

Theorem 2.8 *Suppose that there exist a number $n_0 \geq 0$ and a sequence $\{\lambda_n\}_{n=n_0}^{\infty}$, where all $\lambda_n > 1$, such that*

$$\sum_{k=1}^{m} a_k^+(t) \leq \lambda_n e^{-(\lambda_{n-1}(n\tau - t) + \lambda_n(t-(n-1)\tau))}, \ (n-1)\tau \leq t \leq n\tau, \ n \geq n_0, \tag{2.5.5}$$

where $\tau = \max_k \tau_k$.

Then (2.5.4) has a positive fundamental function $X(t,s)$ for $t \geq s \geq t_0 = n_0\tau$.

Proof Let us demonstrate that the function

$$u(t) = \lambda_n, \ (n-1)\tau \leq t \leq n\tau, \ n \geq n_0,$$

is a solution of (2.4.5) for $t \geq n_0$.

In the interval $(n-1)\tau \leq t \leq n\tau$, we have

$$\begin{aligned}
&\sum_{k=1}^{m} a_k^+(t) \exp\left\{\int_{t-\tau_k}^{t} u(s)ds\right\} \\
&\quad \leq \sum_{k=1}^{m} a_k^+(t) \exp\left\{\int_{t-\tau}^{t} u(s)ds\right\} \\
&\quad = \sum_{k=1}^{m} a_k^+(t) \exp\left\{\int_{t-\tau}^{(n-1)\tau} \lambda_{n-1} ds + \int_{(n-1)\tau}^{t} \lambda_n ds\right\} \\
&\quad = \sum_{k=1}^{m} a_k^+(t) \exp\left\{\lambda_{n-1}(n\tau - t) + \lambda_n\big(t-(n-1)\tau\big)\right\} \leq \lambda_n = u(t).
\end{aligned}$$

Hence (2.4.5) is equivalent to (2.5.5), which completes the proof. □

By Theorem 2.4, we obtain a more general result.

Corollary 2.7 *Suppose there exists $\tau_k > 0$ such that $t - h_k(t) \leq \tau_k$. If all the conditions of Theorem 2.8 hold, then (2.2.1) has a positive fundamental function $X(t,s)$ for $t \geq s \geq t_0$.*

Example 2.1 Consider the equation

$$\dot{x}(t) + a(t)x(t-\tau) = 0, \tag{2.5.6}$$

where $\tau = 1$ and

$$a(t) = \begin{cases} e^{-(2n-t+1)}, & 2n-1 \leq t < 2n, \ n \geq 1, \\ 2e^{-(t-2n+1)}, & 2n \leq t < 2n+1, \ n \geq 0. \end{cases}$$

Denote

$$\lambda_{2n} = 1, \ \lambda_{2n-1} = 2.$$

Then all the conditions of Theorem 2.8 hold, and thus (2.5.6) has a positive fundamental function.

In addition, we have

$$\limsup_{t\to\infty} \int_{t-1}^{t} a(s)ds = 2\left(e^{-1} - e^{-2}\right) > \frac{1}{e}.$$

Hence (2.5.2) does not hold for (2.5.6). Thus (2.5.2) is not necessary for nonoscillation of (2.2.1).

We apply the idea of Example 2.1 to prove the following theorem.

Theorem 2.9 *For any* $\alpha \in (1/e, 1)$, *there exists nonoscillatory equation* (2.5.6) *with* $a(t) \ge 0$ *such that*

$$\sup_{t\ge\tau} \int_{t-\tau}^{t} a(s)ds = \alpha. \qquad (2.5.7)$$

Proof It is sufficient to prove the theorem for $\tau = 1$. Suppose $\lambda > 0$, $a > 1$. Consider (2.5.6), where $\tau = 1$ and

$$a(t) = \begin{cases} \lambda e^{-(2\lambda(a-1)n-\lambda(a-1)t+\lambda)}, & 2n-1 \le t < 2n, \ n \ge 1, \\ \lambda a e^{-(\lambda(a-1)t-2\lambda(a-1)n+\lambda)}, & 2n \le t < 2n+1, \ n \ge 0. \end{cases}$$

Denote

$$\lambda_{2n} = \lambda, \ \lambda_{2n-1} = \lambda a.$$

Then all the conditions of Theorem 2.8 hold, and hence (2.5.6) has a positive fundamental function.

We have

$$\sup_{t\ge 1} \int_{t-1}^{t} a(s)ds = \frac{a}{a-1}\left(e^{-\lambda} - e^{-\lambda a}\right).$$

The function

$$f(\lambda) = \frac{a}{a-1}\left(e^{-\lambda} - e^{-\lambda a}\right)$$

has the maximum $\max f(\lambda) = f(\lambda_0) = e^{-\lambda_0}$ at the point $\lambda_0 = \frac{\ln a}{a-1}$. Let us note that

$$\lim_{a\to 1} \frac{\ln a}{a-1} = 1, \ \lim_{a\to\infty} \frac{\ln a}{a-1} = 0,$$

and take $\lambda = \frac{\ln a}{a-1}$ in the definition of function $a(t)$. Then

$$\sup_{t\ge 1} \int_{t-1}^{t} a(s)ds = e^{-\ln a/(a-1)}.$$

Since

$$\lim_{a\to 1} \sup_{t\geq 1} \int_{t-1}^{t} a(s)ds = 1/e, \quad \lim_{a\to\infty} \sup_{t\geq 1} \int_{t-1}^{t} a(s)ds = 1,$$

the continuous function $\sup_{t\geq 1} \int_{t-1}^{t} a(s)ds$ of a takes all the values from the interval $(1/e, 1)$, which completes the proof. □

Now we proceed to an integral nonoscillation condition similar to Theorem 2.8.

Theorem 2.10 *Suppose that there exist $n_0 \geq 0$ and a sequence $\{\lambda_n\}_{n=n_0}^{\infty}$, where all $\lambda_n > 1$, such that*

$$\lambda_{n-1} \int_{t-\tau}^{(n-1)\tau} \sum_{k=1}^{m} a_k^+(s)ds + \lambda_n \int_{(n-1)\tau}^{t} \sum_{k=1}^{m} a_k^+(s)ds \leq \ln \lambda_n, \ (n-1)\tau \leq t \leq n\tau, \tag{2.5.8}$$

$n \geq n_0$, where $\tau = \max_k \tau_k$. Then (2.5.4) has a positive fundamental function for $t \geq s \geq t_0 = n_0\tau$.

Proof The proof is similar to the proof of the previous theorem if we put

$$u(t) = \lambda_n a(t), \ (n-1)\tau \leq t \leq n\tau, \ n \geq n_0,$$

where $a(t) = \sum_{k=1}^{m} a_k^+(s)ds$. □

Corollary 2.8 *Suppose there exists $\tau_k > 0$ such that $t - h_k(t) \leq \tau_k$. If all the conditions of Theorem 2.10 hold, then (2.2.1) has a positive fundamental function for $t \geq s \geq t_0$.*

Let us note that if in (2.5.6) we substitute the maximum delay by the minimum delay

$$\limsup_{t\to\infty} \int_{\min_k h_k(t)}^{t} \sum_{j=1}^{m} a_j(s)ds \leq 1, \tag{2.5.9}$$

this condition is not necessary for nonoscillation.

Example 2.2 The equation

$$x'(t) + 0.01x(t-10) + 0.3x(t) = 0 \tag{2.5.10}$$

is nonoscillatory since the characteristic equation $\lambda + 0.01e^{-10\lambda} + 0.3 = 0$ has two real roots, $\lambda_1 \approx -0.3261$ and $\lambda_1 \approx -0.5536$. However, (2.5.9) is not satisfied since $10(0.01 + 0.3) = 3.01 > 1$.

2.6 Nonoscillation Conditions, Part 2

The explicit nonoscillation condition in (2.5.1) is easily checked but contains only "the worst delay". To give a sharper result, where all delays are included, denote

$$A_{ij} = \sup_{t \geq t_1} \int_{h_i(t)}^{t} a_j^+(s)ds, \ 1 \leq i, j \leq m. \tag{2.6.1}$$

Theorem 2.11 *Suppose there exist a point $t_1 \geq 0$ and positive numbers $x_i, i = 1, \cdots, m$ such that $A_{ij} < \infty$, $t \geq t_1$ and*

$$\ln x_i \geq \sum_{j=1}^{m} A_{ij} x_j, \ i = 1, \cdots, m. \tag{2.6.2}$$

Then (2.2.1) has a positive fundamental function $X(t, s)$ for $t \geq s \geq t_1$.

Proof Inequality (2.6.2) implies that for any $t \geq t_1$

$$x_k \geq \exp\left\{ \sum_{j=1}^{m} \int_{h_k(t)}^{t} x_j a_j^+(s)ds \right\}.$$

After introducing the function

$$u(t) = \sum_{j=1}^{m} x_j a_j^+(t), \ t \geq t_1, \ u(t) = 0, \ t \leq t_1,$$

we obtain

$$\begin{aligned} &\sum_{k=1}^{m} a_k^+(t) \exp\left\{ \int_{h_k(t)}^{t} u(s)ds \right\} \\ &\quad = \sum_{k=1}^{m} a_k^+(t) \exp\left\{ \int_{h_k(t)}^{t} \sum_{j=1}^{m} x_j a_j^+(s)ds \right\} \\ &\quad \leq \sum_{k=1}^{m} a_k^+(t) x_k = u(t). \end{aligned}$$

Then all the conditions of Part 2 of Corollary 2.2 are satisfied. Hence (2.2.1) has a positive fundamental function for $t \geq t_1$. □

Theorem 2.11 contains only implicit nonoscillation conditions. To derive explicit conditions from this theorem, we consider first the equation with two delays

$$\dot{x}(t) + a(t)x\big(h(t)\big) + b(t)x\big(g(t)\big) = 0, \tag{2.6.3}$$

where

$$a(t) \geq 0, \ b(t) \geq 0, \ h(t) \leq t, \ g(t) \leq t.$$

Similar to (2.6.1), we denote (and assume A, B, C, D are finite)

$$A = \sup_{t \ge t_1} \int_{h(t)}^{t} a^+(s)ds, \; B = \sup_{t \ge t_1} \int_{h(t)}^{t} b^+(s)ds,$$
$$C = \sup_{t \ge t_1} \int_{g(t)}^{t} a^+(s)ds, \; D = \sup_{t \ge t_1} \int_{g(t)}^{t} b^+(s)ds. \tag{2.6.4}$$

By Theorem 2.11, the existence of positive solutions of the system

$$\ln x_1 \ge Ax_1 + Bx_2, \; \ln x_2 \ge Cx_1 + Dx_2, \tag{2.6.5}$$

implies nonoscillation of (2.6.3).

Theorem 2.12 *Let*

$$\limsup_{t \to \infty} \int_{g(t)}^{t} a^+(s)ds = \limsup_{t \to \infty} \int_{g(t)}^{t} b^+(s)ds = 0 \tag{2.6.6}$$

and

$$Ae^B < \frac{1}{e}. \tag{2.6.7}$$

Then (2.6.3) *has an eventually positive fundamental function.*

Proof It is sufficient to prove the existence of a positive solution (x_1, x_2), $x_1 > 0$, $x_2 > 0$ for the system

$$\ln x_1 > A_0 x_1 + B_0 x_2, \; \ln x_2 > 0, \tag{2.6.8}$$

where

$$A_0 = \limsup_{t \to \infty} \int_{h(t)}^{t} a^+(s)ds, \; B_0 = \limsup_{t \to \infty} \int_{h(t)}^{t} b^+(s)ds.$$

Assume first $A_0 > 0$, $B_0 > 0$, and put $x_1 = \frac{1}{A_0}$. Then (2.6.8) takes the form

$$B_0 < B_0 x_2 < -1 - \ln A_0.$$

By (2.6.7), there exists $C > 0$ such that $B_0 < C < -1 - \ln A_0$. Therefore the pair

$$(x_1, x_2) = \left(\frac{1}{A_0}, \frac{C}{B_0} \right)$$

will be a solution of the system (2.6.8).

If $A_0 > 0$, $B_0 = 0$, then the pair (x_1, x_2), where $x_1 = e$, $x_2 > 1$, is a solution of (2.6.8).

The case $A_0 = 0$, $B_0 > 0$ is treated similarly.

Existence of a positive solution of (2.6.8) in the case $A_0 = B_0 = 0$ is obvious. □

Example 2.3 Consider the equation

$$\dot{x}(t) + ax(t-1) + bx\big(g(t)\big) = 0, \tag{2.6.9}$$

where a, b are positive numbers, $\lim_{t\to\infty}(t-g(t))=0$. We have $A=a$, $B=b$, $C=D=0$. Then the condition

$$ae^b < \frac{1}{e}$$

implies nonoscillation of (2.6.9).

Example 2.4 Consider the equation

$$\dot{x}(t) + \frac{a}{t}x\left(\frac{t}{\mu}\right) + \frac{b}{t}x(t-\tau) = 0,\ t \ge t_0 > 0, \tag{2.6.10}$$

where $a>0$, $b>0$, $\mu>1$, $\tau>0$. We have $A=a\ln\mu$, $B=b\ln\mu$, $C=D=0$. Hence, if the condition

$$a\mu^b < \frac{1}{e\ln\mu}$$

holds, then (2.6.10) has a nonoscillatory solution.

Example 2.5 Consider the equation

$$\dot{x}(t) + \frac{a}{t\ln t}x\left(t^{\alpha}\right) + \frac{b}{t}x(t-\tau) = 0,\ t \ge t_0 > 1, \tag{2.6.11}$$

where $a>0$, $b>0$, $1>\alpha>0$, $\tau>0$. We have $A=a\ln\frac{1}{\alpha}$, $B=b\ln\frac{1}{\alpha}$, $C=D=0$. Hence, if the condition

$$a\alpha^{-b} < \frac{1}{e\ln\frac{1}{\alpha}}$$

holds, then (2.6.11) has a nonoscillatory solution.

Theorem 2.13 *Suppose that for some $t_1 \ge 0$ at least one of the following conditions holds*:

1) $0<A\le\frac{1}{e}$, $B>0$, *and there exists a number $y_0>0$ such that*

$$y_0 \le -\frac{1+\ln A}{B},\ \ \frac{C}{A} + Dy_0 \le \ln y_0;$$

2) $C>0$, $0<D\le\frac{1}{e}$, *and there exists a number $x_0>0$ such that*

$$x_0 \le -\frac{1+\ln D}{C},\ \ \frac{B}{D} + Ax_0 \le \ln x_0.$$

Then the fundamental function $X(t,s)$ of (2.6.3) *is positive for $t \ge s \ge t_1$.*

Proof Suppose the inequalities in 1) hold. The function $y=(\ln x - Ax)/B$ has the unique maximum $y_{\max} = -\frac{1+\ln A}{B}$ at the point $x_{\max}=\frac{1}{A}$. The inequality

$$-(1+\ln A) \ge By_0 > 0$$

implies $y_{\max}>0$, while $y_0 \le -\frac{1+\ln A}{B}$ yields that the point $(x_{\max}, y_0)$ satisfies the first inequality in (2.6.5) in the case $y_0<y_{\max}$. Since $\frac{C}{A}+Dy_0<\ln y_0$, this point also satisfies the second inequality in (2.6.5). If $y_0 = y_{\max}$, then there exists $y_1 < y_0$ for which the inequality $\frac{C}{A} + Dy_1 < \ln y_1$ still holds. Then $(x_{\max}, y_1)$ is a solution of (2.6.5). If 2) holds, the proof is similar. □

Corollary 2.9 *Suppose that there exists a point $t_1 \geq 0$ such that at least one of the following conditions holds*:

$$0 < A \leq \frac{1}{e},\ B > 0,\ \frac{C}{A} - \frac{D(1+\ln A)}{B} \leq \ln\left(-\frac{1+\ln A}{B}\right), \tag{2.6.12}$$

$$C > 0,\ 0 < D \leq \frac{1}{e},\ \frac{B}{D} - \frac{A(1+\ln D)}{C} \leq \ln\left(-\frac{1+\ln D}{C}\right). \tag{2.6.13}$$

Then the fundamental function $X(t,s)$ of (2.6.3) is positive for $t \geq s \geq t_1$.

Proof If (2.6.12) holds, then there exists $\varepsilon > 0$ such that for $y_0 = -\frac{1+\ln A}{B} - \varepsilon$ the first condition of Theorem 2.13 is satisfied. Similarly, (2.6.13) implies the second condition. □

Remark 2.2 In Theorem 2.13, it is assumed that either $A > 0$, $B > 0$ or $C > 0$, $D > 0$. Including the cases where these conditions are not satisfied, by analyzing (2.6.5) we immediately obtain the following sufficient nonoscillation conditions:

1. $B = 0$, $D > 0$, $A < 1/e$, $1 + \ln D + C/e < 0$;
2. $C = 0$, $A > 0$, $D < 1/e$, $1 + \ln A + B/e < 0$;
3. $A = 0$, $D > 0$, $Ce^{B/D} + 1 + \ln D < 0$;
4. $D = 0$, $A > 0$, $be^{C/A} + \ln A + 1 < 0$;
5. $B = 0$, $C = 0$, $A < 1/e$, $D < 1/e$;
6. $A = 0$, $C = 0$, $D < 1/e$;
7. $B = 0$, $D = 0$, $A < 1/e$.

For $A = D = 0$, the situation is a little bit more complicated in that there exists an eventually positive solution if the following condition is satisfied:

8. $A = D = 0$, and there exists either $x > 0$ such that $\ln x > Be^{Cx}$ or $y > 0$ such that $\ln y > Ce^{By}$.

Example 2.6 Consider the equation

$$\dot{x}(t) + \frac{0.2}{\pi}\sin^2 t x(t-\pi) + \frac{0.2}{\pi}\cos^2 t x(t-2\pi) = 0. \tag{2.6.14}$$

By simple calculations, we have $A = B = 0.1$, $C = D = 0.2$. Condition (2.6.12) in Corollary 2.9 is not satisfied, but inequality (2.6.13) holds. Hence (2.6.14) has an eventually positive solution.

Figure 2.1 illustrates the domain for (x, y) where the inequalities of type (2.6.5) hold:

$$\ln x \geq 0.1x + 0.1y,\ \ln y \geq 0.2x + 0.2y. \tag{2.6.15}$$

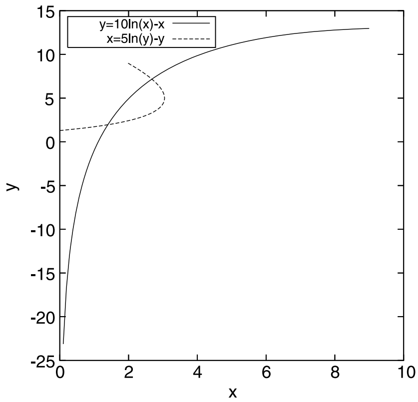

Fig. 2.1 In the *domain between the curves*, the system of inequalities (2.6.15) has a positive solution, so (2.6.14) has an eventually positive solution. Here $A = B = 0.1$, $C = D = 0.2$

We observe that the maximum of $f(x) = 10\ln(x) - x$ is not in the domain between the curves (thus, (2.6.12) is not satisfied), while the maximum of the function $g(y) = 5\ln(y) - y$ is in the intersection domain, so (2.6.13) holds. It should be noted that Theorem 2.7 fails for this equation.

Let us present different sufficient conditions for the existence of positive solutions.

Theorem 2.14 *Suppose that there exists a point* $t_1 \geq 0$ *such that at least one of the following conditions holds*:

1) *There exists* $y_0 > 0$ *such that* $y_0 < (1 - Ae)/B$, $Ce + Dy_0 < \ln y_0$.
2) *There exists* $x_0 > 0$ *such that* $x_0 < (1 - De)/C$, $Ax_0 + Be < \ln x_0$.

Then the fundamental function of (2.6.3) *is positive*.

Proof Suppose 1) holds. Then $Ae < 1$ and (e, y_0) is a solution of the system of inequalities (2.6.5). Similarly, if 2) holds, then (x_0, e) is a solution of (2.6.5). □

Remark 2.3 In Theorem 2.14, the value $x = e$ was chosen to minimize the coefficient of x in the first inequality of the system

$$\left(A - \frac{\ln x}{x}\right)x + By < 0, \;\; Cx + \left(D - \frac{\ln y}{y}\right)y < 0,$$

which is equivalent to (2.6.5), and $y = e$ minimizes the coefficient of y in the second inequality.

Corollary 2.10 *Suppose that there exists a point* $t_1 \geq 0$ *such that at least one of the following inequalities holds*:

$$Ce + \frac{D}{B}(1 - Ae) \leq \ln\left(\frac{1 - Ae}{B}\right), \tag{2.6.16}$$

$$Be + \frac{A}{C}(1 - De) \le \ln\left(\frac{1 - De}{C}\right). \tag{2.6.17}$$

Then the fundamental function of (2.6.3) *is positive.*

Let us modify Example 2.6 to demonstrate that there are cases where either Theorem 2.13 or Theorem 2.14 can be applied while the other one fails.

Example 2.7 Consider the following modified version of (2.6.14):

$$\dot{x}(t) + \frac{0.5}{\pi}\sin^2 t\, x(t - \pi) + \frac{0.08}{\pi}\cos^2 t\, x(t - 2\pi) = 0. \tag{2.6.18}$$

Then $A = 0.25$, $B = 0.04$, $C = 0.5$, $D = 0.08$ and (2.6.16) becomes

$$0.5e + 2(1 - 0.25e) = 2 < 2.08 \approx \ln\left(\frac{1 - 0.25e}{0.04}\right);$$

i.e., (2.6.16) is satisfied and there exists an eventually positive solution of (2.6.18). Theorem 2.7 fails for (2.6.18) since $0.5 + 0.08 > 1/e$. Simple computations demonstrate that (2.6.12), (2.6.13) and (2.6.17) also fail for (2.6.18).

On the other hand, for the equation

$$\dot{x}(t) + \frac{0.2}{\pi}\sin^2 t\, x(t - \pi) + \frac{0.25}{\pi}\cos^2 t\, x(t - 2\pi) = 0 \tag{2.6.19}$$

with $A = 0.1$, $B = 0.125$, $C = 0.2$, $D = 0.25$, inequality (2.6.13) is satisfied. This implies existence of an eventually positive solution for (2.6.19), while Theorem 2.7, (2.6.16), (2.6.17) and (2.6.12) fail.

Next, consider (2.2.1) with several delays.

Theorem 2.15 *Suppose that there exists a point* $t_1 \ge 0$ *and an index* k, $1 \le k \le m$, *such that*

$$B_i := \sum_{j \ne k} A_{ij} \le \frac{1}{e}, \ i = 1, 2, \cdots, m, \tag{2.6.20}$$

where A_{ij} *are defined in* (2.6.1), *and there exists* $z > 0$ *satisfying the inequalities*

$$z \le \min_{i \ne k} \frac{1 - B_i e}{A_{ik}}, \ \sum_{j \ne k} A_{kj} e + A_{kk} z \le \ln z. \tag{2.6.21}$$

Then the fundamental function of (2.2.1) *is positive.*

Proof Suppose that such k exists. Let $x_i = e, i \ne k; x_i = z, i = k$. Then the first inequality in (2.6.21) implies all inequalities in (2.6.2) but the k-th one, which is a corollary of the latter inequality in (2.6.21). Thus (2.6.2) has a positive solution, so (2.2.1) has an eventually positive solution, which completes the proof. □

Corollary 2.11 *Suppose there exist a point $t_1 \ge 0$ and an index k, $1 \le k \le m$, such that*

$$e \sum_{j \neq k} A_{kj} + A_{kk} B \le \ln B, \tag{2.6.22}$$

where $B = \min_{i \neq k} \frac{1-B_i e}{A_{ik}}$ and A_{kj} are denoted by (2.6.1). Then the fundamental function of (2.2.1) is positive.

Proof Due to the continuity of the function $\ln x - A_{kk}x$, there exists $\varepsilon > 0$ such that if we substitute $z = B - \varepsilon$ instead of B, the inequality (2.6.22) is still valid; i.e., the second inequality in (2.6.21) is satisfied. Then $z \le \frac{1-B_i e}{A_{ik}}$ for any $i \neq k$, where B_i are defined in (2.6.20), so the first inequality in (2.6.21) is also satisfied. By Theorem 2.15, (2.2.1) has an eventually positive solution. □

Using Theorem 2.4, we can also apply Theorem 2.13 to general equations with several delays.

Theorem 2.16 *Suppose $a_k(t) \ge 0$, $k = 1, \cdots, m$, and let $I_1 \subset I = \{1, \cdots, m\}$, $I_2 = I \backslash I_1$. Denote*

$$a(t) = \sum_{k \in I_1} a_k(t),\ \ b(t) = \sum_{k \in I_2} a_k(t),\ \ h(t) = \min_{k \in I_1} h_k(t),\ \ g(t) = \min_{k \in I_2} h_k(t).$$

Here we assume $h(t) \equiv t$ or $g(t) \equiv t$ if $I_1 = \emptyset$ or $I_2 = \emptyset$, respectively. Suppose that there exists a point $t_1 \ge 0$ such that the hypotheses of Theorem 2.13 or Remark 2.2 are satisfied, where A, B, C, D are defined in (2.6.4). Then the fundamental function of (2.2.1) is positive.

Proof Nonoscillation of (2.6.3) and Theorem 2.4 imply nonoscillation of (2.2.1). □

Remark 2.4 Theorem 2.16 contains 2^m different nonoscillation conditions. In particular, if $I_1 = I$, $I_2 = \emptyset$, then Remark 2.2 implies Theorem 2.7. Indeed, in this case we have $a(t) = \sum_{k=1}^{m} a_k(t)$, $b(t) \equiv 0$, $h(t) = \min_{k \in I} h_k(t)$, $g(t) \equiv t$. We have

$$A = \sup_{t \ge t_1} \int_{h(t)}^{t} \sum_{k=1}^{m} a_k(s) ds,\ \ B = C = D = 0.$$

If we take $x_1 = e$, $x_2 > 1$, then inequalities (2.6.5) have the form $A \le \frac{1}{e}$, $\ln x_2 > 0$, which is equivalent to (2.5.1).

2.7 Oscillation Conditions

There are many explicit oscillation conditions for equations with one delay and only a few for equations with several delays (2.2.1). We present here some explicit oscillation tests. First, let us mention two known oscillation conditions.

Lemma 2.2 [192] *Suppose* $a_k(t) \geq 0$ *and at least one of the following conditions holds*:

1)

$$\liminf_{t\to\infty} \int_{\max_k h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds > \frac{1}{e},$$

2) h_k *are nondecreasing and*

$$\limsup_{t\to\infty} \int_{\max_k h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds > 1.$$

Then all solutions of (2.2.1) *are oscillatory.*

Lemma 2.3 [192] *Suppose* $a_k(t) \geq 0$, $k = 1, \cdots, m$ *and*

$$\liminf_{t\to\infty} \sum_{k=1}^{m} a_k(t)\big(t - h_k(t)\big) > \frac{1}{e}.$$

Then all solutions of (2.2.1) *are oscillatory.*

The conditions of Lemma 2.2 are given in the integral form but contain only the worst delay function. The inequality of Lemma 2.3 contains all the delays but is presented in the pointwise form. The following result contains all the delays and has the integral form.

Theorem 2.17 *Suppose* $a_k(t) \geq 0$, $k = 1, \cdots, m$ *and there exists a set of indices* $J \subset \{1, \cdots, m\}$ *such that* $\sum_{k\in J} a_k(t) \neq 0$ *almost everywhere*, $\int_{t_0}^{\infty} \sum_{i=1}^{m} a_i(s)ds = \infty$ *and*

$$\liminf_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i\in J} a_i(t)} \int_{h_k(t)}^{t} \sum_{i\in J} a_i(s)ds > \frac{1}{e}. \tag{2.7.1}$$

Then all solutions of (2.2.1) *are oscillatory.*

Proof After the substitution

$$s = \int_{t_0}^{t} \sum_{k\in J} a_k(\tau)d\tau, \;\; y(s) = x(t), \;\; l_k(s) = \int_{t_0}^{h_k(t)} \sum_{k\in J} a_k(\tau)d\tau,$$

(2.2.1) has the form

$$\dot{y}(s) + \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i\in J} a_i(t)} y\big(l_k(s)\big) = 0. \tag{2.7.2}$$

Evidently oscillation of (2.2.1) is equivalent to oscillation of (2.7.2).

Since $s - l_k(s) = \int_{h_k(t)}^{t} \sum_{i\in J} a_i(s)ds$, Lemma 2.3 and condition (2.7.1) imply this theorem. □

Remark 2.5 The first part of Lemma 2.2 can be obtained as a corollary of Theorem 2.17 for $J = \{1, \cdots, m\}$.

Consider now (2.2.1) with two delays:

$$\dot{x}(t) + a(t)x\big(h(t)\big) + b(t)x\big(g(t)\big) = 0. \tag{2.7.3}$$

Corollary 2.12 *Suppose $a(t) \geq 0$, $b(t) \geq 0$ and at least one of the following conditions holds*:

1. $a(t) \neq 0$ *almost everywhere* (*a.e.*) *and*
$$\liminf_{t\to\infty}\left(\int_{h(t)}^{t} a(s)ds + \frac{b(t)}{a(t)}\int_{g(t)}^{t} a(s)ds\right) > \frac{1}{e};$$
2. $b(t) \neq 0$ *a.e. and*
$$\liminf_{t\to\infty}\left(\frac{a(t)}{b(t)}\int_{h(t)}^{t} b(s)ds + \int_{g(t)}^{t} b(s)ds\right) > \frac{1}{e};$$
3. $a(t) + b(t) \neq 0$ *a.e. and*
$$\liminf_{t\to\infty}\left(\frac{a(t)}{a(t)+b(t)}\int_{h(t)}^{t}\big[a(s)+b(s)\big]ds + \frac{b(t)}{a(t)+b(t)}\int_{g(t)}^{t}\big[a(s)+b(s)\big]ds\right) > \frac{1}{e}.$$

Then all solutions of (2.7.3) *are oscillatory.*

Proof We fix the sets of indices $J = \{1\}$, $J = \{2\}$ and $J = \{1, 2\}$, respectively. □

Consider (2.7.3) with a nondelay term,

$$\dot{x}(t) + a(t)x(t) + b(t)x\big(g(t)\big) = 0. \tag{2.7.4}$$

Corollary 2.13 *Suppose $a(t) \geq 0$, $b(t) \geq 0$, $a(t) \neq 0$ a.e. and*

$$\liminf_{t\to\infty}\frac{b(t)}{a(t)}\int_{g(t)}^{t} a(s)ds > \frac{1}{e}.$$

Then all solutions of (2.7.3) *are oscillatory.*

Example 2.8 By Part 3 of Corollary 2.12, the equation

$$\dot{x}(t) + \big[1+\sin(2\pi t)\big]x\big(h(t)\big) + \gamma\big[1+\sin(2\pi t)\big]x(t-1) = 0, \tag{2.7.5}$$

where $h(t) \leq t$, $\lim_{t\to\infty} h(t) = t$, is oscillatory if

$$\gamma > \frac{1}{e} \tag{2.7.6}$$

since

$$\int_{t-1}^{t}\big[1+\sin(2\pi s)\big]ds = 1 \text{ for any } t,\ \frac{b(t)}{a(t)} = \gamma.$$

However, Lemma 2.2 cannot be applied to establish oscillation since $\max\{t, t-1\} = t$, and the condition of Lemma 2.3 is not satisfied since $\liminf_{t\to\infty} \gamma[1 + \sin(2\pi t)] = 0$ for any γ.

2.8 Estimations of Solutions

First let us obtain a lower estimation of the fundamental function.

Theorem 2.18 *Suppose conditions of Theorem 2.7 hold. Then*

$$X(t,s) \geq \exp\left\{-e\int_s^t \sum_{k=1}^m a_k^+(s)ds\right\},\ t \geq s \geq t_1.$$

Proof Suppose first that $a_k(t) \geq 0, t \geq t_0$ and conditions of Theorem 2.1, Part 2, hold. In the proof of Theorem 2.1, it was shown that

$$X(t,t_1) \geq \exp\left\{-\int_{t_1}^t u(s)ds\right\},\ t \geq t_1,$$

where the function $u(t)$ was denoted in Part 2 of Theorem 2.1.

The same calculations lead to the estimate

$$X(t,s) \geq \exp\left\{-\int_s^t u(s)ds\right\},\ t \geq s \geq t_1.$$

By the proof of Theorem 2.7, the function $u(t) = e\sum_{k=1}^m a_k^+(t)$ satisfies all the conditions of Part 2 of Theorem 2.1. Hence the theorem is true for the case $a_k(t) \geq 0$.

Consider now the general case and denote by $X^+(t,s)$ the fundamental function of the equation

$$\dot{x}(t) + \sum_{k=1}^m a_k^+(t)x\big(h_k(t)\big) = 0.$$

As was proven before, $X^+(t,s) \geq \exp\{-e\int_s^t \sum_{k=1}^m a_k^+(s)ds\}, t \geq s \geq t_1$.

The fundamental function $X(t,s)$ of (2.2.1) is the solution of the initial value problem

$$\dot{x}(t) + \sum_{k=1}^m a_k^+(t)x\big(h_k(t)\big) - \sum_{k=1}^m a_k^-(t)x\big(h_k(t)\big) = 0,\ t \geq s,$$
$$x(t) = 0,\ t < s,\ x(s) = 1.$$

Hence, by solution representation formula (2.2.5) for $t \geq t_1$,

$$X(t,s) = X^+(t,s) + \int_s^t X^+(t,\tau)\sum_{k=1}^m a_k^-(\tau)X\big(h_k(\tau),s\big)d\tau.$$

By Theorem 2.7, we have $X(t,s) > 0$, $t \ge s \ge t_1$. Then

$$X(t,s) \ge X^+(t,s) \ge \exp\left\{-e \int_s^t \sum_{k=1}^m a_k^+(s)ds\right\}. \qquad \square$$

Now let us proceed to upper estimates of the fundamental function.

Theorem 2.19 *In* (2.2.1), *let*

$$a_k(t) \ge 0,\ X(t,s) > 0,\ t \ge s \ge t_0,\ t - h_k(t) \le H,\ t \ge t_0.$$

Then, for $t > s \ge t_0$,

$$0 < X(t,s) \le Y(t,s) := \begin{cases} \exp\{-\int_{s+H}^t \sum_{k=1}^m a_k(\tau)d\tau\}, & t \ge s+H, \\ 1, & s \le t \le s+H. \end{cases}$$

Proof It is sufficient to prove the theorem for $s = t_0$ since the general case is considered similarly. Denote

$$x(t) = X(t,t_0),\ y(t) = Y(t,t_0),\ t > t_0 + H,\ y(t) = X(t,t_0),\ t \le t_0 + H.$$

Then, $x(t) = y(t)$ for $t \le t_0 + H$, and for $t \ge t_0 + H$ we have

$$\begin{aligned} &\dot{y}(t) + \sum_{k=1}^m a_k(t) y\big(h_k(t)\big) \\ &\quad = -\sum_{k=1}^m a_k(t) \exp\left\{-\int_{t_0+H}^t \sum_{k=1}^m a_k(\tau)d\tau\right\} + \sum_{k=1}^m a_k(t) r_k(t) \\ &\quad = \sum_{k=1}^m a_k(t)\left[r_k(t) - \exp\left\{-\int_{t_0+H}^t \sum_{k=1}^m a_k(\tau)d\tau\right\}\right] \ge 0, \end{aligned}$$

where

$$r_k(t) = \begin{cases} \exp\{-\int_{t_0+H}^{h_k(t)} \sum_{k=1}^m a_k(\tau)d\tau\}, & h_k(t) \ge t_0 + H, \\ X(t,t_0), & t_0 \le h_k(t) \le t_0 + H. \end{cases}$$

Theorem 2.5 implies $y(t) \ge x(t)$. Hence $Y(t,t_0) \ge X(t,t_0)$, $t \ge t_0 + H$.

Inequality $X(t,t_0) \le 1$ is valid since $X(t_0,t_0) = 1$ and $X_t'(t,t_0) \le 0$. Hence $1 = Y(t,t_0) \ge X(t,t_0)$ for $t_0 \le t \le t_0 + H$. $\square$

Corollary 2.14 *Let*

$$a_k(t) \ge 0,\ \sum_{k=1}^m a_k(t) \ge a > 0,\ X(t,s) > 0,\ t \ge s \ge t_0,\ t - h_k(t) \le H,\ t \ge t_0.$$

Then, for $t > s \ge t_0$,

$$0 < X(t,s) \le Y(t,s) := \begin{cases} \exp\{-a(t-s-H)\}, & t \ge s+H, \\ 1, & s \le t \le s+H. \end{cases}$$

Theorem 2.20 *Suppose* $a_k(t) \geq 0$, $\sum_{k=1}^m a_k(t) \geq a > 0$, $X(t,s) > 0$, $t \geq s \geq t_0$, $t - h_k(t) \leq H$, $t \geq t_0$. *Then, for the solution of problem* (2.2.2), (2.2.3), *we have the estimates*

$$\left|x(t)\right| \leq \left|x(t_0)\right| + \left(\|f\| + \sum_{k=1}^m \|a_k\|\|\varphi\|\right)(t-t_0),\ t_0 \leq t < t_0 + H,$$

$$\left|x(t)\right| \leq \left(\left|x(t_0)\right| + \frac{1}{a}\left(e^{aH} - 1\right)\sum_{k=1}^m \|a_k\|\|\varphi\|\right)e^{-a(t-t_0-H)} + \frac{\|f\|}{a}e^{aH},$$
$$t \geq t_0 + H,$$

where

$$\|\varphi\| = \sup_{t_0 - H \leq t \leq t_0}\left|\varphi(t)\right|,\ \|f\| = \sup_{t \geq t_0}\left|f(t)\right|,\ \|a_k\| = \sup_{t\geq t_0}\left|a_k(t)\right|.$$

Proof By solution representation formula (2.2.5), we have

$$\left|x(t)\right| \leq X(t,t_0)\left|x(t_0)\right| + \int_{t_0}^t X(t,s)\left(\sum_{k=1}^m a_k(s)\left|\varphi\left(h_k(s)\right)\right| + \left|f(s)\right|\right)ds,$$

where $\varphi(t) = 0$, $t \geq t_0$.

Suppose first $t_0 \leq t \leq t_0 + H$. Then

$$\begin{aligned}\left|x(t)\right| &\leq \left|x(t_0)\right| + \int_{t_0}^t \left(\sum_{k=1}^m a_k(s)\left|\varphi\left(h_k(s)\right)\right| + \left|f(s)\right|\right)ds \\ &\leq \left|x(t_0)\right| + \left(\|f\| + \sum_{k=1}^m \|a_k\|\|\varphi\|\right)(t-t_0).\end{aligned}$$

Next, let $t \geq t_0 + H$. Then

$$\begin{aligned}\left|x(t)\right| &\leq \left|x(t_0)\right|e^{-a(t-t_0-H)} + \int_{t_0}^{t_0+H} e^{-a(t-s-H)}\sum_{k=1}^m a_k(s)\left|\varphi\left(h_k(s)\right)\right|ds \\ &\quad + \int_{t_0}^t e^{-a(t-s-H)}\left|f(s)\right|ds \\ &\leq \left|x(t_0)\right|e^{-a(t-t_0-H)} + \sum_{k=1}^m \|a_k\|\|\varphi\|\frac{1}{a}\left(e^{aH} - 1\right)e^{-a(t-t_0-H)} + \frac{\|f\|}{a}e^{aH}.\end{aligned}$$

□

Another integral estimation of the fundamental function can be obtained using the following result.

Theorem 2.21 *Suppose* $a_k(t) \geq 0$ *and the fundamental function of* (2.2.1) *is positive*: $X(t,s) > 0$, $t \geq s \geq t_0$. *Then there exists* $t_1 \geq t_0$ *such that*

$$0 \leq \int_{t_1}^t X(t,s)\sum_{k=1}^m a_k(s)ds \leq 1. \tag{2.8.1}$$

Proof The function $x(t) \equiv 1$, $t > t_0$, is a solution of the problem

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x\big(h_k(t)\big) = \sum_{k=1}^{m} a_k(t) \chi_{(t_0,\infty)}\big(h_k(t)\big), \; x(t) = 0, \; t \le t_0,$$

where $\chi_{(t_0,\infty)}(t)$ is the characteristic function of the interval (t_0, ∞).

Hence, by (2.2.5) we have

$$1 = \int_{t_0}^{t} X(t,s) \sum_{k=1}^{m} a_k(s) \chi_{(t_0,\infty)}\big(h_k(s)\big) ds.$$

There exists $t_1 \ge t_0$ such that all $h_k(t) \ge t_0$ for $t \ge t_1$, and thus for any $t \ge t_1$

$$\int_{t_0}^{t_1} X(t,s) \sum_{k=1}^{m} a_k(s) \chi_{(t_0,\infty)}\big(h_k(s)\big) ds + \int_{t_1}^{t} X(t,s) \sum_{k=1}^{m} a_k(s) ds = 1,$$

which implies inequality (2.8.1). □

Remark 2.6 If $t - h_k(t) \le H$, then we can take $t_1 = t_0 + H$.

2.9 Positivity of Solutions

Now we proceed to the analysis of positivity for solutions of problem (2.2.2), (2.2.3). We will show that if the inequality (2.3.2) has a nonnegative solution and the condition

$$0 \le \varphi(t) \le x(t_0), \; t \le t_0, \; x(t_0) > 0, \tag{2.9.1}$$

holds, then the solution of the initial value problem (2.2.1), (2.2.3) is positive. This result supplements some statements in [192].

Theorem 2.22 *Suppose $a_k(t) \ge 0$, $f(t) \ge 0$ and there exists a nonnegative solution of the inequality*

$$u(t) \ge \sum_{k=1}^{m} a_k(t) \int_{\max\{t_0, h_k(t)\}}^{t} u(s) ds, \; t \ge t_0, \tag{2.9.2}$$

for a certain $t_0 \ge 0$ and conditions (2.9.1) hold. Then the solution of problem (2.2.2), (2.2.3) is positive for $t \ge t_0$.

Proof Let $u(t) \ge 0$, $t \ge t_0$ be a solution of (2.9.2). Denote $u(t) = 0$, $t < t_0$. Then

$$u(t) \ge \sum_{k=1}^{m} a_k(t) \int_{h_k(t)}^{t} u(s) ds, \; t \ge t_0.$$

Hence all conditions of Theorem 2.1 are satisfied, and thus the fundamental function $X(t,s)$ is positive: $X(t,s) > 0$ for $t \ge s \ge t_0$.

First assume $f \equiv 0$. Consider the auxiliary problem

$$\dot{z}(t) + \sum_{k=1}^{m} a_k(t) z\big(h_k(t)\big) = 0,\ t \geq t_0,\ z(t) = x_0,\ t \leq t_0. \tag{2.9.3}$$

Denote

$$v(t) = \begin{cases} x_0 \exp\{-\int_{t_0}^{t} u(s)ds\}, & t \geq t_0, \\ x_0, & t < t_0, \end{cases}$$

and for a fixed $t \geq t_0$ define the sets

$$N_1(t) = \big\{k : h_k(t) \geq t_0\big\},\ N_2(t) = \big\{k : h_k(t) < t_0\big\}.$$

We obtain

$$\begin{aligned}
&\dot{v}(t) + \sum_{k=1}^{m} a_k(t) v\big(h_k(t)\big) \\
&\quad = -x_0 u(t) \exp\left\{-\int_{t_0}^{t} u(s)ds\right\} \\
&\qquad + x_0 \sum_{k \in N_1(t)} a_k(t) \exp\left\{-\int_{t_0}^{h_k(t)} u(s)ds\right\} + x_0 \sum_{k \in N_2(t)} a_k(t) \\
&\quad = -x_0 \exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\left[u(t) - \sum_{k \in N_1(t)} a_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}\right. \\
&\qquad \left. - \sum_{k \in N_2(t)} a_k(t) \exp\left\{\int_{t_0}^{t} u(s)ds\right\}\right] \\
&\quad = -x_0 \exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\left[u(t) - \sum_{k \in N_1(t)} a_k(t) \exp\left\{\int_{\max\{t_0, h_k(t)\}}^{t} u(s)ds\right\}\right. \\
&\qquad \left. - \sum_{k \in N_2(t)} a_k(t) \exp\left\{\int_{\max\{t_0, h_k(t)\}}^{t} u(s)ds\right\}\right] \\
&\quad = -x_0 \exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\left[u(t) - \sum_{k=1}^{m} a_k(t) \exp\left\{\int_{\max\{t_0, h_k(t)\}}^{t} u(s)ds\right\}\right] \leq 0.
\end{aligned}$$

Hence $v(t)$ is a solution of the problem

$$\dot{v}(t) + \sum_{k=1}^{m} a_k(t) v\big(h_k(t)\big) = g(t),\ t \geq t_0,\ v(t) = x_0,\ t \leq t_0,$$

with $g(t) \leq 0$. Theorem 2.5 implies that $z(t) \geq v(t) > 0$.

Conditions (2.9.1) and Corollary 2.3 imply $x(t) \geq z(t) > 0$, $t \geq t_0$. For the case $f \equiv 0$, the theorem is proven. The general case also follows from Theorem 2.5 since $f(t) \geq 0$. □

Corollary 2.15 *Suppose* $a_k(t) \geq 0,\ f(t) \geq 0$ *and*

$$\int_{\max\{t_0, \min_k h_k(t)\}}^{t} \sum_{j=1}^{m} a_j^+(s)ds \leq \frac{1}{e},\ t \geq t_0,$$

for a certain $t_0 \geq 0$ *and conditions* (2.9.1) *hold. Then the solution of the problem* (2.2.2), (2.2.3) *is positive for* $t \geq t_0$.

Proof As demonstrated in the proof of Theorem 2.7, the function $u(t) = e\sum_{k=1}^{m} a_k(t)$ is a solution of inequality (2.9.2). Application of Theorem 2.22 completes the proof. □

2.10 Slowly Oscillating Solutions for Delay Differential Equations

Definition 2.4 A solution x of (2.2.1) is said to be *slowly oscillating* if for every $t_0 \geq 0$ there exist $t_1 > t_0$, $t_2 > t_1$ such that $h_k(t) \geq t_1$ for $t \geq t_2$, $x(t_1) = x(t_2) = 0$, $x(t) > 0$, $t \in (t_1, t_2)$.

In particular, if $h_k(t) = t - \tau_k$, $\tau_k > 0$ and for every $t_0 \geq 0$ there exist $t_1 > t_0$, $t_2 > t_1$ such that $x(t_1) = x(t_2) = 0$, $x(t) > 0$, $t \in (t_1, t_2)$, $t_2 - t_1 \geq \max_k \tau_k$, then $x(t)$ is slowly oscillating.

Theorem 2.23 *Let* $a_k(t) \geq 0$. *If there exists a slowly oscillating solution of* (2.2.1) (*inequality* (2.3.1)), *then all solutions of this equation* (*inequality*) *are oscillatory.*

Proof Denote by x a slowly oscillating solution of (2.2.1). Suppose that this equation has a nonoscillatory solution. Then, by Theorem 2.1, for a certain $t_0 \geq 0$ the fundamental function satisfies $X(t, s) > 0$ if $t \geq s > t_0$.

There exist $t_1 > t_0$, $t_2 > t_1$ such that

$$h_k(t) \geq t_1 \text{ for } t \geq t_2,\ x(t_1) = x(t_2) = 0,\ x(t) > 0,\ t \in (t_1, t_2). \tag{2.10.1}$$

Due to solution representation formula (2.2.5), for $t \geq t_2$, solution $x(t)$ has the form

$$x(t) = -\int_{t_2}^{t} X(t, s) \sum_{k=1}^{m} a_k(s) x\big(h_k(s)\big) ds, \tag{2.10.2}$$

where $x(h_k(s)) = 0$ if $h_k(s) > t_2$. The inequality $h_k(t) \geq t_1$ for $t \geq t_2$ yields that the expression under the integral in (2.10.2) can differ from zero only if $t_1 < h_k(s) < t_2$. Therefore (2.10.1) yields that in (2.10.2) we have $x(h_k(s)) > 0$. Consequently, (2.10.2) implies $x(t) \leq 0$ for each $t \geq t_2$. This contradicts the assumption that x is an oscillatory solution. □

Corollary 2.16 *Suppose* $a_k(t) \geq 0$ *and there exists a nonoscillatory solution of* (2.2.1). *Then* (2.2.1) *has no slowly oscillating solutions.*

2.11 Stability and Nonoscillation

In this section, we present a corollary of Theorem 9.18 that will later be obtained for systems of linear delay differential equations.

Theorem 2.24 *Suppose that* $a_k(t) \ge 0$, $\sum_{k=1}^{m} a_k(t) \ge a_0 > 0$, $t - h_k(t) \le h_0$, $k = 1, \cdots, m$, *and there exists an eventually positive solution of* (2.2.1). *Then* (2.2.1) *is exponentially stable.*

Corollary 2.17 *Suppose that* $a_k > 0$, $k = 1, \cdots, m$, *and there exists a positive solution* λ *of the equation*

$$\lambda = \sum_{k=1}^{m} a_k e^{\lambda \tau_k}.$$

Then the autonomous equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k x(t - \tau_k) = 0$$

is exponentially stable.

Corollary 2.18 *Suppose that* $a_k(t) \ge 0$, $\sum_{k=1}^{m} a_k(t) \ge a_0 > 0$, $t - h_k(t) \le h_0$, $k = 1, \cdots, m$, *and the conditions of anyone of Theorems* 2.7, 2.8, 2.10, 2.11, 2.15 *and* 2.16 *hold. Then* (2.2.1) *is exponentially stable.*

Corollary 2.19 *Suppose* $a(t) \ge 0$, $b(t) \ge 0$, $a(t) + b(t) \ge a_0 > 0$, $t - h(t) \le h_0$, $t - g(t) \le g_0$ *and conditions of one of either Theorems* 2.12 *or* 2.14 *or Corollaries* 2.9 *or* 2.10 *hold. Then* (2.6.3) *is exponentially stable.*

In particular, all equations in Examples 2.1, 2.6 and 2.7 are exponentially stable.

2.12 Discussion and Open Problems

This chapter deals with some properties of a scalar delay differential equation that are equivalent to nonoscillation. For most classes of autonomous functional differential equations, nonoscillation is equivalent to existence of a real root of the characteristic equation [192].

As was demonstrated in [192], for a nonautonomous scalar linear delay differential equation, nonoscillation is equivalent to existence of a nonnegative solution for a certain nonlinear integral equation that was called “the generalized characteristic equation”. In [80], for a neutral scalar differential equation, an integral nonlinear inequality was constructed that has a nonnegative solution if and only if the fundamental function of this equation is positive.

For a scalar differential equation with one delay, the equivalence of nonoscillation and the existence of a nonnegative solution of the same inequality as in [80] was justified in [154, 232]. Unlike [154, 192, 232], it is assumed in this monograph that coefficients of the equation, delays and the initial function are not necessarily continuous but Lebesgue measurable. Such weak constraints on equation parameters are sufficient if a solution is an absolutely continuous function. Besides, a solution is not assumed to be a continuous extension of the initial function, which is a natural assumption for impulsive differential equations, considered further in Chaps. 12–14.

The main result of this chapter is Theorem 2.1, where it is demonstrated that nonoscillation is equivalent to the three other properties of (2.2.1). Such theorems are very popular for delay equations (see, for example, [192, Theorem 3.1.1]). However, in contrast to [192], we also show the equivalence of nonoscillation and positivity of the fundamental function. This property of the fundamental function is very important in stability theory, boundary value problems, control theory and generally in the qualitative theory of differential equations; we apply here positivity of the fundamental function, in particular, to prove comparison theorems.

Comparison theorems appear to be an efficient tool in oscillation theory [154, 167, 192, 228, 289]. In paper [193], a rather general comparison result was presented for a nonlinear delay differential equation. In Sect. 2.3, a similar result is obtained for a linear equation using a different technique based on the equivalence of nonoscillation and positivity of the fundamental function. Here we follow the paper [41], where some results of [193] were improved and extended to a more general class of equations.

Explicit nonoscillation conditions were obtained in Sects. 2.4 and 2.5. Theorems 2.8 and 2.10 outline the fact that for the equation

$$\dot{x}(t) + a(t)x\big(h(t)\big) = 0, \tag{2.12.1}$$

where $a(t) \geq 0$, the condition

$$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)ds < \frac{1}{e} \tag{2.12.2}$$

is not necessary for nonoscillation.

Some nonoscillation conditions of Sect. 2.5 were taken from the paper [41], while Sect. 2.6 is based on [64].

Consider the equation

$$\dot{x}(t) + a(t)x(t-\tau) = 0, \tag{2.12.3}$$

where $a(t) \geq 0$, $\tau \geq 0$ and $a(t)$ is a continuous function.

For (2.12.3), the situation where

$$\liminf_{t\to\infty} \left[a(t) - \frac{1}{\tau e} \right] = 0$$

is called *the critical case* (see, for example, [104]) because a small perturbation can change oscillation properties of (2.12.3). In [104, 109], nonoscillation and oscillation results were obtained for (2.12.3) in the critical case.

Theorem 2.25 [104] *Let us assume that for some number $k \in \mathbb{N}$ and large t we have $a(t) \le a_k(t)$, where*

$$a_k(t) := \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \frac{\tau}{8e(t\ln t\ln_2 t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\cdots\ln_k t)^2},$$

$\ln_k t = \ln\ln\cdots\ln t$.

Then there exists a positive solution of (2.12.3) *such that*

$$x(t) < e^{-t/\tau}\sqrt{t\ln t\ln_2 t\cdots\ln_k t}.$$

If

$$a(t) > a_{k-2} + \frac{\theta\tau}{8e(t\ln t\ln_2 t\cdots\ln_k t)^2}$$

for some $\theta > 1$, then all solutions of (2.12.3) *are oscillatory.*

In [109], *the authors extend this result for* (2.12.3) *with continuous delay $\tau(t)$ and obtain the following result.*

Theorem 2.26 [109] *Suppose in* (2.12.3) *that $\tau = \tau(t)$ is a nonnegative continuous function. If for large t*

$$a(t) \le \frac{1}{\tau(t)}\exp\left\{-\int_{t-\tau(t)}^{t}\frac{ds}{\tau(s)}\right\},$$

then (2.12.3) *has a positive solution such that*

$$x(t) < \exp\left\{-\int_{t_0-\tau(t_0)}^{t}\frac{ds}{\tau(s)}\right\}$$

for some $t_0 \ge 0$.

Some other nonoscillation and oscillation results in the critical case can be found in the papers [35, 105, 144, 317, 324].

For the noncritical case, a summary of some other nonoscillation results is presented in the following theorem.

Theorem 2.27 *Suppose at least one of the following conditions holds*:

1. [302, 336] *For sufficiently large T and for some $\lambda > 0$,*

$$-\lambda + \sup_{t\ge T}\sum_{i=1}^{m} a_i^+(t)\exp\{\lambda(t - h_i(t))\} \le 0.$$

2. [303] $t - h_i(t) = \tau_i > 0$, *and there exist $\lambda > 0$ and a sufficiently large T such that*

$$-\lambda + \sup_{t\ge T}\max_{j=1,\cdots,m}\sum_{k=1}^{n} p_{jk}(t)e^{\lambda\tau_k} \le 0,$$

where $p_{jk}(t) = \frac{1}{\tau_j}\int_{t-\tau_j}^{t} a_k^+(s)ds$.

Then there exists a nonoscillatory solution of (2.2.1).

There are a lot of papers devoted to explicit oscillation conditions for (2.2.1). A review of these results for equations with one delay is presented in the paper [320] and for equations with several delays in [170]. In [323, 352], a connection between oscillation properties of a linear differential equation with several constant delays and an explicitly constructed linear second-order ordinary differential equation was established.

In the following theorem, some explicit oscillation conditions for (2.2.1) with several delays are outlined.

Theorem 2.28 *Let $a_k(t) \geq 0$, $k = 1, \cdots, m$. All solutions of (2.2.1) are oscillatory if any of the following conditions hold:*

1. [18] *Let $h_i(t) := t - \tau_i$, $\tau_i > 0$,*

$$\liminf_{t\to\infty} \int_t^{t+\tau_i} a_i(s)ds > 0,$$

   and at least one of the following three inequalities holds:
   a. $p_{ij}^* := \liminf_{t\to\infty} \int_{t-\tau_i}^{t} a_j(s)ds > 1/e$ *for some i, j;*
   b. $[\prod_{i=1}^{n} \sum_{j=1}^{n} p_{ij}^*]^{1/n} > 1/e$;
   c. $\sum_{i=1}^{m} p_{ij}^* + 2\sum_{i<j}^{m} (p_{ij}^* p_{ji}^*)^{1/2} > m/2$ *for some j.*
2. [208] *Coefficients and delays satisfy $a_k(t) > 0$, $0 < t - h_k(t) < \sigma$, $k = 1, \cdots, m$, and*

$$\liminf_{t\to\infty} \sum_{k=1}^{m} a_k(t)\big(t - h_k(t)\big) > \frac{1}{e}.$$

3. [180] *There exist indices $i_l \in \{1, \cdots, m\}$ such that*

$$\liminf_{t\to\infty}\big(t - h_{i_l}(t)\big) > 0, \ \liminf_{t\to\infty} \sum_{i=1}^{m} a_{i_l}(t) > 0$$

   and at least one of the following inequalities holds:
   a.

$$\liminf_{t\to\infty}\left[\inf_{\lambda>0}\left\{\frac{1}{\lambda}\sum_{i=1}^{m} a_i(t)\exp\big\{\lambda\big(t - h_i(t)\big)\big\}\right\}\right] > 1,$$

   b.

$$\liminf_{t\to\infty}\left\{\left[\prod_{i=1}^{m} a_i(t)\right]^{1/m}\left[\sum_{i=1}^{m}\big(t - h_i(t)\big)\right]\right\} > \frac{1}{e}.$$

4. [302] *There exist a nonempty set $I \subset \{1, \cdots, m\}$ and constants $\tau_0, \tau_1, \tau_0 > \tau_1 > 0$, such that*

$$t - h_i(t) \geq \tau_0, \ i \in I, \ \liminf_{t\to\infty} \int_t^{t+\tau_1} \sum_{t\in I} a_i(s)ds > 0,$$

$$\limsup_{t\to\infty}\left\{\max_k \int_{h_k(t)}^t \sum_{k=1}^m a_k(s)ds\right\} < \infty,$$

and at least one of the following inequalities holds:

a. *For all* $\lambda > 0$ *and some* $T > 0$,

$$-\lambda + \inf_{t\ge T} \frac{\sum_{k=1}^m a_k(t)\exp\{\lambda \int_{h_k(t)}^t \sum_{i=1}^m a_i(s)ds\}}{\sum_{k=1}^m a_k(t)} > 0.$$

b.

$$\liminf_{t\to\infty} \frac{\sum_{k=1}^m a_k(t)\int_{h_k(t)}^t \sum_{i=1}^m a_i(s)ds}{\sum_{k=1}^m a_k(t)} > \frac{1}{e}.$$

5. [303] *Let* $t - h_i(t) = \tau_i > 0$, *and at least one of the following conditions holds*:

a. *For every* $\lambda > 0$ *and sufficiently large* T,

$$-\lambda + \inf_{t\ge T} \min_{j=1,\cdots,m} \sum_{k=1}^m p_{jk}(t)e^{\lambda\tau_k} > 0,$$

where

$$p_{jk}(t) = \frac{1}{\tau_j}\int_{t-\tau_j}^t a_k(s)ds.$$

b.

$$\liminf_{t\to\infty} \min_{j=1,\cdots,m} \sum_{k=1}^m p_{jk}(t) > \frac{1}{e}.$$

6. [153] *There exist indices* $i_l \in \{1,\cdots,m\}$ *such that*

$$\liminf_{t\to\infty}\big(t - h_{i_l}(t)\big) > 0,\ \liminf_{t\to\infty}\sum_{i=1}^m a_{i_l}(t) > 0,$$

and at least one of the following inequalities holds:

a. *For every* $\lambda > 0$ *and* $i = 1,\cdots,m$,

$$\liminf_{t\to\infty} \frac{1}{\lambda\tau_i(t)} \sum_{k=1}^m \int_t^{t+\tau_k(t)} a_k(s)e^{\lambda\tau_k(s)}ds > 1.$$

b. *For every* $i = 1,\cdots,m$,

$$\liminf_{t\to\infty} \frac{1}{\tau_i(t)} \sum_{k=1}^m \int_t^{t+\tau_k(t)} a_k(s)\tau_k(s)ds > \frac{1}{e}.$$

Some other oscillation results were obtained in the papers [229, 257, 258, 290].

In Sect. 2.8, we obtain lower and upper estimates of the fundamental function for a nonoscillatory equation. Applying these bounds, we can estimate a solution of the initial value problem for such equations. Moreover, we obtain here an estimation of the integral of the fundamental function; such estimations are very useful in stability theory. The results of this section were partially published in [62, 63].

In [193], several sufficient conditions on equation parameters and initial functions were established that yield that the solution of the initial value problem is positive. We supplement the results of [193] in Sect. 2.9. Namely, as is demonstrated in Sect. 2.7, if the nonlinear integral inequality has a nonnegative solution, then under certain conditions on the initial function (the same as in [193]) the solution of the initial value problem is positive. We used here the results of paper [41].

For ordinary linear differential equations of the second order, the following oscillation criterion is known: if the equation has an oscillatory solution, then all its solutions oscillate. As is well known, for delay differential equations this is not true. Y. Domshlak [140] revised the result above for differential equation (2.2.1) with monotone delays. He demonstrated that if an *associated* equation has a *slowly oscillating solution*, then every solution of (2.2.1) is oscillating. In [78, 142, 144], several new explicit sufficient conditions of oscillation were obtained by explicit construction of such slowly oscillating solutions.

In particular, the following theorem was obtained in [78].

Theorem 2.29 [78] *Let $A + D > 0$ and the system*

$$\begin{cases} (AD - BC)x_1x_2 - Ax_1 - Dx_2 + 1 = 0, \\ \ln x_1 - Ax_1 - Bx_2 < 0, \\ \ln x_2 - Cx_1 - Dx_2 < 0, \end{cases} \tag{2.12.4}$$

have a positive solution $\{x_1, x_2\}$, where A, B, C, D are defined by (2.6.4). *Then all solutions of* (2.6.3) *are oscillatory.*

Application of Theorem 2.29 to (2.6.9) gives the sufficient condition $ae^b > \frac{1}{e}$ for oscillation of all solutions. Note that in Example 2.3 (by application of Theorem 2.12) the inequality $ae^b < \frac{1}{e}$ implies nonoscillation of (2.6.9). Thus Theorems 2.12 and 2.29 give sharp nonoscillation and oscillation conditions for the equation with two delays and nonnegative coefficients.

Similarly, if $\mu^b < \frac{1}{e \ln \mu}$, then (2.6.10) has a nonoscillatory solution; if $a\mu^b > \frac{1}{e \ln \mu}$, then all solutions of (2.6.10) are oscillatory. If $a\alpha^{-b} < \frac{1}{e \ln(\alpha^{-1})}$, then (2.6.11) has a nonoscillatory solution. If $a\alpha^{-b} > \frac{1}{e \ln(\alpha^{-1})}$, then all solutions of (2.6.10) are oscillatory.

In Sect. 2.10, we present an oscillation criterion similar to Domshlak's result, where the existence of a slowly oscillating solution is assumed for (2.2.1) and not for the associated equation; moreover, the delays are not necessarily monotone. We prove the following result: if an equation has a nonoscillatory solution, then it has no slowly oscillating solutions; this result was also first obtained in [41].

Some other nonoscillation results for scalar delay differential equations can be found in the papers [66, 74, 106, 107, 152, 156, 284, 317, 352]. For example, in the papers [317, 352], oscillation properties of first-order delay differential equations are compared with second-order ordinary differential equations. In particular, the following theorems were obtained.

Theorem 2.30 [317] *Assume that*

$$t-\tau(t)\ge \frac{1}{e},\ \limsup_{t\to\infty}\left(t-\tau(t)-\frac{1}{e}\right)e^{e(t-\tau(t))}<a<1,$$

and the second-order ordinary differential equation

$$\ddot{x}(t)+\frac{2}{1-a}e^{2+e(t-\tau(t))}\left(t-\tau(t)-\frac{1}{e}\right)x(t)=0$$

has an eventually positive solution. Then the equation $\dot{x}(t)+x(\tau(t))=0$ *also has an eventually positive solution.*

Theorem 2.31 [352] *Suppose* $p_i\in C([t_0,\infty),\mathbb{R}^+)$, $\tau_i>0$. *Then the equation*

$$\dot{x}(t)+\sum_{i=1}^{m}p_i(t)x(t-\tau_i)=0$$

has a nonoscillatory solution if and only if the equation

$$\ddot{x}(t)+\frac{2em}{\sum_{i=1}^{m}\tau_i}\sum_{i=1}^{m}\left[p_i(t)-\frac{1}{em\tau_i}\right]x(t)=0$$

has a nonoscillatory solution.

Finally, let us formulate some open problems and topics for research and discussion.

1. Prove or disprove that inequality (2.12.2) implies nonoscillation of (2.12.1) without the assumption that $a(t)\ge 0$.
2. Find necessary and/or sufficient nonoscillation conditions for some partial cases of (2.2.1): equations with periodic coefficients and delays $t-h_k(t)$ and equations with delays of the form $h_k(t)=\lambda_k t^{\beta_k}$, $t\ge 1$, $0<\lambda_k<1$, $0<\beta_k\le 1$.
3. Find lower and upper bounds of the fundamental function of nonoscillatory equation (2.2.1) without the assumption that $a_k(t)\ge 0$.
4. Is it possible to extend Theorems 2.22 and 2.23 to equations with oscillatory coefficients?
5. Can Lemmas 2.2 and 2.3 be generalized to equations with positive and negative coefficients?

http://www.springer.com/978-1-4614-3454-2

Nonoscillation Theory of Functional Differential Equations with Applications
Agarwal, R.P.; Berezansky, L.; Braverman, E.; Domoshnitsky, A.
2012, XVI, 520 p., Hardcover
ISBN: 978-1-4614-3454-2