# Contents

1 **Introduction** ...................................................... 1
   1.1 Control Systems, Models and Algorithms ............... 2
   1.2 Repetition and Iteration ................................. 3
      1.2.1 Periodic Demand Signals ...................... 3
      1.2.2 Repetitive Control and Multipass Systems .... 4
      1.2.3 Iterative Control Examples .................... 6
   1.3 Dynamical Properties of Iteration: A Review of Ideas ... 9
   1.4 So What Do We Need? .................................... 12
      1.4.1 An Overview of Mathematical Techniques ...... 13
      1.4.2 The Conceptual Basis for Algorithms .......... 15
   1.5 Discussion and Further Background Reading .......... 16

2 **Mathematical Methods** ........................................ 19
   2.1 Elements of Matrix Theory ............................... 19
   2.2 Quadratic Optimization and Quadratic Forms ........... 27
      2.2.1 Completing the Square ......................... 27
      2.2.2 Singular Values, Lagrangians and Matrix Norms ... 28
   2.3 Banach Spaces, Operators, Norms and Convergent
      Sequences ................................................ 29
      2.3.1 Vector Spaces ..................................... 29
      2.3.2 Normed Spaces .................................. 31
      2.3.3 Convergence, Closure, Completeness and Banach
                  Spaces ........................................ 33
      2.3.4 Linear Operators and Dense Subsets ............. 34
   2.4 Hilbert Spaces .............................................. 37
      2.4.1 Inner Products and Norms ....................... 37
      2.4.2 Norm and Weak Convergence ...................... 39
      2.4.3 Adjoint and Self-adjoint Operators
                  in Hilbert Space ................................ 41
2.5 Real Hilbert Spaces, Convex Sets and Projections ............... 46
2.6 Optimal Control Problems in Hilbert Space .................. 48
  2.6.1 Proof by Completing the Square ......................... 50
  2.6.2 Proof Using the Projection Theorem ..................... 51
  2.6.3 Discussion ........................................... 52
2.7 Further Discussion and Bibliography .......................... 53

3 State Space Models ............................................. 55
  3.1 Models of Continuous State Space Systems ................. 57
    3.1.1 Solution of the State Equations .................... 58
    3.1.2 The Convolution Operator and the Impulse Response .... 59
    3.1.3 The System as an Operator Between Function Spaces .... 59
  3.2 Laplace Transforms ....................................... 60
  3.3 Transfer Function Matrices, Poles, Zeros and Relative Degree 61
  3.4 The System Frequency Response .......................... 63
  3.5 Discrete Time, Sampled Data State Space Models .......... 64
    3.5.1 State Space Models as Difference Equations .......... 64
    3.5.2 Solution of Linear, Discrete Time State Equations .... 65
    3.5.3 The Discrete Convolution Operator and the Discrete Impulse Response Sequence .... 66
  3.6 $\mathcal{Z}$-Transforms and the Discrete Transfer Function Matrix 67
    3.6.1 Discrete Transfer Function Matrices, Poles, Zeros and the Relative Degree .... 68
    3.6.2 The Discrete System Frequency Response ................ 69
  3.7 Multi-rate Discrete Time Systems .......................... 70
  3.8 Controllability, Observability, Minimal Realizations and Pole Allocation .......... 70
  3.9 Inverse Systems ......................................... 72
    3.9.1 The Case of $m = \ell$, Zeros and $\nu^*$ .......... 72
    3.9.2 Left and Right Inverses When $m \neq \ell$ .......... 74
  3.10 Quadratic Optimal Control of Linear Continuous Systems .... 76
    3.10.1 The Relevant Operators and Spaces ................. 76
    3.10.2 Computation of the Adjoint Operator ............... 78
    3.10.3 The Two Point Boundary Value Problem ............. 81
    3.10.4 The Riccati Equation and a State Feedback Plus Feedforward Representation ...... 82
    3.10.5 An Alternative Riccati Representation .............. 84
  3.11 Further Reading and Bibliography ........................ 85
4 Matrix Models, Supervectors and Discrete Systems .............. 87
4.1 Supervectors and the Matrix Model. ....................... 87
4.2 The Algebra of Series and Parallel Connections .......... 88
4.3 The Transpose System and Time Reversal ............... 89
4.4 Invertibility, Range and Relative Degrees ............ 90
4.4.1 The Relative Degree and the Kernel
and Range of $G$ ........................................ 92
4.4.2 The Range of $G$ and Decoupling Theory ......... 93
4.5 The Range and Kernel and the Use of the Inverse System ... 96
4.5.1 A Partition of the Inverse ......................... 96
4.5.2 Ensuring Stability of $P^{-1}(z)$ .................. 98
4.6 The Range, Kernel and the $\mathcal{H}^*$ Canonical Form .... 99
4.6.1 Factorization Using State Feedback
and Output Injection ...................................... 99
4.6.2 The $\mathcal{H}^*$ Canonical Form .................... 100
4.6.3 The Special Case of Uniform Rank Systems ....... 102
4.7 Quadratic Optimal Control of Linear Discrete Systems ... 104
4.7.1 The Adjoint and the Discrete Two Point
Boundary Value Problem ............................... 105
4.7.2 A State Feedback/Feedforward Solution .......... 106
4.8 Frequency Domain Relationships ........................ 107
4.8.1 Bounding Norms on Finite Intervals ............... 108
4.8.2 Computing the Norm Using the Frequency
Response .................................................... 109
4.8.3 Quadratic Forms and Positive Real Transfer
Function Matrices ....................................... 110
4.8.4 Frequency Dependent Lower Bounds .......... 112
4.9 Discussion and Further Reading ........................ 116

5 Iterative Learning Control: A Formulation .................. 119
5.1 Abstract Formulation of a Design Problem ............ 119
5.1.1 The Design Problem ............................... 120
5.1.2 Input and Error Update Equations:
The Linear Case ........................................... 123
5.1.3 Robustness and Uncertainty Models ............ 124
5.2 General Conditions for Convergence of Linear Iterations ... 128
5.2.1 Spectral Radius and Norm Conditions .......... 129
5.2.2 Infinite Dimensions with $r(L) = \|L\| = 1$
and $L = L^*$ .............................................. 132
5.2.3 Relaxation, Convergence and Robustness .......... 134
5.2.4 Eigenstructure Interpretation .................. 138
5.2.5 Formal Computation of the Eigenvalues
and Eigenfunctions .................................... 139
5.3 Robustness, Positivity and Inverse Systems ........................................... 141
5.4 Discussion and Further Reading ............................................................... 143

6 Control Using Inverse Model Algorithms .................................................. 145
6.1 Inverse Model Control: A Benchmark Algorithm ..................................... 145
  6.1.1 Use of a Right Inverse of the Plant .................................................. 145
  6.1.2 Use of a Left Inverse of the Plant .................................................... 147
  6.1.3 Why the Inverse Model Is Important ................................................ 149
  6.1.4 Inverse Model Algorithms for State Space Models ............................. 151
  6.1.5 Robustness Tests and Multiplicative Error Models ............................ 152
6.2 Frequency Domain Robustness Criteria ................................................. 156
  6.2.1 Discrete System Robust Monotonicity Tests .................................... 156
  6.2.2 Improving Robustness Using Relaxation ......................................... 158
  6.2.3 Discrete Systems: Robustness and Non-monotonic Convergence .......... 159
6.3 Discussion and Further Reading ................................................................ 161

7 Monotonicity and Gradient Algorithms ..................................................... 165
7.1 Steepest Descent: Achieving Minimum Energy Solutions .......................... 166
7.2 Application to Discrete Time State Space Systems ................................... 168
  7.2.1 Algorithm Construction ................................................................. 169
  7.2.2 Eigenstructure Interpretation: Convergence in Finite Iterations ........... 171
  7.2.3 Frequency Domain Attenuation ....................................................... 174
7.3 Steepest Descent for Continuous Time State Space Systems ..................... 178
7.4 Monotonic Evolution Using General Gradients ....................................... 180
7.5 Discrete State Space Models Revisited .................................................... 183
  7.5.1 Gradients Using the Adjoint of a State Space System ......................... 183
  7.5.2 Why the Case of $m = \ell$ May Be Important in Design ..................... 192
  7.5.3 Robustness Tests in the Frequency Domain ..................................... 194
  7.5.4 Robustness and Relaxation ............................................................. 197
  7.5.5 Non-monotonic Gradient-Based Control and $\varepsilon$-Weighted Norms . 198
  7.5.6 A Steepest Descent Algorithm Using $\varepsilon$-Norms ........................... 203
7.6 Discussion, Comments and Further Generalizations ............................... 203
  7.6.1 Bringing the Ideas Together? ......................................................... 204
  7.6.2 Factors Influencing Achievable Performance ................................. 206
  7.6.3 Notes on Continuous State Space Systems ..................................... 207
8 Combined Inverse and Gradient Based Design

8.1 Inverse Algorithms: Robustness and Bi-directional Filtering
8.2 General Issues in Design
8.2.1 Pre-conditioning Control Loops
8.2.2 Compensator Structures
8.2.3 Stable Inversion Algorithms
8.2.4 All-Pass Networks and Non-minimum-phase Systems
8.3 Gradients, Compensation and Feedback Design Methods
8.3.1 Feedback Design: The Discrete Time Case
8.3.2 Feedback Design: The Continuous Time Case
8.4 Discussion and Further Reading

9 Norm Optimal Iterative Learning Control

9.1 Problem Formulation and Formal Algorithm
9.1.1 The Choice of Objective Function
9.1.2 Relaxed Versions of NOILC
9.1.3 NOILC for Discrete-Time State Space Systems
9.1.4 Relaxed NOILC for Discrete-Time State Space Systems
9.1.5 A Note on Frequency Attenuation:
9.1.6 NOILC: The Case of Continuous-Time State Space Systems
9.1.7 Convergence, Eigenstructure, $\varepsilon^2$ and Spectral Bandwidth
9.1.8 Convergence: General Properties of NOILC Algorithms
9.2 Robustness of NOILC: Feedforward Implementation
9.2.1 Computational Aspects of Feedforward NOILC
9.2.2 The Case of Right Multiplicative Modelling Errors
9.2.3 Discrete State Space Systems with Right Multiplicative Errors
9.2.4 The Case of Left Multiplicative Modelling Errors
9.2.5 Discrete Systems with Left Multiplicative Modelling Errors
9.2.6 Monotonicity in $\mathcal{Y}$ with Respect to the Norm $\| \cdot \|_{\mathcal{Y}}$
9.3 Non-minimum-phase Properties and Flat-Lining
9.4 Discussion and Further Reading
9.4.1 Background Comments
9.4.2 Practical Observations
9.4.3 Performance
### 10 NOILC: Natural Extensions

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.1 Filtering Using Input and Error Weighting</td>
<td>277</td>
</tr>
<tr>
<td>10.2 Multi-rate Sampled Discrete Time Systems</td>
<td>279</td>
</tr>
<tr>
<td>10.3 Initial Conditions as Control Signals</td>
<td>280</td>
</tr>
<tr>
<td>10.4 Problems with Several Objectives</td>
<td>284</td>
</tr>
<tr>
<td>10.5 Intermediate Point Problems</td>
<td>286</td>
</tr>
<tr>
<td>10.5.1 Continuous Time Systems: An Intermediate Point Problem</td>
<td>286</td>
</tr>
<tr>
<td>10.5.2 Discrete Time Systems: An Intermediate Point Problem</td>
<td>290</td>
</tr>
<tr>
<td>10.5.3 IPNOILC: Additional Issues and Robustness</td>
<td>290</td>
</tr>
<tr>
<td>10.6 Multi-task NOILC</td>
<td>293</td>
</tr>
<tr>
<td>10.6.1 Continuous State Space Systems</td>
<td>294</td>
</tr>
<tr>
<td>10.6.2 Adding Initial Conditions as Controls</td>
<td>299</td>
</tr>
<tr>
<td>10.6.3 Discrete State Space Systems</td>
<td>300</td>
</tr>
<tr>
<td>10.7 Multi-models and Predictive NOILC</td>
<td>301</td>
</tr>
<tr>
<td>10.7.1 Predictive NOILC—General Theory and a Link to Inversion</td>
<td>301</td>
</tr>
<tr>
<td>10.7.2 A Multi-model Representation</td>
<td>304</td>
</tr>
<tr>
<td>10.7.3 The Case of Linear, State Space Models</td>
<td>305</td>
</tr>
<tr>
<td>10.7.4 Convergence and Other Algorithm Properties</td>
<td>308</td>
</tr>
<tr>
<td>10.7.5 The Special Cases of $M = 2$ and $M = \infty$</td>
<td>313</td>
</tr>
<tr>
<td>10.7.6 A Note on Robustness of Feedforward Predictive NOILC</td>
<td>315</td>
</tr>
<tr>
<td>10.8 Discussion and Further Reading</td>
<td>319</td>
</tr>
</tbody>
</table>

### 11 Iteration and Auxiliary Optimization

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.1 Models with Auxiliary Variables and Problem Formulation</td>
<td>323</td>
</tr>
<tr>
<td>11.2 A Right Inverse Model Solution</td>
<td>325</td>
</tr>
<tr>
<td>11.3 Solutions Using Switching Algorithms</td>
<td>327</td>
</tr>
<tr>
<td>11.3.1 Switching Algorithm Construction</td>
<td>327</td>
</tr>
<tr>
<td>11.3.2 Properties of the Switching Algorithm</td>
<td>328</td>
</tr>
<tr>
<td>11.3.3 Characterization of Convergence Rates</td>
<td>331</td>
</tr>
<tr>
<td>11.3.4 Decoupling Minimum Energy Representations from NOILC</td>
<td>333</td>
</tr>
<tr>
<td>11.3.5 Intermediate Point Tracking and the Choice $G_1 = G_e$</td>
<td>334</td>
</tr>
<tr>
<td>11.3.6 Restructuring the NOILC Spectrum</td>
<td>335</td>
</tr>
</tbody>
</table>
## Contents

11.4 A Note on Robustness of Switching Algorithms .................. 338
11.5 The Switching Algorithm When $G_r G_e^r$ Is Invertible ............ 341
11.6 Discussion and Further Reading .................................... 344

### 12 Iteration as Successive Projection .......................... 347
12.1 Convergence Versus Proximity .................................... 347
12.2 Successive Projection and Proximity Algorithms .................. 349
12.3 Iterative Control with Constraints ............................... 354
   12.3.1 NOILC with Input Constraints ............................... 355
   12.3.2 General Analysis ............................................ 358
   12.3.3 Intermediate Point Control with Input
         and Output Constraints ..................................... 362
   12.3.4 Iterative Control to Satisfy Auxiliary
         Variable Bounds ............................................ 364
   12.3.5 An Overview and Summary ................................... 366
12.4 “Iteration Management” by Operator Intervention ................. 367
12.5 What Happens If $S_1$ and $S_2$ Do Not Intersect? ................ 370
12.6 Discussion and Further Reading ................................... 373

### 13 Acceleration and Successive Projection ....................... 377
13.1 Replacing Plant Iterations by Off-Line Iterations ................. 378
13.2 Accelerating Algorithms Using Extrapolation .................... 378
   13.2.1 Successive Projection and Extrapolation
         Algorithms .................................................. 379
   13.2.2 NOILC: Acceleration Using Extrapolation .................. 381
13.3 A Notch Algorithm Using Parameterized Sets ..................... 383
   13.3.1 Creating a Spectral Notch: Computation
         and Properties ............................................ 383
   13.3.2 The Notch Algorithm and Iterative Control
         Using Successive Projection .............................. 389
   13.3.3 A Notch Algorithm for Discrete State
         Space Systems ............................................. 393
   13.3.4 Robustness of the Notch Algorithm
         in Feedforward Form .................................... 396
13.4 Discussion and Further Reading ................................... 401

### 14 Parameter Optimal Iterative Control .......................... 403
14.1 Parameterizations and Norm Optimal Iteration .................... 403
14.2 Parameter Optimal Control: The Single Parameter Case .......... 408
   14.2.1 Alternative Objective Functions ............................ 408
   14.2.2 Problem Definition and Convergence
         Characterization ........................................... 410
   14.2.3 Convergence Properties: Dependence
         on Parameters ............................................ 413
14.2.4 Choosing the Compensator ........................................ 415
14.2.5 Computing $tr[I_0^T I_0]:$ Discrete State
Space Systems ...................................................... 416
14.2.6 Choosing Parameters in $J(\beta)$ ................................ 418
14.2.7 Iteration Dynamics ............................................. 420
14.2.8 Plateauing/Flatlining Phenomena .............................. 420
14.2.9 Switching Algorithms ........................................... 425
14.3 Robustness of POILC: The Single Parameter Case ............ 429
  14.3.1 Robustness Using the Right Inverse ........................... 429
  14.3.2 Robustness: A More General Case ............................. 431
14.4 Multi-Parameter Learning Control ................................. 433
  14.4.1 The Form of the Parameterization ............................. 433
  14.4.2 Alternative Forms for $\Omega_\Gamma$
      and the Objective Function ................................... 434
  14.4.3 The Multi-parameter POILC Algorithm ......................... 437
  14.4.4 Choice of Multi-parameter Parameterization .................. 439
14.5 Discussion and Further Reading ................................. 441
  14.5.1 Chapter Overview ............................................ 441
  14.5.2 High Order POILC: A Brief Summary ......................... 443

References ........................................................... 445

Index ................................................................. 451
Iterative Learning Control
An Optimization Paradigm
Owens, D.H.
2016, XXVIII, 456 p., Hardcover