Chapter 2
Numbers

2.1 Introduction

This chapter revises some basic ideas about counting and number systems, and how they are employed in the context of mathematics for computer graphics.

2.2 Background

Over the centuries, mathematicians have realised that in order to progress, they must give precise definitions to their discoveries, ideas and concepts, so that they can be built upon and referenced by new mathematical inventions. In the event of any new discovery, these definitions have to be occasionally changed or extended. For example, once upon a time, integers, rational and irrational numbers, satisfied all the needs of mathematicians, until imaginary quantities were invented. Today, complex numbers have helped shape the current number system hierarchy. Consequently, there must be clear definitions for numbers, and the operators that act upon them. Therefore, we need to identify the types of numbers that exist, what they are used for, and any problems that arise when they are stored in a computer.

2.3 Counting

Our brain's visual cortex possesses some incredible image processing features. For example, children know instinctively when they are given less sweets than another child, and adults know instinctively when they are short-changed by a Parisian taxi driver, or driven around the Arc de Triumph several times, on the way to the airport! Intuitively, we can assess how many donkeys are in a field without counting them, and generally, we seem to know within a second or two, whether there are just a few,

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dozens, or hundreds of something. But when accuracy is required, one can't beat counting. But what is counting?

Well normally, we are taught to count by our parents by memorising first, the counting words 'one, two, three, four, five, six, seven, eight, nine, ten, ...' and second, associating them with our fingers, so that when asked to count the number of donkeys in a picture book, each donkey is associated with a counting word. When each donkey has been identified, the number of donkeys equals the last word mentioned. However, this still assumes that we know the meaning of 'one, two, three, four, ...' etc. Memorising these counting words is only part of the problem – getting them in the correct sequence is the real challenge. The incorrect sequence 'one, two, five, three, nine, four, ...' etc., introduces an element of randomness into any calculation, but practice makes perfect, and it's useful to master the correct sequence before going to university!

2.4 Sets of Numbers

A *set* is a collection of arbitrary objects called its *elements* or *members*. For example, each system of number belongs to a set with given a name, such as $\mathbb{N}$ for the natural numbers, $\mathbb{R}$ for real numbers, and $\mathbb{Q}$ for rational numbers. When we want to indicate that something is whole, real or rational, etc., we use the notation:

$$n \in \mathbb{N}$$

which reads 'n is a member of ($\in$) the set $\mathbb{N}$', i.e. n is a whole number. Similarly:

$$x \in \mathbb{R}$$

stands for 'x is a real number.'

A *well-ordered set* possesses a unique order, such as the natural numbers $\mathbb{N}$. Therefore, if P is the well-ordered set of prime numbers and $\mathbb{N}$ is the well-ordered set of natural numbers, we can write:

$$P = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \dots\}$$
$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, \dots\}.$$

By pairing the prime numbers in P with the numbers in $\mathbb{N}$, we have:

$$\{2, 1\}, \{3, 2\}, \{5, 3\}, \{7, 4\}, \{11, 5\}, \{13, 6\}, \{17, 7\}, \{19, 8\}, \{23, 9\}, \dots$$

and we can reason that 2 is the 1$^{\text{st}}$ prime, and 3 is the 2$^{\text{nd}}$ prime, etc. However, we still have to declare what we mean by $1, 2, 3, 4, 5, \dots$ etc., and without getting too philosophical, I like the idea of defining them as follows. The word 'one', represented by 1, stands for ‘oneness’ of anything: one finger, one house, one tree, one donkey, etc. The word ‘two’, represented by 2, is ‘one more than one’. The word ‘three’, represented by 3, is ‘one more than two’, and so on.

We are now in a position to associate some mathematical notation with our numbers by introducing the $+$ and $=$ signs. We know that $+$ means add, but it also can stand for ‘more’. We also know that $=$ means equal, and it can also stand for ‘is the same as’. Thus the statement:

$$2 = 1 + 1$$

is read as ‘two is the same as one more than one.’

We can also write

$$3 = 1 + 2$$

which is read as ‘three is the same as one more than two.’ But as we already have a definition for 2, we can write

$$\begin{aligned} 3 &= 1 + 2 \\ &= 1 + 1 + 1. \end{aligned}$$

Developing this idea, and including some extra combinations, we have:

$$\begin{aligned} 2 &= 1 + 1 \\ 3 &= 1 + 2 \\ 4 &= 1 + 3 = 2 + 2 \\ 5 &= 1 + 4 = 2 + 3 \\ 6 &= 1 + 5 = 2 + 4 = 3 + 3 \\ 7 &= 1 + 6 = 2 + 5 = 3 + 4 \\ &\text{etc.} \end{aligned}$$

and can be continued without limit. These numbers, 1, 2, 3, 4, 5, 6, etc., are called *natural numbers*, and are the set $\mathbb{N}$.

2.5 Zero

The concept of zero has a well-documented history, which shows that it has been used by different cultures over a period of two-thousand years or more. It was the Indian mathematician and astronomer Brahmagupta (598-c.–670) who argued that zero was just as valid as any natural number, with the definition: *the result of subtracting any number from itself*. However, even today, there is no universal agreement as to whether zero belongs to the set $\mathbb{N}$, consequently, the set $\mathbb{N}^0$ stands for the set of natural numbers including zero.

In today's positional decimal system, which is a *place value system*, the digit 0 is a placeholder. For example, 203 stands for: two hundreds, no tens and three units. Although $0 \in \mathbb{N}^0$, it does have special properties that distinguish it from other members of the set, and Brahmagupta also gave rules showing this interaction.

If $x \in \mathbb{N}^0$, then the following rules apply:

$$\begin{aligned}
\text{addition:} &\quad x + 0 = x \\
\text{subtraction:} &\quad x - 0 = x \\
\text{multiplication:} &\quad x \times 0 = 0 \times x = 0 \\
\text{division:} &\quad 0/x = 0 \\
\text{undefined division:} &\quad x/0.
\end{aligned}$$

The expression $0/0$ is called an *indeterminate form*, as it is possible to show that under different conditions, especially limiting conditions, it can equal anything. So for the moment, we will avoid using it until we cover calculus.

2.6 Negative Numbers

When negative numbers were first proposed, they were not accepted with open arms, as it was difficult to visualise -5 of something. For instance, if there are 5 donkeys in a field, and they are all stolen to make salami, the field is now empty, and there is nothing we can do in the arithmetic of donkeys to create a field of -5 donkeys. However, in applied mathematics, numbers have to represent all sorts of quantities such as temperature, displacement, angular rotation, speed, acceleration, etc., and we also need to incorporate ideas such as left and right, up and down, before and after, forwards and backwards, etc. Fortunately, negative numbers are perfect for representing all of the above quantities and ideas.

Consider the expression $4 - x$, where $x \in \mathbb{N}^0$. When x takes on certain values, we have

$$\begin{aligned}
4 - 1 &= 3 \\
4 - 2 &= 2 \\
4 - 3 &= 1 \\
4 - 4 &= 0
\end{aligned}$$

and unless we introduce negative numbers, we are unable to express the result of $4 - 5$. Consequently, negative numbers are visualised as shown in Fig. 2.1, where the *number line* shows negative numbers to the left of the natural numbers, which are *positive*, although the $+$ sign is omitted for clarity.

Moving from left to right, the number line provides a numerical continuum from large negative numbers, through zero, towards large positive numbers. In any

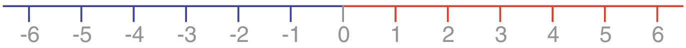


Fig. 2.1 The *number line* showing negative and positive numbers

calculations, we could agree that angles above the horizon are positive, and angles below the horizon, negative. Similarly, a movement forwards is positive, and a movement backwards is negative. So now we are able to write:

$$\begin{aligned}4-5&=-1\\4-6&=-2\\4-7&=-3\\&\text{etc.,}\end{aligned}$$

without worrying about creating impossible conditions.

2.6.1 *The Arithmetic of Positive and Negative Numbers*

Once again, Brahmagupta compiled all the rules supporting the addition, subtraction, multiplication and division of positive and negative numbers, Tables 2.1 and 2.2. The real fly in the ointment, being negative numbers, which cause problems for children, math teachers and occasional accidents for mathematicians. Perhaps, the one rule we all remember from our school days is that *two negatives make a positive*.

Table 2.1 Rules for adding and subtracting positive and negative numbers

$+$	b	$-b$
a	$a+b$	$a-b$
$-a$	$b-a$	$-(a+b)$
$-$	b	$-b$
a	$a-b$	$a+b$
$-a$	$-(a+b)$	$b-a$

Table 2.2 Rules for multiplying and dividing positive and negative numbers

$\times$	b	$-b$
a	ab	$-ab$
$-a$	$-ab$	ab
$/$	b	$-b$
a	a/b	$-a/b$
$-a$	$-a/b$	a/b

Another problem with negative numbers arises when we employ the square-root function. As the product of two positive or negative numbers results in a positive result, the square-root of a positive number gives rise to a positive **and** a negative answer. For example, $\sqrt{4} = \pm 2$. This means that the square-root function only applies to positive numbers. Nevertheless, it did not stop the invention of the *imaginary* object i, where $i^2 = -1$. However, i is not a number, but behaves like an operator, and is described later.

2.7 Observations and Axioms

The following *axioms* or laws provide a formal basis for mathematics, and in the following descriptions a *binary operation* is an arithmetic operation such as $+$, $-$, $\times$, $/$ which operates on two operands.

2.7.1 Commutative Law

The *commutative law* in algebra states that when two elements are linked through some binary operation, the result is independent of the order of the elements. The commutative law of addition is

$$
\begin{gathered}
a + b = b + a \\
\text{e.g. } 1 + 2 = 2 + 1.
\end{gathered}
$$

The commutative law of multiplication is

$$
\begin{gathered}
a \times b = b \times a \\
\text{e.g. } 1 \times 2 = 2 \times 1.
\end{gathered}
$$

Note that subtraction is not commutative:

$$
\begin{gathered}
a - b \neq b - a \\
\text{e.g. } 1 - 2 \neq 2 - 1.
\end{gathered}
$$

2.7.2 Associative Law

The *associative law* in algebra states that when three or more elements are linked together through a binary operation, the result is independent of how each pair of elements is grouped. The associative law of addition is

$$
\begin{aligned}
a + (b + c) &= (a + b) + c\\
\text{e.g. } 1 + (2 + 3) &= (1 + 2) + 3.
\end{aligned}
$$

The associative law of multiplication is

$$
\begin{aligned}
a \times (b \times c) &= (a \times b) \times c\\
\text{e.g. } 1 \times (2 \times 3) &= (1 \times 2) \times 3.
\end{aligned}
$$

However, note that subtraction is not associative:

$$
\begin{aligned}
a - (b - c) &\neq (a - b) - c\\
\text{e.g. } 1 - (2 - 3) &\neq (1 - 2) - 3.
\end{aligned}
$$

which may seem surprising, but at the same time confirms the need for clear axioms.

2.7.3 Distributive Law

The *distributive law* in algebra describes an operation which when performed on a combination of elements is the same as performing the operation on the individual elements. The distributive law does not work in all cases of arithmetic. For example, multiplication over addition holds:

$$
\begin{aligned}
a(b + c) &= ab + ac\\
\text{e.g. } 2(3 + 4) &= 6 + 8,
\end{aligned}
$$

whereas addition over multiplication does not:

$$
\begin{aligned}
a + (b \times c) &\neq (a + b) \times (a + c)\\
\text{e.g. } 3 + (4 \times 5) &\neq (3 + 4) \times (3 + 5).
\end{aligned}
$$

Although these laws are natural for numbers, they do not necessarily apply to all mathematical objects. For instance, the vector product, which multiplies two vectors together, is not commutative. The same applies for matrix multiplication.

2.8 The Base of a Number System

2.8.1 Background

Over recent millennia, mankind has invented and discarded many systems for representing number. People have counted on their fingers and toes, used pictures (hieroglyphics), cut marks on clay tablets (cuneiform symbols), employed Greek symbols (Ionic system) and struggled with, and abandoned Roman numerals (I, V, X, L, C, D, M, etc.), until we reach today's decimal place system, which has Hindu-Arabic and Chinese origins. And since the invention of computers, we have witnessed the emergence of binary, octal and hexadecimal number systems, where 2, 8 and 16 respectively, replace the 10 in our decimal system.

The decimal number 23 means 'two tens and three units', and in English is written 'twenty-three', in French 'vingt-trois' (twenty-three), and in German 'dreiundzwanzig' (three and twenty). Let's investigate the algebra behind the decimal system and see how it can be used to represent numbers to any base. The expression:

$$a \times 1000 + b \times 100 + c \times 10 + d \times 1$$

where a, b, c, d take on any value between 0 and 9, describes any whole number between 0 and 9999. By including

$$e \times 0.1 + f \times 0.01 + g \times 0.001 + h \times 0.0001$$

where e, f, g, h take on any value between 0 and 9, any decimal number between 0 and 9999.9999 can be represented.

Indices bring the notation alive and reveal the true underlying pattern:

$$\dots a10^3 + b10^2 + c10^1 + d10^0 + e10^{-1} + f10^{-2} + g10^{-3} + h10^{-4} \dots.$$

Remember that any number raised to the power 0 equals 1. By adding extra terms, both left and right, any number can be accommodated.

In this example, 10 is the base, which means that the values of a to h range between 0 and 9, 1 less than the base. Therefore, by substituting B for the base we have

$$\dots aB^3 + bB^2 + cB^1 + dB^0 + eB^{-1} + fB^{-2} + gB^{-3} + hB^{-4} \dots$$

where the values of a to h range between 0 and $B - 1$.

2.8.2 Octal Numbers

The octal number system has $B = 8$, and a to h range between 0 and 7:

$$\dots a8^3 + b8^2 + c8^1 + d8^0 + e8^{-1} + f8^{-2} + g8^{-3} + h8^{-4} \dots$$

and the first 17 octal numbers are:

$$1_8, 2_8, 3_8, 4_8, 5_8, 6_8, 7_8, 10_8, 11_8, 12_8, 13_8, 14_8, 15_8, 16_8, 17_8, 20_8, 21_8.$$

The subscript 8, reminds us that although we may continue to use the words 'twenty-one', it is an octal number, and not a decimal. But what is 14_8 in decimal? Well, it stands for:

$$1 \times 8^1 + 4 \times 8^0 = 12.$$

Thus 356.4_8 in decimal, equals:

$$\begin{gathered}(3 \times 8^2) + (5\times8^1) + (6 \times 8^0) + (4 \times 8^{-1}) \\ (3 \times 64) + (5\times8) + (6 \times 1) + (4 \times 0.125) \\ (192 + 40 + 6) + (0.5) \\ 238.5.\end{gathered}$$

Counting in octal appears difficult, simply because we have never been exposed to it, like the decimal system. If we had evolved with 8 fingers, instead of 10, we would be counting in octal!

2.8.3 Binary Numbers

The binary number system has $B = 2$, and a to h are 0 or 1:

$$\dots a2^3 + b2^2 + c2^1 + d2^0 + e2^{-1} + f2^{-2} + g2^{-3} + h2^{-4} \dots$$

and the first 13 binary numbers are:

$$1_2, 10_2, 11_2, 100_2, 101_2, 110_2, 111_2, 1000_2, 1001_2, 1010_2, 1011_2, 1100_2, 1101_2.$$

Thus 11011.11_2 in decimal, equals:

$$\begin{gathered}(1 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0) + (1 \times 2^{-1}) + (1 \times 2^{-2}) \\ (1 \times 16) + (1 \times 8) + (0\times4) + (1 \times 2) + (1 \times 0.5) + (1 \times 0.25) \\ (16 + 8 + 2) + (0.5 + 0.25) \\ 26.75.\end{gathered}$$

The reason why computers work with binary numbers – rather than decimal – is due to the difficulty of designing electrical circuits that can store decimal numbers in a stable fashion. A switch, where the open state represents 0, and the closed state represents 1, is the simplest electrical component to emulate. No matter how often it is used, or how old it becomes, it will always behave like a switch. The main advantage of electrical circuits is that they can be switched on and off trillions of times a second, and the only disadvantage is that the encoded binary numbers and characters contain a large number of bits, and humans are not familiar with binary.

2.8.4 Hexadecimal Numbers

The hexadecimal number system has $B = 16$, and a to h can be 0 to 15, which presents a slight problem, as we don't have 15 different numerical characters. Consequently, we use 0 to 9, and the letters A, B, C, D, E, F to represent 10, 11, 12, 13, 14, 15 respectively:

$$\dots a16^3 + b16^2 + c16^1 + d16^0 + e16^{-1} + f16^{-2} + g16^{-3} + h16^{-4} \dots$$

and the first 17 hexadecimal numbers are:

$$1_{16}, 2_{16}, 3_{16}, 4_{16}, 5_{16}, 6_{16}, 7_{16}, 8_{16}, 9_{16}, A_{16}, B_{16}, C_{16}, D_{16}, E_{16}, F_{16}, 10_{16}, 11_{16}.$$

Thus $1E.8_{16}$ in decimal, equals

$$\begin{gathered}(1 \times 16) + (E \times 1) + (8 \times 16^{-1})\\ (16{+}14) + (8/16)\\ 30.5.\end{gathered}$$

Although it is not obvious, binary, octal and hexadecimal numbers are closely related, which is why they are part of a programmer's toolkit. Even though computers work with binary, it's the last thing a programmer wants to use. So to simplify the man-machine interface, binary is converted into octal or hexadecimal. To illustrate this, let's convert the 16-bit binary code 1101011000110001 into octal.

Using the following general binary integer

$$a2^8 + b2^7 + c2^6 + d2^5 + e2^4 + f2^3 + g2^2 + h2^1 + i2^0$$

we group the terms into threes, starting from the right, because $2^3 = 8$:

$$(a2^8 + b2^7 + c2^6) + (d2^5 + e2^4 + f2^3) + (g2^2 + h2^1 + i2^0).$$

Simplifying:

$$\begin{gathered}2^6(a2^2 + b2^1 + c2^0) + 2^3(d2^2 + e2^1 + f2^0) + 2^0(g2^2 + h2^1 + i2^0)\\ 8^2(a2^2 + b2^1 + c2^1) + 8^1(d2^2 + e2^1 + f2^0) + 8^0(g2^2 + h2^1 + i2^0)\\ 8^2R + 8^1S + 8^0T\end{gathered}$$

where

$$\begin{aligned}R &= a2^2 + b2^1 + c\\ S &= d2^2 + e2^1 + f\\ T &= g2^2 + h2^1 + i\end{aligned}$$

and the values of R, S, T vary between 0 and 7. Therefore, given 1101011000110001, we divide the binary code into groups of three, starting at the right, and adding two zeros on the left:

$$(001)(101)(011)(000)(110)(001).$$

For each group, multiply the zeros and ones by 4, 2, 1, right to left:

$$\begin{gathered}(0+0+1)(4+0+1)(0+2+1)(0+0+0)(4+2+0)(0+0+1)\\(1)(5)(3)(0)(6)(1)\\153061_8.\end{gathered}$$

Therefore, $1101011000110001_2 \equiv 153061_8$, ($\equiv$ stands for 'equivalent to') which is much more compact. The secret of this technique is to memorise the patterns:

$$\begin{aligned}000_2 &\equiv 0_8\\001_2 &\equiv 1_8\\010_2 &\equiv 2_8\\011_2 &\equiv 3_8\\100_2 &\equiv 4_8\\101_2 &\equiv 5_8\\110_2 &\equiv 6_8\\111_2 &\equiv 7_8.\end{aligned}$$

Here are a few more examples, with the binary digits grouped in threes:

$$\begin{aligned}111_2 &\equiv 7_8\\101\ 101_2 &\equiv 55_8\\100\ 000_2 &\equiv 40_8\\111\ 000\ 111\ 000\ 111_2 &\equiv 70707_8.\end{aligned}$$

It's just as easy to reverse the process, and convert octal into binary. Here are some examples:

$$\begin{aligned}567_8 &\equiv 101\ 110\ 111_2\\23_8 &\equiv 010\ 011_2\\1741_8 &\equiv 001\ 111\ 100\ 001_2.\end{aligned}$$

A similar technique is used to convert binary to hexadecimal, but this time we divide the binary code into groups of four, because $2^4 = 16$, starting at the right, and adding leading zeros, if necessary. To illustrate this, let's convert the 16-bit binary code 1101 0110 0011 0001 into hexadecimal.

Using the following general binary integer number

$$a2^{11} + b2^{10} + c2^{9} + d2^{8} + e2^{7} + f2^{6} + g2^{5} + h2^{4} + i2^{3} + j2^{2} + k2^{1} + l2^{0}$$

from the right, we divide the binary code into groups of four:

$$(a2^{11} + b2^{10} + c2^{9} + d2^{8}) + (e2^{7} + f2^{6} + g2^{5} + h2^{4}) + (i2^{3} + j2^{2} + k2^{1} + l2^{0}).$$

Simplifying:

$$\begin{gathered} 2^{8}(a2^{3} + b2^{2} + c2^{1} + d2^{0}) + 2^{4}(e2^{3} + f2^{2} + g2^{1} + h2^{0}) + 2^{0}(i2^{3} + j2^{2} + k2^{1} + l2^{0}) \\ 16^{2}(a2^{3} + b2^{2} + c2^{1} + d) + 16^{1}(e2^{3} + f2^{2} + g2^{1} + h) + 16^{0}(i2^{3} + j2^{2} + k2^{1} + l) \\ 16^{2}R + 16^{1}S + 16^{0}T \end{gathered}$$

where

$$\begin{aligned} R &= a2^{3} + b2^{2} + c2^{1} + d \\ S &= e2^{3} + f2^{2} + g2^{1} + h \\ T &= i2^{3} + j2^{2} + k2^{1} + l \end{aligned}$$

and the values of R, S, T vary between 0 and 15. Therefore, given 1101011000 110001_2, we divide the binary code into groups of fours, starting at the right:

$$(1101)(0110)(0011)(0001)$$

For each group, multiply the zeros and ones by 8, 4, 2, 1 respectively, right to left:

$$\begin{gathered} (8 + 4 + 0 + 1)(0 + 4+2 + 0)(0 + 0 + 2 + 1)(0 + 0 + 0 + 1) \\ (13)(6)(3)(1) \\ D631_{16}. \end{gathered}$$

Therefore, 1101 0110 0011 $0001_2 \equiv D631_{16}$, which is even more compact than its octal value 153061_8.

I have deliberately used whole numbers in the above examples, but they can all be extended to include a fractional part. For example, when converting a binary number such as 11.1101_2 to octal, the groups are formed about the binary point:

$$(011).(110)(100) \equiv 3.64_8.$$

Similarly, when converting a binary number such as 101010.100110_2 to hexadecimal, the groups are also formed about the binary point:

$$(0010)(1010).(1001)(1000) \equiv 2A.98_{16}.$$

Table 2.3 The first twenty decimal, binary, octal, and hexadecimal numbers

Decimal	Binary	Octal	Hex	Decimal	Binary	Octal	Hex
1	1	1	1	11	1011	13	B
2	10	2	2	12	1100	14	C
3	11	3	3	13	1101	15	D
4	100	4	4	14	1110	16	E
5	101	5	5	15	1111	17	F
6	110	6	6	16	10000	20	10
7	111	7	7	17	10001	21	11
8	1000	10	8	18	10010	22	12
9	1001	11	9	19	10011	23	13
10	1010	12	A	20	10100	24	14

Table 2.3 shows the first twenty decimal, binary, octal and hexadecimal numbers.

2.8.5 *Adding Binary Numbers*

When we are first taught the addition of integers containing several digits, we are advised to solve the problem digit by digit, working from right to left. For example, to add 254 to 561 we write:

$$\begin{array}{r} 561 \\ 254 \\ \hline 815 \\ \hline \end{array}$$

where $4 + 1 = 5$, $5 + 6 = 1$ with a $carry = 1$, $2 + 5 + carry = 8$.

Table 2.4 shows all the arrangements for adding two digits with the $carry$ shown as ${}^{carry}n$. However, when adding binary numbers, the possible arrangements collapse to the four shown in Table 2.5, which greatly simplifies the process.

For example, to add 124 to 188 as two 16-bit binary integers, we write, showing the status of the $carry$ bit:

$$\begin{array}{r} 0000000011111000\ carry \\ 0000000010111100 = 188 \\ 0000000001111100 = 124 \\ \hline 0000000100111000 = 312 \\ \hline \end{array}$$

Table 2.4 Addition of two decimal integers showing the *carry*

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	$^{1}0$
2	2	3	4	5	6	7	8	9	$^{1}0$	$^{1}1$
3	3	4	5	6	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$
4	4	5	6	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$
5	5	6	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$
6	6	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$	$^{1}5$
7	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$	$^{1}5$	$^{1}6$
8	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$	$^{1}5$	$^{1}6$	$^{1}7$
9	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$	$^{1}5$	$^{1}6$	$^{1}7$	$^{1}8$

Table 2.5 Addition of two binary integers showing the *carry*

+	0	1
0	0	1
1	1	$^{1}0$

Such addition is easily undertaken by digital electronic circuits, and instead of having separate circuitry for subtraction, it is possible to perform subtraction using the technique of *two's complement*.

2.8.6 Subtracting Binary Numbers

Two's complement is a technique for converting a binary number into a form such that when it is added to another binary number, it results in a subtraction. There are two stages to the conversion: inversion, followed by the addition of 1. For example, 24 in binary is 0000000000110000, and is inverted by switching every 1 to 0, and *vice versa*: 1111111111100111. Next, we add 1: 1111111111101000, which now represents -24. If this is added to binary 36: 0000000000100100, we have

$$\begin{array}{r} 0000000000100100 = +36 \\ 1111111111101000 = -24 \\ \hline 0000000000001100 = +12 \\ \hline \end{array}$$

Note that the last high-order addition creates a *carry* of 1, which is ignored. Here is another example, $100 - 30$:

	0000000000011110 = +30
inversion	1111111111100001
add 1	0000000000000001
	1111111111100010 = −30
add 100	0000000001100100 = +100
	0000000001000110 = +70

2.9 Types of Numbers

As mathematics evolved, mathematicians introduced different types of numbers to help classify equations and simplify the language employed to describe their work. These are the various types and their set names.

2.9.1 Natural Numbers

The *natural numbers* $\{1,\ 2,\ 3,\ 4,\ \ldots\}$ are used for counting, ordering and labelling and represented by the set $\mathbb{N}$. When zero is included, $\mathbb{N}^0$ or $\mathbb{N}_0$ is used:

$$\mathbb{N}^0 = \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

Note that negative numbers are not included. Natural numbers are used to subscript a quantity to distinguish one element from another, e.g. $x_1,\ x_2,\ x_3,\ x_4,\ \ldots.$

2.9.2 Integers

Integer numbers include the natural numbers, both positive and negative, and zero, and are represented by the set $\mathbb{Z}$:

$$\mathbb{Z} = \{\ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3, \ldots\}.$$

The reason for using $\mathbb{Z}$ is because the German for whole number is *ganzen Zahlen*. Leopold Kronecker apparently criticised Georg Cantor for his work on set theory with the jibe: '*Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk*', which translates: '*God made the integers, and all the rest is man's work*', implying that the rest are artificial. However, Cantor's work on set theory and transfinite numbers proved to be far from artificial.

2.9.3 Rational Numbers

Any number that equals the quotient of one integer divided by another non-zero integer, is a *rational number*, and represented by the set $\mathbb{Q}$. For example, 2, $\sqrt{16}$, 0.25 are rational numbers because

$$
\begin{aligned}
2 &= 4/2 \\
\sqrt{16} &= 4 = 8/2 \\
0.25 &= 1/4.
\end{aligned}
$$

Some rational numbers can be stored accurately inside a computer, but many others can only be stored approximately. For example, 4/3 produces an infinite sequence of threes 1.333333 ... and is truncated when stored as a binary number.

2.9.4 Irrational Numbers

An *irrational number* cannot be expressed as the quotient of two integers. Irrational numbers never terminate, nor contain repeated sequences of digits, consequently, they are always subject to a small error when stored within a computer. Examples are:

$$
\begin{aligned}
\sqrt{2} &= 1.41421356\ldots \\
\phi &= 1.61803398\ldots \text{(golden section)} \\
\mathrm{e} &= 2.71828182\ldots \\
\pi &= 3.14159265\ldots
\end{aligned}
$$

2.9.5 Real Numbers

Rational and irrational numbers comprise the set of *real numbers* $\mathbb{R}$. Examples are 1.5, 0.004, 12.999 and 23.0.

2.9.6 Algebraic and Transcendental Numbers

Polynomial equations with rational coefficients have the form:

$$
f(x) = ax^n + bx^{n-1} + cx^{n-2} \cdots + C
$$

such as

$$y = 3x^2 + 2x - 1$$

and their roots belong to the set of *algebraic numbers* $\mathbb{A}$. A consequence of this definition implies that all rational numbers are algebraic, since if

$$x = \frac{p}{q}$$

then

$$qx - p = 0$$

which is a polynomial. Numbers that are not roots to polynomial equations are *transcendental numbers* and include most irrational numbers, but not $\sqrt{2}$, since if

$$x = \sqrt{2}$$

then

$$x^2 - 2 = 0$$

which is a polynomial.

2.9.7 Imaginary Numbers

Imaginary numbers were invented to resolve problems where an equation such as $x^2 + 16 = 0$, has no real solution (roots). The simple idea of declaring the existence of an object i, such that $i^2 = -1$, permits the solution to be expressed as

$$x = \pm 4i.$$

For example, if $x = 4i$ we have

$$\begin{aligned} x^2 + 16 &= 16i^2 + 16 \\ &= -16 + 16 \\ &= 0 \end{aligned}$$

and if $x = -4i$ we have

$$\begin{aligned} x^2 + 16 &= 16i^2 + 16 \\ &= -16 + 16 \\ &= 0. \end{aligned}$$

Table 2.6 Increasing powers of i

i^0	i^1	i^2	i^3	i^4	i^5	i^6
1	i	-1	$-i$	1	i	-1

But what is i? In 1637, the French mathematician René Descartes (1596–1650), published *La Géométrie*, in which he stated that numbers incorporating $\sqrt{-1}$ were '*imaginary*', and for centuries this label has stuck. Unfortunately, it was a derogatory remark, as there is nothing '*imaginary*' about i – it *simply* is an object that when introduced into various algebraic expressions, reveals some amazing underlying patterns. i is not a number in the accepted sense, it is a mathematical object or construct that squares to -1. In some respects it is like time, which probably does not really exist, but is useful in describing the universe. However, i does lose its mystery when interpreted as a rotational operator, which we investigate below.

The set of imaginary numbers is represented by $\mathbb{I}$, which permits us to define an imaginary number bi as

$$bi \in \mathbb{I}, \quad b \in \mathbb{R}, \quad i^2 = -1.$$

As $i^2 = -1$ then it must be possible to raise i to other powers. For example,

$$i^4 = i^2 i^2 = 1$$

and

$$i^5 = i i^4 = i.$$

Table 2.6 shows the sequence up to i^6.

This cyclic pattern is quite striking, and reminds one of a similar pattern:

$$(x, y, -x, -y, x, ...)$$

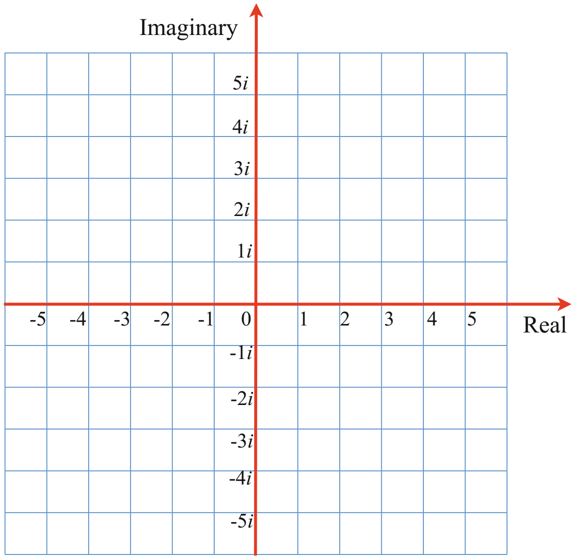
that arises when rotating around the Cartesian axes in a anticlockwise direction. Such a similarity cannot be ignored, for when the real number line is combined with a vertical imaginary axis, it creates the *complex plane*, as shown in Fig. 2.2.

The above sequence is summarised as

$$\begin{aligned} i^{4n} &= 1 \\ i^{4n+1} &= i \\ i^{4n+2} &= -1 \\ i^{4n+3} &= -i \end{aligned}$$

where $n \in \mathbb{N}^0$.

Fig. 2.2 The complex plane



But what about negative powers? Well they, too, are also possible. Consider i^{-1}, which is evaluated as follows:

$$i^{-1} = \frac{1}{i} = \frac{1(-i)}{i(-i)} = \frac{-i}{1} = -i.$$

Similarly,

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$$

and

$$i^{-3} = i^{-1}i^{-2} = -i(-1) = i.$$

Table 2.7 shows the sequence down to i^{-6}.

This time the cyclic pattern is reversed and is similar to the pattern

$$(x, -y, -x, y, x, \ldots)$$

that arises when rotating around the Cartesian axes in a clockwise direction.

Perhaps the strangest power of all is i^i, which happens to equal $e^{-\pi/2} = 0.207879576\ldots$.

Table 2.7 Decreasing powers of i

i^0	i^{-1}	i^{-2}	i^{-3}	i^{-4}	i^{-5}	i^{-6}
1	$-i$	-1	i	1	$-i$	-1

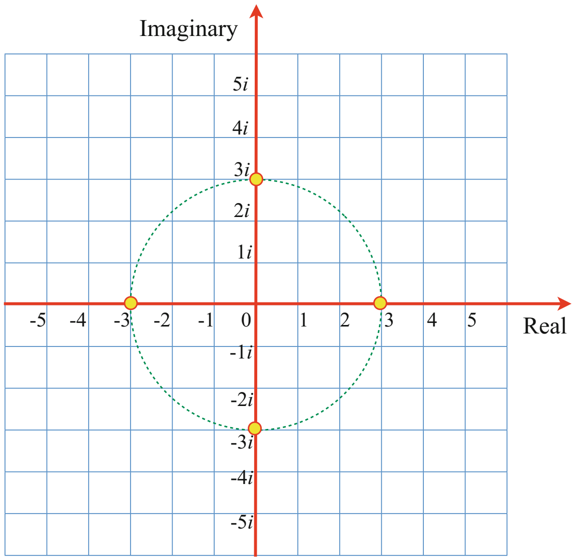


Fig. 2.3 The cycle of points created by repeatedly multiplying 3 by i

Now let's investigate how a real number behaves when it is repeatedly multiplied by i. Starting with the number 3, we have:

$$\begin{aligned} i \times 3 &= 3i \\ i \times 3i &= -3 \\ i \times (-3) &= -3i \\ i \times (-3)i &= 3. \end{aligned}$$

So the cycle is $(3, 3i, -3, -3i, 3, 3i, -3, -3i, 3, \ldots)$, which has four steps, as shown in Fig. 2.3.

Imaginary objects occur for all sorts of reasons. For example, consider the statements

$$\begin{aligned} AB &= -BA \\ BA &= -AB \end{aligned}$$

where A and B are two undefined objects that obey the associative law, but not the commutative law, and $A^2 = B^2 = 1$. The operation $(AB)^2$ reveals

$$\begin{aligned} (AB)(AB) &= A(BA)B \\ &= -A(AB)B \\ &= -(A^2)(B^2) \\ &= -1 \end{aligned}$$

which means that the product AB is imaginary. Such objects, which can be matrices, are useful in describing the behaviour of sub-atomic particles.

2.9.8 Complex Numbers

A *complex number* has a real and imaginary part: $z = a + ib$, and represented by the set $\mathbb{C}$:

$$z = a + bi \qquad z \in \mathbb{C}, \quad a, b \in \mathbb{R}, \quad i^2 = -1.$$

Some examples are

$$\begin{aligned} z &= 1 + i \\ z &= 3 - 2i \\ z &= -23 + \sqrt{23}i. \end{aligned}$$

Complex numbers obey all the normal laws of algebra. For example, if we multiply $(a + bi)$ by $(c + di)$ we have

$$(a + bi)(c + di) = ac + adi + bci + bdi^2.$$

Collecting up like terms and substituting -1 for i^2 we get

$$(a + bi)(c + di) = ac + (ad + bc)i - bd$$

which simplifies to

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

which is another complex number.

Something interesting happens when we multiply a complex number by its *complex conjugate*, which is the same complex number but with the sign of the imaginary part reversed:

$$(a + bi)(a - bi) = a^2 - abi + bai - b^2i^2.$$

Collecting up like terms and simplifying we obtain

$$(a + bi)(a - bi) = a^2 + b^2$$

which is a real number, as the imaginary part has been cancelled out by the action of the complex conjugate.

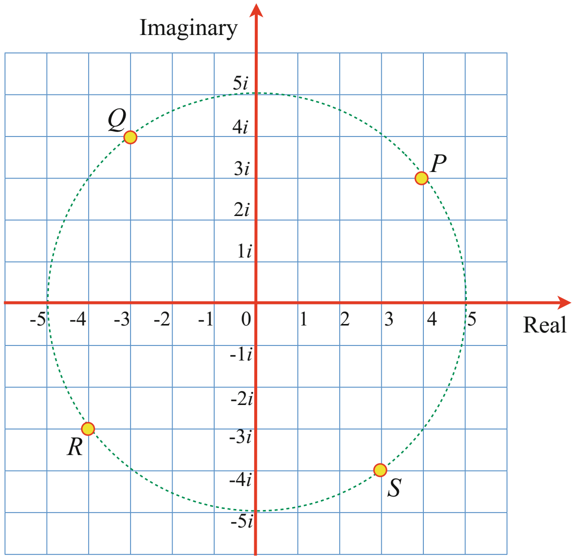


Fig. 2.4 The complex plane showing four complex numbers

Figure 2.4 shows how complex numbers are represented graphically using the complex plane. For example, the complex number $P = 4 + 3i$ in Fig. 2.4 is rotated $90°$ to Q by multiplying it by i. Let's do this, and remember that $i^2 = -1$:

$$\begin{aligned} i(4 + 3i) &= 4i + 3i^2 \\ &= 4i - 3 \\ &= -3 + 4i. \end{aligned}$$

The point $Q = -3 + 4i$ is rotated $90°$ to R by multiplying it by i:

$$\begin{aligned} i(-3 + 4i) &= -3i + 4i^2 \\ &= -3i - 4 \\ &= -4 - 3i. \end{aligned}$$

The point $R = -4 - 3i$ is rotated $90°$ to S by multiplying it by i:

$$\begin{aligned} i(-4 - 3i) &= -4i - 3i^2 \\ &= -4i + 3 \\ &= 3 - 4i. \end{aligned}$$

Finally, the point $S = 3 - 4i$ is rotated $90°$ back to P by multiplying it by i:

$$\begin{aligned} i(3-4i) &= 3i - 4i^2 \\ &= 3i + 4 \\ &= 4 + 3i. \end{aligned}$$

As you can see, complex numbers are intimately related to Cartesian coordinates, in that the ordered pair $(x, \ y) \equiv (x + yi)$.

2.9.9 Transcendental and Algebraic Numbers

Given a polynomial built from integers, for example

$$y = 3x^3 - 4x^2 + x + 23,$$

if the result is an integer, it is called an *algebraic number*, otherwise it is a *transcendental number*. Familiar examples of the latter being $\pi = 3.141\ 592\ 653\ldots$, and $e = 2.718\ 281\ 828\ldots$, which can be represented as various continued fractions:

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ldots}}}}}$$

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ldots}}}}}$$

2.9.10 Infinity

The term *infinity* is used to describe the size of unbounded systems. For example, there is no end to prime numbers: i.e. they are infinite; so too, are the sets of other numbers. Consequently, no matter how we try, it is impossible to visualise the size of infinity. Nevertheless, this did not stop Georg Cantor from showing that one infinite set could be infinitely larger than another.

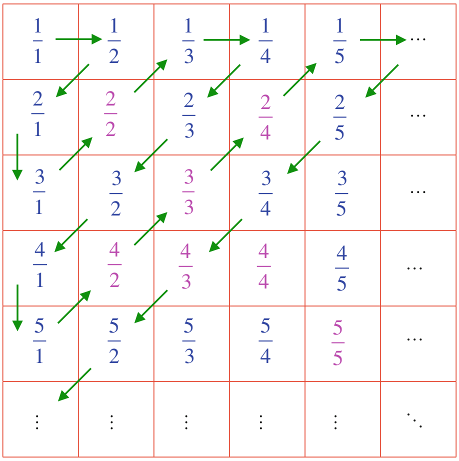


Fig. 2.5 Rational number grid

Cantor distinguished between those infinite number sets that could be 'counted', and those that could not. For Cantor, counting meant the one-to-one correspondence of a natural number with the members of another infinite set. If there was a clear correspondence, without leaving any gaps, then the two sets shared a common infinite size, called its *cardinality* using the first letter of the Hebrew alphabet aleph: $\aleph$. The cardinality of the natural numbers $\mathbb{N}$ is $\aleph_0$, called aleph-zero.

Cantor discovered a way of representing the rational numbers as a grid, which is traversed diagonally, back and forth, as shown in Fig. 2.5. Some ratios appear several times, such as $\frac{2}{2}$, $\frac{3}{3}$ etc., which are not counted. Nevertheless, the one-to-one correspondence with the natural numbers means that the cardinality of rational numbers is also $\aleph_0$.

A real surprise was that there are infinitely more transcendental numbers than natural numbers. Furthermore, there are an infinite number of cardinalities rising to $\aleph_\aleph$. Cantor had been alone working in this esoteric area, and as he published his results, he shook the very foundations of mathematics, which is why he was treated so badly by his fellow mathematicians.

2.10 Summary

Apart from the natural numbers, integers, rational, irrational, prime, real and complex numbers, there are also Fermat, Mersenne, amicable, chromic, cubic, Fibonacci, pentagonal, perfect, random, square and tetrahedral numbers, which although equally interesting, don't concern us in this text.

Now that we know something about some important number sets, let's revise some ideas behind algebra.

2.11 Worked Examples

2.11.1 Algebraic Expansion

Expand $(a+b)(c+d)$, $(a-b)(c+d)$, and $(a-b)(c-d)$.

$$\begin{aligned}(a+b)(c+d) &= a(c+d)+b(c+d)\\ &= ac+ad+bc+bd.\\ (a-b)(c+d) &= a(c+d)-b(c+d)\\ &= ac+ad-bc-bd.\\ (a-b)(c-d) &= a(c-d)-b(c-d)\\ &= ac-ad-bc+bd.\end{aligned}$$

2.11.2 Binary Subtraction

Using two's complement, subtract 12 from 50.

	0000000000001100	$=+12$
inversion	1111111111110011	
add 1	0000000000000001	
	1111111111110100	$=-12$
add 50	0000000000110010	$=+50$
	0000000000100110	$=+38$

2.11.3 Complex Numbers

Compute $(3+2i)+(2+2i)+(5-3i)$ and $(3+2i)(2+2i)(5-3i)$.

$$(3+2i)+(2+2i)+(5-3i)=10+i.$$

$$\begin{aligned}(3+2i)(2+2i)(5-3i) &= (3+2i)(10-6i+10i+6)\\ &= (3+2i)(16+4i)\\ &= 48+12i+32i-8\\ &= 40+44i.\end{aligned}$$

2.11.4 Complex Rotation

Rotate the complex point $(3 + 2i)$ by $\pm 90°$ and $\pm 180°$.
To rotate $+90°$ (anticlockwise) multiply by i.

$$i(3 + 2i) = (3i - 2) = (-2 + 3i).$$

To rotate $-90°$ (clockwise) multiply by $-i$.

$$-i(3 + 2i) = (-3i + 2) = (2 - 3i).$$

To rotate $+180°$ (anticlockwise) multiply by -1.

$$-1(3 + 2i) = (-3 - 2i).$$

To rotate $-180°$ (clockwise) multiply by -1.

$$-1(3 + 2i) = (-3 - 2i).$$

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