Chapter 2
Inequalities of the Jensen Type

2.1 Introduction

Jensen’s type inequalities in their various settings ranging from discrete to continuous case play an important role in different branches of Modern Mathematics. A simple search in the MathSciNet database of the American Mathematical Society with the key words “jensen” and “inequality” in the title reveals that there are more than 300 items intimately devoted to this famous result. However, the number of papers where this inequality is applied is a lot larger and far more difficult to find. It can be a good project in itself for someone to write a monograph devoted to Jensen’s inequality in its different forms and its applications across Mathematics.

In the introductory chapter we have recalled a number of Jensen’s type inequalities for convex and operator convex functions of selfadjoint operators in Hilbert spaces. In this chapter we present some recent results obtained by the author that deal with different aspects of this well-researched inequality than those recently reported in the book [19]. They include but are not restricted to the operator version of the Dragomir–Ionescu inequality, Slater’s type inequalities for operators and its inverses, Jensen’s inequality for twice differentiable functions whose second derivatives satisfy some upper and lower bounds conditions, Jensen’s type inequalities for log-convex functions and for differentiable log-convex functions and their applications to Ky Fan’s inequality.

Finally, some Hermite–Hadamard’s type inequalities for convex functions and Hermite–Hadamard’s type inequalities for operator convex functions are presented as well.

All the above results are exemplified for some classes of elementary functions of interest such as the power function and the logarithmic function.
2.2 Reverses of the Jensen Inequality

2.2.1 An Operator Version of the Dragomir–Ionescu Inequality

The following result holds:

**Theorem 2.1 (Dragomir, 2008, [8]).** Let $I$ be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\hat{I}$ (the interior of $I$) whose derivative $f'$ is continuous on $\hat{I}$. If $A$ is a selfadjoint operators on the Hilbert space $H$ with $\text{Sp}(A) \subseteq [m, M] \subset \hat{I}$, then

\[
(0 \leq) \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \leq \langle f'(A)Ax, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A)x, x \rangle \tag{2.1}
\]

for any $x \in H$ with $\|x\| = 1$.

**Proof.** Since $f$ is convex and differentiable, we have that

\[
f(t) - f(s) \leq f'(t) \cdot (t - s)
\]

for any $t, s \in [m, M]$.

Now, if we chose in this inequality $s = \langle Ax, x \rangle \in [m, M]$ for any $x \in H$ with $\|x\| = 1$ since $\text{Sp}(A) \subseteq [m, M]$, then we have

\[
f(t) - f(\langle Ax, x \rangle) \leq f'(t) \cdot (t - \langle Ax, x \rangle) \tag{2.2}
\]

for any $t \in [m, M]$ any $x \in H$ with $\|x\| = 1$.

If we fix $x \in H$ with $\|x\| = 1$ in (2.2) and apply property (P) then we get

\[
\langle [f(A) - f(\langle Ax, x \rangle)]_{1H} \rangle x, x \leq \langle f'(A) \cdot (A - \langle Ax, x \rangle)_{1H} \rangle x, x
\]

for each $x \in H$ with $\|x\| = 1$, which is clearly equivalent to the desired inequality (2.1). \qed

**Corollary 2.2 (Dragomir, 2008, [8]).** Assume that $f$ is as in Theorem 2.1. If $A_j$ are selfadjoint operators with $\text{Sp} (A_j) \subseteq [m, M] \subset \hat{I}$, $j \in \{1, \ldots, n\}$ and $x_j \in H$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then

\[
(0 \leq) \sum_{j=1}^{n} \langle f'(A_j)A_j x_j, x_j \rangle - f \left( \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \\
\leq \sum_{j=1}^{n} \langle f'(A_j)A_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle. \tag{2.3}
\]
Corollary 2.3 (Dragomir, 2008, [8]). Assume that $f$ is as in Theorem 2.1. If $A_j$ are selfadjoint operators with $\text{Sp} (A_j) \subseteq [m, M] \subseteq \hat{I}$, $j \in \{1, \ldots, n\}$ and $p_j \geq 0$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$, then

$$
(0 \leq) \left\{ \sum_{j=1}^{n} p_j f (A_j) x, x \right\} - f \left( \left( \sum_{j=1}^{n} p_j A_j x, x \right) \right)
$$

$$
\leq \left\{ \sum_{j=1}^{n} p_j f' (A_j) A_j x, x \right\} - \left\{ \sum_{j=1}^{n} p_j A_j x, x \right\} \cdot \left\{ \sum_{j=1}^{n} p_j f' (A_j) x, x \right\}, \quad (2.4)
$$

for each $x \in H$ with $\|x\| = 1$.

Remark 2.4. Inequality (2.4), in the scalar case, namely

$$
(0 \leq) \sum_{j=1}^{n} p_j f (x_j) - f \left( \sum_{j=1}^{n} p_j x_j \right)
$$

$$
\leq \sum_{j=1}^{n} p_j f' (x_j) x_j - \sum_{j=1}^{n} p_j x_j \cdot \sum_{j=1}^{n} p_j f' (x_j), \quad (2.5)
$$

where $x_j \in \hat{I}$, $j \in \{1, \ldots, n\}$, has been obtained by the first time in 1994 by Dragomir and Ionescu, see [16].

### 2.2.2 Further Reverses

In applications would be perhaps more useful to find upper bounds for the quantity

$$
(f (A)x, x) - f (\langle Ax, x \rangle), \quad x \in H \quad \text{with} \quad \|x\| = 1,
$$

that are in terms of the spectrum margins $m, M$ and of the function $f$.

The following result may be stated:

Theorem 2.5 (Dragomir, 2008, [8]). Let $I$ be an interval and $f : I \rightarrow \mathbb{R}$ be a convex and differentiable function on $\hat{I}$ (the interior of $I$) whose derivative $f'$ is continuous on $\hat{I}$. If $A$ is a selfadjoint operator on the Hilbert space $H$ with $\text{Sp}(A) \subseteq [m, M] \subseteq \hat{I}$, then
\[(0 \leq) \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \leq \left\{ \begin{array}{c} \frac{1}{2} \cdot (M - m) \left[ \|f'(A)x\|^2 - \langle f'(A)x, x \rangle^2 \right]^{1/2} \\
\frac{1}{2} \cdot (f'(M) - f'(m)) \left[ \|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \end{array} \right. \]
\[\leq \frac{1}{4} (M - m) (f'(M) - f'(m)) \tag{2.6} \]

for any \(x \in H\) with \(\|x\| = 1\).

We also have the inequality
\[(0 \leq) \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \leq \frac{1}{4} (M - m) (f'(M) - f'(m)) \]
\[\leq \left\{ \begin{array}{c} \sqrt{\|Ax\|^2 - \langle Ax, x \rangle^2} \|f'(A)x\|^2 - \langle f'(A)x, x \rangle^2 \ \\
\|Ax\|^2 - \langle Ax, x \rangle^2 \left[ \langle f'(A)x, x \rangle - f'(M) + f'(m) \right] \ \\
\leq \frac{1}{4} (M - m) (f'(M) - f'(m)) \tag{2.7} \]

for any \(x \in H\) with \(\|x\| = 1\).

Moreover, if \(m > 0\) and \(f'(m) > 0\), then we also have
\[(0 \leq) \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \leq \frac{1}{4} (M - m) (f'(M) - f'(m)) \]
\[\leq \left\{ \begin{array}{c} \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{M+m} f'(m)} \langle Ax, x \rangle \langle f'(A)x, x \rangle, \\
\left( \sqrt{M} - \sqrt{m} \right) \left( \sqrt{f'(M)} - \sqrt{f'(m)} \right) \left[ \langle Ax, x \rangle \langle f'(A)x, x \rangle \right]^{3/2} \tag{2.8} \]

for any \(x \in H\) with \(\|x\| = 1\).

Proof. We use the following Grüss’ type result we obtained in [6]:

Let \(A\) be a selfadjoint operator on the Hilbert space \((H; \langle \ldots \rangle)\) and assume that \(\text{Sp}(A) \subseteq [m, M]\) for some scalars \(m < M\). If \(h\) and \(g\) are continuous on \([m, M]\) and \(\gamma := \min_{t \in [m, M]} h(t)\) and \(\Gamma := \max_{t \in [m, M]} h(t)\), then

\[|\langle h(A)g(A)x, x \rangle - \langle h(A)x, x \rangle \cdot \langle g(A)x, x \rangle| \leq \frac{1}{2} \cdot (\Gamma - \gamma) \left[ \|g(A)x\|^2 - \langle g(A)x, x \rangle^2 \right]^{1/2} \]
\[\leq \frac{1}{4} (\Gamma - \gamma) (\Delta - \delta) \tag{2.9} \]

for each \(x \in H\) with \(\|x\| = 1\), where \(\delta := \min_{t \in [m, M]} g(t)\) and \(\Delta := \max_{t \in [m, M]} g(t)\).
Therefore, we can state that
\[
\langle Af'(A)x, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A)x, x \rangle \\
\leq \frac{1}{2} \cdot (M - m) \left[ \| f'(A)x \|^2 - \langle f'(A)x, x \rangle \right]^{1/2} \\
\leq \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right)
\]
(2.10)

and
\[
\langle Af'(A)x, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A)x, x \rangle \\
\leq \frac{1}{2} \cdot (f'(M) - f'(m)) \left[ \| Ax \|^2 - \langle Ax, x \rangle \right]^{1/2} \\
\leq \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right)
\]
(2.11)

for each \( x \in H \) with \( \| x \| = 1 \), which together with (2.1) provide the desired result (2.6).

On making use of the inequality obtained in [7]:
\[
|\langle h(A)g(A)x, x \rangle - \langle h(A)x, x \rangle \cdot \langle g(A)x, x \rangle| \\
\leq \frac{1}{4} \cdot (\Gamma - \gamma) (\Delta - \delta) \\
- \left\{ \frac{\left[ \langle \Gamma x - h(A)x, f(A)x - \gamma x \rangle \cdot \langle \Delta x - g(A)x, g(A)x - \delta x \rangle \right]^{1/2}}{2}, \right. \\
\left. \frac{\left| \langle h(A)x, x \rangle - \frac{\Gamma + \gamma}{2} \right| \left| \langle g(A)x, x \rangle - \frac{\Delta + \delta}{2} \right|}{2} \right. \\
\right) \right)
(2.12)

for each \( x \in H \) with \( \| x \| = 1 \), we can state that
\[
\langle Af'(A)x, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A)x, x \rangle \\
\leq \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right) \\
- \left\{ \left[ \langle Mx - Ax, Ax - mx \rangle \cdot (f'(M)x - f'(A)x, f'(A)x - f'(m)x) \right]^{1/2}, \\
\frac{\left| \langle Ax, x \rangle - \frac{M + m}{2} \right| \left| \langle f'(A)x, x \rangle - \frac{f'(M) + f'(m)}{2} \right|}{2} \right. \\
\right)
\]

for each \( x \in H \) with \( \| x \| = 1 \), which together with (2.1) provide the desired result (2.7).

Further, in order to prove the third inequality, we make use of the following result of Grüss type obtained in [7]:
If \( \gamma \) and \( \delta \) are positive, then

\[
|\langle h(A)g(A)x, x \rangle - \langle h(A)x, x \rangle \langle g(A)x, x \rangle| \\
\leq \left\{ \frac{1}{4} \cdot \frac{(T-\gamma)(A-\delta)}{\sqrt{T\gamma A\delta}} \right\} \langle h(A)x, x \rangle \langle g(A)x, x \rangle, \\
\left( \sqrt{T} - \sqrt{\gamma} \right) \left( \sqrt{A} - \sqrt{\delta} \right) [\langle h(A)x, x \rangle \langle g(A)x, x \rangle]^{1/2}
\]

(2.13)

for each \( x \in H \) with \( \|x\| = 1 \).

Now, on making use of (2.13) we can state that

\[
\langle Af'(A)x, x \rangle - \langle Ax, x \rangle \cdot \{ f'(A)x, x \} \\
\leq \left\{ \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mm}f'(M)f'(m)} \right\} \langle Ax, x \rangle \{ f'(A)x, x \}, \\
\left( \sqrt{M} - \sqrt{m} \right) \left( \sqrt{f'(M)} - \sqrt{f'(m)} \right) [\langle Ax, x \rangle \{ f'(A)x, x \}]^{1/2}
\]

(2.14)

for each \( x \in H \) with \( \|x\| = 1 \), which together with (2.1) provide the desired result (2.8).

\[\tag{2.14}\]

**Corollary 2.6 (Dragomir, 2008, [8]).** Assume that \( f \) is as in Theorem 2.5. If \( A_j \) are selfadjoint operators with \( \text{Sp} (A_j) \subseteq [m, M] \subset \bar{I}, j \in \{1, \ldots, n\} \), then

\[
(0 \leq \sum_{j=1}^{n} \langle f (A_j) x_j, x_j \rangle - f \left( \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \\
\leq \left\{ \frac{1}{2} \cdot (M - m) \left[ \sum_{j=1}^{n} || f' (A_j) x_j ||^2 - \left( \sum_{j=1}^{n} \langle f' (A_j) x_j, x_j \rangle \right) \right]^{1/2} \\
\leq \frac{1}{2} \cdot (f' (M) - f' (m)) \right\} \left[ \sum_{j=1}^{n} || A_j x_j ||^2 - \left( \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \right]^{1/2}, \\
\leq \frac{1}{4} (M - m) (f' (M) - f' (m))
\]

(2.14)

for any \( x_j \in H, j \in \{1, \ldots, n\} \) with \( \sum_{j=1}^{n} || x_j ||^2 = 1 \).

We also have the inequality

\[
(0 \leq \sum_{j=1}^{n} \langle f (A_j) x_j, x_j \rangle - f \left( \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \\
\leq \frac{1}{4} (M - m) (f' (M) - f' (m))
\]

(2.14)
\[ \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right) \] (2.15)

for any \( x_j \in H, j \in \{1, \ldots, n\} \) with \( \sum_{j=1}^{n} \| x_j \|^2 = 1 \).

Moreover, if \( m > 0 \) and \( f'(m) > 0 \), then we also have

\[ \left( \sqrt{M - \sqrt{m}} \right) \left( \sqrt{f'(M)} - \sqrt{f'(m)} \right) \]

(2.16)

for any \( x_j \in H, j \in \{1, \ldots, n\} \) with \( \sum_{j=1}^{n} \| x_j \|^2 = 1 \).

The following corollary also holds:

**Corollary 2.7 (Dragomir, 2008, [8]).** Assume that \( f \) is as in Theorem 2.1. If \( A_j \) are selfadjoint operators with \( \text{Sp} (A_j) \subseteq [m, M] \subseteq I, j \in \{1, \ldots, n\} \) and \( p_j \geq 0, j \in \{1, \ldots, n\} \) with \( \sum_{j=1}^{n} p_j = 1 \), then

\[ \left( 0 \leq \sum_{j=1}^{n} p_j f (A_j) x, x \right) - f \left( \sum_{j=1}^{n} p_j A_j x, x \right) \]

\[ \leq \frac{1}{2} \left( M - m \right) \left( \sum_{j=1}^{n} p_j \| f'(A_j) x \|^2 - \left( \sum_{j=1}^{n} p_j f'(A_j) x, x \right) \right)^{1/2}, \]

\[ \leq \frac{1}{2} \left( f'(M) - f'(m) \right) \left( \sum_{j=1}^{n} p_j \| A_j x \|^2 - \left( \sum_{j=1}^{n} p_j A_j x, x \right) \right)^{1/2}, \]

\[ \leq \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right) \] (2.17)

for any \( x \in H \) with \( \| x \| = 1 \).
We also have the inequality

\[
(0 \leq) \left( \sum_{j=1}^{n} p_j f(A_j) x, x \right) - f \left( \left( \sum_{j=1}^{n} p_j A_j x, x \right) \right) \\
\leq \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right) \\
\leq \frac{1}{4} \left[ \left( \sum_{j=1}^{n} p_j \left( M x - A_j x, A_j x - m x \right) \right)^{1/2} \\
- \left( \sum_{j=1}^{n} p_j \left( f'(M)x - f'(A_j)x, f'(A_j)x - f'(m)x \right) \right)^{1/2} \right] \\
\left( \left( \sum_{j=1}^{n} p_j A_j x, x \right) - \frac{M + m}{2} \right) \left( \left( \sum_{j=1}^{n} p_j f'(A_j)x, x \right) - \frac{f'(M) + f'(m)}{2} \right) \\
\leq \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right)
\tag{2.18}
\]

for any \( x \in H \) with \( \|x\| = 1 \).

Moreover, if \( m > 0 \) and \( f'(m) > 0 \), then we also have

\[
(0 \leq) \left( \sum_{j=1}^{n} p_j f(A_j) x, x \right) - f \left( \left( \sum_{j=1}^{n} p_j A_j x, x \right) \right) \\
\leq \frac{1}{4} \left( \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mm f'(M) f'(m)}} \right) \left( \sum_{j=1}^{n} p_j A_j x, x \right) \left( \sum_{j=1}^{n} p_j f'(A_j) x, x \right), \\
\leq \frac{1}{4} \left( \sqrt{M - \sqrt{m}} \right) \left( \sqrt{f'(M) - \sqrt{f'(m)}} \right) \\
\times \left[ \left( \sum_{j=1}^{n} p_j A_j x, x \right) \left( \sum_{j=1}^{n} p_j f'(A_j) x, x \right) \right]^{1/2}
\tag{2.19}
\]

for any \( x \in H \) with \( \|x\| = 1 \).

Remark 2.8. Some of the inequalities in Corollary 2.7 can be used to produce reverse norm inequalities for the sum of positive operators in the case when the convex function \( f \) is non-negative and monotonic non-decreasing on \([0, M]\).

For instance, if we use inequality (2.17), then we have

\[
(0 \leq) \left\| \sum_{j=1}^{n} p_j f(A_j) \right\| - f \left( \left\| \sum_{j=1}^{n} p_j A_j \right\| \right) \\
\leq \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right)
\tag{2.20}
\]
Moreover, if we use inequality (2.19), then we obtain

\[
(0 \leq) \left\| \sum_{j=1}^{n} p_j f(A_j) \right\| - f\left(\left\| \sum_{j=1}^{n} p_j A_j \right\|\right) \\
\leq \left\{ \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mm/f'(m)}} \left\| \sum_{j=1}^{n} p_j A_j \right\| \left\| \sum_{j=1}^{n} p_j f'(A_j) \right\|, \\
\left( \sqrt{M-m}\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right) \right) \left\| \sum_{j=1}^{n} p_j A_j \right\| \left\| \sum_{j=1}^{n} p_j f'(A_j) \right\| \right\}^{\frac{1}{2}}.
\]

(2.21)

### 2.3 Some Slater Type Inequalities

#### 2.3.1 Slater Type Inequalities for Functions of Real Variables

Suppose that \( I \) is an interval of real numbers with interior \( \mathring{I} \) and \( f : I \to \mathbb{R} \) is a convex function on \( I \). Then \( f \) is continuous on \( \mathring{I} \) and has finite left and right derivatives at each point of \( \mathring{I} \). Moreover, if \( x, y \in \mathring{I} \) and \( x < y \), then \( f'_-(x) \leq f'_-(y) \leq f'_+(y) \) which shows that both \( f'_- \) and \( f'_+ \) are non-decreasing function on \( \mathring{I} \). It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function \( f : I \to \mathbb{R} \), the sub-differential of \( f \) denoted by \( \partial f \) is the set of all functions \( \varphi : I \to [-\infty, \infty] \) such that \( \varphi(x) \leq f(x) - f(a) - \varphi(a) \) for any \( x, a \in I \).

It is also well known that if \( f \) is convex on \( I \), then \( \partial f \) is non-empty, \( f'_-, f'_+ \in \partial f \) and if \( \varphi \in \partial f \), then

\[
f'_-(x) \leq \varphi(x) \leq f'_+(x) \quad \text{for any } x \in \mathring{I}.
\]

In particular, \( \varphi \) is a non-decreasing function.

If \( f \) is differentiable and convex on \( \mathring{I} \), then \( \partial f = \{ f' \} \).

The following result is well known in the literature as the Slater inequality:

**Theorem 2.9 (Slater, 1981, [28]).** If \( f : I \to \mathbb{R} \) is a non-increasing (non-decreasing) convex function, \( x_i \in I, p_i \geq 0 \) with \( P_n := \sum_{i=1}^{n} p_i > 0 \) and \( \sum_{i=1}^{n} p_i \varphi(x_i) \neq 0 \), where \( \varphi \in \partial f \), then

\[
\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leq f\left( \frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \right).
\]

(2.22)
As pointed out in [5, p. 208], the monotonicity assumption for the derivative $\varphi$ can be replaced with the condition
\[
\frac{\sum_{i=1}^{n} p_i x_i \varphi (x_i)}{\sum_{i=1}^{n} p_i \varphi (x_i)} \in I,
\] (2.23)
which is more general and can hold for suitable points in $I$ and for not necessarily monotonic functions.

### 2.3.2 Some Slater Type Inequalities for Operators

The following result holds:

**Theorem 2.10 (Dragomir, 2008, [9]).** Let $I$ be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\bar{I}$ (the interior of $I$) whose derivative $f'$ is continuous on $\bar{I}$. If $A$ is a selfadjoint operator on the Hilbert space $H$ with $\text{Sp}(A) \subseteq [m, M] \subseteq \bar{I}$ and $f'(A)$ is a positive definite operator on $H$ then
\[
\begin{align*}
0 \leq f & \left( \frac{\langle Af'(A)x, x \rangle}{f'(A)x, x} \right) - \langle f(A)x, x \rangle \\
& \leq f' \left( \frac{\langle Af'(A)x, x \rangle}{f'(A)x, x} \right) \left[ \frac{\langle Af'(A)x, x \rangle - \langle Ax, x \rangle \langle f'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \right]
\end{align*}
\] (2.24)
for any $x \in H$ with $\|x\| = 1$.

**Proof.** Since $f$ is convex and differentiable on $\bar{I}$, then we have that
\[
f'(s) \cdot (t - s) \leq f(t) - f(s) \leq f'(t) \cdot (t - s)
\] (2.25)
for any $t, s \in [m, M]$.

Now, if we fix $t \in [m, M]$ and apply property (P) for the operator $A$, then for any $x \in H$ with $\|x\| = 1$ we have
\[
\begin{align*}
\langle f'(A) \cdot (t \cdot 1_h - A)x, x \rangle & \leq \langle [f(t) \cdot 1_h - f(A)]x, x \rangle \\
& \leq \langle f'(t) \cdot (t \cdot 1_h - A)x, x \rangle
\end{align*}
\] (2.26)
for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

Inequality (2.26) is equivalent with
\[
\begin{align*}
t \langle f'(A)x, x \rangle - \langle f'(A)Ax, x \rangle & \leq f(t) - \langle f(A)x, x \rangle \\
& \leq f'(t) t - f'(t) \langle Ax, x \rangle
\end{align*}
\] (2.27)
for any $t \in [m, M]$ any $x \in H$ with $\|x\| = 1$. 
Now, since \( A \) is selfadjoint with \( mI \leq A \leq MI \) and \( f'(A) \) is positive definite, then \( mf'(A) \leq Af'(A) \leq Mf'(A) \), i.e. \( m \langle f'(A)x, x \rangle \leq \langle Af'(A)x, x \rangle \leq M \langle f'(A)x, x \rangle \) for any \( x \in H \) with \( \|x\| = 1 \), which shows that

\[
t_0 := \frac{\langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \in [m, M] \quad \text{for any } x \in H \quad \text{with} \quad \|x\| = 1.
\]

Finally, if we put \( t = t_0 \) in (2.27), then we get the desired result (2.24).

**Remark 2.11.** It is important to observe that, the condition that \( f'(A) \) is a positive definite operator on \( H \) can be replaced with the more general assumption that

\[
\frac{\langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \in \hat{I} \quad \text{for any } x \in H \quad \text{with} \quad \|x\| = 1,
\]

which may be easily verified for particular convex functions \( f \).

**Remark 2.12.** Now, if the functions are concave on \( \hat{I} \) and condition (2.28) holds, then we have the inequality

\[
0 \leq \langle f(A)x, x \rangle - f \left( \frac{\langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \right) \\
\leq f' \left( \frac{\langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \right) \left[ \frac{\langle Ax, x \rangle \langle f'(A)x, x \rangle - \langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \right]
\]

(2.29)

for any \( x \in H \) with \( \|x\| = 1 \).

### 2.3.3 Further Reverses

The following results that provide perhaps more useful upper bounds for the non-negative quantity:

\[
f \left( \frac{\langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \right) - \langle f(A)x, x \rangle \quad \text{for } x \in H \quad \text{with} \quad \|x\| = 1
\]

can be stated:

**Theorem 2.13 (Dragomir, 2008, [9]).** Let \( I \) be an interval and \( f : I \to \mathbb{R} \) be a convex and differentiable function on \( \hat{I} \) (the interior of \( I \)) whose derivative \( f' \) is continuous on \( \hat{I} \). Assume that \( A \) is a selfadjoint operator on the Hilbert space \( H \) with \( \text{Sp}(A) \subseteq [m, M] \subset \hat{I} \) and \( f'(A) \) is a positive definite operator on \( H \). If we define

\[
B \left( f', A; x \right) := \frac{1}{\langle f'(A)x, x \rangle} \cdot f' \left( \frac{\langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \right).
\]
then

\[
(0 \leq) f \left( \frac{\langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \right) - \langle f(A)x, x \rangle
\]

\[
\leq B \left( f', A; x \right) \times \left\{ \frac{1}{2} \cdot (M - m) \left[ \| f'(A)x \|^2 - \langle f'(A)x, x \rangle^2 \right]^{1/2} \right.
\]

\[
- \left\{ \frac{1}{2} \cdot (f'(M) - f'(m)) \left[ \| Ax \|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \right\}
\]

\[
\leq \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right) B \left( f', A; x \right)
\]

(2.30)

and

\[
(0 \leq) f \left( \frac{\langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \right) - \langle f(A)x, x \rangle
\]

\[
\leq B \left( f', A; x \right) \times \left[ \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right) \right.
\]

\[
- \left. \left\{ \left[ (M - Ax, Ax - mx) \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x \rangle \right]^{1/2} \right\}
\]

\[
\leq \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right) B \left( f', A; x \right)
\]

(2.31)

for any \( x \in H \) with \( \| x \| = 1 \), respectively.

Moreover, if \( A \) is a positive definite operator, then

\[
(0 \leq) f \left( \frac{\langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \right) - \langle f(A)x, x \rangle
\]

\[
\leq B(f', A; x) \times \left\{ \frac{1}{4} \cdot \frac{(M - m)(f'(M) - f'(m))}{\sqrt{M+m}/f'(m)} \langle Ax, x \rangle \langle f'(A)x, x \rangle, \right.
\]

\[
\left. \left[ \left( \sqrt{M - m} \right) \left( \sqrt{f'(M) - \sqrt{f'(m)}} \right) \right] \langle Ax, x \rangle \langle f'(A)x, x \rangle \right\}
\]

(2.32)

for any \( x \in H \) with \( \| x \| = 1 \).

Proof. We use the following Grüss’ type result we obtained in [6]:

Let \( A \) be a selfadjoint operator on the Hilbert space \( (H; \langle \cdot, \cdot \rangle) \) and assume that \( \text{Sp}(A) \subseteq [m, M] \) for some scalars \( m < M \). If \( h \) and \( g \) are continuous on \([m, M]\) and \( \gamma := \min_{t \in [m,M]} h(t) \) and \( \Gamma := \max_{t \in [m,M]} h(t) \), then
\[
\langle h(A)g(A)x, x \rangle - \langle h(A)x, x \rangle \cdot \langle g(A)x, x \rangle \leq \frac{1}{2} \cdot (\Gamma - \gamma) \left[ \|g(A)x\|^2 - \langle g(A)x, x \rangle^2 \right]^{1/2}
\]

\[
\quad \left( \leq \frac{1}{4} (\Gamma - \gamma) (\Delta - \delta) \right) \quad (2.33)
\]

for each \( x \in H \) with \( \|x\| = 1 \), where \( \delta := \min_{t \in [m,M]} g(t) \) and \( \Delta := \max_{t \in [m,M]} g(t) \). Therefore, we can state that

\[
\langle Af'(A)x, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A)x, x \rangle \leq \frac{1}{2} \cdot (M - m) \left[ \|f'(A)x\|^2 - \langle f'(A)x, x \rangle^2 \right]^{1/2}
\]

\[
\quad \leq \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right) \quad (2.34)
\]

and

\[
\langle Af'(A)x, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A)x, x \rangle \leq \frac{1}{2} \cdot (f'(M) - f'(m)) \left[ \|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2}
\]

\[
\quad \leq \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right) \quad (2.35)
\]

for each \( x \in H \) with \( \|x\| = 1 \), which together with (2.24) provide the desired result (2.30).

On making use of the inequality obtained in [7]

\[
\langle h(A)g(A)x, x \rangle - \langle h(A)x, x \rangle \cdot \langle g(A)x, x \rangle \leq \frac{1}{4} (\Gamma - \gamma) (\Delta - \delta)
\]

\[
\quad \left[ \left( \langle h(A)x, x \rangle - \frac{\Gamma + \gamma}{2} \right) \left( \langle g(A)x, x \rangle - \frac{\Delta + \delta}{2} \right) \right]^{1/2}
\]

\[
\quad \leq \frac{1}{2} \cdot \left( \langle h(A)x, x \rangle - \frac{\Gamma + \gamma}{2} \right) \left( \langle g(A)x, x \rangle - \frac{\Delta + \delta}{2} \right)
\]

\[
\quad \left[ \langle h(A)x, x \rangle - \frac{\Gamma + \gamma}{2} \right] \left( \langle g(A)x, x \rangle - \frac{\Delta + \delta}{2} \right) \quad (2.36)
\]

for each \( x \in H \) with \( \|x\| = 1 \), we can state that

\[
\langle Af'(A)x, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A)x, x \rangle \leq \frac{1}{4} (M - m) \left( f'(M) - f'(m) \right)
\]

\[
\quad - \left[ \left( \langle Ax, x \rangle - \frac{M + m}{2} \right) \left( \langle f'(A)x, x \rangle - \frac{f'(M) + f'(m)}{2} \right) \right]^{1/2}
\]
for each \( x \in H \) with \( \|x\| = 1 \), which together with (2.24) provide the desired result (2.31).

Further, in order to prove the third inequality, we make use of the following result of Grüss type obtained in [7]:

If \( \gamma \) and \( \delta \) are positive, then

\[
|\langle h(A)g(A)x, x \rangle - \langle h(A)x, x \rangle \langle g(A)x, x \rangle| \\
\leq \left\{ \frac{1}{4} \cdot \frac{(\gamma-\delta)(\Delta-\delta)}{\sqrt{\gamma \Delta}} |\langle h(A)x, x \rangle \langle g(A)x, x \rangle|, \right. \\
\left. \left( \sqrt{\gamma} - \sqrt{\delta} \right) \left( \sqrt{\Delta} - \sqrt{\delta} \right) \right\}^{1/2} (2.37)
\]

for each \( x \in H \) with \( \|x\| = 1 \).

Now, on making use of (2.37) we can state that

\[
\langle Af'(A)x, x \rangle - \langle Ax, x \rangle \langle f'(A)x, x \rangle \\
\leq \left\{ \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mm}f'(M)f'(m)} |\langle Ax, x \rangle \langle f'(A)x, x \rangle|, \right. \\
\left. \left( \sqrt{M} - \sqrt{m} \right) \left( \sqrt{f'(M)} - \sqrt{f'(m)} \right) \right\}^{1/2} (2.38)
\]

for each \( x \in H \) with \( \|x\| = 1 \), which together with (2.24) provide the desired result (2.32). \( \square \)

**Remark 2.14.** We observe, from the first inequality in (2.32), that

\[
(1 \leq) \frac{\langle Af'(A)x, x \rangle}{\langle Ax, x \rangle \langle f'(A)x, x \rangle} \leq \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mm}f'(M)f'(m)} + 1
\]

which implies that

\[
f' \left( \frac{\langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \right) \leq f' \left( \left\{ \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mm}f'(M)f'(m)} + 1 \right\} \langle Ax, x \rangle \right)
\]

for each \( x \in H \) with \( \|x\| = 1 \), since \( f' \) is monotonic non-decreasing and \( A \) is positive definite.

Now, the first inequality in (2.32) implies the following result:

\[
(0 \leq) f \left( \frac{\langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \right) - \langle f(A)x, x \rangle \\
\leq \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mm}f'(M)f'(m)} \\
\times f' \left( \left\{ \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mm}f'(M)f'(m)} + 1 \right\} \langle Ax, x \rangle \right) \langle Ax, x \rangle \quad (2.39)
\]

for each \( x \in H \) with \( \|x\| = 1 \).
From the second inequality in (2.32) we also have

\[ (0 \leq f \left( \frac{\langle Af'(A)x, x \rangle}{\langle f'(A)x, x \rangle} \right) - \langle f(A)x, x \rangle \]

\[ \leq \left( \sqrt{M} - \sqrt{m} \right) \left( \sqrt{f'(M)} - \sqrt{f'(m)} \right) \]

\[ \times f' \left( \left[ \frac{1}{4} \cdot \frac{(M - m) (f'(M) - f'(m))}{\sqrt{M m f'(M) f'(m)}} + 1 \right] \langle Ax, x \rangle \right) \left[ \frac{\langle Ax, x \rangle}{\langle f'(A)x, x \rangle} \right]^{\frac{1}{2}} \]

(2.40)

for each \( x \in H \) with \( \|x\| = 1 \).

**Remark 2.15.** If the condition that \( f'(A) \) is a positive definite operator on \( H \) from Theorem 2.13 is replaced by condition (2.28), then inequalities (2.30) and (2.33) will still hold. Similar inequalities for concave functions can be stated. However, the details are not provided here.

## 2.4 Other Inequalities for Convex Functions

### 2.4.1 Some Inequalities for Two Operators

The following result holds:

**Theorem 2.16 (Dragomir, 2008, [10]).** Let \( I \) be an interval and \( f : I \to \mathbb{R} \) be a convex and differentiable function on \( \hat{I} \) (the interior of \( I \)) whose derivative \( f' \) is continuous on \( \hat{I} \). If \( A \) and \( B \) are selfadjoint operators on the Hilbert space \( H \) with \( Sp(A), Sp(B) \subseteq [m, M] \subseteq \hat{I} \), then

\[ \langle f'(A)x, x \rangle \langle By, y \rangle - \langle f'(A)Ax, x \rangle \]

\[ \leq \langle f(B)y, y \rangle - \langle f(A)x, x \rangle \]

\[ \leq \langle f'(B)By, y \rangle - \langle Ax, x \rangle \left\{ f'(B)y, y \right\} \]

(2.41)

for any \( x, y \in H \) with \( \|x\| = \|y\| = 1 \).

In particular, we have

\[ \langle f'(A)x, x \rangle \langle Ay, y \rangle - \langle f'(A)Ax, x \rangle \]

\[ \leq \langle f(A)y, y \rangle - \langle f(A)x, x \rangle \]

\[ \leq \langle f'(A)Ay, y \rangle - \langle Ax, x \rangle \left\{ f'(A)y, y \right\} \]

(2.42)

for any \( x, y \in H \) with \( \|x\| = \|y\| = 1 \) and
\[
\begin{aligned}
\{ f'(A)x, x \} \langle Bx, x \rangle - \{ f'(A)Ax, x \} \\
\leq \{ f(B)x, x \} - \{ f(A)x, x \} \\
\leq \{ f'(B)Bx, x \} - \{ Ax, x \} \{ f'(B)x, x \}
\end{aligned}
\]  
(2.43)

for any \( x \in H \) with \( \|x\| = 1 \).

**Proof.** Since \( f \) is convex and differentiable on \( \hat{I} \), then we have that
\[
f'(s) \cdot (t - s) \leq f(t) - f(s) \leq f'(t) \cdot (t - s)
\]  
(2.44)

for any \( t, s \in [m, M] \).

Now, if we fix \( t \in [m, M] \) and apply property (P) for the operator \( A \), then for any \( x \in H \) with \( \|x\| = 1 \) we have
\[
\begin{aligned}
\{ f'(A) \cdot (t \cdot 1_H - A)x, x \} &\leq \{ [f(t) \cdot 1_H - f(A)]x, x \} \\
&\leq \{ f'(t) \cdot (t \cdot 1_H - A)x, x \}
\end{aligned}
\]  
(2.45)

for any \( t \in [m, M] \) and any \( x \in H \) with \( \|x\| = 1 \).

Inequality (2.45) is equivalent with
\[
\begin{aligned}
t \{ f'(A)x, x \} - \{ f'(A)Ax, x \} &\leq f(t) - \{ f(A)x, x \} \\
&\leq f'(t)t - f'(t) \langle Ax, x \rangle
\end{aligned}
\]  
(2.46)

for any \( t \in [m, M] \) and any \( x \in H \) with \( \|x\| = 1 \).

If we fix \( x \in H \) with \( \|x\| = 1 \) in (2.46) and apply property (P) for the operator \( B \), then we get
\[
\begin{aligned}
\{ \{ f'(A)x, x \} B - \{ f'(A)Ax, x \} 1_H \} y, y \} &\leq \{ [f(B) - \{ f(A)x, x \} 1_H] y, y \} \\
&\leq \{ [f'(B)B - \{ Ax, x \} f'(B)] y, y \}
\end{aligned}
\]  

for each \( y \in H \) with \( \|y\| = 1 \), which is clearly equivalent to the desired inequality (2.41).

**Remark 2.17.** If we fix \( x \in H \) with \( \|x\| = 1 \) and choose \( B = \langle Ax, x \rangle \cdot 1_H \), then we obtain from the first inequality in (2.41) the reverse of the Mond–Pečarić inequality obtained by the author in [8]. The second inequality will provide the Mond–Pečarić inequality for convex functions whose derivatives are continuous.

The following corollary is of interest:

**Corollary 2.18.** Let \( I \) be an interval and \( f : I \to \mathbb{R} \) be a convex and differentiable function on \( \hat{I} \) whose derivative \( f' \) is continuous on \( \hat{I} \). Also, suppose that \( A \) is a selfadjoint operator on the Hilbert space \( H \) with \( \text{Sp}(A) \subseteq [m, M] \subset \hat{I} \). If \( g \) is non-increasing and continuous on \([m, M]\) and
\[
f'(A) [g(A) - A] \geq 0
\]  
(2.47)
in the operator order of $B(H)$, then

$$(f \circ g)(A) \geq f(A)$$

in the operator order of $B(H)$.

The following result may be stated as well:

**Theorem 2.19 (Dragomir, 2008, [10]).** Let $I$ be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\hat{I}$ (the interior of $I$) whose derivative $f'$ is continuous on $\hat{I}$. If $A$ and $B$ are selfadjoint operators on the Hilbert space $H$ with $\text{Sp}(A), \text{Sp}(B) \subseteq [m, M] \subset \hat{I}$, then

$$f'((Ax, x))((By, y) - (Ax, x))$$

$$\leq (f(B)y, y) - f((Ax, x))$$

$$\leq \left\{ f'(B)y, y \right\} - \left\{ f'(B)x, x \right\}$$

(2.49)

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$f'((Ax, x))((Ay, y) - (Ax, x))$$

$$\leq (f(A)y, y) - f((Ax, x))$$

$$\leq \left\{ f'(A)y, y \right\} - \left\{ f'(A)x, x \right\}$$

(2.50)

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$f'((Ax, x))((Bx, x) - (Ax, x)) \leq (f(B)x, x) - f((Ax, x))$$

$$\leq \left\{ f'(B)x, x \right\} - \left\{ f'(B)x, x \right\}$$

(2.51)

for any $x \in H$ with $\|x\| = 1$.

**Proof.** Since $f$ is convex and differentiable on $\hat{I}$, then we have that

$$f'(s) \cdot (t - s) \leq f(t) - f(s) \leq f'(t) \cdot (t - s)$$

(2.52)

for any $t, s \in [m, M]$.

If we choose $s = (Ax, x) \in [m, M]$, with a fix $x \in H$ with $\|x\| = 1$, then we have

$$f'((Ax, x))\cdot (t - (Ax, x)) \leq f(t) - f((Ax, x))$$

$$\leq f'(t) \cdot (t - (Ax, x))$$

(2.53)

for any $t \in [m, M]$.  

Now, if we apply property (P) to inequality (2.53) and the operator $B$, then we get

$$\left\{ f'((Ax, x)) \cdot (B - (Ax, x) \cdot 1_H) \cdot y, y \right\}$$

$$\leq \left\{ [f(B) - f((Ax, x)) \cdot 1_H] \cdot y, y \right\}$$

$$\leq \left\{ f'(B) \cdot (B - (Ax, x) \cdot 1_H) \cdot y, y \right\}$$

(2.54)

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, which is equivalent with the desired result (2.49).

**Remark 2.20.** We observe that if we choose $B = A$ in (2.51) or $y = x$ in (2.50) then we recapture the Mond–Pečarić inequality and its reverse from (2.1).

The following particular case of interest follows from Theorem 2.19:

**Corollary 2.21 (Dragomir, 2008, [10]).** Assume that $f, A$ and $B$ are as in Theorem 2.19. If, either $f$ is increasing on $[m, M]$ and $B \geq A$ in the operator order of $B (H)$ or $f$ is decreasing and $B \leq A$, then we have the Jensen’s type inequality

$$\langle f(B)x, x \rangle \geq f((Ax, x))$$

(2.55)

for any $x \in H$ with $\|x\| = 1$.

The proof is obvious by the first inequality in (2.51) and the details are omitted.

### 2.5 Some Jensen Type Inequalities for Twice Differentiable Functions

#### 2.5.1 Jensen’s Inequality for Twice Differentiable Functions

The following result may be stated:

**Theorem 2.22 (Dragomir, 2008, [11]).** Let $A$ be a positive definite operator on the Hilbert space $H$ and assume that $\text{Sp}(A) \subseteq [m, M]$ for some scalars $m, M$ with $0 < m < M$. If $f$ is a twice differentiable function on $(m, M)$ and for $p \in (-\infty, 0) \cup (1, \infty)$ we have for some $\gamma < \Gamma$ that

$$\gamma \leq \frac{t^{2-p}}{p \cdot (p - 1)} \cdot f''(t) \leq \Gamma \quad \text{for any} \quad t \in (m, M),$$

(2.56)

then

$$\gamma \left( (A^p x, x) - (Ax, x)^p \right) \leq \langle f(A)x, x \rangle - f((Ax, x))$$

$$\leq \Gamma \left( (A^p x, x) - (Ax, x)^p \right)$$

(2.57)

for each $x \in H$ with $\|x\| = 1$. 
If
\[ \delta \leq \frac{t^{2-p}}{p(1-p)} \cdot f''(t) \leq \Delta \quad \text{for any} \quad t \in (m, M) \] (2.58)
and for some \( \delta < \Delta \), where \( p \in (0, 1) \), then
\[ \delta \left( \langle Ax, x \rangle^p - \langle A^p x, x \rangle \right) \leq \left( f(A)x, x \right) - f \left( \langle Ax, x \rangle \right) \]
\[ \leq \Delta \left( \langle Ax, x \rangle^p - \langle A^p x, x \rangle \right) \] (2.59)
for each \( x \in H \) with \( \|x\| = 1 \).

**Proof.** Consider the function \( g_{y,p} : (m, M) \to \mathbb{R} \) given by \( g_{y,p}(t) = f(t) - \gamma t^p \)
where \( p \in (-\infty, 0) \cup (1, \infty) \). The function \( g_{y,p} \) is twice differentiable,
\[ g_{y,p}''(t) = f''(t) - \gamma p(p - 1)t^{p-2} \]
for any \( t \in (m, M) \) and by (2.56) we deduce that \( g_{y,p} \) is convex on \( (m, M) \). Now, applying the Mond and Pečarić inequality for \( g_{y,p} \) we have
\[ 0 \leq \left( (f(A) - \gamma A^p)x, x \right) - \left[ f \left( \langle Ax, x \rangle \right) - \gamma \left( A^p x, x \right) \right] \]
\[ = \left( f(A)x, x \right) - f \left( \langle Ax, x \rangle \right) - \gamma \left( A^p x, x \right) \]
which is equivalent with the first inequality in (2.57).

By defining the function \( g_{r,p} : (m, M) \to \mathbb{R} \) given by \( g_{r,p}(t) = rt^p - f(t) \)
and applying the same argument we deduce the second part of (2.57).

The rest goes likewise and the details are omitted. \( \square \)

**Remark 2.23.** We observe that if \( f \) is a twice differentiable function on \( (m, M) \) and \( \varphi := \inf_{t \in (m, M)} f''(t) \), \( \Phi := \sup_{t \in (m, M)} f''(t) \), then by (2.57) we get the inequality
\[ \frac{1}{2} \varphi \left[ \langle A^2 x, x \rangle - \langle Ax, x \rangle^2 \right] \leq \left( f(A)x, x \right) - f \left( \langle Ax, x \rangle \right) \]
\[ \leq \frac{1}{2} \Phi \left[ \langle A^2 x, x \rangle - \langle Ax, x \rangle^2 \right] \] (2.60)
for each \( x \in H \) with \( \|x\| = 1 \).

We observe that inequality (2.60) holds for selfadjoint operators that are not necessarily positive.

The next result provides some inequalities for the function \( f \) which replace the cases \( p = 0 \) and \( p = 1 \) that were not allowed in Theorem 2.22.
Theorem 2.24 (Dragomir, 2008, [10]). Let $A$ be a positive definite operator on the Hilbert space $H$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m, M$ with $0 < m < M$. If $f$ is a twice differentiable function on $(m, M)$ and we have for some $\gamma < \Gamma$ that

$$\gamma \leq t^2 \cdot f''(t) \leq \Gamma \quad \text{for any} \quad t \in (m, M),$$

then

$$\gamma (\ln (\langle Ax, x \rangle) - \langle \ln Ax, x \rangle) \leq \langle f(A)x, x \rangle - f(\langle Ax, x \rangle)$$

$$\leq \Gamma (\ln (\langle Ax, x \rangle) - \langle \ln Ax, x \rangle)$$

(2.62)

for each $x \in H$ with $\|x\| = 1$.

If

$$\delta \leq t \cdot f''(t) \leq \Delta \quad \text{for any} \quad t \in (m, M)$$

(2.63)

for some $\delta < \Delta$, then

$$\delta (\langle A \ln Ax, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle))$$

$$\leq \langle f(A)x, x \rangle - f(\langle Ax, x \rangle)$$

$$\leq \Delta (\langle A \ln Ax, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle))$$

(2.64)

for each $x \in H$ with $\|x\| = 1$.

Proof. Consider the function $g_{\gamma,0} : (m, M) \to \mathbb{R}$ given by $g_{\gamma,0}(t) = f(t) + \gamma \ln t$. The function $g_{\gamma,0}$ is twice differentiable,

$$g_{\gamma,0}''(t) = f''(t) - \gamma t^{-2}$$

for any $t \in (m, M)$ and by (2.61) we deduce that $g_{\gamma,0}$ is convex on $(m, M)$. Now, applying the Mond and Pečarić inequality for $g_{\gamma,0}$ we have

$$0 \leq \langle f(A) + \gamma \ln A \rangle x, x \rangle - \left[ f(\langle Ax, x \rangle) + \gamma \ln (\langle Ax, x \rangle) \right]$$

$$= \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) + \gamma \ln (\langle Ax, x \rangle) \rangle - \langle \ln Ax, x \rangle \rangle$$

which is equivalent with the first inequality in (2.62).

By defining the function $g_{\Gamma,0} : (m, M) \to \mathbb{R}$ given by $g_{\Gamma,0}(t) = -\Gamma \ln t - f(t)$ and applying the same argument we deduce the second part of (2.62).

The rest goes likewise for the functions

$$g_{\delta,1}(t) = f(t) - \delta t \ln t \quad \text{and} \quad g_{\Delta,0}(t) = \Delta t \ln t - f(t)$$

and the details are omitted.
2.6 Some Jensen’s Type Inequalities for Log-Convex Functions

2.6.1 Preliminary Results

The following result that provides an operator version for the Jensen inequality for convex functions is due to Mond and Pečarić [25] (see also [19, p. 5]):

**Theorem 2.25 (Mond–Pečarić, 1993, [25]).** Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $\text{Sp}(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $f$ is a convex function on $[m, M]$, then

$$f ((Ax, x)) \leq \langle f(A)x, x \rangle$$

(MP)

for each $x \in H$ with $\|x\| = 1$.

Taking into account the above result and its applications for various concrete examples of convex functions, it is therefore natural to investigate the corresponding results for the case of log-convex functions, namely functions $f : I \to (0, \infty)$ for which $\ln f$ is convex.

We observe that such functions satisfy the elementary inequality

$$f ((1 - t) a + tb) \leq \left[ f(a) \right]^{1-t} \left[ f(b) \right]^t$$

(2.65)

for any $a, b \in I$ and $t \in [0, 1]$. Also, due to the fact that the weighted geometric mean is less than the weighted arithmetic mean, it follows that any log-convex function is a convex function. However, obviously, there are functions that are convex but not log-convex.

As an immediate consequence of the Mond–Pečarić inequality above, we can provide the following result:

**Theorem 2.26 (Dragomir, 2010, [14]).** Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $\text{Sp}(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $g : [m, M] \to (0, \infty)$ is log-convex, then

$$g ((Ax, x)) \leq \exp \langle \ln g(A)x, x \rangle \leq \langle g(A)x, x \rangle$$

(2.66)

for each $x \in H$ with $\|x\| = 1$.

**Proof.** Consider the function $f := \ln g$, which is convex on $[m, M]$. Writing (MP) for $f$ we get $\ln [g ((Ax, x))] \leq \langle \ln g(A)x, x \rangle$, for each $x \in H$ with $\|x\| = 1$, which, by taking the exponential, produces the first inequality in (2.66).

If we also use (MP) for the exponential function, we get

$$\exp \langle \ln g(A)x, x \rangle \leq \langle \exp [\ln g(A)] x, x \rangle = \langle g(A)x, x \rangle$$

for each $x \in H$ with $\|x\| = 1$ and the proof is complete.
The case of sequences of operators may be of interest and is embodied in the following corollary:

**Corollary 2.27 (Dragomir, 2010, [14]).** Assume that \( g \) is as in Theorem 2.26. If \( A_j \) are selfadjoint operators with \( \text{Sp} (A_j) \subseteq [m, M] \), \( j \in \{1, \ldots, n \} \) and \( x_j \in H, j \in \{1, \ldots, n \} \) with \( \sum_{j=1}^{n} \|x_j\|^2 = 1 \), then

\[
g \left( \sum_{j=1}^{n} (A_j x_j, x_j) \right) \leq \exp \left( \sum_{j=1}^{n} \ln g (A_j) x_j, x_j \right)
\]

\[
\leq \sum_{j=1}^{n} g (A_j) x_j, x_j \cdot \exp \left( \sum_{j=1}^{n} \ln g (A_j) x_j, x_j \right)
\]

(2.67)

**Proof.** Follows from Theorem 2.26 and we omit the details.

In particular we have:

**Corollary 2.28 (Dragomir, 2010, [14]).** Assume that \( g \) is as in Theorem 2.26. If \( A_j \) are selfadjoint operators with \( \text{Sp} (A_j) \subseteq [m, M] \subseteq I, j \in \{1, \ldots, n \} \) and \( p_j \geq 0, j \in \{1, \ldots, n \} \) with \( \sum_{j=1}^{n} p_j = 1 \), then

\[
g \left( \sum_{j=1}^{n} p_j A_j x, x \right) \leq \prod_{j=1}^{n} [g (A_j)]^{p_j} x, x
\]

\[
\leq \sum_{j=1}^{n} p_j g (A_j) x, x
\]

(2.68)

for each \( x \in H \) with \( \|x\| = 1 \).

**Proof.** Follows from Corollary 2.27 by choosing \( x_j = \sqrt{p_j} \cdot x, j \in \{1, \ldots, n\} \) where \( x \in H \) with \( \|x\| = 1 \).

The following reverse for the Mond–Pečarić inequality that generalizes the scalar Lah–Ribarić inequality for convex functions is well known, see for instance [19, p.57]:

**Theorem 2.29.** Let \( A \) be a selfadjoint operator on the Hilbert space \( H \) and assume that \( \text{Sp}(A) \subseteq [m, M] \) for some scalars \( m, M \) with \( m < M \). If \( f \) is a convex function on \([m, M] \), then

\[
\langle f(A)x, x \rangle \leq \frac{M - \langle Ax, x \rangle}{M - m} \cdot f(m) + \frac{\langle Ax, x \rangle - m}{M - m} \cdot f(M)
\]

(2.69)

for each \( x \in H \) with \( \|x\| = 1 \).
This result can be improved for log-convex functions as follows:

**Theorem 2.30 (Dragomir, 2010, [14]).** Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $\text{Sp}(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $g : [m, M] \rightarrow (0, \infty)$ is log-convex, then

$$
\langle g(A)x, x \rangle \leq \left( g(m) \right)^{M/m} \left( g(M) \right)^{t-m} \langle x, x \rangle
$$

and

$$
g(\langle Ax, x \rangle) \leq \left( g(m) \right)^{M/m} \left( g(M) \right)^{t-m} \langle x, x \rangle
$$

for each $x \in H$ with $\|x\| = 1$.

**Proof.** Observe that, by the log-convexity of $g$, we have

$$
g(t) = g \left( \frac{M-t}{M-m} \cdot m + \frac{t-m}{M-m} \cdot M \right)
$$

$$
\leq \left( g(m) \right)^{M/m} \left( g(M) \right)^{t-m}
$$

for any $t \in [m, M]$.

Applying property (P) for the operator $A$, we have that

$$
\langle g(A)x, x \rangle \leq \langle \Psi(A)x, x \rangle
$$

for each $x \in H$ with $\|x\| = 1$, where $\Psi(t) := \left( g(m) \right)^{M/m} \left( g(M) \right)^{t-m}$, $t \in [m, M]$. This proves the first inequality in (2.70).

Now, observe that, by the weighted arithmetic mean–geometric mean inequality we have

$$
\left( g(m) \right)^{M/m} \left( g(M) \right)^{t-m} \leq \frac{M-t}{M-m} \cdot g(m) + \frac{t-m}{M-m} \cdot g(M)
$$

for any $t \in [m, M]$.

Applying property (P) for the operator $A$, we deduce the second inequality in (2.70).

Further on, if we use inequality (2.72) for $t = \langle Ax, x \rangle \in [m, M]$, then we deduce the first part of (2.71).

Now, observe that the function $\Psi$ introduced above can be rearranged to read as

$$
\Psi(t) = g(m) \left[ \frac{g(M)}{g(m)} \right]^{t-m} \langle x, x \rangle, t \in [m, M]
$$

showing that $\Psi$ is a convex function on $[m, M]$. 
Applying Mond–Pečarić’s inequality for $\Psi$ we deduce the second part of (2.71) and the proof is complete.

### 2.6.2 Jensen’s Inequality for Differentiable Log-Convex Functions

The following result provides a reverse for the Jensen type inequality (MP):

**Theorem 2.31 (Dragomir, 2008, [8]).** Let $J$ be an interval and $f : J \to \mathbb{R}$ be a convex and differentiable function on $\bar{J}$ (the interior of $J$) whose derivative $f'$ is continuous on $\bar{J}$. If $A$ is a selfadjoint operator on the Hilbert space $H$ with $\text{Sp}(A) \subseteq [m, M] \subset J$, then

\[
0 \leq f(A)x, x - f(\langle Ax, x \rangle)
\]

\[
\leq f'(A)Ax, x - \langle Ax, x \rangle \cdot f'(A)x, x \tag{2.73}
\]

for any $x \in H$ with $\|x\| = 1$.

The following result may be stated:

**Proposition 2.32 (Dragomir, 2010, [14]).** Let $J$ be an interval and $g : J \to \mathbb{R}$ be a differentiable log-convex function on $\bar{J}$ whose derivative $g'$ is continuous on $\bar{J}$. If $A$ is a selfadjoint operator on the Hilbert space $H$ with $\text{Sp}(A) \subseteq [m, M] \subset J$, then

\[
1 \leq \exp \frac{\ln g(A)x, x}{g(\langle Ax, x \rangle)} \leq \exp \left[ \frac{g'(A)[g(A)]^{-1}Ax, x}{g(\langle Ax, x \rangle)} - \langle Ax, x \rangle \cdot \frac{g'(A)}{g(A)}[g(A)]^{-1}x, x \right] \tag{2.74}
\]

for each $x \in H$ with $\|x\| = 1$.

**Proof.** It follows by inequality (2.73) written for the convex function $f = \ln g$ that

\[
\langle \ln g(A)x, x \rangle \leq \ln g(\langle Ax, x \rangle)
\]

\[
+ \left\{ g'(A)[g(A)]^{-1}Ax, x \right\} - \langle Ax, x \rangle \cdot \left\{ g'(A)[g(A)]^{-1}x, x \right\}
\]

for each $x \in H$ with $\|x\| = 1$.

Now, taking the exponential and dividing by $g(\langle Ax, x \rangle) > 0$ for each $x \in H$ with $\|x\| = 1$, we deduce the desired result (2.74).

The following result that provides both a refinement and a reverse of the multiplicative version of Jensen’s inequality can be stated as well:

**Theorem 2.33 (Dragomir, 2010, [14]).** Let $J$ be an interval and $g : J \to \mathbb{R}$ be a log-convex differentiable function on $\bar{J}$ whose derivative $g'$ is continuous on $\bar{J}$. If $A$ is a selfadjoint operator on the Hilbert space $H$ with $\text{Sp}(A) \subseteq [m, M] \subset J$, then
\[ 1 \leq \exp \left[ \frac{g'(\langle Ax, x \rangle)}{g(\langle Ax, x \rangle)} (A - \langle Ax, x \rangle 1_H) \right] x, x \]
\[ \leq \frac{g(A)x, x}{g(\langle Ax, x \rangle)} \leq \left( \exp \left[ g'(A) [g(A)]^{-1} (A - \langle Ax, x \rangle 1_H) \right] \right) x, x \]  
(2.75)

for each \( x \in H \) with \( \|x\| = 1 \), where \( 1_H \) denotes the identity operator on \( H \).

**Proof.** It is well known that if \( h : J \rightarrow \mathbb{R} \) is a convex differentiable function on \( \tilde{J} \), then the following gradient inequality holds:

\[ h(t) - h(s) \geq h'(s)(t - s) \]

for any \( t, s \in \tilde{J} \).

Now, if we write this inequality for the convex function \( h = \ln g \), then we get

\[ \ln g(t) - \ln g(s) \geq \frac{g'(s)}{g(s)} (t - s) \]  
(2.76)

which is equivalent with

\[ g(t) \geq g(s) \exp \left[ \frac{g'(s)}{g(s)} (t - s) \right] \]  
(2.77)

for any \( t, s \in \tilde{J} \).

Further, if we take \( s := \langle Ax, x \rangle \in [m, M] \subset \tilde{J} \), for a fixed \( x \in H \) with \( \|x\| = 1 \), in inequality (2.77), then we get

\[ g(t) \geq g(\langle Ax, x \rangle) \exp \left[ \frac{g'(\langle Ax, x \rangle)}{g(\langle Ax, x \rangle)} (t - \langle Ax, x \rangle) \right] \]

for any \( t \in \tilde{J} \).

Utilizing property (P) for the operator \( A \) and the Mond–Pečarić inequality for the exponential function, we can state the following inequality that is of interest in itself as well:

\[ \langle g(A)y, y \rangle \]
\[ \geq g(\langle Ax, x \rangle) \exp \left[ \frac{g'(\langle Ax, x \rangle)}{g(\langle Ax, x \rangle)} (A - \langle Ax, x \rangle 1_H) \right] y, y \]
\[ \geq g(\langle Ax, x \rangle) \exp \left[ \frac{g'(\langle Ax, x \rangle)}{g(\langle Ax, x \rangle)} ((Ay, y) - \langle Ax, x \rangle) \right] \]  
(2.78)

for each \( x, y \in H \) with \( \|x\| = \|y\| = 1 \).

Further, if we put \( y = x \) in (2.78), then we deduce the first and the second inequality in (2.75).
Now, if we replace $s$ with $t$ in (2.77) we can also write the inequality

$$g(t) \exp \left[ \frac{g'(t)}{g(t)} (s - t) \right] \leq g(s)$$

which is equivalent with

$$g(t) \leq g(s) \exp \left[ \frac{g'(t)}{g(t)} (t - s) \right]$$

for any $t, s \in \mathbf{J}$.

Further, if we take $s := \langle Ax, x \rangle \in [m, M] \subset \mathbf{J}$, for a fixed $x \in H$ with $\|x\| = 1$, in inequality (2.79), then we get

$$g(t) \leq g \left( \langle Ax, x \rangle \right) \exp \left[ \frac{g'(t)}{g(t)} (t - \langle Ax, x \rangle) \right]$$

for any $t \in \mathbf{J}$.

Utilizing property (P) for the operator $A$, then we can state the following inequality that is of interest in itself as well:

$$\langle g(A) y, y \rangle \leq g \left( \langle Ax, x \rangle \right) \exp \left[ g'(A) \left[ g(A) \right]^{-1} (A - \langle Ax, x \rangle 1_H) \right] y, y \right)$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Finally, if we put $y = x$ in (2.80), then we deduce the last inequality in (2.75).

The following reverse inequality may be proven as well:

**Theorem 2.34 (Dragomir, 2010, [14]).** Let $J$ be an interval and $g : J \to \mathbb{R}$ be a log-convex differentiable function on $\mathbf{J}$ whose derivative $g'$ is continuous on $\mathbf{J}$. If $A$ is a selfadjoint operators on the Hilbert space $H$ with $\text{Sp}(A) \subseteq [m, M] \subset \mathbf{J}$, then

$$\langle g(A) x, x \rangle \leq \left\langle \left[ g(M)^{\frac{A - m}{M - m}} g(m) \right]^{\frac{M^1_H - A}{M - m}} x, x \right\rangle$$

$$\leq \left\langle \left\{ \frac{(M^1_H - A)(A - m)^1_H}{M - m} \left( \frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right\} x, x \right\rangle$$

$$\leq \exp \left[ \frac{1}{4} (M - m) \left( \frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]$$

for each $x \in H$ with $\|x\| = 1$. 

\[ \text{for each } x \in H \text{ with } \|x\| = 1. \]
Proof. Utilizing inequality (2.76) we have successively

\[
\frac{g ((1 - \lambda) t + \lambda s)}{g(s)} \geq \exp \left[ (1 - \lambda) \frac{g'(s)}{g(s)} (t - s) \right] \tag{2.82}
\]

and

\[
\frac{g ((1 - \lambda) t + \lambda s)}{g(t)} \geq \exp \left[ -\lambda \frac{g'(t)}{g(t)} (t - s) \right] \tag{2.83}
\]

for any \( t, s \in \mathbb{J} \) and any \( \lambda \in [0, 1] \).

Now, if we take the power \( \lambda \) in inequality (2.82) and the power \( 1 - \lambda \) in (2.83) and multiply the obtained inequalities, we deduce

\[
\frac{[g(t)]^{1-\lambda} [g(s)]^\lambda}{g ((1 - \lambda) t + \lambda s)} \leq \exp \left[ (1 - \lambda) \lambda \left( \frac{g'(t)}{g(t)} - \frac{g'(s)}{g(s)} \right) (t - s) \right] \tag{2.84}
\]

for any \( t, s \in \mathbb{J} \) and any \( \lambda \in [0, 1] \).

Further on, if we choose in (2.84) \( t = M, s = m \) and \( \lambda = \frac{M - u}{M - m} \), then, from (2.84) we get the inequality

\[
\frac{[g(M)]^{\frac{u-m}{M-m}} [g(m)]^{\frac{M-u}{M-m}}}{g (u)} \leq \exp \left[ \frac{(M - u) (u - m)}{M - m} \left( \frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right] \tag{2.85}
\]

which, together with the inequality

\[
\frac{(M - u) (u - m)}{M - m} \leq \frac{1}{4} (M - m)
\]

produce

\[
[g(M)]^{\frac{u-m}{M-m}} [g(m)]^{\frac{M-u}{M-m}} \leq g (u) \exp \left[ \frac{(M - u) (u - m)}{M - m} \left( \frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right] \leq g (u) \exp \left[ \frac{1}{4} (M - m) \left( \frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right] \tag{2.86}
\]

for any \( u \in [m, M] \).

If we apply property (P) to inequality (2.86) and for the operator \( A \) we deduce the desired result. \( \Box \)
2.6.3 More Inequalities for Differentiable Log-Convex Functions

The following results providing companion inequalities for the Jensen inequality for differentiable log-convex functions obtained above hold:

**Theorem 2.35 (Dragomir, 2010, [15]).** Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $\text{Sp}(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $g : J \to (0, \infty)$ is a differentiable log-convex function with the derivative continuous on $\mathcal{J}$ and $[m, M] \subset \mathcal{J}$, then

$$
\exp \left[ \frac{(g'(A)x, x)}{(g(A)x, x)} - \frac{(g(A)x, x)}{(g(A)x, x)} \cdot \frac{(g'(A)x, x)}{(g'(A)x, x)} \right] \\
\geq \exp \left[ \frac{(g(A)\ln g(A)x, x)}{(g(A)x, x)} \right] \geq 1 \text{ } (2.87)
$$

for each $x \in H$ with $\|x\| = 1$.

If

$$
\frac{(g'(A)x, x)}{(g'(A)x, x)} \in \mathcal{J} \text{ for each } x \in H \text{ with } \|x\| = 1, \text{ (C)}
$$

then

$$
\exp \left[ \frac{g'}{g} \left( \frac{(g'(A)x, x)}{(g'(A)x, x)} \right) \left( \frac{(g'(A)x, x)}{(g(A)x, x)} - \frac{(A g(A)x, x)}{(g(A)x, x)} \right) \right] \\
\geq \exp \left( \frac{(g(A)\ln g(A)x, x)}{(g(A)x, x)} \right) \geq 1 \text{ } (2.88)
$$

for each $x \in H$ with $\|x\| = 1$.

**Proof.** By the gradient inequality for the convex function $\ln g$ we have

$$
g'(t) \frac{g}{g(t)} (t - s) \geq \ln g(t) - \ln g(s) \geq \frac{g'(s)}{g(s)} (t - s) \text{ } (2.89)
$$

for any $t, s \in \mathcal{J}$, which by multiplication with $g(t) > 0$ is equivalent with

$$
g'(t)(t - s) \geq g(t) \ln g(t) - g(t) \ln g(s) \geq \frac{g'(s)}{g(s)} (tg(t) - sg(t)) \text{ } (2.90)
$$

for any $t, s \in \mathcal{J}$.
Fix \( s \in J \) and apply property (P) to get that
\[
\left\langle g'(A)x, x \right\rangle - s \left\langle g(A)x, x \right\rangle \\
\geq \langle g(A) \ln g(A)x, x \rangle - \left\langle g(A)x, x \right\rangle \ln g(s) \\
\geq \frac{g'(s)}{g(s)} \left( \langle Ag(A)x, x \rangle - s \langle g(A)x, x \rangle \right)
\] (2.91)
for any \( x \in H \) with \( \|x\| = 1 \), which is an inequality of interest in itself as well.

Since
\[
\frac{\langle g(A)x, x \rangle}{\langle g(A)x, x \rangle} \in [m, M] \text{ for any } x \in H \text{ with } \|x\| = 1
\]
then on choosing \( s := \frac{\langle g(A)x, x \rangle}{\langle g(A)x, x \rangle} \) in (2.91) we get
\[
\left\langle g'(A)x, x \right\rangle - \frac{\langle g(A)x, x \rangle}{\langle g(A)x, x \rangle} \left\langle g(A)x, x \right\rangle \\
\geq \langle g(A) \ln g(A)x, x \rangle - \langle g(A)x, x \rangle \ln \left( \frac{\langle g(A)x, x \rangle}{\langle g(A)x, x \rangle} \right) \\
\geq 0,
\]
which, by division with \( \langle g(A)x, x \rangle > 0 \), produces
\[
\frac{\langle g'(A)x, x \rangle}{\langle g(A)x, x \rangle} - \frac{\langle g(A)x, x \rangle}{\langle g(A)x, x \rangle} \cdot \frac{\langle g'(A)x, x \rangle}{\langle g(A)x, x \rangle} \\
\geq \frac{\langle g(A) \ln g(A)x, x \rangle}{\langle g(A)x, x \rangle} - \ln \left( \frac{\langle g(A)x, x \rangle}{\langle g(A)x, x \rangle} \right) \geq 0
\] (2.92)
for any \( x \in H \) with \( \|x\| = 1 \).

Taking the exponential in (2.92) we deduce the desired inequality (2.87).

Now, assuming that condition (C) holds, then by choosing \( s := \frac{\langle g'(A)x, x \rangle}{\langle g'(A)x, x \rangle} \) in (2.91) we get
\[
0 \geq \langle g(A) \ln g(A)x, x \rangle - \langle g(A)x, x \rangle \ln \left( \frac{\langle g'(A)x, x \rangle}{\langle g'(A)x, x \rangle} \right) \\
\geq g' \left( \frac{\langle g'(A)x, x \rangle}{\langle g'(A)x, x \rangle} \right) \left( \langle Ag(A)x, x \rangle - \frac{\langle g'(A)x, x \rangle}{\langle g'(A)x, x \rangle} \langle g(A)x, x \rangle \right)
\]
which, by dividing with \( \langle g(A)x, x \rangle > 0 \) and rearranging, is equivalent with
\[
\frac{g' \left( \frac{\langle g'(A)x, x \rangle}{\langle g'(A)x, x \rangle} \right)}{g \left( \frac{\langle g'(A)x, x \rangle}{\langle g'(A)x, x \rangle} \right)} \left( \frac{\langle g'(A)x, x \rangle}{\langle g'(A)x, x \rangle} - \langle Ag(A)x, x \rangle \right) \\
= \frac{g' \left( \frac{\langle g'(A)x, x \rangle}{\langle g'(A)x, x \rangle} \right)}{g \left( \frac{\langle g'(A)x, x \rangle}{\langle g'(A)x, x \rangle} \right)} \left( \langle Ag(A)x, x \rangle - \frac{\langle g'(A)x, x \rangle}{\langle g'(A)x, x \rangle} \langle g(A)x, x \rangle \right)
\]
\[
\ln g \left( \frac{g'(A)Ax}{g'(A)x,x} \right) \geq \ln g \left( \frac{g(A)Ax}{g(A)x,x} \right) \geq 0 \tag{2.93}
\]
for any \( x \in H \) with \( \|x\| = 1 \).

Finally, on taking the exponential in (2.93) we deduce the desired inequality (2.88).

Remark 2.36. We observe that a sufficient condition for (C) to hold is that either \( g'(A) \) or \( -g'(A) \) is a positive definite operator on \( H \).

Corollary 2.37 (Dragomir, 2010, [15]). Assume that \( A \) and \( g \) are as in Theorem 2.35. If condition (C) holds, then we have the double inequality

\[
\ln g \left( \frac{g'(A)Ax}{g'(A)x,x} \right) \geq \ln g \left( \frac{g(A)Ax}{g(A)x,x} \right) \geq 0 \tag{2.94}
\]
for any \( x \in H \) with \( \|x\| = 1 \).

The following result providing different inequalities also holds:

Theorem 2.38 (Dragomir, 2010, [15]). Let \( A \) be a selfadjoint operator on the Hilbert space \( H \) and assume that \( \text{Sp}(A) \subseteq [m, M] \) for some scalars \( m, M \) with \( m < M \). If \( g : J \to (0, \infty) \) is a differentiable log-convex function with the derivative continuous on \( J \) and \( [m, M] \subset J \), then

\[
\left\langle \exp \left[ g'(A) \left( A - \frac{g(A)Ax}{g(A)x,x} \right) 1_H \right] x, x \right\rangle \geq \left\langle \frac{g(A)}{g \left( \frac{g(A)Ax}{g(A)x,x} \right)} x, x \right\rangle \geq 1 \tag{2.95}
\]
for each \( x \in H \) with \( \|x\| = 1 \).

If condition (C) from Theorem 2.35 holds, then

\[
\left\langle \exp \left[ \frac{g'(A)Ax}{g \left( \frac{g'(A)Ax}{g'(A)x,x} \right)} \right] \left( \frac{g'(A)Ax}{g'(A)x,x} - g(A) - Ag(A) \right) x, x \right\rangle \geq \left\langle \left( g \left( \frac{g'(A)Ax}{g'(A)x,x} \right) \right) \left( g(A) - Ag(A) \right)^{-1} g(A) x, x \right\rangle
\]
2.6 Some Jensen’s Type Inequalities for Log-Convex Functions

\[ \geq \exp \left[ g'(A) \left( \frac{g'(A)Ax}{g'(A)x,x} 1_H - A \right) \right] x,x \geq 1 \]  

(2.96)

for each \( x \in H \) with \( \|x\| = 1 \).

**Proof.** By taking the exponential in (2.90) we have the following inequality:

\[ \exp \left[ g'(t)(t-s) \right] \geq \left( \frac{g(t)}{g(s)} \right)^{g(t)} \geq \exp \left[ g'(s) \left( (tg(t) - sg(t)) \right) \right] \]

(2.97)

for any \( t, s \in \mathcal{J} \).

If we fix \( s \in \mathcal{J} \) and apply property (P) to inequality (2.97), we deduce

\[ \exp \left[ g'(A) (A-s1_H) \right] x,x \geq \exp \left[ g'(s) \left( (Ag(A) - sg(A)) \right) \right] x,x \]

(2.98)

for each \( x \in H \) with \( \|x\| = 1 \), where \( 1_H \) is the identity operator on \( H \).

By Mond–Pečarić’s inequality applied for the convex function \( \exp \), we also have

\[ \exp \left[ \frac{g'(s)}{g(s)} (sg(A)) \right] x,x \geq \exp \left( \frac{g'(s)}{g(s)} ( Ag(A) - sg(A) ) \right) \]

(2.99)

for each \( s \in \mathcal{J} \) and \( x \in H \) with \( \|x\| = 1 \).

Now, if we choose \( s := \frac{g(A)x,x}{g'(A)x,x} \in [m, M] \) in (2.98) and (2.99) we deduce the desired result (2.95).

Observe that, inequality (2.97) is equivalent with

\[ \exp \left[ \frac{g'(s)}{g(s)} (sg(t) - tg(t)) \right] \geq \left( \frac{g(s)}{g(t)} \right)^{g(t)} \geq \exp \left[ g'(t) (s-t) \right] \]

(2.100)

for any \( t, s \in \mathcal{J} \).

If we fix \( s \in \mathcal{J} \) and apply property (P) to inequality (2.100) we deduce

\[ \exp \left[ g'(A) (s1_H - A) \right] x,x \geq \exp \left[ \left( \frac{g(s)}{g(A)} \right)^{g(A)} \right] x,x \]

(2.101)

for each \( x \in H \) with \( \|x\| = 1 \).
By Mond–Pečarić’s inequality we also have

$$
\langle \exp \left[ g'(A) (s1_H - A) \right] x, x \rangle \geq \exp \left[ s \langle g'(A)x, x \rangle - \langle g'(A)Ax, x \rangle \right]
$$

(2.102)

for each \( s \in \tilde{J} \) and \( x \in H \) with \( \|x\| = 1 \).

Taking into account that condition (C) is valid, then we can choose in (2.101) and (2.102) \( s := \frac{\langle g'(A)Ax, x \rangle}{\langle g'(A)x, x \rangle} \) to get the desired result (2.96).

\[ \square \]

### 2.6.4 A Reverse Inequality

The following reverse inequality is also of interest:

**Theorem 2.39 (Dragomir, 2010, [15]).** Let \( A \) be a selfadjoint operator on the Hilbert space \( H \) and assume that \( \text{Sp}(A) \subseteq [m, M] \) for some scalars \( m, M \) with \( m < M \). If \( g : J \to (0, \infty) \) is a differentiable log-convex function with the derivative continuous on \( \tilde{J} \) and \( [m, M] \subset \tilde{J} \), then

$$
1 \leq \exp \left[ \frac{g(m)}{g(M)} \right] \frac{g^\prime(M) - g^\prime(m)}{g^\prime(M) - g^\prime(m)} \frac{\langle Ax, x \rangle}{M - m} \left( \frac{g^\prime(M)}{g(M)} - \frac{g^\prime(m)}{g(m)} \right)
$$

(2.103)

for each \( x \in H \) with \( \|x\| = 1 \).

**Proof.** Utilizing inequality (2.89) we have successively

$$
\ln g ((1 - \lambda) t + \lambda s) - \ln g(s) \geq (1 - \lambda) \frac{g'(s)}{g(s)} (t - s)
$$

(2.104)

and

$$
\ln g ((1 - \lambda) t + \lambda s) - \ln g(t) \geq -\lambda \frac{g'(t)}{g(t)} (t - s)
$$

(2.105)

for any \( t, s \in \tilde{J} \) and any \( \lambda \in [0, 1] \).

Now, if we multiply (2.104) by \( \lambda \) and (2.105) by \( 1 - \lambda \) and sum the obtained inequalities, we deduce...
\[(1 - \lambda) \ln g(t) + \lambda \ln g(s) - \ln g ((1 - \lambda) t + \lambda s) \]
\[
\leq (1 - \lambda) \lambda \left[ \left( \frac{g'(t)}{g(t)} - \frac{g'(s)}{g(s)} \right) (t - s) \right]
\] (2.106)

for any \( t, s \in I \) and any \( \lambda \in [0, 1] \).

Now, if we choose \( \lambda := \frac{M - u}{M - m} \), \( s := m \) and \( t := M \) in (2.106) then we get the inequality
\[
\frac{u - m}{M - m} \ln g(M) + \frac{M - u}{M - m} \ln g(m) - \ln g(u)
\]
\[
\leq \left[ \frac{(M - u) (u - m)}{M - m} \left( \frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]
\] (2.107)

for any \( u \in [m, M] \).

If we use property (P) for the operator \( A \) we get
\[
\langle Ax, x \rangle - m \frac{M - \langle Ax, x \rangle}{M - m} \ln g(M) - \langle \ln g(A) x, x \rangle
\]
\[
\leq \left[ \frac{((M1_H - A)(A - m1_H)) x, x)}{M - m} \left( \frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]
\] (2.108)

for each \( x \in H \) with \( \|x\| = 1 \).

Taking the exponential in (2.108) we deduce the first inequality in (2.103).

Now, consider the function \( h : [m, M] \rightarrow \mathbb{R}, h(t) = (M - t) (t - m) \). This function is concave in \([m, M]\) and by Mond–Pečarić’s inequality we have
\[
((M1_H - A)(A - m1_H)) x, x) \leq (M - \langle Ax, x \rangle) (\langle Ax, x \rangle - m)
\]

for each \( x \in H \) with \( \|x\| = 1 \), which proves the second inequality in (2.103).

For the last inequality, we observe that
\[
(M - \langle Ax, x \rangle) (\langle Ax, x \rangle - m) \leq \frac{1}{4} (M - m)^2,
\]
and the proof is complete. \( \square \)

### 2.7 Hermite–Hadamard’s Type Inequalities

#### 2.7.1 Scalar Case

If \( f : I \rightarrow \mathbb{R} \) is a convex function on the interval \( I \), then for any \( a, b \in I \) with \( a \neq b \) we have the following double inequality:
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\[ f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(t) \, dt \leq \frac{f(a) + f(b)}{2}. \]  

(HH)

This remarkable result is well known in the literature as the \textit{Hermite–Hadamard inequality} [24].

For various generalizations, extensions, reverses and related inequalities, see [1, 2, 18, 20–24] the monograph [17] and the references therein.

### 2.7.2 Some Inequalities for Convex Functions

The following inequality related to the Mond–Pečarić result also holds:

**Theorem 2.40 (Dragomir, 2010, [13]).** Let \( A \) be a selfadjoint operator on the Hilbert space \( H \) and assume that \( \mathrm{Sp}(A) \subseteq [m, M] \) for some scalars \( m, M \) with \( m < M \).

If \( f \) is a convex function on \( [m, M] \), then

\[
\frac{f(m) + f(M)}{2} \geq \left\{ \frac{f(A) + f((m + M)1_H - A)_{x,x}}{2} \right\} \\
\geq \frac{f(\langle Ax, x \rangle) + f(m + M - \langle Ax, x \rangle)}{2} \\
\geq f \left( \frac{m + M}{2} \right)
\]  

(2.109)

for each \( x \in H \) with \( \|x\| = 1 \).

In addition, if \( x \in H \) with \( \|x\| = 1 \) and \( \langle Ax, x \rangle \neq \frac{m + M}{2} \), then also

\[
\frac{f(\langle Ax, x \rangle) + f(m + M - \langle Ax, x \rangle)}{2} \geq \frac{2}{m + M - \langle Ax, x \rangle} \int_{\langle Ax, x \rangle}^{m + M - \langle Ax, x \rangle} f(u) \, du \geq f \left( \frac{m + M}{2} \right).
\]  

(2.110)

**Proof.** Since \( f \) is convex on \( [m, M] \) then for each \( u \in [m, M] \) we have the inequalities

\[
\frac{M - u}{M - m} f(m) + \frac{u - m}{M - m} f(M) \\
\geq f \left( \frac{M - u}{M - m} m + \frac{u - m}{M - m} M \right) = f(u)
\]  

(2.111)
2.7 Hermite–Hadamard’s Type Inequalities

and

\[
\frac{M - u}{M - m} f(M) + \frac{u - m}{M - m} f(m) \geq f \left( \frac{M - u}{M - m} M + \frac{u - m}{M - m} m \right) = f(M + m - u). \tag{2.112}
\]

If we add these two inequalities we get

\[f(m) + f(M) \geq f(u) + f(M + m - u)\]

for any \(u \in [m, M]\), which, by property (P) applied for the operator \(A\), produces the first inequality in (2.109).

By the Mond–Pečarić inequality we have

\[\langle f((m + M)1_H - A)x, x\rangle \geq f(m + M - \langle Ax, x\rangle),\]

which together with the same inequality produces the second inequality in (2.109).

The third part follows by the convexity of \(f\).

In order to prove (2.110), we use the Hermite–Hadamard inequality (HH) for the convex functions \(f\) and the choices \(a = \langle Ax, x\rangle\) and \(b = m + M - \langle Ax, x\rangle\).

The proof is complete. \(\Box\)

**Remark 2.41.** We observe that, from inequality (2.109) we have the following inequality in the operator order of \(B(H)\):

\[
\left[ \frac{f(m) + f(M)}{2} \right] 1_H \geq \frac{f(A) + f((m + M)1_H - A)}{2} \geq f\left(\frac{m + M}{2}\right) 1_H, \tag{2.113}
\]

where \(f\) is a convex function on \([m, M]\) and \(A\) a selfadjoint operator on the Hilbert space \(H\) with \(\text{Sp}(A) \subseteq [m, M]\) for some scalars \(m, M\) with \(m < M\).

The case of log-convex functions may be of interest for applications and therefore is stated in:

**Corollary 2.42 (Dragomir, 2010, [13]).** If \(g\) is a log-convex function on \([m, M]\), then

\[
\sqrt{g(m)g(M)} \geq \exp \left\{ \ln [g(A)g((m + M)1_H - A)]^{1/2} x, x \right\} \geq \sqrt{g(\langle Ax, x\rangle) g(m + M - \langle Ax, x\rangle)} \geq g\left(\frac{m + M}{2}\right) \tag{2.114}
\]

for each \(x \in H\) with \(\|x\| = 1\).
In addition, if \( x \in H \) with \( \|x\| = 1 \) and \( \langle Ax, x \rangle \neq \frac{m + M}{2} \), then also

\[
\sqrt{g \left( \langle Ax, x \rangle \right)} g \left( m + M - \langle Ax, x \rangle \right) \\
\geq \exp \left[ \frac{2}{m + M - \langle Ax, x \rangle} \int_{\langle Ax, x \rangle}^{m + M - \langle Ax, x \rangle} \ln g \left( u \right) \, du \right] \\
\geq g \left( \frac{m + M}{2} \right). \tag{2.115}
\]

The following result also holds:

**Theorem 2.43 (Dragomir, 2010, [13]).** Let \( A \) and \( B \) selfadjoint operators on the Hilbert space \( H \) and assume that \( \text{Sp}(A), \text{Sp}(B) \subseteq [m, M] \) for some scalars \( m, M \) with \( m < M \).

If \( f \) is a convex function on \([m, M]\), then

\[
f \left( \left\langle \frac{A + B}{2} x, x \right\rangle \right) \\
\leq \frac{1}{2} \left[ f \left( (1-t) \langle Ax, x \rangle + t \langle Bx, x \rangle \right) + f \left( t \langle Ax, x \rangle + (1-t) \langle Bx, x \rangle \right) \right] \\
\leq \left\langle \frac{1}{2} \left[ f \left((1-t) A + t B \right) + f \left( tA + (1-t) B \right) \right] x, x \right\rangle \\
\leq \frac{M - \left\langle \frac{A + B}{2} x, x \right\rangle}{M - m} f(m) + \frac{\left\langle \frac{A + B}{2} x, x \right\rangle - m}{M - m} f(M) \tag{2.116}
\]

for any \( t \in [0, 1] \) and each \( x \in H \) with \( \|x\| = 1 \).

Moreover, we have the Hermite–Hadamard’s type inequalities:

\[
f \left( \left\langle \frac{A + B}{2} x, x \right\rangle \right) \\
\leq \int_{0}^{1} f \left( (1-t) \langle Ax, x \rangle + t \langle Bx, x \rangle \right) \, dt \\
\leq \left\langle \int_{0}^{1} f \left((1-t) A + t B \right) \, dt \right\rangle x, x \rangle \\
\leq \frac{M - \left\langle \frac{A + B}{2} x, x \right\rangle}{M - m} f(m) + \frac{\left\langle \frac{A + B}{2} x, x \right\rangle - m}{M - m} f(M) \tag{2.117}
\]

for each \( x \in H \) with \( \|x\| = 1 \).
In addition, if we assume that $B - A$ is a positive definite operator, then

$$f \left( \left( \frac{A + B}{2} x, x \right) \right) \langle (B - A) x, x \rangle$$

$$\leq \int_{\langle A, x \rangle}^{\langle B, x \rangle} f(u) \, du \leq \langle (B - A) x, x \rangle \left[ \int_{0}^{1} f \left( (1 - t) A + tB \right) \, dt \right] x, x \rangle$$

$$\leq \langle (B - A) x, x \rangle \left[ \frac{M - \langle \frac{A + B}{2} x, x \rangle}{M - m} f(m) + \frac{\langle \frac{A + B}{2} x, x \rangle - m}{M - m} f(M) \right]. \quad (2.118)$$

**Proof.** It is obvious that for any $t \in [0, 1]$ we have

$$Sp ((1 - t) A + tB), Sp (tA + (1 - t) B) \subseteq [m, M].$$

On making use of the Mond–Pečarić inequality we have

$$f \left( (1 - t) \langle Ax, x \rangle + t \langle Bx, x \rangle \right) \leq \langle f \left( (1 - t) A + tB \right) x, x \rangle \quad (2.119)$$

and

$$f \left( t \langle Ax, x \rangle + (1 - t) \langle Bx, x \rangle \right) \leq \langle f \left( tA + (1 - t) B \right) x, x \rangle \quad (2.120)$$

for any $t \in [0, 1]$ and each $x \in H$ with $\|x\| = 1$.

Adding (2.119) with (2.120) and utilizing the convexity of $f$ we deduce the first two inequalities in (2.116).

By the Lah–Ribarić inequality (2.69) we also have

$$\langle f \left( (1 - t) A + tB \right) x, x \rangle \leq \frac{M - (1 - t) \langle Ax, x \rangle - t \langle Bx, x \rangle}{M - m} \cdot f(m)$$

$$+ \frac{(1 - t) \langle Ax, x \rangle + t \langle Bx, x \rangle - m}{M - m} \cdot f(M) \quad (2.121)$$

and

$$\langle f \left( tA + (1 - t) B \right) x, x \rangle \leq \frac{M - t \langle Ax, x \rangle - (1 - t) \langle Bx, x \rangle}{M - m} \cdot f(m)$$

$$+ \frac{t \langle Ax, x \rangle + (1 - t) \langle Bx, x \rangle - m}{M - m} \cdot f(M) \quad (2.122)$$

for any $t \in [0, 1]$ and each $x \in H$ with $\|x\| = 1$.

Now, if we add inequalities (2.121) with (2.122) and divide by two, we deduce the last part in (2.116).
Integrating the inequality over $t \in [0, 1]$, utilizing the continuity property of the inner product and the properties of the integral of operator-valued functions we have

$$f \left( \frac{A + B}{2}, x, x \right) \leq \frac{1}{2} \left[ \int_0^1 f \left( (1 - t) \langle Ax, x \rangle + t \langle Bx, x \rangle \right) \, dt 
+ \int_0^1 f \left( t \langle Ax, x \rangle + (1 - t) \langle Bx, x \rangle \right) \, dt \right]$$

$$\leq \frac{1}{2} \left[ \int_0^1 f \left( (1 - t) A + tB \right) \, dt 
+ \int_0^1 f \left( tA + (1 - t) B \right) \, dt \right] x, x \rangle$$

$$\leq \frac{M - \langle A + B, x, x \rangle}{M - m} f(m) + \frac{\langle A + B, x, x \rangle - m}{M - m} f(M). \quad (2.123)$$

Since

$$\int_0^1 f \left( (1 - t) \langle Ax, x \rangle + t \langle Bx, x \rangle \right) \, dt = \int_0^1 f \left( t \langle Ax, x \rangle + (1 - t) \langle Bx, x \rangle \right) \, dt$$

and

$$\int_0^1 f \left( (1 - t) A + tB \right) \, dt = \int_0^1 f \left( tA + (1 - t) B \right) \, dt$$

then, by (2.123), we deduce inequality (2.117).

Inequality (2.118) follows from (2.117) by observing that for $\langle Bx, x \rangle > \langle Ax, x \rangle$ we have

$$\int_0^1 f \left( (1 - t) \langle Ax, x \rangle + t \langle Bx, x \rangle \right) \, dt = \frac{1}{\langle Bx, x \rangle - \langle Ax, x \rangle} \int_{\langle Ax, x \rangle}^{\langle Bx, x \rangle} f(u) \, du$$

for each $x \in H$ with $\|x\| = 1$. \hfill \blacksquare

**Remark 2.44.** We observe that, from inequalities (2.116) and (2.117) we have the following inequalities in the operator order of $B(H)$:

$$\frac{1}{2} \left[ f \left( (1 - t) A + tB \right) + f \left( tA + (1 - t) B \right) \right] \leq f(m) \frac{M1_H - A + B}{M - m} + f(M) \frac{A + B - m1_H}{M - m}, \quad (2.124)$$

where $f$ is a convex function on $[m, M]$ and $A, B$ are selfadjoint operator on the Hilbert space $H$ with $Sp(A), Sp(B) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. 

The case of log-convex functions is as follows:

**Corollary 2.45 (Dragomir, 2010, [13]).** If $g$ is a log-convex function on $[m, M]$, then

$$
g \left( \frac{A + B}{2}, x \right) \leq g \left( (1 - t) \langle Ax, x \rangle + t \langle Bx, x \rangle \right) g \left( t \langle Ax, x \rangle + (1 - t) \langle Bx, x \rangle \right)
$$

$$
\leq \exp \left\{ \frac{1}{2} \left[ \ln g \left( (1 - t) A + t B \right) + \ln g \left( t A + (1 - t) B \right) \right] x, x \right\}
$$

$$
\leq g(m) \frac{M - \frac{A + B}{M - m}}{g(M) \frac{A + B}{M - m}}
$$

for any $t \in [0, 1]$ and each $x \in H$ with $\|x\| = 1$.

Moreover, we have the Hermite–Hadamard’s type inequalities:

$$
g \left( \frac{A + B}{2}, x \right) \leq \exp \left[ \int_0^1 \ln g \left( ((1 - t) A + t B) \right) dt \right] x, x \right\}
$$

$$
\leq \exp \left[ \left[ \int_0^1 \ln g \left( (1 - t) A + t B \right) \right] x, x \right\}
$$

$$
\leq g(m) \frac{M - \frac{A + B}{M - m}}{g(M) \frac{A + B}{M - m}}
$$

for each $x \in H$ with $\|x\| = 1$.

In addition, if we assume that $B - A$ is a positive definite operator, then

$$
g \left( \frac{A + B}{2}, x \right) \leq \exp \left[ \int \ln g \left( (B - A) x \right) du \right] x, x \right\}
$$

$$
\leq \exp \left[ \left[ \int_0^1 \ln g \left( (1 - t) A + t B \right) dt \right] x, x \right\}
$$

$$
\leq \left[ g(m) \frac{M - \frac{A + B}{M - m}}{g(M) \frac{A + B}{M - m}} \right]^{(B - A)x, x}
$$

for each $x \in H$ with $\|x\| = 1$. 

From a different perspective, we have the following result as well:

**Theorem 2.46 (Dragomir, 2010, [13]).** Let $A$ and $B$ selfadjoint operators on the Hilbert space $H$ and assume that $\text{Sp}(A), \text{Sp}(B) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $f$ is a convex function on $[m, M]$, then

$$f \left( \frac{(Ax, x) + (By, y)}{2} \right)$$

$$\leq \int_0^1 f \left( (1 - t) (Ax, x) + t (By, y) \right) dt$$

$$\leq \left\langle \left[ \int_0^1 f \left( (1 - t) A + t (By, y) 1_H \right) dt \right] x, x \right\rangle$$

$$\leq \frac{1}{2} \left[ f(A)x, x \right] + f \left( (By, y) \right)$$

$$\leq \frac{1}{2} \left[ f(A)x, x \right] + f \left( (B)x, y \right)$$

and

$$f \left( \frac{(Ax, x) + (By, y)}{2} \right) \leq \left\langle f \left( \frac{A + (By, y) 1_H}{2} \right) \right. x, x \right\rangle$$

$$\leq \left\langle \left[ \int_0^1 f \left( (1 - t) A + t (By, y) 1_H \right) dt \right] x, x \right\rangle$$

(2.128)

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

**Proof.** For a convex function $f$ and any $u, v \in [m, M]$ and $t \in [0, 1]$, we have the double inequality:

$$f \left( \frac{u + v}{2} \right) \leq \frac{1}{2} \left[ f \left( (1 - t) u + t v \right) + f \left( (1 - t) v \right) \right]$$

$$\leq \frac{1}{2} \left[ f \left( u \right) + f \left( v \right) \right].$$

(2.130)

Utilizing the second inequality in (2.130) we have

$$\frac{1}{2} \left[ f \left( (1 - t) u + t (By, y) \right) + f \left( (1 - t) (By, y) \right) \right]$$

$$\leq \frac{1}{2} \left[ f \left( u \right) + f \left( (By, y) \right) \right]$$

(2.131)

for any $u \in [m, M], t \in [0, 1]$ and $y \in H$ with $\|y\| = 1$. 
Now, on applying property (P) to inequality (2.131) for the operator $A$ we have
\[
\frac{1}{2} \left[ (f ((1 - t) A + t (B y, y)) x, x) + (f (t A + (1 - t) (B y, y)) x, x) \right] \\
\leq \frac{1}{2} \left[ (f(A) x, x) + f ((B y, y)) \right]
\]
for any $t \in [0, 1]$ and $x, y \in H$ with $\|x\| = \|y\| = 1$.  
On applying the Mond–Pečarić inequality we also have
\[
\frac{1}{2} \left[ f ((1 - t) (A x, x) + t (B y, y)) + f (t (A x, x) + (1 - t) (B y, y)) \right] \\
\leq \frac{1}{2} \left[ (f((1 - t) A + t (B y, y) 1_H)) x, x) + (f (t A + (1 - t) (B y, y) 1_H)) x, x) \right]
\]
for any $t \in [0, 1]$ and $x, y \in H$ with $\|x\| = \|y\| = 1$.  
Now, integrating over $t$ on $[0, 1]$ inequalities (2.132) and (2.133) and taking into account that
\[
\int_0^1 \langle f ((1 - t) A + t (B y, y) 1_H) x, x \rangle \, dt \\
= \int_0^1 \langle f (t A + (1 - t) (B y, y) 1_H) x, x \rangle \, dt \\
= \langle \left[ \int_0^1 f ((1 - t) A + t (B y, y) 1_H) \, dt \right] x, x \rangle
\]
and
\[
\int_0^1 f ((1 - t) (A x, x) + t (B y, y)) \, dt = \int_0^1 f (t (A x, x) + (1 - t) (B y, y)) \, dt.
\]
we obtain the second and the third inequality in (2.128).  
Further, on applying the Jensen integral inequality for the convex function $f$ we also have
\[
\int_0^1 f ((1 - t) (A x, x) + t (B y, y)) \, dt \\
\geq f \left( \int_0^1 [(1 - t) (A x, x) + t (B y, y)] \, dt \right) \\
= f \left( \frac{(A x, x) + (B y, y)}{2} \right)
\]
for each $x, y \in H$ with $\|x\| = \|y\| = 1$, proving the first part of (2.128).
Now, on utilizing the first part of \((2.130)\) we can also state that
\[
\frac{1}{2} \left[ f ((1-t) u + t \langle By, y \rangle) + f (tu + (1-t) \langle By, y \rangle) \right]
\] (2.134)
for any \(u \in [m, M]\), \(t \in [0, 1]\) and \(y \in H\) with \(\|y\| = 1\).

Further, on applying property (P) to inequality (2.134) and for the operator \(A\) we get
\[
\langle f \left( \frac{A + \langle By, y \rangle 1_H}{2} \right) x, x \rangle
\]
\[
\leq \frac{1}{2} \left[ \{f ((1-t) A + t \langle By, y \rangle 1_H) x, x \} + \langle f (tA + (1-t) \langle By, y \rangle 1_H) x, x \rangle \right]
\]
for each \(x, y \in H\) with \(\|x\| = \|y\| = 1\), which, by integration over \(t\) in \([0, 1]\) produces the second inequality in (2.129). The first inequality is obvious.

**Remark 2.47.** It is important to remark that, from inequalities (2.128) and (2.129) we have the following Hermite–Hadamard’s type results in the operator order of \(B (H)\) and for the convex function \(f : [m, M] \to \mathbb{R}\):
\[
f \left( \frac{A + \langle By, y \rangle 1_H}{2} \right) \leq \int_0^1 f ((1-t) A + t \langle By, y \rangle 1_H) dt
\]
\[
\leq \frac{1}{2} \left[ f(A) + f (\langle By, y \rangle 1_H) \right]
\] (2.135)
for any \(y \in H\) with \(\|y\| = 1\) and any selfadjoint operators \(A, B\) with spectra in \([m, M]\).

In particular, we have from (2.135)
\[
f \left( \frac{A + \langle Ay, y \rangle 1_H}{2} \right) \leq \int_0^1 f ((1-t) A + t \langle Ay, y \rangle 1_H) dt
\]
\[
\leq \frac{1}{2} \left[ f(A) + f (\langle Ay, y \rangle 1_H) \right]
\] (2.136)
for any \(y \in H\) with \(\|y\| = 1\) and
\[
f \left( \frac{A + s1_H}{2} \right) \leq \int_0^1 f ((1-t) A + ts1_H) dt
\]
\[
\leq \frac{1}{2} \left[ f(A) + f(s)1_H \right]
\] (2.137)
for any \(s \in [m, M]\).
As a particular case of the above theorem, we have the following refinement of the Mond–Pečarić inequality:

**Corollary 2.48 (Dragomir, 2010, [13]).** Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $f$ is a convex function on $[m, M]$, then

$$f \left( \langle Ax, x \rangle \right) \leq \left( f \left( \frac{A + \langle Ax, x \rangle 1_H}{2} \right) \right) x, x$$

$$\leq \left( \int_0^1 f \left( (1 - t) A + t \langle Ax, x \rangle 1_H \right) dt \right) x, x$$

$$\leq \frac{1}{2} \left( [f(A)x, x] + f \left( \langle Ax, x \rangle \right) \right) \leq \langle f(A)x, x \rangle . \quad (2.138)$$

Finally, the case of log-convex functions is as follows:

**Corollary 2.49 (Dragomir, 2010, [13]).** If $g$ is a log-convex function on $[m, M]$, then

$$g \left( \frac{\langle Ax, x \rangle + \langle By, y \rangle}{2} \right)$$

$$\leq \exp \left( \int_0^1 \ln g \left( (1 - t) \langle Ax, x \rangle + t \langle By, y \rangle \right) dt \right) x, x$$

$$\leq \exp \left( \int_0^1 \ln g \left( (1 - t) A + t \langle By, y \rangle 1_H \right) dt \right) x, x$$

$$\leq \exp \left[ \frac{1}{2} \left( [\ln g(A)x, x] + \ln g \left( \langle By, y \rangle \right) \right) \right]$$

$$\leq \exp \left[ \frac{1}{2} \left( [\ln g(A)x, x] + \langle \ln g(B)y, y \rangle \right) \right] \quad (2.139)$$

and

$$g \left( \frac{\langle Ax, x \rangle + \langle By, y \rangle}{2} \right) \leq \exp \left( \ln g \left( \frac{A + \langle By, y \rangle 1_H}{2} \right) \right) x, x$$

$$\leq \exp \left( \int_0^1 \ln g \left( (1 - t) A + t \langle By, y \rangle 1_H \right) dt \right) x, x$$

$$\quad (2.140)$$
and
\[
g((Ax, x)) \leq \exp \left( \ln g \left( \frac{A + \langle Ax, x \rangle 1_H}{2} \right) x, x \right)
\]
\[
\leq \exp \left( \int_0^1 \ln g \left( (1 - t) A + t \langle Ax, x \rangle 1_H \right) dt \right) x, x \]
\[
\leq \exp \left[ \frac{1}{2} \left( \ln g(A)x, x + \ln g \left( \langle Ax, x \rangle \right) \right) \right]
\]
\[
\leq \exp \left( \ln g(A)x, x \right)
\]
respectively, for each \( x \in H \) with \( \|x\| = 1 \) and \( A, B \) selfadjoint operators with spectra in \([m, M] \).

It is obvious that all the above inequalities can be applied for particular convex or log-convex functions of interest. The details are left to the interested reader.

2.8 Hermite–Hadamard’s Type Inequalities for Operator Convex Functions

2.8.1 Introduction

The following inequality holds for any convex function \( f \) defined on \( \mathbb{R} \):
\[
(b - a) f \left( \frac{a + b}{2} \right) < \int_a^b f(x)dx < (b - a) \frac{f(a) + f(b)}{2}, \quad a, b \in \mathbb{R}. \quad (2.142)
\]

It was first discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [24]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite’s result [27].

E.F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [3]. In 1974, D.S. Mitrinović found Hermite’s note in *Mathesis* [24]. Since (2.142) was known as Hadamard’s inequality, the inequality is now commonly referred as the Hermite–Hadamard inequality [27].

Let \( X \) be a vector space, \( x, y \in X, \ x \neq y. \) Define the segment
\[
[x, y] := \{(1 - t)x + ty, \ t \in [0, 1]\}.\]
We consider the function \(f : [x, y] \to \mathbb{R}\) and the associated function
\[g(x, y) : [0, 1] \to \mathbb{R}, \quad g(x, y)(t) := f[(1 - t)x + ty], \quad t \in [0, 1].\]

Note that \(f\) is convex on \([x, y]\) if and only if \(g(x, y)\) is convex on \([0, 1]\).

For any convex function defined on a segment \([x, y] \subset X\), we have the Hermite–Hadamard integral inequality (see [4, p. 2])
\[
f \left( \frac{x + y}{2} \right) \leq \int_0^1 f[(1 - t)x + ty]dt \leq \frac{f(x) + f(y)}{2},
\]
which can be derived from the classical Hermite–Hadamard inequality (2.142) for the convex function \(g(x, y) : [0, 1] \to \mathbb{R}\).

Since \(f(x) = \|x\|^p\) (\(x \in X\) and \(1 \leq p < \infty\)) is a convex function, we have the following norm inequality from (2.143) (see [26, p. 106]):
\[
\left\| \frac{x + y}{2} \right\|^p \leq \int_0^1 \| (1 - t)x + ty \|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2},
\]
for any \(x, y \in X\).

Motivated by the above results, we investigate in this section the operator version of the Hermite–Hadamard inequality for operator convex functions. The operator quasi-linearity of some associated functionals are also provided.

A real-valued continuous function \(f\) on an interval \(I\) is said to be operator convex (operator concave) if
\[
f ((1 - \lambda) A + \lambda B) \leq (\geq) (1 - \lambda) f(A) + \lambda f(B)
\]
in the operator order, for all \(\lambda \in [0, 1]\) and for every selfadjoint operator \(A\) and \(B\) on a Hilbert space \(H\) whose spectra are contained in \(I\). Notice that a function \(f\) is operator concave if \(-f\) is operator convex.

A real-valued continuous function \(f\) on an interval \(I\) is said to be operator monotone if it is monotone with respect to the operator order, i.e. \(A \leq B\) with \(Sp(A), Sp(B) \subset I\) imply \(f(A) \leq f(B)\).

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [19] and the references therein.

As examples of such functions, we note that \(f(t) = t^r\) is operator monotone on \([0, \infty)\) if and only if \(0 \leq r \leq 1\). The function \(f(t) = t^r\) is operator convex on \((0, \infty)\) if either \(1 \leq r \leq 2\) or \(-1 \leq r \leq 0\) and is operator concave on \((0, \infty)\) if \(0 \leq r \leq 1\). The logarithmic function \(f(t) = \ln t\) is operator monotone and operator concave on \((0, \infty)\). The entropy function \(f(t) = -t \ln t\) is operator concave on \((0, \infty)\). The exponential function \(f(t) = e^t\) is neither operator convex nor operator monotone.
2.8.2 Some Hermite–Hadamard’s Type Inequalities

We start with the following result:

**Theorem 2.50 (Dragomir, 2010, [12]).** Let $f : I \to \mathbb{R}$ be an operator convex function on the interval $I$. Then for any selfadjoint operators $A$ and $B$ with spectra in $I$ we have the inequality

$$
\left( f \left( \frac{A + B}{2} \right) \right) \leq \frac{1}{2} \left[ f \left( \frac{3A + B}{4} \right) + f \left( \frac{A + 3B}{4} \right) \right]
$$

$$
\leq \int_0^1 f((1-t)A + tB)dt
$$

$$
\leq \frac{1}{2} \left[ f \left( \frac{A + B}{2} \right) + \frac{f(A) + f(B)}{2} \right] \left( \leq \frac{f(A) + f(B)}{2} \right). \tag{2.145}
$$

**Proof.** First of all, since the function $f$ is continuous, the operator-valued integral $\int_0^1 f((1-t)A + tB)dt$ exists for any selfadjoint operators $A$ and $B$ with spectra in $I$.

We give here two proofs, the first using only the definition of operator convex functions and the second using the classical Hermite–Hadamard inequality for real-valued functions.

1. By the definition of operator convex functions we have the double inequality:

$$
f \left( \frac{C + D}{2} \right) \leq \frac{1}{2} \left[ f \left( (1-t)C + tD \right) + f \left( (1-t)D + tC \right) \right]
$$

$$
\leq \frac{1}{2} \left[ f(C) + f(D) \right] \tag{2.146}
$$

for any $t \in [0, 1]$ and any selfadjoint operators $C$ and $D$ with the spectra in $I$.

Integrating inequality (2.146) over $t \in [0, 1]$ and taking into account that

$$
\int_0^1 f \left( (1-t)C + tD \right) dt = \int_0^1 f \left( (1-t)D + tC \right) dt
$$

then we deduce the Hermite–Hadamard inequality for operator convex functions

$$
f \left( \frac{C + D}{2} \right) \leq \int_0^1 f \left( (1-t)C + tD \right) dt \leq \frac{1}{2} \left[ f(C) + f(D) \right] \tag{HHO}
$$

that holds for any selfadjoint operators $C$ and $D$ with the spectra in $I$. 

Now, on making use of the change of variable $u = 2t$ we have
\[
\int_0^{1/2} f((1 - t)A + tB)dt = \frac{1}{2} \int_0^{1} f \left( (1 - u) A + u \frac{A + B}{2} \right) du
\]
and by the change of variable $u = 2t - 1$ we have
\[
\int_{1/2}^{1} f((1 - t)A + tB)dt = \frac{1}{2} \int_0^{1} f \left( (1 - u) \frac{A + B}{2} + uB \right) du.
\]
Utilizing the Hermite–Hadamard inequality (HHO) we can write
\[
f \left( \frac{3A + B}{4} \right) \leq \int_0^{1} f \left( (1 - u) A + u \frac{A + B}{2} \right) du
\]
\[
\leq \frac{1}{2} \left[ f(A) + f \left( \frac{A + B}{2} \right) \right]
\]
and
\[
f \left( \frac{A + 3B}{4} \right) \leq \int_0^{1} f \left( (1 - u) \frac{A + B}{2} + uB \right) du
\]
\[
\leq \frac{1}{2} \left[ f(A) + f \left( \frac{A + B}{2} \right) \right],
\]
which by summation and division by two produces the desired result (2.145).

2. Consider now $x \in H$, $\|x\| = 1$ and two selfadjoint operators $A$ and $B$ with spectra in $I$. Define the real-valued function $\varphi_{x,A,B} : [0, 1] \rightarrow \mathbb{R}$ given by
\[
\varphi_{x,A,B}(t) = \langle f ((1 - t)A + tB)x, x \rangle.
\]
Since $f$ is operator convex, then for any $t_1, t_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we have
\[
\varphi_{x,A,B} (\alpha t_1 + \beta t_2)
\]
\[
= (f ((1 - (\alpha t_1 + \beta t_2)) A + (\alpha t_1 + \beta t_2) B) x, x)
\]
\[
= (f (\alpha [(1 - t_1) A + t_1 B] + \beta [(1 - t_2) A + t_2 B]) x, x)
\]
\[
\leq \alpha \langle f ((1 - t_1) A + t_1 B) x, x \rangle + \beta \langle f ((1 - t_2) A + t_2 B) x, x \rangle
\]
\[
= \alpha \varphi_{x,A,B} (t_1) + \beta \varphi_{x,A,B} (t_2)
\]
showing that $\varphi_{x,A,B}$ is a convex function on $[0, 1]$. 
Now we use the Hermite–Hadamard inequality for real-valued convex functions
\[
g\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b g(s) \, ds \leq \frac{g(a) + g(b)}{2}
\]
to get that
\[
\varphi_{x,A,B}\left(\frac{1}{4}\right) \leq 2 \int_0^{1/2} \varphi_{x,A,B}(t) \, dt \leq \frac{\varphi_{x,A,B}(0) + \varphi_{x,A,B}\left(\frac{1}{2}\right)}{2}
\]
and
\[
\varphi_{x,A,B}\left(\frac{3}{4}\right) \leq 2 \int_{1/2}^{1} \varphi_{x,A,B}(t) \, dt \leq \frac{\varphi_{x,A,B}\left(\frac{1}{2}\right) + \varphi_{x,A,B}(1)}{2}
\]
which by summation and division by two produces
\[
\frac{1}{2} \left[ \left\{ f\left(\frac{3A + B}{4}\right) + f\left(\frac{A + 3B}{4}\right) \right\} x, x \right] \leq \int_0^{1} \langle f((1 - t)A + tB)x, x \rangle \, dt \leq \frac{1}{2} \left[ \left\{ f\left(\frac{A + B}{2}\right) + \frac{f(A) + f(B)}{2} \right\} x, x \right].
\]
Finally, since by the continuity of the function \( f \) we have
\[
\int_0^{1} \langle f((1 - t)A + tB)x, x \rangle \, dt = \left( \int_0^{1} f((1 - t)A + tB) dt \right) x, x
\]
for any \( x \in H, \|x\| = 1 \) and any two selfadjoint operators \( A \) and \( B \) with spectra in \( I \), we deduce from (2.147) the desired result (2.145).

A simple consequence of the above theorem is that the integral is closer to the left bound than to the right, namely we can state:

**Corollary 2.51 (Dragomir, 2010, [12]).** With the assumptions in Theorem 2.50 we have the inequality
\[
(0 \leq) \int_0^{1} f((1 - t)A + tB) dt - f\left(\frac{A + B}{2}\right) \leq \frac{f(A) + f(B)}{2} - \int_0^{1} f((1 - t)A + tB) dt.
\]

**Remark 2.52.** Utilizing different examples of operator convex or concave functions, we can provide inequalities of interest.
If $r \in [-1, 0] \cup [1, 2]$ then we have the inequalities for powers of operators
\[
\left(\frac{A + B}{2}\right)^r \leq \frac{1}{2} \left[\left(\frac{3A + B}{4}\right)^r + \left(\frac{A + 3B}{4}\right)^r\right]
\]
\[
\leq \int_0^1 ((1 - t)A + tB)^r dt
\]
\[
\leq \frac{1}{2} \left[\left(\frac{A + B}{2}\right)^r + \frac{A' + B'}{2}\right] \left(\leq \frac{A' + B'}{2}\right)
\]
for any two selfadjoint operators $A$ and $B$ with spectra in $(0, \infty)$.

If $r \in (0, 1)$ the inequalities in (2.149) hold with “$\geq$” instead of “$\leq$”.

We also have the following inequalities for logarithm:
\[
\left(\ln\left(\frac{A + B}{2}\right)^r \geq \frac{1}{2} \left[\ln\left(\frac{3A + B}{4}\right) + \ln\left(\frac{A + 3B}{4}\right)\right]\right]
\]
\[
\geq \int_0^1 \ln((1 - t)A + tB) dt
\]
\[
\geq \frac{1}{2} \left[\ln\left(\frac{A + B}{2}\right) + \ln(A) + \ln(B)\right] \left(\geq \frac{\ln(A) + \ln(B)}{2}\right)
\]
for any two selfadjoint operators $A$ and $B$ with spectra in $(0, \infty)$.

### 2.8.3 Some Operator Quasi-linearity Properties

Consider an operator convex function $f : I \subset \mathbb{R} \to \mathbb{R}$ defined on the interval $I$ and two distinct selfadjoint operators $A, B$ with the spectra in $I$. We denote by $[A, B]$ the closed operator segment defined by the family of operators $(1 - t)A + tB$, $t \in [0, 1]$. We also define the operator-valued functional
\[
\Delta_f (A, B; t) := (1 - t)f(A) + tf(B) - f((1 - t)A + tB) \geq 0
\]
in the operator order, for any $t \in [0, 1]$.

The following result concerning an operator quasi-linearity property for the functional $\Delta_f (\cdot, \cdot; t)$ may be stated:

**Theorem 2.53 (Dragomir, 2010, [12]).** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an operator convex function on the interval $I$. Then for each $A, B$ two distinct selfadjoint operators with the spectra in $I$ and $C \in [A, B]$ we have
\[
(0 \leq) \Delta_f (A, C; t) + \Delta_f (C, B; t) \leq \Delta_f (A, B; t)
\]
for each $t \in [0, 1]$, i.e. the functional $\Delta_f (\cdot, \cdot; t)$ is operator super-additive as a function of interval.

If $[C, D] \subset [A, B]$, then

$$
(0 \leq) \Delta_f (C, D; t) \leq \Delta_f (A, B; t)
$$

(2.153)

for each $t \in [0, 1]$, i.e. the functional $\Delta_f (\cdot, \cdot; t)$ is operator non-decreasing as a function of interval.

**Proof.** Let $C = (1 - s)A + sB$ with $s \in (0, 1)$. For $t \in (0, 1)$ we have

$$
\Delta_f (C, B; t) = (1 - t) f ((1 - s)A + sB) + t f (B) - f ((1 - t) [(1 - s)A + sB] + tB)
$$

and

$$
\Delta_f (A, C; t) = (1 - t) f (A) + t f ((1 - s)A + sB) - f ((1 - t)A + t [((1 - s)A + sB])
$$

giving that

$$
\Delta_f (A, C; t) + \Delta_f (C, B; t) - \Delta_f (A, B; t)
$$

$$
= f ((1 - s)A + sB) + f ((1 - t)A + tB) - f ((1 - t) (1 - s)A + [(1 - t)s + t] B) - f ((1 - ts) A + tsB). 
$$

(2.154)

Now, for a convex function $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, where $I$ is an interval, and any real numbers $t_1, t_2, s_1$ and $s_2$ from $I$ and with the properties that $t_1 \leq s_1$ and $t_2 \leq s_2$ we have that

$$
\varphi (t_1) - \varphi (t_2) \leq \varphi (s_1) - \varphi (s_2). 
$$

(2.155)

Indeed, since $\varphi$ is convex on $I$ then for any $a \in I$ the function $\psi : I \setminus \{a\} \to \mathbb{R}$

$$
\psi (t) := \frac{\varphi (t) - \varphi (a)}{t - a}
$$

is monotonic non-decreasing where is defined. Utilizing this property repeatedly we have

$$
\frac{\varphi (t_1) - \varphi (t_2)}{t_1 - t_2} \leq \frac{\varphi (s_1) - \varphi (t_2)}{s_1 - t_2} = \frac{\varphi (t_2) - \varphi (s_1)}{t_2 - s_1}
$$

$$
\leq \frac{\varphi (s_2) - \varphi (s_1)}{s_2 - s_1} = \frac{\varphi (s_1) - \varphi (s_2)}{s_1 - s_2},
$$

which proves inequality (2.155).
For a vector $x \in H$, with $\|x\| = 1$, consider the function $\varphi_x : [0, 1] \to \mathbb{R}$ given by $\varphi_x(t) := \langle f((1-t)A + tB) x, x \rangle$. Since $f$ is operator convex on $I$ it follows that $\varphi_x$ is convex on $[0, 1]$. Now, if we consider, for given $t, s \in (0, 1)$,

$$t_1 := ts < s =: s_1 \text{ and } t_2 := t + (1-t)s =: s_2,$$

then we have

$$\varphi_x(t_1) = \langle f((1-ts)A + tsB) x, x \rangle$$

and

$$\varphi_x(t_2) = \langle f((1-t)A + tB)x, x \rangle$$

giving that

$$\varphi_x(t_1) - \varphi_x(t_2) = \left\langle \left[ f((1-ts)A + tsB) - f((1-t)A + tB) \right] \frac{t}{(s-1)} x, x \right\rangle.$$

Also

$$\varphi_x(s_1) = \langle f((1-s)A + sB) x, x \rangle$$

and

$$\varphi_x(s_2) = \langle f((1-t)(1-s)A + [(1-t)s + t]B) x, x \rangle$$

giving that

$$\varphi_x(s_1) - \varphi_x(s_2) = \left\langle \left[ f((1-s)A + sB) - f((1-t)(1-s)A + [(1-t)s + t]B) \right] \frac{t}{(s-1)} x, x \right\rangle.$$

Utilizing inequality (2.155) and multiplying with $t(s-1) < 0$, we deduce the following inequality in the operator order:

$$f((1-ts)A + tsB) - f((1-t)A + tB) \geq f((1-s)A + sB) - f((1-t)(1-s)A + [(1-t)s + t]B). \quad (2.156)$$

Finally, by (2.154) and (2.156) we get the desired result (2.152).

Applying repeatedly the superadditivity property we have for $[C, D] \subset [A, B]$ that

$$\Delta_f(A, C; t) + \Delta_f(C, D; t) + \Delta_f(D, B; t) \leq \Delta_f(A, B; t)$$

giving that

$$0 \leq \Delta_f(A, C; t) + \Delta_f(D, B; t) \leq \Delta_f(A, B; t) - \Delta_f(C, D; t)$$

which proves (2.153).
For $t = \frac{1}{2}$ we consider the functional

$$\Delta_f (A, B) := \Delta_f \left( A, B; \frac{1}{2} \right) = \frac{f(A) + f(B)}{2} - f \left( \frac{A + B}{2} \right),$$

which obviously inherits the superadditivity and monotonicity properties of the functional $\Delta_f (\cdot, \cdot; t)$. We are able then to state the following:

**Corollary 2.54 (Dragomir, 2010, [12]).** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an operator convex function on the interval $I$. Then for each $A, B$ two distinct selfadjoint operators with the spectra in $I$ we have the following bounds in the operator order:

$$\inf_{C \in [A, B]} \left[ f \left( \frac{A + C}{2} \right) + f \left( \frac{C + B}{2} \right) - f(C) \right] = f \left( \frac{A + B}{2} \right) \quad (2.157)$$

and

$$\sup_{C, D \in [A, B]} \left[ \frac{f(C) + f(D)}{2} - f \left( \frac{C + D}{2} \right) \right] = \frac{f(A) + f(B)}{2} - f \left( \frac{A + B}{2} \right). \quad (2.158)$$

**Proof.** By the superadditivity of the functional $\Delta_f (\cdot, \cdot)$ we have for each $C \in [A, B]$ that

$$\frac{f(A) + f(B)}{2} - f \left( \frac{A + B}{2} \right) \geq \frac{f(A) + f(C)}{2} - f \left( \frac{A + C}{2} \right) + \frac{f(C) + f(B)}{2} - f \left( \frac{C + B}{2} \right)$$

which is equivalent with

$$f \left( \frac{A + C}{2} \right) + f \left( \frac{C + B}{2} \right) - f(C) \geq f \left( \frac{A + B}{2} \right). \quad (2.159)$$

Since the equality case in (2.159) is realized for either $C = A$ or $C = B$ we get the desired bound (2.157).

The bound (2.158) is obvious by the monotonicity of the functional $\Delta_f (\cdot, \cdot)$ as a function of interval.

Consider now the following functional:

$$\Gamma_f (A, B; t) := f(A) + f(B) - f((1 - t)A + tB) - f ((1 - t)B + tA),$$

where, as above, \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) is a convex function on the convex set \( I \) and \( A, B \) two distinct selfadjoint operators with the spectra in \( I \) while \( t \in [0, 1] \).

We notice that
\[
\Gamma_f (A, B; t) = \Gamma_f (B, A; t) = \Gamma_f (A, B; 1 - t)
\]
and
\[
\Gamma_f (A, B; t) = \Delta_f (A, B; t) + \Delta_f (A, B; 1 - t) \geq 0
\]
for any \( A, B \) and \( t \in [0, 1] \).

Therefore, we can state the following result as well:

**Corollary 2.55 (Dragomir, 2010, [12]).** Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be an operator convex function on the interval \( I \). Then for each \( A, B \) two distinct selfadjoint operators with the spectra in \( I \), the functional \( \Gamma_f (\cdot, \cdot; t) \) is operator superadditive and operator non-decreasing as a function of interval.

In particular, if \( C \in [A, B] \) then we have the inequality
\[
\begin{align*}
\frac{1}{2} \left[ f((1 - t)A + tB) + f((1 - t)B + tA) \right] \\
\leq \frac{1}{2} \left[ f((1 - t)A + tC) + f((1 - t)C + tA) \right] \\
+ \frac{1}{2} \left[ f((1 - t)C + tB) + f((1 - t)B + tC) \right] - f(C).
\end{align*}
\] (2.160)

Also, if \( C, D \in [A, B] \) then we have the inequality
\[
\begin{align*}
f(A) + f(B) - f((1 - t)A + tB) - f((1 - t)B + tA) \\
\geq f(C) + f(D) - f((1 - t)C + tD) - f((1 - t)C + tD)
\end{align*}
\] (2.161)
for any \( t \in [0, 1] \).

Perhaps the most interesting functional we can consider is the following one:
\[
\Theta_f (A, B) = \frac{f(A) + f(B)}{2} - \int_0^1 f((1 - t)A + tB) dt.
\] (2.162)

Notice that, by the second Hermite–Hadamard inequality for operator convex functions we have that \( \Theta_f (A, B) \geq 0 \) in the operator order.

We also observe that
\[
\Theta_f (A, B) = \int_0^1 \Delta_f (A, B; t) dt = \int_0^1 \Delta_f (A, B; 1 - t) dt.
\] (2.163)
Utilizing this representation, we can state the following result as well:

**Corollary 2.56 (Dragomir, 2010, [12]).** Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be an operator convex function on the interval \( I \). Then for each \( A, B \) two distinct selfadjoint operators with the spectra in \( I \), the functional \( \Theta_f \) \((\cdot,\cdot)\) is operator superadditive and operator non-decreasing as a function of interval. Moreover, we have the bounds in the operator order

\[
\inf_{C \in [A,B]} \left[ \int_0^1 \left[ f ((1 - t)A + tC) + f ((1 - t)B + tD) \right] P dt - f(C) \right] = \int_0^1 f((1 - t)A + tB) dt \tag{2.164}
\]

and

\[
\sup_{C,D \in [A,B]} \left[ \frac{f(C) + f(D)}{2} - \int_0^1 f((1 - t)C + tD) dt \right] = \frac{f(A) + f(B)}{2} - \int_0^1 f((1 - t)A + tB) dt. \tag{2.165}
\]

**Remark 2.57.** The above inequalities can be applied to various concrete operator convex function of interest.

If we choose for instance inequality (2.165), then we get the following bounds in the operator order:

\[
\sup_{C,D \in [A,B]} \left[ \frac{C^r + D^r}{2} - \int_0^1 ((1 - t)C + tD)^r dt \right] = \frac{A^r + B^r}{2} - \int_0^1 ((1 - t)A + tB)^r dt, \tag{2.166}
\]

where \( r \in [-1,0] \cup [1,2] \) and \( A, B \) are selfadjoint operators with spectra in \((0, \infty)\).

If \( r \in (0,1) \) then

\[
\sup_{C,D \in [A,B]} \left[ \int_0^1 ((1 - t)C + tD)^r dt - \frac{C^r + D^r}{2} \right] = \int_0^1 ((1 - t)A + tB)^r dt - \frac{A^r + B^r}{2} \tag{2.167}
\]

and \( A, B \) are selfadjoint operators with spectra in \((0, \infty)\).
We also have the operator bound for the logarithm

\[
\sup_{C, D \in [A, B]} \left[ \int_0^1 \ln((1 - t)C + tD) \, dt - \frac{\ln(C) + \ln(D)}{2} \right]
= \int_0^1 \ln((1 - t)A + tB) \, dt - \frac{\ln(A) + \ln(B)}{2},
\]

(2.168)

where \(A, B\) are selfadjoint operators with spectra in \((0, \infty)\).

References

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