

Contents

1	Introduction	1
1.1	Mathematics for Computer Graphics	1
1.2	Understanding Mathematics	1
1.3	What Makes Mathematics Difficult?	2
1.4	Does Mathematics Exist Outside Our Brains?	2
1.5	Symbols and Notation	3
2	Numbers	5
2.1	Introduction	5
2.2	Background	5
2.3	Counting	5
2.4	Sets of Numbers	6
2.5	Zero	7
2.6	Negative Numbers	8
2.6.1	The Arithmetic of Positive and Negative Numbers	9
2.7	Observations and Axioms	10
2.7.1	Commutative Law	10
2.7.2	Associative Law	10
2.7.3	Distributive Law	11
2.8	The Base of a Number System	11
2.8.1	Background	11
2.8.2	Octal Numbers	12
2.8.3	Binary Numbers	13
2.8.4	Hexadecimal Numbers	14
2.8.5	Adding Binary Numbers	17
2.8.6	Subtracting Binary Numbers	18
2.9	Types of Numbers	19
2.9.1	Natural Numbers	19
2.9.2	Integers	19
2.9.3	Rational Numbers	20

2.9.4	Irrational Numbers	20
2.9.5	Real Numbers	20
2.9.6	Algebraic and Transcendental Numbers	20
2.9.7	Imaginary Numbers.	21
2.9.8	Complex Numbers	25
2.9.9	Transcendental and Algebraic Numbers	27
2.9.10	Infinity	27
2.10	Summary	28
2.11	Worked Examples	29
2.11.1	Algebraic Expansion	29
2.11.2	Binary Subtraction	29
2.11.3	Complex Numbers	29
2.11.4	Complex Rotation.	30
3	Algebra	31
3.1	Introduction	31
3.2	Background	32
3.2.1	Solving the Roots of a Quadratic Equation.	33
3.3	Indices	37
3.3.1	Laws of Indices.	37
3.4	Logarithms	38
3.5	Further Notation.	40
3.6	Functions	40
3.6.1	Explicit and Implicit Equations.	40
3.6.2	Function Notation	41
3.6.3	Intervals	42
3.6.4	Function Domains and Ranges	43
3.6.5	Odd and Even Functions.	44
3.6.6	Power Functions	45
3.7	Summary	46
3.8	Worked Examples	46
3.8.1	Algebraic Manipulation.	46
3.8.2	Solving a Quadratic Equation	47
3.8.3	Factorising	49
4	Trigonometry	51
4.1	Introduction	51
4.2	Background	51
4.3	Units of Angular Measurement	51
4.4	The Trigonometric Ratios.	52
4.4.1	Domains and Ranges	55
4.5	Inverse Trigonometric Ratios	55
4.6	Trigonometric Identities	57
4.7	The Sine Rule	58
4.8	The Cosine Rule	58

4.9	Compound-Angle Identities	59
4.9.1	Double-Angle Identities	60
4.9.2	Multiple-Angle Identities.	61
4.9.3	Half-Angle Identities.	62
4.10	Perimeter Relationships	62
4.11	Summary	63
5	Coordinate Systems	65
5.1	Introduction	65
5.2	Background	65
5.3	The Cartesian Plane	66
5.4	Function Graphs.	66
5.5	Shape Representation	67
5.5.1	2D Polygons	67
5.5.2	Area of a Shape	68
5.6	Theorem of Pythagoras in 2D	69
5.7	3D Cartesian Coordinates.	69
5.7.1	Theorem of Pythagoras in 3D.	70
5.8	Polar Coordinates.	70
5.9	Spherical Polar Coordinates	71
5.10	Cylindrical Coordinates	72
5.11	Summary	73
5.12	Worked Examples	73
5.12.1	Area of a Shape	73
5.12.2	Distance Between Two Points.	74
5.12.3	Polar Coordinates	74
5.12.4	Spherical Polar Coordinates	75
5.12.5	Cylindrical Coordinates.	75
6	Determinants	77
6.1	Introduction	77
6.2	Linear Equations with Two Variables	78
6.3	Linear Equations with Three Variables.	81
6.3.1	Sarrus’s Rule.	87
6.4	Mathematical Notation	88
6.4.1	Matrix.	88
6.4.2	Order of a Determinant	89
6.4.3	Value of a Determinant.	89
6.4.4	Properties of Determinants	90
6.5	Summary	91
6.6	Worked Examples	91
6.6.1	Determinant Expansion	91
6.6.2	Complex Determinant	92
6.6.3	Simple Expansion	92
6.6.4	Simultaneous Equations	93

7	Vectors	95
7.1	Introduction	95
7.2	Background	95
7.3	2D Vectors	96
7.3.1	Vector Notation	96
7.3.2	Graphical Representation of Vectors	97
7.3.3	Magnitude of a Vector	98
7.4	3D Vectors	99
7.4.1	Vector Manipulation	100
7.4.2	Scaling a Vector	100
7.4.3	Vector Addition and Subtraction	101
7.4.4	Position Vectors	102
7.4.5	Unit Vectors	102
7.4.6	Cartesian Vectors	103
7.4.7	Products	103
7.4.8	Scalar Product	104
7.4.9	The Dot Product in Lighting Calculations	105
7.4.10	The Scalar Product in Back-Face Detection	106
7.4.11	The Vector Product	107
7.4.12	The Right-Hand Rule	112
7.5	Deriving a Unit Normal Vector for a Triangle	112
7.6	Surface Areas	113
7.6.1	Calculating 2D Areas	114
7.7	Summary	115
7.8	Worked Examples	115
7.8.1	Position Vector	115
7.8.2	Unit Vector	115
7.8.3	Vector Magnitude	116
7.8.4	Angle Between Two Vectors	116
7.8.5	Vector Product	116
8	Matrix Algebra	119
8.1	Introduction	119
8.2	Background	119
8.3	Matrix Notation	122
8.3.1	Matrix Dimension or Order	122
8.3.2	Square Matrix	123
8.3.3	Column Vector	123
8.3.4	Row Vector	123
8.3.5	Null Matrix	123
8.3.6	Unit Matrix	124
8.3.7	Trace	124
8.3.8	Determinant of a Matrix	125
8.3.9	Transpose	125
8.3.10	Symmetric Matrix	127
8.3.11	Antisymmetric Matrix	128

8.4	Matrix Addition and Subtraction	130
8.4.1	Scalar Multiplication	130
8.5	Matrix Products	131
8.5.1	Row and Column Vectors	131
8.5.2	Row Vector and a Matrix	132
8.5.3	Matrix and a Column Vector	133
8.5.4	Square Matrices	133
8.5.5	Rectangular Matrices	134
8.6	Inverse Matrix	135
8.6.1	Inverting a Pair of Matrices	141
8.7	Orthogonal Matrix	142
8.8	Diagonal Matrix	143
8.9	Summary	143
8.10	Worked Examples	144
8.10.1	Matrix Inversion	144
8.10.2	Identity Matrix	144
8.10.3	Solving Two Equations Using Matrices	145
8.10.4	Solving Three Equations Using Matrices	146
8.10.5	Solving Two Complex Equations	147
8.10.6	Solving Three Complex Equations	147
8.10.7	Solving Two Complex Equations	148
8.10.8	Solving Three Complex Equations	149
9	Geometric Transforms	153
9.1	Introduction	153
9.2	Background	153
9.3	2D Transforms	154
9.3.1	Translation	154
9.3.2	Scaling	154
9.3.3	Reflection	155
9.4	Transforms as Matrices	156
9.4.1	Systems of Notation	156
9.5	Homogeneous Coordinates	156
9.5.1	2D Translation	158
9.5.2	2D Scaling	158
9.5.3	2D Reflections	159
9.5.4	2D Shearing	161
9.5.5	2D Rotation	162
9.5.6	2D Scaling	164
9.5.7	2D Reflection	165
9.5.8	2D Rotation About an Arbitrary Point	166
9.6	3D Transforms	167
9.6.1	3D Translation	167
9.6.2	3D Scaling	167
9.6.3	3D Rotation	168

- 9.6.4 Gimbal Lock 172
- 9.6.5 Rotating About an Axis 173
- 9.6.6 3D Reflections 174
- 9.7 Change of Axes 174
 - 9.7.1 2D Change of Axes 174
 - 9.7.2 Direction Cosines 176
 - 9.7.3 3D Change of Axes 177
- 9.8 Positioning the Virtual Camera 177
 - 9.8.1 Direction Cosines 178
 - 9.8.2 Euler Angles 181
- 9.9 Rotating a Point About an Arbitrary Axis 183
 - 9.9.1 Matrices 183
 - 9.9.2 Quaternions 190
 - 9.9.3 Adding and Subtracting Quaternions 191
 - 9.9.4 Multiplying Quaternions 192
 - 9.9.5 Pure Quaternion 192
 - 9.9.6 The Inverse Quaternion 193
 - 9.9.7 Unit Quaternion 193
 - 9.9.8 Rotating Points About an Axis 193
 - 9.9.9 Roll, Pitch and Yaw Quaternions 197
 - 9.9.10 Quaternions in Matrix Form 199
 - 9.9.11 Frames of Reference 200
- 9.10 Transforming Vectors 201
- 9.11 Determinants 202
- 9.12 Perspective Projection 204
- 9.13 Summary 206
- 9.14 Worked Examples 206
 - 9.14.1 2D Scaling Transform 206
 - 9.14.2 2D Scale and Translate 206
 - 9.14.3 3D Scaling Transform 207
 - 9.14.4 2D Rotation 208
 - 9.14.5 2D Rotation About a Point 209
 - 9.14.6 Determinant of the Rotate Transform 209
 - 9.14.7 Determinant of the Shear Transform 209
 - 9.14.8 Yaw, Pitch and Roll Transforms 210
 - 9.14.9 3D Rotation About an Axis 210
 - 9.14.10 3D Rotation Transform Matrix 211
 - 9.14.11 2D Change of Axes 211
 - 9.14.12 3D Change of Axes 212
 - 9.14.13 Rotate a Point About an Axis 213
 - 9.14.14 Perspective Projection 214
- 10 Interpolation 217**
 - 10.1 Introduction 217
 - 10.2 Background 217

10.3	Linear Interpolation	218
10.4	Non-linear Interpolation	220
	10.4.1 Trigonometric Interpolation	220
	10.4.2 Cubic Interpolation	221
10.5	Interpolating Vectors	227
10.6	Interpolating Quaternions	230
10.7	Summary	232
11	Curves and Patches	233
11.1	Introduction	233
11.2	Background	233
11.3	The Circle	234
11.4	The Ellipse	234
11.5	Bézier Curves	235
	11.5.1 Bernstein Polynomials	235
	11.5.2 Quadratic Bézier Curves	238
	11.5.3 Cubic Bernstein Polynomials	239
11.6	A Recursive Bézier Formula	242
11.7	Bézier Curves Using Matrices	243
	11.7.1 Linear Interpolation	244
11.8	B-Splines	247
	11.8.1 Uniform B-Splines	247
	11.8.2 Continuity	250
	11.8.3 Non-uniform B-Splines	251
	11.8.4 Non-uniform Rational B-Splines	251
11.9	Surface Patches	251
	11.9.1 Planar Surface Patch	251
	11.9.2 Quadratic Bézier Surface Patch	253
	11.9.3 Cubic Bézier Surface Patch	255
11.10	Summary	257
12	Analytic Geometry	259
12.1	Introduction	259
12.2	Background	259
	12.2.1 Angles	260
	12.2.2 Intercept Theorems	260
	12.2.3 Golden Section	261
	12.2.4 Triangles	261
	12.2.5 Centre of Gravity of a Triangle	262
	12.2.6 Isosceles Triangle	262
	12.2.7 Equilateral Triangle	263
	12.2.8 Right Triangle	263
	12.2.9 Theorem of Thales	263
	12.2.10 Theorem of Pythagoras	264
	12.2.11 Quadrilateral	265

12.2.12	Trapezoid	265
12.2.13	Parallelogram	265
12.2.14	Rhombus	266
12.2.15	Regular Polygon	266
12.2.16	Circle	267
12.3	2D Analytic Geometry	269
12.3.1	Equation of a Straight Line	269
12.3.2	The Hessian Normal Form	270
12.3.3	Space Partitioning	272
12.3.4	The Hessian Normal Form from Two Points	272
12.4	Intersection Points	273
12.4.1	Intersecting Straight Lines	273
12.4.2	Intersecting Line Segments	274
12.5	Point Inside a Triangle	276
12.5.1	Area of a Triangle	276
12.5.2	Hessian Normal Form	278
12.6	Intersection of a Circle with a Straight Line	280
12.7	3D Geometry	282
12.7.1	Equation of a Straight Line	282
12.7.2	Intersecting Two Straight Lines	283
12.8	Equation of a Plane	286
12.8.1	Cartesian Form of the Plane Equation	286
12.8.2	General Form of the Plane Equation	289
12.8.3	Parametric Form of the Plane Equation	289
12.8.4	Converting from the Parametric to the General Form	291
12.8.5	Plane Equation from Three Points	293
12.9	Intersecting Planes	295
12.9.1	Intersection of Three Planes	298
12.9.2	Angle Between Two Planes	301
12.9.3	Angle Between a Line and a Plane	302
12.9.4	Intersection of a Line with a Plane	304
12.10	Summary	306
13	Barycentric Coordinates	307
13.1	Introduction	307
13.2	Background	307
13.3	Ceva's Theorem	308
13.4	Ratios and Proportion	309
13.5	Mass Points	310
13.6	Linear Interpolation	316
13.7	Convex Hull Property	323
13.8	Areas	324
13.9	Volumes	333

13.10	Bézier Curves and Patches	335
13.11	Summary	336
14	Geometric Algebra	337
14.1	Introduction	337
14.2	Background	337
14.3	Symmetric and Antisymmetric Functions	338
14.4	Trigonometric Foundations	339
14.5	Vectorial Foundations	341
14.6	Inner and Outer Products	341
14.7	The Geometric Product in 2D	343
14.8	The Geometric Product in 3D	345
14.9	The Outer Product of Three 3D Vectors	347
14.10	Axioms	348
14.11	Notation	349
14.12	Grades, Pseudoscalars and Multivectors	349
14.13	Redefining the Inner and Outer Products	351
14.14	The Inverse of a Vector	351
14.15	The Imaginary Properties of the Outer Product	353
14.16	Duality	355
14.17	The Relationship Between the Vector Product and the Outer Product	356
14.18	The Relationship Between Quaternions and Bivectors	357
14.19	Reflections and Rotations	358
	14.19.1 2D Reflections	358
	14.19.2 3D Reflections	359
	14.19.3 2D Rotations	360
14.20	Rotors	362
14.21	Worked Examples	365
	14.21.1 The Sine Rule	365
	14.21.2 The Cosine Rule	366
	14.21.3 A Point Perpendicular to a Line	367
	14.21.4 Reflecting a Vector About a Vector	369
	14.21.5 A Point Above or Below a Plane	370
14.22	Summary	372
15	Calculus: Derivatives	373
15.1	Introduction	373
15.2	Background	373
15.3	Small Numerical Quantities	373
15.4	Equations and Limits	375
	15.4.1 Quadratic Function	375
	15.4.2 Cubic Equation	376
	15.4.3 Functions and Limits	378
	15.4.4 Graphical Interpretation of the Derivative	380

15.4.5	Derivatives and Differentials	381
15.4.6	Integration and Antiderivatives	382
15.5	Function Types	384
15.6	Differentiating Groups of Functions	385
15.6.1	Sums of Functions	385
15.6.2	Function of a Function	387
15.6.3	Function Products	391
15.6.4	Function Quotients	395
15.7	Differentiating Implicit Functions	397
15.8	Differentiating Exponential and Logarithmic Functions	400
15.8.1	Exponential Functions	400
15.8.2	Logarithmic Functions	403
15.9	Differentiating Trigonometric Functions	404
15.9.1	Differentiating tan	404
15.9.2	Differentiating csc	406
15.9.3	Differentiating sec	406
15.9.4	Differentiating cot	407
15.9.5	Differentiating arcsin, arccos and arctan	408
15.9.6	Differentiating arcsc, arcsec and arccot	409
15.10	Differentiating Hyperbolic Functions	410
15.10.1	Differentiating sinh, cosh and tanh	412
15.11	Higher Derivatives	413
15.12	Higher Derivatives of a Polynomial	413
15.13	Identifying a Local Maximum or Minimum	416
15.14	Partial Derivatives	418
15.14.1	Visualising Partial Derivatives	422
15.14.2	Mixed Partial Derivatives	423
15.15	Chain Rule	426
15.16	Total Derivative	428
15.17	Summary	429
16	Calculus: Integration	431
16.1	Introduction	431
16.2	Indefinite Integral	431
16.3	Integration Techniques	432
16.3.1	Continuous Functions	432
16.3.2	Difficult Functions	432
16.3.3	Trigonometric Identities	434
16.3.4	Exponent Notation	436
16.3.5	Completing the Square	437
16.3.6	The Integrand Contains a Derivative	439
16.3.7	Converting the Integrand into a Series of Fractions	440
16.3.8	Integration by Parts	441

16.3.9 Integration by Substitution 446

16.3.10 Partial Fractions 450

16.4 Area Under a Graph. 453

16.5 Calculating Areas. 453

16.6 Positive and Negative Areas. 462

16.7 Area Between Two Functions 463

16.8 Areas with the y-Axis 465

16.9 Area with Parametric Functions 466

16.10 The Riemann Sum 468

16.11 Summary 470

17 Worked Examples. 471

17.1 Introduction 471

17.2 Area of Regular Polygon 471

17.3 Area of Any Polygon 472

17.4 Dihedral Angle of a Dodecahedron 473

17.5 Vector Normal to a Triangle 474

17.6 Area of a Triangle Using Vectors 475

17.7 General Form of the Line Equation from Two Points 475

17.8 Angle Between Two Straight Lines 476

17.9 Test if Three Points Lie on a Straight Line 477

17.10 Position and Distance of the Nearest Point on a Line
to a Point 478

17.11 Position of a Point Reflected in a Line. 480

17.12 Intersection of a Line and a Sphere 483

17.13 Sphere Touching a Plane 487

17.14 Summary 489

18 Conclusion. 491

Appendix A: Limit of $(\sin\theta)/\theta$ 493

Appendix B: Integrating $\cos^n \theta$ 497

Index 499



<http://www.springer.com/978-1-4471-7334-2>

Mathematics for Computer Graphics

Vince, J.

2017, XIX, 505 p. 292 illus. in color., Softcover

ISBN: 978-1-4471-7334-2