Chapter 2
Laplace's Equation

2.1 Introduction

Potential theory has its origins in gravitational theory and electromagnetic theory. The common element of these two is the inverse square law governing the interaction of two bodies. A potential function relates to the work done in moving a unit charge from one point of space to another in the presence of another charged body. A basic potential function $1/r$, the reciprocal distance function, has the important property that it satisfies Laplace's equation except when $r = 0$. A function u on a set $\Omega \subset R^n$ satisfies Laplace's equation if

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 \tag{2.1}$$

on Ω and is said to be harmonic if some additional continuity properties are satisfied. Since the potential energy at a point due to a distribution of charge can be regarded as the sum of a large number of potential energies due to point charges, the corresponding potential function should also satisfy Laplace's equation. Applications to electromagnetic theory led to the problem of determining a harmonic function on a region having prescribed values on the boundary of the region. This problem came to be known as the Dirichlet problem. A similar problem, connected with steady-state heat distribution, asks for a harmonic function with prescribed flux or normal derivative at each point of the boundary. This problem is known as the Neumann problem. Another problem, Robin's problem, asks for a harmonic function satisfying a condition at points of the boundary which is a linear combination of prescribed values and prescribed flux.

In this chapter, explicit formulas will be developed for solving the Dirichlet problem for a ball in n-space, uniqueness of the solution will be demonstrated, and the solution will be proved to have the right "boundary values." Not nearly as much is possible for the Neumann problem. Explicit formulas for solving the Neumann problem for a ball are known only for the $n = 2$ and $n = 3$ cases.

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2.1.1 Exercises for Sect. 2.1

1. If u is a function of the polar coordinates (r, θ) on an open set $\Omega \subset R^2$, show that

$$\Delta u(r, \theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

2. If u satisfies Laplace's equation $\Delta u = 0$ on R^n, $a, b \in R$, $a \neq 0$, and $v(y) = u(x)$ where $y = ax + b$, $x \in R^n$, show that v satisfies Laplace's equation on the image of Ω under the map $x \to y$.
3. Find all polynomials of the form $ax^2 + bxy + cy^2$ that satisfy Laplace's equation on R^2.
4. Find all harmonic functions u on R^n of the form

$$u(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j.$$

2.2 Green's Theorem

In this section, only real-valued functions on a bounded open set $\Omega \subset R^n$ will be considered. The Ω will be required to have a smooth boundary, this being the case when Ω is a ball in R^n or the interior of the region between two spheres, of which one is inside the other. A precise definition of "smooth boundary" can be found in Sect. 12.3.

Let $v = (v_1, \ldots, v_n)$ be a vector-valued function on Ω whose components v_j have continuous extensions to Ω^- and continuous first partials on Ω. The divergence of v is defined by

$$\text{div } v = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}.$$

The outer unit normal to the surface $\partial\Omega$ at a point $x \in \partial\Omega$ will be denoted by $\mathbf{n}(x)$. The starting point for this chapter is the following theorem which can be found in any advanced calculus textbook (e.g., [1]).

Theorem 2.2.1 (Gauss' Divergence Theorem) *If* $v \in C^0(\Omega^-) \cap C^1(\Omega)$, *then*

$$\int_{\Omega} \text{div } v \, dx = \int_{\partial\Omega} (v \cdot \mathbf{n}) \, d\sigma$$

where $d\sigma$ *denotes integration with respect to surface area.*

By convention, whenever "**n**" appears in an integral over a smooth surface it is understood to be the outer unit normal to the surface.

Suppose now that u is a function defined on a neighborhood of Ω^- and has continuous second partials on that neighborhood. The **Laplacian** of u, Δu, is defined by

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

If u is also a function of variables other than x and it necessary to clarify the meaning of the Laplacian, $\Delta_{(x)}$ will signify that the Laplacian is relative to the coordinates of x. Let v be a second such function. Then

$$u\nabla v = (uD_1 v, \dots, uD_n v)$$

and

$$\operatorname{div}(u\nabla v) = u\Delta v + (\nabla u \cdot \nabla v).$$

It follows from the divergence theorem that

$$\begin{aligned} \textstyle\int_\Omega u\Delta v\, dx + \int_\Omega (\nabla u \cdot \nabla v)\, dx &= \textstyle\int_{\partial\Omega} (u\nabla v \cdot \mathbf{n})\, d\sigma \\ &= \textstyle\int_{\partial\Omega} u D_{\mathbf{n}} v\, d\sigma \end{aligned} \tag{2.2}$$

since $(u\nabla v, \mathbf{n}) = u\,(\nabla v, \mathbf{n})$ and the latter inner product is just the directional derivative $D_{\mathbf{n}}v$ of v in the direction $\mathbf{n}$. The following important identity is obtained by interchanging u and v and subtracting.

Theorem 2.2.2 (Green's Identity) *If $u, v \in C^2(\Omega^-)$, then*

$$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_{\partial\Omega} (uD_{\mathbf{n}} v - vD_{\mathbf{n}} u)\, d\sigma. \tag{2.3}$$

2.3 Fundamental Harmonic Function

A real-valued function u on R^n having continuous second partials is called a **harmonic function** if $\Delta u = 0$ on R^n.

Example 2.3.1 If f is an analytic function of a complex variable z, then the real and imaginary parts of f are harmonic functions. This fact follows from the Cauchy-Riemann equations. Thus, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$ are harmonic functions since they are the real and imaginary parts, respectively, of the analytic function $f(z) = e^z$.

Definition 2.3.2 The real-valued function u on an open set $\Omega \subset R^n$ is **harmonic** on Ω if $u \in C^2(\Omega)$ and $\Delta u = 0$ on Ω.

Remark 2.3.3 Note that harmonicity is preserved under rigid motions of R^n; that is, if $\tau : R^n \to R^n$ is an orthogonal transformation followed by a translation, $\tau(\Omega) = \{\tau x; x \in \Omega\}$, and $u^\tau(y) = u(\tau^{-1}y)$, $y \in \tau(\Omega)$, then u^τ is harmonic on Ω^τ whenever u is harmonic on Ω. This is easy to verify if τ is a translation given by $\tau x = x + a$, $x \in R^n$, for some fixed $a \in R^n$. Suppose τ is defined by the equation $y = Ax$ where A is an orthogonal matrix. Since $A^{-1} = A^T$,

$$\frac{\partial u(x)}{\partial x_i}(x) = \sum_{j=1}^{n} \frac{\partial u(A^T y)}{\partial y_j} \frac{\partial y_j}{\partial x_i} = \sum_{j=1}^{n} a_{ji} \frac{\partial u(A^T y)}{\partial y_j}$$

and

$$\frac{\partial^2 u}{\partial x_i^2}(x) = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{ji} a_{ki} \frac{\partial^2 u(A^T y)}{\partial y_j \partial y_k}.$$

Since $\sum_{i=1}^{n} a_{ji} a_{ki}$ is 1 or 0 according as $j = k$ or $j \neq k$, respectively,

$$\Delta_{(x)} u(x) = \Delta_{(y)} u(A^T y) = \Delta_{(y)} u(A^{-1} y) = \Delta_{(y)} u^\tau(y).$$

Therefore, $\Delta_{(y)} u^\tau(y) = 0$ whenever $y = Ax$ and $\Delta_{(x)} u(x) = 0$.

Remark 2.3.4 Note that directional derivatives are preserved under translation since this is true of ordinary derivatives. Using properties of orthogonal transformations, as in the above remark, it is easy to see that directional derivatives are preserved under orthogonal transformations and, in fact, that inner products are also preserved.

Suppose u is harmonic on the open set Ω and $y \in \Omega$. Then $u(x)$ can be regarded as a function of the spherical coordinates (r, θ) relative to y where $r = |x - y|$ and θ is the point of intersection of the line segment joining x and y and a unit sphere with center at y. Suppose that u is a function of r only. Then Δu, as a function of spherical coordinates, is easily seen to be given by

$$\Delta u = \frac{d^2 u}{dr^2} + \frac{(n-1)}{r} \frac{du}{dr}, \qquad r \neq 0.$$

The only functions that are harmonic on $R^n \sim \{y\}$ and a function of r only are those that satisfy the equation $\Delta u = 0$ on $R^n \sim \{y\}$. If $n = 2$, the general solution of this equation is $A \log r + B$, where A and B are arbitrary constants. The particular solution $u(r) = -\log r$ is harmonic on $R^n \sim \{y\}$ and is called the **fundamental harmonic function** for R^2 with pole y. If $n \geq 3$, the general solution is $Ar^{-n+2} + B$; the particular solution $u(r) = r^{-n+2}$ is called the **fundamental harmonic function** for R^n with pole y. The notations $u_y(x)$ and $u(|x - y|)$ will be used interchangeably.

2.3.1 Exercises for Sect. 2.3

1. Show that the directional derivative is invariant under orthogonal transformations.
2. Show that the inner product of two vector-valued functions $u(x) = (u_1(x), \ldots, u_n(x))$ and $v(x) = (v_1(x), \ldots, v_n(x))$ is invariant under orthogonal transformations.
3. If u and v are harmonic on the open set $\Omega \subset R^n$, show that uv is harmonic on Ω if and only if $\nabla u \cdot \nabla v = 0$.
4. If Ω is an open connected subset of R^n and both u and u^2 are harmonic on Ω, show that u is a constant function.
5. If u is a harmonic function on $B_{0,\delta}$ and is a function of the polar coordinate r only, show that u is a constant function on $B_{0,\delta}$.

2.4 The Mean Value Property

Most of the theorems about harmonic functions have their origin in the works of Gauss and Green concerning electromagnetic and gravitational forces.

Theorem 2.4.1 (Gauss' Integral Theorem) *If u is harmonic on the ball B and $u \in C^1(B^-)$, then*

$$\int_{\partial B} D_{\mathbf{n}} u \, d\sigma = 0.$$

Proof Put $v = 1$ in Green's Identity, Theorem 2.2.2. □

Much more can be obtained from Green's Identity.

Theorem 2.4.2 (Green's Representation Theorem) *If u has continuous second partial derivatives on a closed ball $B^- = B^-_{y,\rho}$, then*

(i) for $n = 2$ and $x \in B$,

$$\begin{aligned} u(x) = &\ \frac{1}{2\pi} \int_{\partial B} ((-\log r) D_{\mathbf{n}} u - u D_{\mathbf{n}}(-\log r)) \, d\sigma(z) \\ &- \frac{1}{2\pi} \int_B \Delta u (-\log r) \, dz; \end{aligned}$$

(ii) for $n \geq 3$ and $x \in B$,

$$\begin{aligned} u(x) = &\ \frac{1}{\sigma_n (n-2)} \int_{\partial B} \left(r^{-n+2} D_{\mathbf{n}} u - u D_{\mathbf{n}} r^{-n+2} \right) \, d\sigma(z) \\ &- \frac{1}{\sigma_n (n-2)} \int_B \Delta u \cdot r^{-n+2} \, dz, \end{aligned}$$

where $r = |x - z|, z \in B^-$.

Proof of (ii) Consider a fixed $x \in B$, and let $v(z) = |x - z|^{-n+2} = r^{-n+2}$ for $z \neq x$. Then v is harmonic on $R^n \sim \{x\}$. Consider $\delta > 0$ for which $B^-_{x,\delta} \subset B_{y,\rho}$, and let Ω be the open set $B_{y,\rho} \sim B^-_{x,\delta}$. By Green's Identity,

$$\int_\Omega (u\Delta v - v\Delta u)\,dz = \int_{\partial\Omega} (uD_{\mathbf{n}}v - vD_{\mathbf{n}}u)\,d\sigma(z).$$

Since v is harmonic on Ω and $\partial\Omega = \partial B_{y,\rho} \cup \partial B_{x,\delta}$,

$$\begin{aligned} -\int_\Omega v\Delta u\,dz = &\int_{\partial B_{y,\rho}} (uD_{\mathbf{n}}v - vD_{\mathbf{n}}u)\,d\sigma(z) \\ &- \int_{\partial B_{x,\delta}} (uD_{\mathbf{n}}v - vD_{\mathbf{n}}u)\,d\sigma(z). \end{aligned} \tag{2.4}$$

The minus sign precedes the second term on the right because the outer unit normal to $\partial\Omega$ at a point of $\partial B_{x,\delta}$ is the negative of the outer unit normal to $\partial B_{x,\delta}$. The next step is to let $\delta \to 0$ in Eq. (2.4). If it can be shown that v is integrable over $B_{y,\rho}$, it would follow from the boundedness of Δu that $v\Delta u$ is integrable and that

$$\lim_{\delta\to 0}\int_\Omega v\Delta u\,dz = \int_{B_{y,\rho}} v\Delta u\,dz \tag{2.5}$$

by the Lebesgue dominated convergence theorem, applied sequentially. Since v is bounded on Ω, it need only be shown that v is integrable on $B_{x,\delta}$. By transforming to spherical coordinates with pole x,

$$\int_{B_{x,\delta}} |v|\,dz = \sigma_n \int_0^\delta r^{-n+2} r^{n-1}\,dr = \sigma_n \delta^2/2.$$

Therefore, v is integrable on $B_{x,\delta}$ and Eq. (2.5) holds. Consider the right side of Eq. (2.4). The first integral does not depend upon δ and only the second must be considered. Since

$$|D_{\mathbf{n}}u| = |\nabla u \cdot \mathbf{n}| \leq |\mathbf{n}||\nabla u| = |\nabla u| = \left(\sum_{i=1}^n u^2_{x_i}\right)^{1/2},$$

and the first partials of u are bounded on $B^-_{y,\rho}$, $|D_{\mathbf{n}}u| \leq m$ on $\partial B_{x,\delta}$ for some constant m. Thus,

$$\left|\int_{\partial B_{x,\delta}} vD_{\mathbf{n}}u\,d\sigma(z)\right| \leq m\int_{\partial B_{x,\delta}} r^{-n+2}\,d\sigma(z) = m\sigma_n\delta$$

and therefore,

$$\lim_{\delta\to 0}\int_{\partial B_{x,\delta}} v D_{\mathbf{n}} u \, d\sigma(z) = 0.$$

Now consider

$$\lim_{\delta\to 0}\int_{\partial B_{x,\delta}} u D_{\mathbf{n}} v \, d\sigma(z).$$

Since $D_{\mathbf{n}} v = D_r v = (-n+2)\delta^{-n+1}$ at points of $\partial B_{x,\delta}$,

$$\int_{\partial B_{x,\delta}} u D_{\mathbf{n}} v \, d\sigma(z) = -\sigma_n(n-2)\left(\frac{1}{\sigma_n \delta^{n-1}}\int_{\partial B_{x.\delta}} u \, d\sigma(z)\right).$$

The factor in parentheses is just the average of the continuous function u on $\partial B_{x,\delta}$, and therefore has the limit $u(x)$ as $\delta \to 0$. This shows that

$$\lim_{\delta\to 0}\int_{\partial B_{x,\delta}} u D_{\mathbf{n}} v \, d\sigma(z) = -(n-2)\sigma_n u(x).$$

Taking the limit as $\delta \to 0$ in Eq. (2.4),

$$-\int_B v\Delta u \, dz = \int_{\partial B}(u D_{\mathbf{n}} v - v D_{\mathbf{n}} u)\, d\sigma(z) + (n-2)\sigma_n u(x),$$

which is the equation in (ii). The proof of (i) is basically the same. □

According to the following theorem, the value of a harmonic function at a point y is equal to the average of its values over a sphere centered at y.

Theorem 2.4.3 (Mean Value Property) *If u is harmonic on a neighborhood of the closed ball $B^-_{y,\rho}$, then*

$$u(y) = \frac{1}{\sigma_n \rho^{n-1}}\int_{\partial B_{y,\rho}} u(z)\, d\sigma(z).$$

Proof Only the $n \geq 3$ case will be proved since the $n = 2$ case is similar. By Theorem 2.4.2 and the fact that $\Delta u = 0$ on $B = B_{y,\rho}$,

$$u(y) = \frac{1}{\sigma_n(n-2)}\int_{\partial B}\left(|y-z|^{-n+2} D_{\mathbf{n}} u - u D_{\mathbf{n}}|y-z|^{-n+2}\right) d\sigma(z).$$

For $z \in \partial B$, $|y-z|^{-n+2} = \rho^{-n+2}$ and

$$D_{\mathbf{n}}|y-z|^{-n+2} = D_r r^{-n+2}|_{r=\rho} = -(n-2)\rho^{-n+1}.$$

After substituting in the above integral,

$$u(y) = \frac{\rho^{-n+2}}{\sigma_n(n-2)} \int_{\partial B} D_{\mathbf{n}} u \, d\sigma(z) + \frac{1}{\sigma_n \rho^{n-1}} \int_{\partial B} u \, d\sigma(z).$$

The first integral on the right is zero by Gauss' Integral Theorem. □

Averaging over a sphere in this theorem can be replaced by averaging over a solid ball.

Theorem 2.4.4 *If u is harmonic on the ball $B = B_{y,\rho}$, then*

$$u(y) = \frac{1}{\nu_n \rho^n} \int_B u(z) \, dz.$$

Proof Using spherical coordinates (r, θ) relative to the pole y,

$$\frac{1}{\nu_n \rho^n} \int_B u(z) \, dz = \frac{1}{\nu_n \rho^n} \int_0^\rho r^{n-1} \left(\int_{|\theta|=1} u(r, \theta) \, d\sigma(\theta) \right) dr.$$

The integral within the parentheses is just the integal over a sphere of radius r relative to a uniformly distributed measure of total mass σ_n and is equal to $\sigma_n u(y)$ by the mean value property. Therefore,

$$\frac{1}{\nu_n \rho^n} \int_B u(z) \, dz = \frac{1}{\nu_n \rho^n} \int_0^\rho r^{n-1} \sigma_n u(y) \, dr = \frac{\sigma_n}{n \nu_n} u(y).$$

The result follows from the fact that $\nu_n = \sigma_n / n$. □

The content of the two preceding theorems is summarized by saying that the harmonic functions are **mean valued** or satisfy the **averaging principle**. Averages occur so frequently that a notation for them will be introduced. If u is a function integrable relative to surface area on the boundary ∂B of $B = B_{y,\rho}$, let

$$\mathbf{L}(u : y, \rho) = \frac{1}{\sigma_n \rho^{n-1}} \int_{\partial B} u \, d\sigma.$$

If u is integrable on B relative to Lebesgue measure, let

$$\mathbf{A}(u : y, \rho) = \frac{1}{\nu_n \rho^n} \int_B u \, dx.$$

Using this notation, if h is harmonic on the open set R, then

$$h(y) = \mathbf{L}(h : y, \rho) = \mathbf{A}(h : y, \rho)$$

whenever $B^-_{y,\rho} \subset R$. A partial converse will be proved later.

2.4.1 Exercises for Sect. 2.4

1. If the function ϕ is convex on the open interval (a, b), show that ϕ cannot attain a maximum value at a point of (a, b) unless it is a constant function.
2. Show that the function $u = 2x(1 - y)$ is harmonic on R^2 and that its average over $\partial B_{0,1}$ is equal to its value at the center of the ball.
3. Verify that the value of the function $u = 2x(1 - y)$ at the point $(\frac{1}{2}, \frac{1}{2})$ is equal to its average over the the unit square $[0, 1] \times [0, 1]$ and equal to its average over the boundary of the square.
4. If u is harmonic on $\Omega \subset R^n$ and $B^- = B^-_{x,\rho} \subset \Omega$, use the averaging property and the Divergence Theorem to show that $|Du(x)| \leq (n/\rho) \sup_{y \in \partial B} |u(y)|$.
5. If u is harmonic on the bounded set $\Omega \subset R^n$, Γ is a compact subset of Ω, and α is any multi-index, then

$$\sup_{x \in \Gamma} |D^\alpha u(x)| \leq \left(\frac{n|\alpha|}{d(\Gamma, \sim \Omega)} \right)^{|\alpha|} \sup_{z \in \Omega} |u(z)|.$$

(Hint: For $x \in \Gamma$, consider the nested sequence of balls $B_1 \subset B_2 \subset \ldots \subset B_{|\alpha|}$ where $B_j = B_{x,j\delta}$, $\delta = d(x, \sim \Omega)/|\alpha|$).

2.5 Poisson Integral Formula

In the preceding section, it was shown that the value of a harmonic function at a point y is equal to the average of its values over a sphere centered at y. It is natural to ask if a similar result is true of any point inside the sphere. If u is harmonic on a neighborhood of the closed ball $B^- = B^-_{y,\rho}$, then Green's representation theorem in the $n \geq 3$ cases reduces to

$$u(x) = \frac{1}{\sigma_n(n-2)} \int_{\partial B} (r^{-n+2} D_{\mathbf{n}} u - u D_{\mathbf{n}} r^{-n+2})\, d\sigma(z)$$

where $r = |x - z|, x \in B, z \in \partial B$. This equation suggests the possibility of representing the harmonic function u in terms of the restriction of u to the boundary of B. For instance, if for each $x \in B$ a harmonic function v_x could be found such that $r^{-n+2} + v_x(z)$ vanishes on ∂B, then replacing r^{-n+2} by this sum would result in a representation of u in terms of just $u|_{\partial B}$ and would not involve $D_{\mathbf{n}} u$ on ∂B. This is what will be done in the proof of the next theorem for the $n \geq 3$ case; the proof in the $n = 2$ case is similar. The proof requires a mapping that allows passage back and forth between the interior and exterior of a sphere, called an **inversion**, which was first introduced by Lord Kelvin [2].

Fix $B = B_{y,\rho}$ and $x \in B$, and consider the radial line joining y to x. For $x \neq y$, choose x^* on this radial line so that

$$|x - y||x^* - y| = \rho^2. \tag{2.6}$$

Then

$$x^* = y + \frac{\rho^2}{|y - x|^2}(x - y) \tag{2.7}$$

and is called the **inverse** of x relative to the sphere ∂B. Consider any $z \in \partial B$, and let ϕ be the angle between $z - y$ and $x - y$. Then

$$\begin{aligned} |z - x^*|^2 &= \rho^2 + |x^* - y|^2 - 2\rho|x^* - y|\cos\phi \\ |z - x|^2 &= \rho^2 + |x - y|^2 - 2\rho|x - y|\cos\phi. \end{aligned}$$

Replacing $|x^* - y|$ by $\rho^2/|x - y|$ in the first equation, solving for $\cos\phi$ in the second, and substituting for $\cos\phi$ in the first,

$$|z - x^*|^2 = \rho^2 \frac{|z - x|^2}{|y - x|^2}.$$

Thus, for $x \in B$, $x \neq y$, $z \in \partial B$, and x^* the inverse of x relative to ∂B,

$$\frac{|y - x|}{\rho} \frac{|z - x^*|}{|z - x|} = 1. \tag{2.8}$$

Theorem 2.5.1 *If u is harmonic on a neighborhood of the closure of the ball $B = B_{y,\rho}$, $x \in B$, and $x \neq y$, then*

(i) *for $n = 2$,*

$$u(x) = -\frac{1}{2\pi} \int_{\partial B} u D_{\mathbf{n}} \left(\log \frac{|y - x|}{\rho} \frac{|z - x^*|}{|z - x|} \right) d\sigma(z),$$

(ii) *for $n \geq 3$,*

$$u(x) = -\frac{1}{\sigma_n (n - 2)} \int_{\partial B} u D_{\mathbf{n}} \left(\frac{1}{|z - x|^{n-2}} - \frac{\rho^{n-2}}{|y - x|^{n-2}} \frac{1}{|z - x^*|^{n-2}} \right) d\sigma(z)$$

where x^* is the inverse of x relative to $\partial B_{y,\rho}$.

Proof of (ii) By Green's representation theorem,

$$u(x) = \frac{1}{\sigma_n(n-2)} \int_{\partial B} \left(\frac{1}{|z-x|^{n-2}} D_{\mathbf{n}} u - u D_{\mathbf{n}} \left(\frac{1}{|z-x|^{n-2}} \right) \right) d\sigma(z) \quad (2.9)$$

since $\Delta u = 0$ on B. Since $x^* \notin B^-$, $|z - x^*|^{-n+2}$ is harmonic on a neighborhood of B^- and it follows from Green's Identity that

$$0 = \frac{1}{\sigma_n(n-2)} \int_{\partial B} \left(\frac{1}{|z-x^*|^{n-2}} D_{\mathbf{n}} u - u D_{\mathbf{n}} \left(\frac{1}{|z-x^*|^{n-2}} \right) \right) d\sigma(z). \quad (2.10)$$

Multiplying both sides of this equation by α and subtracting from Eq. (2.9),

$$\begin{aligned} u(x) = \ & \frac{1}{\sigma_n(n-2)} \int_{\partial B} \left[\left(\frac{1}{|z-x|^{n-2}} - \alpha \frac{1}{|z-x^*|^{n-2}} \right) D_{\mathbf{n}} u \right. \\ & \left. - u D_{\mathbf{n}} \left(\frac{1}{|z-x|^{n-2}} - \alpha \frac{1}{|z-x^*|^{n-2}} \right) \right] d\sigma(z). \end{aligned}$$

Letting $\alpha = (\rho/|x-y|)^{n-2}$ and using Eq. (2.8),

$$\alpha = \frac{\rho^{n-2}}{|y-x|^{n-2}} = \frac{|z-x^*|^{n-2}}{|z-x|^{n-2}}, \qquad z \in \partial B,$$

and the integral of the first term on the right is zero. □

The functions within the parentheses in (i) and (ii) of the preceding theorem are called Green functions for the ball B.

Definition 2.5.2 If $n = 2$, the **Green function** for the ball $B = B_{y,\rho}$ is given by

$$G_B(x,z) = \begin{cases} \log \frac{|y-x|}{\rho} \frac{|z-x^*|}{|z-x|} & \text{for } z \in B \sim \{x\}, x \neq y \\ \log \frac{\rho}{|z-x|} & \text{for } z \in B \sim \{x\}, x = y \\ +\infty & \text{for } z = x \end{cases}$$

where x^* is the inverse of x with respect to ∂B. If $n \geq 3$, the **Green function** is given by

$$G_B(x,z) = \begin{cases} \frac{1}{|z-x|^{n-2}} - \frac{\rho^{n-2}}{|x-y|^{n-2}} \frac{1}{|z-x^*|^{n-2}} & \text{for } z \in B \sim \{x\}, x \neq y \\ \frac{1}{|z-x|^{n-2}} - \frac{1}{\rho^{n-2}} & \text{for } z \in B \sim \{x\}, x = y \\ +\infty & \text{for } z = x. \end{cases}$$

Using Eq. (2.8), it is easy to see that $\lim_{z \to z_0} G_B(x,z) = 0$ for all $z_0 \in \partial B$ and fixed $x \in B$. Note also that $G_B(x,z)$ has been defined by continuity for $x = y$ since x^* is not defined when $x = y$. The appropriate value of $G_B(x,z)$ when $x = y$ can be obtained as follows in the $n = 2$ case. By Eqs. (2.7) and (2.8),

$$\frac{|y-x|}{\rho}\frac{|z-x^*|}{|z-x|} = \frac{|y-x|\left|(z-y) - \frac{\rho^2}{|y-x|^2}(x-y)\right|}{\rho|z-x|} \to \frac{\rho}{|z-y|}$$

as $x \to y$, $x \neq y$; that is, $\lim_{x\to y} G_B(x,z) = G_B(y,z)$, $z \in B \sim \{y\}$.

If u is harmonic on a neighborhood of the closure of $B = B_{y,\rho}$, then the equations in (i) and (ii) of the preceding theorem can be written, respectively, as follows. For $n = 2$,

$$u(x) = -\frac{1}{2\pi}\int_{\partial B} u D_{\mathbf{n}} G_B(x,z)\,d\sigma(z), \quad x \in B;$$

for $n \geq 3$

$$u(x) = -\frac{1}{\sigma_n(n-2)}\int_{\partial B} u D_{\mathbf{n}} G_B(x,z)\,d\sigma(z). \tag{2.11}$$

The proof of the following theorem is the same as that of Theorem 2.5.1 except for the inclusion of the term involving Δu in Green's representation theorem.

Theorem 2.5.3 *If u has continuous second partials on closure of the ball $B = B_{y,\rho}$, then*

(i) for each $x \in B$ and $n = 2$,

$$u(x) = -\frac{1}{2\pi}\int_{\partial B} u(z) D_{\mathbf{n}} G_B(x,z)\,d\sigma(z) - \frac{1}{2\pi}\int_B G_B(x,z)\Delta u(z)\,dz;$$

(ii) for each $x \in B$ and $n \geq 3$,

$$u(x) = -\frac{1}{\sigma_n(n-2)}\int_{\partial B} u(z) D_{\mathbf{n}} G_B(x,z)\,d\sigma(z) - \frac{1}{\sigma_n(n-2)}\int_B G_B(x,z)\Delta u(z)\,dz.$$

By evaluating the normal derivatives of the Green functions in these equations, a representation of the harmonic function u can be obtained that does not involve the inverse x^*. The following theorem will be proved only in the $n = 3$ case; the proof of the $n = 2$ case is similar.

Theorem 2.5.4 (Poisson Integral Formula) *If u is harmonic on the ball $B = B_{y,\rho}$, $u \in C^0(B^-)$, and $x \in B$, then*

$$u(x) = \frac{1}{\sigma_n \rho}\int_{\partial B} \frac{\rho^2 - |y-x|^2}{|z-x|^n} u(z)\,d\sigma(z).$$

Proof Suppose first that u is harmonic on a neighborhood of B^-. The gradient of $G_B(x,\cdot)$ is easily seen to be

$$\nabla_{(z)} G_B(x,z) = -(n-2)\left(\frac{1}{|z-x|^n}(z-x) - \frac{\rho^{n-2}}{|x-y|^{n-2}}\frac{1}{|z-x^*|^n}(z-x^*)\right).$$

Using the definition of x^* and Eq. (2.8),

$$\nabla_{(z)} G_B(x,z) = -\frac{(n-2)}{|z-x|^n}\left((z-y) - \frac{|y-x|^2}{\rho^2}(z-y)\right).$$

Since the outer unit normal to ∂B at $z \in \partial B$ is $(z-y)/\rho$,

$$\begin{aligned} D_{\mathbf{n}} G_B(x,z) &= \nabla G_B(x,z) \cdot \frac{z-y}{\rho} \\ &= -\frac{(n-2)}{\rho|z-x|^n}(\rho^2 - |y-x|^2). \end{aligned} \tag{2.12}$$

Substituting for $D_{\mathbf{n}} G_B(x,z)$ in Eq. (2.11),

$$u(x) = \frac{1}{\sigma_n \rho}\int_{\partial B} u(z)\frac{\rho^2 - |y-x|^2}{|z-x|^n}\, d\sigma(z).$$

Suppose now that u is only harmonic on B and continuous on B^-. Fix $x \in B_{y,\rho}$ and let $\{\rho_k\}$ be a sequence of positive numbers increasing to ρ with $x \in B_{y,\rho_k}$ for all $k \geq 1$. Since u is harmonic on a neighborhood of each B_{y,ρ_k},

$$u(x) = \frac{1}{\sigma_n \rho_k}\int_{\partial B_{y,\rho_k}} u(z)\frac{\rho_k^2 - |y-x|^2}{|z-x|^2}\, d\sigma(z).$$

Letting $z = \rho_k \theta$,

$$u(x) = \frac{1}{\sigma_n \rho_k}\int_{|\theta|=1} u(\rho_k\theta)\frac{\rho_k^2 - |y-x|^2}{|\rho_k\theta - x|^2}\rho_k^{n-1}\, d\sigma(\theta).$$

By uniform boundedness of the integrands and continuity,

$$\begin{aligned} u(x) &= \frac{1}{\sigma_n \rho}\int_{|\theta|=1} u(\rho\theta)\frac{\rho^2 - |y-x|^2}{|\rho\theta - x|^2}\rho^{n-1}\, d\sigma(\theta) \\ &= \frac{1}{\sigma_n \rho}\int_{\partial B} u(z)\frac{\rho^2 - |y-x|^2}{|z-x|^2}\, d\sigma(z). \end{aligned}$$

□

The definition of harmonicity of a function u on an open set Ω required that $u \in C^2(\Omega)$. Much more is true as a consequence.

Lemma 2.5.5 *Let U and C be open and compact subsets of R^n, respectively, and let $u = u(x,y)$ be a real-valued function on $U \times C$ with the property that $\partial u/\partial x_j$ is continuous on $U \times C$ for each $j = 1,\ldots,n$. Then*

$$\frac{\partial}{\partial x_j}\int_C u(x,y)\,d\mu(y) = \int_C \frac{\partial u}{\partial x_j}(x,y)\,d\mu(y) \qquad x \in U,\ j = 1,\ldots,n,$$

for any finite measure μ on the Borel subsets of C.

Proof Let $\boldsymbol{\nu}$ be a unit vector. It will be shown now that

$$D_{\boldsymbol{\nu}}\int_C u(x,y)\,d\mu(y) = \int_C D_{\boldsymbol{\nu}}u(x,y)\,d\mu(y).$$

Consider a fixed $x \in U$. The left side of this equation is

$$\lim_{\lambda\to 0+}\int_C \frac{u(x+\lambda\boldsymbol{\nu},y) - u(x,y)}{\lambda}\,d\mu(y).$$

Let B be a ball with center x such that $B^- \subset U$, and consider only those λ for which $x + \lambda\boldsymbol{\nu} \in B$. By the mean value theorem of the calculus, for each such λ and each $y \in C$, there is a point $\xi_{\lambda,y}$ on the line segment joining x to $x + \lambda\boldsymbol{\nu}$ such that

$$\begin{aligned}\int_C \frac{u(x+\lambda\boldsymbol{\nu}) - u(x,y)}{\lambda}\,d\mu(y) &= \int_C \frac{1}{\lambda}(\nabla u(\xi_{\lambda,y})\cdot\lambda\boldsymbol{\nu})\,d\mu(y)\\ &= \int_C D_{\boldsymbol{\nu}}u(\xi_{\lambda,y},y)\,d\mu(y).\end{aligned}$$

Since $D_{\boldsymbol{\nu}}u$ is continuous on $B^- \times C$, the integrand on the right is uniformly bounded for such λ and $y \in C$. Since $\mu(C) < +\infty$, the Lebesgue dominated convergence theorem can be applied as $\lambda \to 0+$, sequentially, to obtain

$$D_{\boldsymbol{\nu}}\int_C u(x,y)\,d\mu(y) = \int_C D_{\boldsymbol{\nu}}u(x,y)\,d\mu(y).$$ □

Theorem 2.5.6 *If u is harmonic on the open set Ω, then all partial derivatives of u are harmonic on Ω.*

Proof If $y \in \Omega$, $B = B_{y,\rho} \subset B^-_{y,\rho} \subset \Omega$, and $x \in B$, then

$$u(x) = \frac{1}{\sigma_n\rho}\int_{\partial B}\frac{\rho^2 - |y-x|^2}{|z-x|^n}u(z)\,d\sigma(z).$$

By the preceding lemma,

$$\frac{\partial u}{\partial x_j}(x) = \frac{1}{\sigma_n\rho}\int_{\partial B}\frac{\partial}{\partial x_j}\left(\frac{\rho^2 - |y-x|^2}{|z-x|^n}\right)u(z)\,d\sigma(z). \tag{2.13}$$

By calculating the partial derivative under the integral sign and using the Lebesgue dominated convergence theorem, it can be shown that $\partial u/\partial x_j$ is continuous on B.

In the same way, it can be shown that partial derivatives of all orders are continuous on B. Since B can be any ball with $B^- \subset \Omega$, partial derivatives of all orders of u are continuous on Ω. Since

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \text{ on } \Omega,$$

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_j} \frac{\partial^2 u}{\partial x_i^2} = 0 \text{ on } \Omega$$

for each $j = 1, \ldots, n$. Using continuity of the third partial derivatives, the order of differentiation can be interchanged to obtain

$$\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \frac{\partial u}{\partial x_j} = 0, \quad j = 1, \ldots, n.$$

Since $\partial u/\partial x_j$ has continuous second partials on Ω, $\partial u/\partial x_j$ is harmonic on Ω, as well as all higher-order partial derivatives. □

Theorem 2.5.7 (Picard) *If u is harmonic on R^n and either bounded above or bounded below, then u is a constant function.*

Proof Since $-u$ is harmonic if u is harmonic, it can be assumed that u is bounded below. Since the sum of a harmonic function and a constant function is harmonic, it can be assumed that $u \geq 0$. Let x and y be distinct points, and consider balls $B_{x,\delta}$ and $B_{y,\epsilon}$ with $B_{y,\epsilon} \supset B_{x,\delta}$. By Theorem 2.4.4,

$$\nu_n \delta^n u(x) = \int_{B_{x,\delta}} u(z)\, dz \leq \int_{B_{y,\epsilon}} u(z)\, dz = \nu_n \epsilon^n u(y)$$

and $u(x) \leq (\epsilon/\delta)^n u(y)$. Now let $\epsilon, \delta \to +\infty$ in such a way that $\epsilon/\delta \to 1$ to obtain $u(x) \leq u(y)$. But since x and y are arbitrary points, $u(y) \leq u(x)$ and u is a constant function. □

According to the Poisson integral formula, if a harmonic function is zero on the boundary of a ball, then it is zero on the ball. This fact suggests the following principle, known as the maximum principle.

Definition 2.5.8 [1] A function u defined on an open connected set Ω obeys the **maximum principle** if $\sup_{x \in \Omega} u(x)$ is not attained on Ω unless u is constant on Ω; the **minimum principle** if $\inf_{x \in \Omega} u(x)$ is not attained on Ω unless u is constant on Ω.

Theorem 2.5.9 *If u is continuous on the open connected set Ω and for each $x \in \Omega$ there is a $\delta_x > 0$ such that*

[1] This version of the maximum principle will be referred to later as the strong maximum principle.

$$u(x) = \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B_{x,\delta}} u(y)\, d\sigma(y)$$

whenever $\delta < \delta_x$, then u obeys the maximum and minimum principles.

Proof Suppose u attains its maximum value at a point of Ω and let $M = \{x \in \Omega; u(x) = \sup_{y \in \Omega} u(y)\} \neq \emptyset$. Since u is continuous, M is relatively closed in Ω. It will be shown now that M is open in Ω. For $x \in M$,

$$u(x) = \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B_{x,\delta}} u(y)\, d\sigma(y)$$

whenever $\delta < \delta_x$. Consider any $y \in B_{x,\delta_x}$ and let $\delta_0 = |y - x| < \delta_x$. Since

$$u(x) = \frac{1}{\sigma_n \delta_0^{n-1}} \int_{\partial B_{x,\delta_0}} u(x)\, d\sigma(y),$$

$$\int_{\partial B_{x,\delta_0}} (u(x) - u(y))\, d\sigma(y) = 0.$$

Since $x \in M$, $u(x) - u \geq 0$ on $\partial B_{x,\delta_0}$ and therefore $u(x) - u = 0$ a.e. on $\partial B_{x,\delta_0}$. By continuity, $u = u(x)$ on $\partial B_{x,\delta_0}$ and, in particular, $u(y) = u(x)$. This shows that $y \in M$, that $B_{x,\delta_x} \subset M$, and that M is both relatively closed and open in Ω. By the connectedness of Ω, either $M = \emptyset$ or $M = \Omega$. Since the first case has been excluded, $M = \Omega$ so that u is constant on Ω. □

Corollary 2.5.10 *If u is harmonic on the open connected set Ω, then u satisfies both the maximum and minimum principles.*

2.5.1 Exercises for Sect. 2.5

1. Show that the inversion $x \to x^*$ relative to the sphere $\partial B_{0,\rho}$ defined by the equation $x^* = (\rho^2/|x|^2)x$, $x = (x_1, x_2, \ldots x_n)$, $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ maps lines and circles into lines and circles, not necessarily respectively; that is, the equation

$$a \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} b_i x_i + c = 0$$

is transformed into the equation

$$a' \sum_{i=1}^{n} x_i^{*2} + \sum_{i=1}^{n} b' x_i^* + c' = 0.$$

2. Using the equation $|x^*||x| = \rho^2$, where x^* is the image of x under the inversion of the preceding problem, determine the image of the sphere $(x_1 - \frac{\rho}{2})^2 + \sum_{i=2}^n x_i^2 = \frac{\rho^2}{4}$ and the image of the sphere $(x_1 - \frac{3}{4}\rho)^2 + \sum_{i=2}^n x_i^2 = \frac{\rho^2}{16}$.
3. Verify that the Green function $G_B(x, z)$ is a symmetric function of x and z. (Hint: Consider $x_0, z_0 \in B$, $\epsilon > 0$, $B^-_{x_0,\epsilon} \subset B$, $B_{z_0,\epsilon} \subset B$, $\Omega_\epsilon = B \sim (B_{x_0,\epsilon} \cup B_{z_0,\epsilon})$, $u(x) = G_B(x, x_0)$, $v(z) = G_B(z, z_0)$. Apply Green's Identity to u and v on Ω_ϵ and let $\epsilon \Rightarrow 0$.)
4. Prove Theorem 2.5.4 in the $n = 2$ case.
5. Under the conditions of Theorem 2.5.4, show that the Poisson Integral in the $n = 2$ and $B = B_{0,1}$ case can be transformed into polar coordinates as

$$u(r, \theta) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{1}{1 + r^2 - r\cos(\theta - \phi)} u(1, \phi)\, d\phi$$

where (r, θ) are the polar coordinates of a point in B.
6. If $f(r, \theta) = 1$ if $0 \le \theta < \pi$ and $f(r, \theta) = 0$ for $\pi \le \theta < 2\pi$, find a function u harmonic on the ball $B_{0,1} \subset R^2$ corresponding to the boundary function $f(r, \theta)$.

2.6 Gauss' Averaging Principle

Among the many contributions of Gauss to potential theory, one of the best known is the assertion that the gravitational potential at a point in space due to a homogeneous spherical body is the same as if the entire mass were concentrated at the center of the body. A classical version of this property will be stated first.

Lemma 2.6.1 *If $B_{x,\delta} \subset R^n$, then*

(i) for $n = 2$ and $y \in R^2$,

$$\frac{1}{2\pi\delta} \int_{\partial B_{x,\delta}} \log \frac{1}{|y - z|}\, d\sigma(z) = \begin{cases} \log \frac{1}{|y-x|} & \text{if } |y - x| > \delta \\ \log \frac{1}{\delta} & \text{if } |y - x| \le \delta \end{cases}$$

(ii) for $n \ge 3$ and $y \in R^n$,

$$\frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B_{x,\delta}} \frac{1}{|y - z|^{n-2}}\, d\sigma(z) = \begin{cases} \frac{1}{|y-x|^{n-2}} & \text{if } |y - x| > \delta \\ \frac{1}{\delta^{n-2}} & \text{if } |y - x| \le \delta. \end{cases}$$

Proof Only (i) will be proved, (ii) being the easier of the two. Fix $x \in R^2$ and $\delta > 0$, and let $u_y(z) = -\log|y - z|$, $z \in R^2$. Three cases will be considered according as $y \in B_{x,\delta}$, $y \in \partial B_{x,\delta}$, or $y \notin B^-_{x,\delta}$. Suppose first that $y \notin B^-_{x,\delta}$. Then u_y is harmonic on a neighborhood of $B^-_{x,\delta}$ and the result follows from the mean value property.

Consider now the case $y \in \partial B_{x,\delta}$. For each $n \geq 1$, let $y_n = x + (1 + \frac{1}{n})(y - x)$. Then $|y_n - x| = (1 + \frac{1}{n})|y - x| = (1 + \frac{1}{n})\delta$, $|y_n - z| \leq 3\delta$ for all $z \in \partial B_{x,\delta}$, and

$$\log \frac{1}{|y_n - z|} \geq \log \frac{1}{3\delta}, \qquad z \in \partial B_{x,\delta}.$$

It follows that the sequence of functions on the left is lower bounded on $\partial B_{x,\delta}$. By the preceding case and Fatou's Lemma,

$$\begin{aligned}\frac{1}{2\pi\delta}\int_{\partial B_{x,\delta}} \log \frac{1}{|y - z|}\, d\sigma(z) &\leq \liminf_{n\to\infty} \frac{1}{2\pi\delta}\int_{\partial B_{x,\delta}} \log \frac{1}{|y_n - z|}\, d\sigma(z)\\ &= \liminf_{n\to\infty}\left(\log \frac{1}{|y_n - x|}\right)\\ &= \log \frac{1}{|y - x|}.\end{aligned}$$

On the other hand, since $|y_n - z| \geq |y - z|$ for $z \in \partial B_{x,\delta}$,

$$\begin{aligned}\log \frac{1}{|y - x|} &= \lim_{n\to\infty} \log \frac{1}{|y_n - x|}\\ &= \lim_{n\to\infty} \frac{1}{2\pi\delta}\int_{\partial B_{x,\delta}} \log \frac{1}{|y_n - z|}\, d\sigma(z)\\ &\leq \frac{1}{2\pi\delta}\int_{\partial B_{x,\delta}} \log \frac{1}{|y - z|}\, d\sigma(z),\end{aligned}$$

and the assertion is true for $y \in \partial B_{x,\delta}$. Lastly, suppose $y \in B_{x,\delta}$. Letting $u_{x,\delta}$ denote the left side of the equation in (i) and applying Lemma 2.5.5 to it twice, it is easily seen that $u_{x,\delta}$ is harmonic on $B_{x,\delta}$. Letting γ denote the angle between the line segment joining x to y and x to z,

$$u_{x,\delta}(y) = \frac{1}{2\pi\delta}\int_0^{2\pi} \log \frac{1}{|y - x|^2 + \delta^2 - 2\delta|y - x|\cos\gamma}\, d\gamma$$

and it is easily seen that $u_{x,\delta}$ is a function of $r = |y - x|$ only and, as in Sect. 2.3, must be of the form $\alpha \log r + \beta$. Since

$$u_{x,\delta}(x) = \frac{1}{2\pi\delta}\int_{\partial B_{x,\delta}} \log \frac{1}{|x - z|}\, d\sigma(z) = \log \frac{1}{\delta},$$

$\alpha = 0$ and $\beta = \log(1/\delta)$; that is, $u_{x,\delta}(y) = \log(1/\delta)$, $y \in B_{x,\delta}$. □

The preceding lemma can be interpreted as a statement about the average value of $\log(1/|y - z|)$ or $1/|y - z|^{n-2}$ for a unit mass concentrated on the point y. Arbitrary mass distributions will be considered now.

Some preliminary calculations will be carried out for the $n = 2$ and $n \geq 3$ cases separately. If μ is a measure on the Borel subsets of R^2 with compact support S_μ, let

$$\mathbf{U}^\mu(y) = \int \log \frac{1}{|y-z|}\, d\mu(z) = \int_{S_\mu} \log \frac{1}{|y-z|}\, d\mu(z), \quad y \in R^2.$$

For a fixed $y \in R^2$, $\log(1/|y-z|)$ is lower bounded on S_μ and the above integral is defined as a real number or $+\infty$. Consider a fixed ball $B_{x,\delta}$. Since $\log(1/|y-z|)$ is lower bounded on $\partial B_{x,\delta} \times S_\mu$, Tonelli's theorem (c.f. [3]) implies that

$$\begin{aligned}\frac{1}{2\pi\delta}\int_{\partial B_{x,\delta}} \mathbf{U}^\mu(y)\, d\sigma(y) &= \frac{1}{2\pi\delta}\int_{\partial B_{x,\delta}}\int_{S_\mu} \log\frac{1}{|y-z|}\, d\mu(z)d\sigma(y)\\ &= \int_{S_\mu}\left(\frac{1}{2\pi\delta}\int_{\partial B_{x,\delta}} \log\frac{1}{|y-z|}\, d\sigma(y)\right)d\mu(z).\end{aligned}$$

If μ is a measure on the Borel subsets of R^n, $n \geq 3$, with compact support S_μ, let

$$\mathbf{U}^\mu(y) = \int \frac{1}{|y-z|^{n-2}}\, d\mu(z) = \int_{S_\mu} \frac{1}{|y-z|^{n-2}}\, d\mu(z).$$

The same argument implies that

$$\frac{1}{\sigma_n\delta^{n-1}}\int_{\partial B_{x,\delta}} \mathbf{U}^\mu(y)\, d\sigma(y) = \int_{S_\mu}\left(\frac{1}{\sigma_n\delta^{n-1}}\int_{\partial B_{x,\delta}} \frac{1}{|y-z|^{n-2}}\, d\sigma(y)\right)d\mu(z).$$

Only the $n = 2$ case of the following theorem will be proved, the $n \geq 3$ case being essentially the same.

Theorem 2.6.2 (Gauss' Averaging Principle [4]) *Let μ be a measure on the Borel subsets of R^n with compact support S_μ and let $B = B_{x,\delta}$. If $S_\mu \cap B^- = \emptyset$, then the average of $\mathbf{U}^\mu$ over ∂B is $\mathbf{U}^\mu(x)$; if $S_\mu \subset B^-$, the average of $\mathbf{U}^\mu$ over ∂B depends only upon the total mass of μ and is equal to $\mu(S_\mu)\log(1/\delta)$ in the $n = 2$ case and is equal to $\mu(S_\mu)/\delta^{n-2}$ in the $n \geq 3$ case.*

Proof ($n = 2$) From the above discussion,

$$\frac{1}{2\pi\delta}\int_{\partial B_{x,\delta}} \mathbf{U}^\mu(y)\, d\sigma(y) = \int_{S_\mu}\left(\frac{1}{2\pi\delta}\int_{\partial B_{x,\delta}} \log\frac{1}{|y-z|}\, d\sigma(y)\right)d\mu(z).$$

If $S_\mu \subset B^-$, it follows from Lemma 2.6.1 that

$$\frac{1}{2\pi\delta}\int_{\partial B_{x,\delta}} \mathbf{U}^{\mu}(y)\,d\sigma(y) = \int_{S_\mu} \log\frac{1}{\delta}\,d\mu(z)$$
$$= \mu(S_\mu)\log(1/\delta);$$

if $S_\mu \cap B^- = \emptyset$, then $\log(1/|y-z|)$ is a harmonic function of y and

$$\frac{1}{2\pi\delta}\int_{\partial B_{x,\delta}} \mathbf{U}^{\mu}(y)\,d\sigma(y) = \int_{S_\mu} \log\frac{1}{|x-z|}\,d\mu(z) = U^{\mu}(x)$$

by the mean value property. □

2.6.1 Exercises for Sect. 2.6

1. Prove (ii) of Lemma 2.6.1
2. If a thin wire with endpoints at $(a, 0, 0)$ and $(b, 0, 0)$ has a uniform density of mass δ, find the potential at any point $P = (x, y, z)$ in R^3. Show that the potential becomes infinite as the point P approaches a point on the wire.
3. If a thin circular disk of radius r in R^3 with center at $(0, 0, 0)$, has the z-axis as its axis, and has a uniform density δ of mass, find the potential $U(P)$ at any point $P = (0, 0, t)$ on the axis. Show that the potential is continuous on the z-axis, the integral is convergent at $z = 0$, and the gravitational force in the direction of the z-axis has a jump of magnitude $4\delta\pi$ at $(0, 0, 0)$.
4. Show that the potential $U(x)$ at a point $x \in R^3$ due to a uniformly distributed mass of density $\delta > 0$ on a thin spherical shell of radius a is $4\pi\delta a^2/|x|$ if x is external to the shell and is $4\pi\delta a$ if x is interior to the sphere.
5. If $B = B_{0,R} \subset R^3$ is a ball with uniform density $\delta > 0$ and P is a point at a distance r from the center of the ball, show that the value of $\mathbf{U}$ at P is $\frac{4}{3}\pi\delta R^3/r$ when $r > R$. What can be said about the value of $\mathbf{U}$ at points interior to B? (Hint: Choose a rectangular coordinate system so that P is a point on the z axis and use spherical coordinates to evaluate the integral.)
6. If μ is a measure on $B_{0,a}$ of density $\delta(\rho) \geq 0$, $0 < \rho < a$, show that

$$\mathbf{U}(\mathbf{x}) = \begin{cases} \frac{4\pi}{|x|}\int_0^{|x|}\delta(\rho)\rho^2\,d\rho + 4\pi\int_{|x|}^{a}\delta(\rho)\,d\rho & \text{if } |x| < a \\ \frac{4\pi}{|x|}\int_0^{a}\delta(\rho)\rho^2\,d\rho & \text{if } |x| \geq a. \end{cases}$$

2.7 The Dirichlet Problem for a Ball

If Ω is a nonempty open subset of R^n with compact closure and f is a real-valued function on $\partial\Omega$, the **Dirichlet problem** is that of finding a harmonic function u on Ω such that $\lim_{y\to x, y\in\Omega} u(y) = f(x)$ for all $x \in \partial\Omega$. As was noted in Sect. 1.1, this limiting behavior of u at the boundary of Ω implies that f is continuous on $\partial\Omega$. Generally speaking, the Dirichlet problem does not have a solution even when Ω is a ball.

Theorem 2.7.1 *The solution of the Dirichlet problem for a nonempty, open connected set Ω with compact closure and a continuous boundary function f is unique if it exists.*

Proof Let u_1 and u_2 be two solutions. Suppose there is a point $z \in \Omega$ such that $u_1(z) > u_2(z)$. Then $\lim_{y\to x, y\in\Omega}[u_1(y) - u_2(y)] = 0$ for all $x \in \partial\Omega$. Let

$$w = \begin{cases} u_1 - u_2 & \text{on } \Omega \\ 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, w is harmonic on Ω, continuous on Ω^-, and zero on $\partial\Omega$. Since $w(z) > 0$ and w is continuous on Ω^-, w must attain a positive supremum at some point of Ω. By the maximum principle, Corollary 2.5.10, w must be constant on Ω. Since $w = 0$ on $\partial\Omega$, $w = 0$ on Ω^-, a contradiction. Therefore, $u_1 \le u_2$ on Ω. Interchanging u_1 and u_2, $u_2 \le u_1$, and the two are equal. □

According to the Poisson integral formula, the value of a harmonic function u at an interior point of a ball B is determined by values of u on ∂B, assuming that u has a continuous extension to ∂B. It is natural to ask if a function f on ∂B determines a function u harmonic on B that has a continuous extension to B^- agreeing with f on ∂B. More generally, it might be asked if a measure on the boundary of B determines a harmonic function on B.

Theorem 2.7.2 (Herglotz [5]) *If μ is a signed measure of bounded variation on the Borel subsets of $\partial B_{y,\rho}$, then*

$$u(x) = \frac{1}{\sigma_n \rho} \int_{\partial B_{y,\rho}} \frac{\rho^2 - |y - x|^2}{|z - x|^n} \, d\mu(z), \qquad x \in B_{y,\rho},$$

is harmonic on $B_{y,\rho}$.

Proof Using Lemma 2.5.5, it can be shown that u has continuous second partials and that

$$\Delta u(x) = \frac{1}{\sigma_n \rho} \int_{\partial B_{y,\rho}} \Delta_{(x)} \frac{\rho^2 - |y - x|^2}{|z - x|^n} \, d\mu(z).$$

A tedious, but straightforward, differentiation shows that the integrand is zero for $x \in B_{y,\rho}$. □

Corollary 2.7.3 *If f is a Borel measurable function on $\partial B_{y,\rho}$ and integrable relative to surface area, then*

$$u(x) = \frac{1}{\sigma_n \rho} \int_{\partial B_{y,\rho}} \frac{\rho^2 - |y - x|^2}{|z - x|^n} f(z)\, d\sigma(z)$$

is harmonic on $B_{y,\rho}$.

Since the Poisson integral formula will be referred to repeatedly, let

$$\mathbf{PI}(\mu : B)(x) = \frac{1}{\sigma_n \rho} \int_{\partial B} \frac{\rho^2 - |y - x|^2}{|z - x|^n}\, d\mu(z),$$

where $B = B_{y,\rho}$ and μ is a signed measure of bounded variation on ∂B. If $B = B_{y,\rho}$, the dependence on the parameters y and ρ will be exhibited by using the notation $\mathbf{PI}(\mu : y, \rho)$. If μ is absolutely continuous relative to surface area on ∂B, then for each Borel set $M \in \partial B$

$$\mu(M) = \int_M f(z)\, d\sigma(z)$$

for some integrable function f on ∂B. In this case, let $\mathbf{PI}(\mu : B) = \mathbf{PI}(f : B)$. This notation will also be used if the domain of f contains ∂B. Note that $\mathbf{PI}(1 : B) = 1$, that $\mathbf{PI}(\mu : B)$ is linear in μ, that $\mathbf{PI}(\mu : B) \geq 0$ if μ is a measure, and that $\mathbf{PI}(f : B) \geq 0$ if f is nonnegative.

According to Corollary 2.7.3, an integrable boundary function f determines a harmonic function u on a ball B. In what way is u related to f? For example, is it true that $\lim_{y \to x, y \in B} u(y) = f(x)$ for $x \in \partial B$? The following three lemmas answer this question for a ball $B = B_{y,\rho}$.

Lemma 2.7.4 *If f is Borel measurable on ∂B, integrable relative to surface area on ∂B, $u = \mathbf{PI}(f : B)$ on B, and there is a constant k such that $f \leq k$ a.e.(σ) on a neighborhood of $x_0 \in \partial B$, then* $\limsup_{x \to x_0, x \in B} u(x) \leq k$.

Proof It can be assumed that $k \geq 0$ for if not, replace f by $f - k$. Choose $\epsilon > 0$ such that $f(z) \leq k$ a.e.(σ) for $z \in B_{x_0,\epsilon} \cap \partial B$. Denoting the indicator function of $B_{x_0,\epsilon}$ by $g_{x_0,\epsilon}$, for $x \in B$

$$u(x) = \mathbf{PI}(g_{x_0,\epsilon} f : B) + \mathbf{PI}((1 - g_{x_0,\epsilon}) f : B).$$

Since $f(z) \leq k$ a.e.(σ) for $z \in B_{x_0,\epsilon} \cap \partial B$,

$$\mathbf{PI}(g_{x_0,\epsilon} f : B)(x) \leq \mathbf{PI}(k : B)(x) = k\mathbf{PI}(1 : B)(x) = k.$$

Suppose $x \in B_{x_0,\epsilon/2}$ and $z \in \partial B$. Then $|z - x| > \epsilon/2$ when $|z - x_0| > \epsilon$ for otherwise, $|z - x_0| \leq |z - x| + |x - x_0| \leq \epsilon$. Thus,

$$|\mathbf{PI}((1 - g_{x_0,\epsilon})f : B)(x)| \leq \frac{1}{\sigma_n \rho} \int_{\sim B_{x_0,\epsilon} \cap \partial B} \frac{\rho^2 - |y - x|^2}{(\epsilon/2)^n} |f(z)| \, d\sigma(z)$$
$$\leq \frac{\rho^2 - |y - x|^2}{\sigma_n \rho (\epsilon/2)^n} \int_{\partial B} |f(z)| \, d\sigma(z).$$

Since $|y - x| \to \rho$ as $x \to x_0$, $\mathbf{PI}((1 - g_{x_0,\epsilon})f : B)(x) \to 0$ as $x \to x_0$. Therefore,

$$\limsup_{x \to x_0} u(x) \leq \limsup_{x \to x_0} \mathbf{PI}(g_{x_0,\epsilon} f : B)(x)$$
$$+ \limsup_{x \to x_0} \mathbf{PI}((1 - g_{x_0,\epsilon})f : B)(x) \leq k.$$ □

Lemma 2.7.5 *If f is Borel measurable on ∂B, integrable relative to surface area on ∂B, and $u = \mathbf{PI}(f : B)$ on B, then for $x_0 \in \partial B$*

$$\limsup_{x \to x_0, x \in B} u(x) \leq \limsup_{x \to x_0, x \in \partial B} f(x).$$

Proof It can be assumed that the right side is finite for otherwise there is nothing to prove. If k is any number greater than the right member of the last inequality, then $f(x) < k$ for all $x \in \partial B$ in a neighborhood of x_0. By the preceding lemma,

$$\limsup_{x \to x_0, x \in B} u(x) \leq k;$$

but since k is any number greater than $\limsup_{x \to x_0, x \in \partial B} f(x)$, the lemma is proved. □

Lemma 2.7.6 *If f is Borel measurable on ∂B, integrable relative to surface area on ∂B, continuous at $x_0 \in \partial B$, and $u = \mathbf{PI}(f : B)$ on B, then*

$$\lim_{x \to x_0, x \in B} u(x) = f(x_0).$$

Proof By the preceding lemma, $\limsup_{x \to x_0, x \in B} u(x) \leq f(x_0)$. Since $\mathbf{PI}(-f : B) = -\mathbf{PI}(f : B)$,

$$\limsup_{x \to x_0, x \in B} -u(x) = \limsup_{x \to x_0, x \in B} \mathbf{PI}(-f : B) \leq -f(x_0),$$

or $\liminf_{x \to x_0, x \in B} u(x) \geq f(x_0)$ and the result follows. □

Theorem 2.7.7 (Schwarz [6])*The Dirichlet problem is uniquely solvable for a ball B and a continuous boundary function f. The solution is given by $\mathbf{PI}(f : B)$.*

Proof Uniqueness was proven in Theorem 2.7.1. □

Remark 2.7.8 Put another way, if $f \in C^0(\partial B)$, then $u = \mathbf{PI}(f : B)$ is harmonic on B, has a continuous extension to B^-, and agrees with f on ∂B.

According to Theorem 2.4.3, if u is harmonic on a neighborhood of a ball, then u has the mean value property. This property characterizes harmonic functions.

Theorem 2.7.9 *A function u on the open set $\Omega \subset R^n$ is harmonic if and only if $u \in C^0(\Omega)$ and*

$$u(x) = \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B_{x,\delta}} u(z)\, d\sigma(z)$$

for every ball $B_{x,\delta} \subset B^-_{x,\delta} \subset \Omega$.

Proof The necessity follows from the continuity of harmonic functions and Theorem 2.4.3. Consider a function $u \in C^0(\Omega)$ that satisfies the above equation for every $B_{x,\delta} \subset B^-_{x,\delta} \subset \Omega$. Fix such a ball. By Lemma 2.7.6, there is a function $v \in C^0(\Omega^-)$ that agrees with u on $\partial B_{x,\delta}$ and is harmonic on $B_{x,\delta}$. The difference $u - v$ is then zero on $\partial B_{x,\delta}$, and since it satisfies the hypothesis of Theorem 2.5.9, $u - v$ satisfies both the maximum and minimum principles on $B_{x,\delta}$. It follows that $u = v$ on $B_{x,\delta}$ and u is harmonic on $B_{x,\delta}$. Since $B_{x,\delta}$ is an arbitrary ball with $B^-_{x,\delta} \subset \Omega$, u is harmonic on Ω □

It is not necessary that the equation of the preceding theorem hold for every ball $B_{x,\delta} \subset B^-_{x,\delta} \subset \Omega$.

Corollary 2.7.10 *A function u on the open set $\Omega \subset R^n$ is harmonic if and only if $u \in C^0(\Omega)$ and for each $x \in \Omega$, there is a $\delta_x > 0$ such that*

$$u(x) = \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B_{x,\delta}} u(z)\, d\sigma(z) \tag{2.14}$$

for all $0 < \delta < \delta_x$, in which case

$$u(x) = \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B_{x,\delta}} u(z)\, d\sigma(z) = \frac{1}{\nu_n \delta^n} \int_{B_{x,\delta}} u(z)\, dz$$

whenever $B^-_{x,\delta} \subset \Omega$.

Proof The necessity follows from the preceding theorem. Suppose $u \in C^0(\Omega)$ and for each $x \in \Omega$, there is a $\delta_x > 0$ such that Eq. (2.14) holds for all $0 < \delta < \delta_x$. If $B = B_{x,\rho} \subset B^-_{x,\rho} \subset \Omega$, then $u|_{\partial B}$ is continuous and the function

$$v(y) = \begin{cases} \mathbf{PI}(u : x, \rho)(y), & y \in B, \\ u(y), & y \in \partial B, \end{cases}$$

is continuous on B^- by Lemma 2.7.6. By definition, $u - v = 0$ on ∂B. Since v is harmonic on B, for each $y \in B$

$$u(y) - v(y) = \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B_{y,\delta}} (u(z) - v(z))\, d\sigma(z)$$

for all sufficiently small $\delta > 0$. By Theorem 2.5.9, $u - v$ satisfies the minimum and maximum principles on B. If $u - v$ attains its maximum at some point of B, then it must be constant and therefore $u - v = 0$ on B; if $u - v$ does not attain its maximum at a point of B, then $u - v \le 0$ on B. In either case, $u \le v$ on B. Since $u - v$ also satisfies the minimum principle on B, $u = v$ on B and u is harmonic on B. Since B is arbitrary, u is harmonic on Ω. Lastly, since $\nu_n n = \sigma_n$

$$\begin{aligned}\frac{1}{\nu_n \delta^n}\int_{B_{x,\delta}} u(z)\,dz &= \frac{1}{\nu_n \delta^n}\int_0^\delta \int_{|\theta|=1} u(x+\rho\theta)\rho^{n-1}\,d\theta\,d\rho \\ &= \frac{\sigma_n}{\nu_n \delta^n}\int_0^\delta \left(\frac{1}{\sigma_n}\int_{|\theta|=1} u(x+\rho\theta)\,d\theta\right)\rho^{n-1}\,d\rho \\ &= \frac{\sigma_n}{\nu_n \delta^n}\int_0^\delta u(x)\rho^{n-1}\,d\rho \\ &= u(x).\end{aligned}$$

□

It is possible to relax the requirement of continuity in the preceding theorem at the expense of replacing surface averages by solid ball averages. In the proof of the following theorem, the symmetric difference of two sets A and B is denoted by $A \triangle B$ and is defined by the equation $A \triangle B = (A \sim B) \cup (B \sim A)$.

Lemma 2.7.11 *If u is locally integrable on the open connected set Ω and*

$$u(x) = \frac{1}{\nu_n \delta^n}\int_{B_{x,\delta}} u(y)\,dy$$

whenever $B_{x,\delta} \subset \Omega$, then u is continuous and obeys the maximum and minimum principles on Ω.

Proof Note first that u is real-valued. It will be shown now that the hypotheses imply that u is continuous on Ω. To see this, consider any $x \in \Omega$ and any $\delta > 0$ such that $B^-_{x,2\delta} \subset \Omega$. For any $y \in B_{x,\delta}$, $B_{y,\delta} \subset B^-_{x,2\delta}$. Since $B_{x,\delta} \triangle B_{y,\delta} \subset B^-_{x,2\delta}$ and u is integrable on $B^-_{x,2\delta}$,

$$|u(x) - u(y)| \le \frac{1}{\nu_n \delta^n}\int_{B_{x,\delta}\triangle B_{y,\delta}} |u(z)|\,dz \to 0$$

as $y \to x$ by the absolute continuity of the Lebesgue integral, which proves that u is continuous at x. Suppose there is a point $x_0 \in \Omega$ such that $u(x_0) = \inf_\Omega u$. Letting $\Sigma = \{y; u(y) = \inf_\Omega u\}$, Σ is a relatively closed subset of Ω by continuity of u. For any $y \in \Sigma$ and $\delta > 0$ with $B_{y,\delta} \subset \Omega$,

$$\frac{1}{\nu_n \delta^n}\int_{B_{y,\delta}} (u(z) - u(y))\,dz = 0.$$

Since $y \in \Sigma$, $u - u(y) \geq 0$ on $B_{y,\delta}$ and it follows that $u - u(y) = 0$ *a.e.* in $B_{y,\delta}$; thus, $u = u(y)$ on $B_{y,\delta}$ by continuity of u. This shows that $B_{y,\delta} \subset \Sigma$ and that Σ is an open subset of Ω. By the connectedness of Ω, $\Sigma = \emptyset$ or $\Sigma = \Omega$. If $\Sigma = \Omega$, then u is constant on Ω; otherwise, u does not attain its minimum at an interior point of Ω. Since $-u$ satisfies the same hypotheses, u satisfies the maximum principle on Ω. □

Theorem 2.7.12 *The function u is harmonic on the open set Ω if and only if u is locally integrable on Ω and*

$$u(x) = \frac{1}{\nu_n \delta^n} \int_{B_{x,\delta}} u(y)\, dy$$

whenever $B_{x,\delta} \subset \Omega$.

Proof The necessity follows from the continuity of harmonic functions and Theorem 2.4.3. As to the sufficiency, let u be locally integrable on Ω and satisfy the above equation. Since it suffices to prove that u is harmonic on each component of Ω, it will be assumed that Ω is connected. Let $B = B_{y,\rho}$ be any ball with $B^- \subset \Omega$. It was shown in the preceding proof that u is continuous on Ω. Consider the harmonic function $h = \mathbf{PI}(u|_{\partial B} : y, \rho)$. By Lemma 2.7.6, $\lim_{z \to x, z \in B}(u(z) - h(z)) = 0$ for all $x \in \partial B$. Since $u - h$ satisfies both the minimum and maximum principles on B by Lemma 2.7.11, $u = h$ on B and u is harmonic on B. Since B is an arbitrary ball with $B^- \subset \Omega$, u is harmonic on Ω. □

The requirement in the preceding theorem that u satisfy a global solid ball averaging condition cannot be relaxed. This can be seen by examining the function u on R^2 defined by

$$u(x, y) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

This function is locally integrable and satisfies a local solid ball averaging principle but is not harmonic.

If $u_1, \ldots, u_p$ are harmonic functions on the open set Ω and $\alpha_1, \ldots, \alpha_p$ are real numbers, then it is clear from the original definition of harmonic function that $u = \sum_{i=1}^p \alpha_i u_i$ is harmonic on Ω. The following theorem extends this result to integrals.

Theorem 2.7.13 *Let U and V be open subsets of R^n, let μ be a measure on U, and let H be a nonnegative function on $U \times V$. If (i) for each $y \in V$, $H(\cdot, y)$ is continuous on U, (ii) for each $x \in U$, $H(x, \cdot)$ is harmonic on V, and (iii) $h(y) = \int_U H(x, y)\, d\mu(x) < +\infty$ for each $y \in V$, then h is harmonic on V.*

Proof By hypothesis, $H(x, y)$ is continuous in each variable separately. This implies that $H(x, y)$ is jointly measurable on $U \times V$. To see this, for each $k \geq 1$ let

$$I_k(j_1, \ldots, j_n) = \{(x_1, \ldots, x_n); \frac{j_i}{2^k} \leq x_i < \frac{j_i + 1}{2^k}, i = 1, \ldots, n\}$$

where each $j_i \in Z$, the set of integers. Define a map $\psi_k : U \to V$ by letting $\psi_k(x)$ be any fixed point of $V \cap I_k(j_1, \dots, j_n)$ whenever $x \in V \cap I_k(j_1, \dots, j_n)$. Note that $\lim_{k\to\infty} \psi_k(x) = x$ for all $x \in U$ and that $\lim_{k\to\infty} H(\psi_k(x), y) = H(x, y)$ for each $x \in U$ and $y \in V$. Since each function $H(\psi_k(x), y)$ is jointly measurable on $U \times V$, $H(x, y)$ is also. Therefore, $H(x, y)$ is a nonnegative jointly measurable function to which Tonelli's theorem can be applied. Suppose $B^-_{y,\delta} \subset V$. Then

$$\begin{aligned}\mathbf{A}(h, y, \delta) &= \mathbf{A}\left(\int_U H(x, \cdot)\, d\mu(x) : y, \delta\right)\\ &= \int_U H(x, y)\, d\mu(x)\\ &= h(y) < +\infty.\end{aligned}$$

This shows that h is locally integrable and satisfies the hypotheses of Theorem 2.7.12. Thus, h is harmonic on V. □

It would appear from Theorem 2.7.2 that the class of functions harmonic on a ball is much more extensive than the class obtained by solving the Dirichlet problem for a Borel measurable boundary function. For example, consider a point $z_0 \in \partial B_{0,\rho}$ and a unit measure μ concentrated on z_0. Then for $0 < \lambda < 1$

$$\begin{aligned}u(\lambda z_0) = \mathbf{PI}(\mu : B)(\lambda z_0) &= \frac{1}{\sigma_n \rho} \int_{\partial B_{0,\rho}} \frac{\rho^2 - |\lambda z_0|^2}{|z - \lambda z_0|^n}\, d\mu(z)\\ &= \frac{1}{\sigma_n \rho^{n-1}} \frac{1 - \lambda^2}{(1 - \lambda)^n} \to +\infty\end{aligned}$$

as $\lambda \to 1-$; that is, $\mathbf{PI}(\mu : B)(x) \to +\infty$ as x approaches z_0 along a radial line. On the other hand, if the measure μ is concentrated on some point z_1 other than z_0, then

$$u(\lambda z_0) = \mathbf{PI}(\mu : B)(\lambda z_0) = \frac{1}{\sigma_n \rho} \frac{\rho^2(1 - \lambda^2)}{|z_1 - \lambda z_0|^n} \to 0$$

as $\lambda \to 1-$. The function u is not determined by the boundary function f that is $+\infty$ at z_0 and is 0 at all other points of $\partial B_{0,\rho}$ since $f = 0$ a.e. (σ) and $\mathbf{PI}(f : B) = 0$ on B.

Lemma 2.7.14 (Herglotz [5]) *If u is harmonic on $B = B_{y,\rho}$ and*

$$\frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B_{y,\delta}} |u|\, d\sigma \le k < +\infty \qquad \textit{for all } \delta < \rho,$$

then there is a signed measure μ of bounded variation on ∂B such that $u = \mathbf{PI}(\mu : B)$ on B.

Proof If $\delta < \rho$ and $x \in B_{y,\delta}$, then

$$u(x) = \frac{1}{\sigma_n \delta} \int_{\partial B_{y,\delta}} \frac{\delta^2 - |x-y|^2}{|z-x|^n} u(z)\, d\sigma(z).$$

If M is a Borel subset of $B_{y,\rho}^-$ and $\delta < \rho$, define

$$\mu_\delta(M) = \int_{M \cap \partial B_{y,\delta}} u(z)\, d\sigma(z).$$

Then μ_δ is concentrated on $\partial B_{y,\delta}$ and

$$\|\mu_\delta\| = \int_{\partial B_{y,\delta}} |u(z)|\, d\sigma(z) \le k\sigma_n \delta^{n-1} \le k\sigma_n \rho^{n-1}$$

for all $\delta < \rho$. Let $\{\delta_j\}$ be a sequence of positive numbers such that $\delta_j \uparrow \rho$. By Theorem 1.2.5, there is a subsequence of the sequence $\{\mu_{\delta_j}\}$ that converges to a signed measure μ in the w^*-topology with $\|\mu\| \le k\sigma_n\rho^{n-1}$. It can be assumed that the sequence $\{\mu_{\delta_j}\}$ converges to μ in the w^*-topology. Since the signed measure μ_{δ_j} is concentrated on $\partial B_{y,\delta_j}$ and $\delta_j \uparrow \rho$, μ is concentrated on $\partial B_{y,\rho}$. Consider any $x \in B$. By dropping a finite number of terms, if necessary, it can be assumed that $|y - x| < \delta_j < \rho$ for all $j \ge 1$. Then

$$u(x) = \frac{1}{\sigma_n \delta_j} \int_{\partial B_{y,\delta_j}} \frac{\delta_j^2 - |y-x|^2}{|z-x|^n}\, d\mu_{\delta_j}(z).$$

Since the signed measures μ_{δ_j} are concentrated on the spherical shell $\{z; \delta_1 \le |z - x| \le \rho\}$ and the sequence of integrands in the last equation converges uniformly to $(\rho^2 - |y-x|^2)/|z-x|^n$ on this shell, by Corollary 1.2.6,

$$u(x) = \frac{1}{\sigma_n \rho} \int_{\partial B_{y,\rho}} \frac{\rho^2 - |y-x|^2}{|z-x|^n}\, d\mu(z).$$ □

Theorem 2.7.15 [Herglotz] *The harmonic function u on $B = B_{y,\rho}$ is a difference of two nonnegative harmonic functions if and only if there is a signed measure μ of bounded variation on ∂B such that $u = \mathbf{PI}(\mu : B)$.*

Proof Suppose $u = u_1 - u_2$, where u_1 and u_2 are nonnegative harmonic functions on B. If $0 < \delta < \rho$, then

$$\begin{aligned}\frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B_{y,\delta}} |u|\, d\sigma &\le \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B_{y,\delta}} |u_1|\, d\sigma + \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B_{y,\delta}} |u_2|\, d\sigma \\ &= u_1(y) + u_2(y) < +\infty\end{aligned}$$

and the necessity follows from Lemma 2.7.14. If $u = \mathbf{PI}(\mu : B)$, where μ is a signed measure of bounded variation on ∂B, then $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are finite measures on ∂B, and so $u = \mathbf{PI}(\mu^+ : B) - \mathbf{PI}(\mu^- : B)$. The sufficiency follows from Theorem 2.7.2. □

2.7.1 Exercises for Sect. 2.7

1. If the function u is harmonic on the set $\Omega = \{y \in R^n; y_n > 0\}$, continuous on Ω^-, and equal to 0 on $\partial\Omega$, show that u has a harmonic extension h to all of R^n.
2. If $x \in B = B_{0,\delta} \subset R^3, z \in \partial B$ have spherical coordinates (ρ, θ, ϕ) and (δ, θ', ϕ'), respectively, and u is harmonic on B and continuous on B^-, show that

$$u(x) = \frac{\delta}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\delta^2 - \rho^2}{(\delta^2 + \rho^2 - 2\delta\rho\cos\gamma)^{\frac{3}{2}}} u(\delta, \theta', \phi') \sin\phi'\, d\phi'\, d\theta'$$

   where $\cos\gamma = \cos\phi\cos\phi' + \sin\phi\sin\phi'\cos(\phi' - \phi)$.
3. Under the hypotheses of Theorem 2.5.4, show that the Poisson Integral in the $n \geq 3$ and $B = B_{y,\rho}$ case can be written

$$u(x) = u(r, \theta) = \frac{1}{\sigma_n}\int_{|\theta'|=1} \frac{\rho^{n-2}(\rho^2 - r^2)}{(\rho^2 + r^2 - 2\theta' \cdot \theta)^{n/2}} u(\rho, \theta')\, d\theta'$$

   where x and z have spherical coordinates $(\theta_1, \ldots, \theta_{n-1}, r)$ and $(\theta'_1, \ldots, \theta'_{n-1}, \rho)$, respectively.
4. Show that the Poisson Integral on a ball $B = B_{0,\rho}$ commutes with orthogonal transformations. That is, if $O = \{o_{ij}\}$ is a real $n \times n$ orthogonal matrix and f is integrable relative to surface area on ∂B, then

$$\mathbf{PI}(f, B) \circ O = \mathbf{PI}(f \circ O, B)$$

   where $f \circ O$ denotes the composition of f with O.
5. If u is a positive harmonic function on the ball $B = B_{0,\rho}$ and for each $0 \leq t < 1$ and Borel set $E \subset \partial B$,

$$\mu_t(E) = \int_E u(tx)\, d\sigma(x),$$

   show that $\sup_{0\leq t<1} \|\mu_t\| < +\infty$, that there is a positive measure μ of bounded variation on the Borel subsets of ∂B, and a sequence $\{t_j\}$ in $[0, 1)$ increasing to 1 such that the sequence $\{\mu_{t_j}; j \geq 1\}$ converges to μ in the w^*-topology; that is,

$$\lim_{j\to\infty}\int_{\partial B} f\, d\mu_{t_j} = \int_{\partial B} f\, d\mu \text{ for all } f \in C^0(\partial B).$$

6. Show that the positive harmonic function of the preceding exercise has the representation

$$u(x) = \frac{1}{\sigma_n \rho} \int_{\partial B} \frac{\rho^2 - |x|^2}{|z - x|^n} \, d\mu(z), \qquad x \in B$$

and that μ is unique.

2.8 Kelvin Transformation

Consider a ball $B_{y,\rho} \subset R^n$. The transformation $x \to x^*$ defined by

$$x^* = y + \frac{\rho^2}{|y - x|^2}(x - y), \qquad x \neq y, \tag{2.15}$$

was used in the derivation of the Poisson integral formula and is called the **Kelvin transformation** or **inversion** with respect to $\partial B_{y,\rho}$. The point x^*, lying on the radial line joining y to x, is called the inverse of x relative to $\partial B_{y,\rho}$. Aside from the derivation of the integral formula, the Kelvin transformation is useful for solving other problems.

The following symmetry property of the Kelvin transformation will be used later in the chapter. If $x, y \in B^-_{z,\rho}$, then

$$|x - z||x^* - y| = |y - z||x - y^*|. \tag{2.16}$$

This equation will be derived assuming that $z = 0$. Let γ denote the angle between the line segments joining 0 to x and 0 to y. Since $x^* = (\rho^2/|x|^2)x$ and $y^* = (\rho^2/|y|^2)y$,

$$\begin{aligned}
|x^* - y|^2 &= \frac{\rho^4}{|x|^2} + |y|^2 - 2\frac{\rho^2}{|x|}|y| \cos \gamma \\
&= \frac{|y|^2}{|x|^2}\left(\frac{\rho^4}{|y|^2} + |x|^2 - 2\frac{\rho^2}{|y|}|x| \cos \gamma\right) \\
&= \frac{|y|^2}{|x|^2}|y^* - x|^2.
\end{aligned}$$

Thus, $|x||x^* - y| = |y||y^* - x|$.

It is easily verified that the Kelvin transformation is a homeomorphism of $R^n \sim \{y\}$ onto itself. As with any other transformation, the effect of the transformation on various geometric regions can be examined. Note first that the transformation maps planes or spheres into planes or spheres, but not necessarily respectively. The calculations will be much simpler if it is assumed that $y = 0$, in which case

$$x^* = \frac{\rho^2}{|x|^2}x \qquad \text{and} \qquad x = \frac{\rho^2}{|x^*|^2}x^*.$$

Starting with the general equation of a plane or sphere in R^n

$$a\sum_{i=1}^n x_i^2 + \sum_{i=1}^n b_i x_i + c = 0,$$

and making the substitution $x_i = (\rho^2/|x^*|^2)x_i^*$, the equation

$$a\rho^4 + \rho^2\sum_{i=1}^n b_i x_i^* + c\sum_{i=1}^n x_i^{*2} = 0$$

is obtained, which is again the general equation of a plane or sphere. It is easily seen that a plane outside the sphere $\partial B_{0,\rho}$ will map into a sphere inside the sphere $\partial B_{0,\rho}$ which passes through the origin, and conversely, a sphere inside the sphere $\partial B_{0,\rho}$, which passes through the origin, will map onto a plane outside the sphere $\partial B_{0,\rho}$.

The effect of the Kelvin transformation on functions will also be examined. Let Ω be an open subset of $R^n \sim \{y\}$ and let Ω^* be the image of Ω under the map $x \to x^*$ relative to $\partial B_{y,\rho}$. If f is a function on Ω, the equation

$$f^*(x) = \frac{\rho^{n-2}}{r^{n-2}}f(y + \frac{\rho^2}{r^2}(x-y)),$$

where $r = |x - y|$ and $r^{n-2} = 1$ if $n = 2$, defines a function on Ω^*. The mapping $f \to f^*$ defined in this way will be called the **Kelvin transform**.

The following theorem can be proved by tedious calculation of the Laplacian of f^*. Instead the proof will be based on the characterization of harmonicity in terms of averaging. To do this it is necessary to make a change of variable in an integral that requires the computation of the Jacobian $J(x^*, x)$ of the transformation $x \to x^*$. To simplify the notation, it will be assumed that the map $x \to x^*$ is relative to the sphere $\partial B_{0,\rho}$. It is left to the exercises to show that

$$J(x^*, x) = \det\left[\frac{\partial x_i^*}{\partial x_j}\right]_{n\times n} = \det\left[\frac{\rho^2}{r^2}\left(\delta_{ij} - \frac{2x_i x_j}{r^2}\right)\right]_{n\times n} = -\frac{\rho^{2n}}{r^{2n}}. \qquad (2.17)$$

where $\delta_{ij} = 1$ or 0 according as $j = i$ or $j \neq i$, respectively, and $r = |x|$.

Theorem 2.8.1 *If Ω is an open subset of $R^n \sim \{0\}$ and f is harmonic (superharmonic) on Ω, then the Kelvin transform f^* of f is harmonic (superharmonic) on Ω^*.*

Proof That positivity is preserved is obvious from the definition. Let $B_{z,\delta}$ be a ball with closure $B^-_{z,\delta} \subset \Omega^*$ Choose $\rho > 0$ such that $B^-_{0,\rho} \cap B^-_{z,\delta} = \emptyset$ and consider the inversion $x \to x^*$ relative to $\partial B_{0,\rho}$. Let $B_{z_0,\epsilon} \subset B_{0,\rho}$ be the pre-image of $B_{z,\delta}$ under the inversion $x \to x^*$. Using Eq. (2.17),

$$\begin{aligned}\frac{1}{\nu_n\delta^n}\int_{B_{z,\delta}} f^*(x^*)\,dx^* &= \frac{1}{\nu_n\delta^n}\int_{B_{z_0,\epsilon}} \frac{\rho^{n-2}}{|x|^{n-2}} f\left(\frac{\rho^2}{|x|^2}x\right)\left(\frac{\rho^{2n}}{|x|^{2n}}\right)dx\\ &= \frac{1}{\nu_n\delta^n}\int_0^\epsilon \frac{\rho^{3n-2}}{t^{3n-2}}\int_{|x|=t} f\left(\frac{\rho^2}{|x|^2}x\right)d\sigma(x)\,dt\\ &= \frac{1}{\nu_n\delta^n}\int_0^\epsilon \frac{\rho^{3n-2}}{t^{3n-2}}\sigma_n t^{n-1} f\left(\frac{\rho^2}{|z_0|^2}z_0\right)dx\,dt\end{aligned}$$

Noting that $f\left(\frac{\rho^2}{|z_0|^2}z_0\right)$ is a constant, reversing the steps

$$\frac{1}{\nu_n\delta^n}\int_{B_{z,\delta}} f^*(x^*)\,dx^* = f\left(\frac{\rho^2}{|z_0|^2}z_0\right) = f^*(z).$$

Since this is true for every $B_{z,\delta} \subset \Omega^*$, it follows from Theorem 2.7.12 that f^* is harmonic on Ω^*. □

2.8.1 Exercises for Sect. 2.8

1. Show that the circle $C \subset R^2$ is invariant under an inversion with respect to the circle $\partial B_{0,\rho}$ if and only if it is orthogonal to $\partial B_{0,\rho}$, that is, the normals to the two curves at the points of intersection must be orthogonal. Formulate the same proposition for spheres in R^n. (Hint:To show that orthogonality implies invariance use Steiner's Theorem.)
2. Show that the Jacobian $J(x^*, x)$, $x \in R^2$, of the inversion $x^* \to x$ relative to $\partial B_{0,\rho}$ is $-\rho^4/|x|^4$.
3. Show that the Jacobian $J(x^*, x)$, $x \in R^3$, of the inversion $x^* \to x$ relative to $\partial B_{0,\rho}$ is $-\rho^6/|x|^6$.
4. If a_i, $i = 1, \ldots n$, are non-zero real numbers, show that

$$\det\left[\delta_{ij} + \frac{1}{a_i}\right]_{n\times n} = \frac{1+\sum_{i=1}^n a_i}{\prod_{i=1}^n a_i}. \qquad \text{(a)}$$

Use this result to show that the Jacobian $J(x^*, x)$, $x \in R^n$, of the inversion $x^* \to x$ relative to $\partial B_{0,\rho}$ is

$$J(x^*, x) = -\frac{\rho^{2n}}{|x|^{2n}}. \qquad \text{(b)}$$

5. Determine the Kelvin transform of the function $f(x) = (|x|^2 + 3|x|)e^{-|x|}$ relative to the sphere $\partial B_{0,\rho}$.
6. The function $u(x) = \log|x - x_0|$ is harmonic on $R^2 \sim \{x_0\}$ and has a pole at x_0. Show that the Kelvin transform u^* of u relative to $\partial B_{0,\rho}$ has poles at 0 and $x_0^* = \frac{\rho^2}{|x_0|^2} x_0$.
7. The function $u(x) = \frac{1}{|x - x_0|^{n-2}}$ is harmonic on $R^n \sim \{x_0\}$, $n \geq 3$, and has a pole at x_0 . Show that the Kelvin transform u^* of u relative to $\partial B_{0,\rho}$ has a pole at $x_0^* = \frac{\rho^2}{|x_0|^2} x_0$.
8. Determine the Kelvin transform u^* of the function $u(x) = 1/|x|$, $x \neq 0$ relative to the sphere $\partial B_{0,\rho}$.

2.9 Poisson Integral for Half-Space

It was shown in the preceding section that a nonnegative harmonic function u on a ball can be represented as the Poisson integral of a measure on the boundary of the ball. A similar result holds for half-spaces. Throughout this section Ω will be the half-space $\{(x_1, \ldots, x_n); x_n > 0\}$.

Theorem 2.9.1 *If u is a nonnegative harmonic function on the open half-space $\Omega = \{(x_1, \ldots, x_n); x_n > 0\}$, then there is a nonnegative constant c and a Borel measure μ on $\partial\Omega$ such that*

$$u(x) = cx_n + \frac{2x_n}{\sigma_n} \int_{\partial\Omega} \frac{1}{|z - x|^n} \, d\mu(z) \qquad \textit{for all } x \in \Omega.$$

Proof Consider an inversion relative to $\partial B_{y,1}$ where $y = (0, \ldots, 0, -1)$. The image of Ω under this map is the open ball $\Omega^* = \{z^*; |z^* - x_0^*| < 1/2\}$ where $x_0^* = (0, \ldots, 0, -\frac{1}{2})$. Let u^* be the image of u under the inversion. Then u^* is a nonnegative, harmonic function on the ball Ω^*. By Theorem 2.7.15, there is a Borel measure μ^* on $\partial\Omega^*$ such that

$$u^*(x^*) = \frac{2}{\sigma_n} \int_{\partial\Omega^*} \frac{\frac{1}{4} - |x^* - x_0^*|^2}{|z^* - x^*|^n} \, d\mu^*(z^*) \qquad \text{for all } x^* \in \Omega^*.$$

Let $c = (2/\sigma_n)\mu^*(\{y\}) \geq 0$ and let $\mu_1^* = \mu^*|_{\partial\Omega^* \sim \{y\}}$. Then

$$u^*(x^*) = c\frac{\frac{1}{4} - |x^* - x_0^*|^2}{|y - x^*|^n} + \frac{2}{\sigma_n} \int_{\partial\Omega^*} \frac{\frac{1}{4} - |x^* - x_0^*|^2}{|z^* - x^*|^n} \, d\mu_1^*(z^*).$$

Letting θ denote the angle between the line segment joining x^* to y and the x_n-axis,

$$\frac{1}{4} - |x^* - x_0^*|^2 = -|x^* - y|^2 + |x^* - y| \cos \theta$$

by the law of cosines. Since $|x^* - y||x - y| = 1$ by definition of x^*,

$$\frac{1}{4} - |x^* - x_0^*|^2 = \frac{1}{|x - y|^2}(-1 + |x - y| \cos \theta).$$

Since $|x - y| \cos \theta = x_n + 1$, where x_n is the nth component of x,

$$\frac{1}{4} - |x^* - x_0^*|^2 = \frac{x_n}{|x - y|^2}.$$

Letting ϕ be the angle between the line segment joining z^* to y, where $z \in \partial\Omega$, and the line segment joining x^* to y,

$$\begin{aligned}
|z^* - x^*|^2 &= |z^* - y|^2 + |x^* - y|^2 - 2|z^* - y||x^* - y| \cos \phi \\
&= \frac{1}{|z - y|^2} + \frac{1}{|x - y|^2} - \frac{2 \cos \phi}{|z - y||x - y|} \\
&= \frac{1}{|z - y|^2 |x - y|^2}(|x - y|^2 + |z - y|^2 - 2|z - y||x - y| \cos \phi) \\
&= \frac{|z - x|^2}{|z - y|^2 |x - y|^2}.
\end{aligned}$$

Therefore,

$$u^*(x^*) = c x_n |x - y|^{n-2} + \frac{2x_n}{\sigma_n} |x - y|^{n-2} \int_{\partial\Omega^*} \frac{|z - y|^n}{|z - x|^n} \, d\mu_1^*(z^*)$$

or

$$\frac{1}{|x - y|^{n-2}} u^* \left(y + \frac{(x - y)}{|x - y|^2} \right) = c x_n + \frac{2x_n}{\sigma_n} \int_{\partial\Omega^*} \frac{|z - y|^n}{|z - x|^n} \, d\mu_1^*(z^*).$$

Note that the left side of this equation is just $u(x)$. Denoting the map $z^* \to z$ by T, the measure μ_1^* on $\partial\Omega^*$ induces a measure μ_1 on $\partial\Omega$ by $\mu_1 = \mu_1^* T^{-1}$. Defining $\mu(E) = \int_E |z - y|^n \, d\mu_1(z)$ for any Borel set E, the last equation can be written

$$u(x) = c x_n + \frac{2x_n}{\sigma_n} \int_{\partial\Omega} \frac{1}{|z - x|^n} \, d\mu(z). \tag{2.18}$$

□

Note There is no reason to believe that the measure in the last equation is finite.

The right side of Eq. (2.18) will be denoted by $\mathbf{PI}(c, \mu, \Omega)(x)$ and is called the Poisson integral of the pair (c, μ) for the half-space Ω. The number c can be any real number and μ any signed measure. As before, if μ is the indefinite integral of a measurable function f on $\partial\Omega$ relative to Lebesgue measure, let $\mathbf{PI}(c, f, \Omega) = \mathbf{PI}(c, \mu, \Omega)$; that is,

$$\mathbf{PI}(c, f, \Omega)(x) = cx_n + \frac{2x_n}{\sigma_n}\int_{\partial\Omega}\frac{f(z)}{|z-x|^n}\,dz, \tag{2.19}$$

where $x = (x_1, \ldots, x_n)$ with $x_n > 0$.

A partial converse of the preceding theorem for the half-space Ω will be taken up now. It will be shown first that

$$\int_{\partial\Omega}\frac{1}{|z-x|^n}\,dz < +\infty \tag{2.20}$$

whenever $n \geq 2$ and $x \in \Omega$. To see this, let $\overline{x}$ be the projection of $x = (x_1, \ldots, x_n)$ onto $\partial\Omega$ and let $z \in \partial\Omega$. Then $|z-x|^2 = x_n^2 + |z-\overline{x}|^2$, and

$$\int_{\partial\Omega}\frac{1}{|z-x|^n}\,dz \leq \int_{\partial\Omega\cap\{|z-\overline{x}|<1\}}\frac{1}{x_n^n}\,dz + \int_{\partial\Omega\cap\{|z-\overline{x}|\geq 1\}}\frac{1}{|z-\overline{x}|^n}\,dz.$$

Since the first integral on the right is finite, only the second need be considered. Transforming to spherical coordinates in the $(n-1)$-dimensional space $\partial\Omega$ relative to the pole $\overline{x}$,

$$\int_{\partial\Omega\cap\{|z-\overline{x}|\geq 1\}}\frac{1}{|z-\overline{x}|^n}\,dz = \int_1^\infty\int_{|\theta|=1}\frac{1}{r^n}r^{n-2}\,d\theta dr = \sigma_{n-1} < +\infty.$$

It will be shown now that $\mathbf{PI}(0, 1, \Omega) = 1$ on Ω. By Eq. (2.20),

$$\mathbf{PI}(0, 1, \Omega)(x) = \frac{2x_n}{\sigma_n}\int_{\partial\Omega}\frac{1}{|z-x|^n}\,dz$$

is finite for each $x \in \Omega$. Equation (2.20) can also be used to show that $\mathbf{PI}(0, 1, \Omega)$ is harmonic on Ω. The argument is straightforward, but tedious, and involves justification of differentiation under the integral. Note that the above integral, as a function of x, is unaffected by adding to x a vector in $\partial\Omega$. Such an operation is a translation of x in a direction normal to the x_n-axis. In other words, $\int_{\partial\Omega}|z-x|^{-n}\,dz$ is a function of x_n only. Let

$$g(x_n) = \mathbf{PI}(0, 1, \Omega)(x) = \frac{2x_n}{\sigma_n}\int_{\partial\Omega}\frac{1}{|z-x|^n}\,dz, \qquad x_n > 0.$$

Since g is harmonic on Ω, $d^2g/x_n^2 = 0$ and $g(x_n) = ax_n + b$. It will be shown now that $a = 0$. Since

$$
\begin{aligned}
g(x_n) &= \frac{2x_n}{\sigma_n}\int_{\partial\Omega}\frac{1}{(|z|^2+x_n^2)^{n/2}}\,dz \\
&= \frac{2}{\sigma_n}\int_{\partial\Omega}\frac{1}{(|z/x_n|^2+1)^{n/2}}\,d\left(\frac{z}{x_n}\right) \\
&= \frac{2}{\sigma_n}\int_{\partial\Omega}\frac{1}{(|z|^2+1)^{n/2}}\,dz,
\end{aligned}
$$

$g(x_n)$ is a constant function and $a = 0$. By transforming to spherical coordinates relative to the origin in $\partial\Omega$, using the formula for the σ_n in Eq. (1.1), and using standard integration techniques,

$$
\begin{aligned}
b = g(1) &= \frac{2}{\sigma_n}\int_{\partial\Omega}\frac{1}{(|z|^2+1)^{n/2}}\,dz \\
&= \frac{2}{\sigma_n}\int_0^{+\infty}\int_{|\theta|=1}\frac{r^{n-2}}{(r^2+1)^{n/2}}\,d\sigma(\theta)dr \\
&= \frac{2\sigma_{n-1}}{\sigma_n}\int_0^{+\infty}\frac{r^{n-2}}{(r^2+1)^{n/2}}\,dr \\
&= 1.
\end{aligned}
$$

Therefore, $g(x_n) = 1$ for all $x_n > 0$; that is,

$$
\mathbf{PI}(0, 1, \Omega) = \frac{2x_n}{\sigma_n}\int_{\partial\Omega}\frac{1}{|z-x|^n}\,dz = 1 \qquad \text{for all } x \in \Omega. \tag{2.21}
$$

The relationship of the harmonic function determined by a boundary function to the boundary function itself will be considered now.

Theorem 2.9.2 *If f is a bounded measurable function on $\partial\Omega$ and c is any real number, then $u = \mathbf{PI}(c, f, \Omega)$ is harmonic on Ω. If f is continuous at $z_0 \in \partial\Omega$, then* $\lim_{x\to z_0} u(x) = f(z_0)$*; moreover,* $\lim_{x_n\to+\infty} u(x)/x_n = c$.

Proof The latter statement will be proved first. It follows from Eq. (2.21) that

$$
\lim_{x_n\to+\infty}\frac{2}{\sigma_n}\int_{\partial\Omega}\frac{1}{|z-x|^n}\,dz = 0.
$$

Returning to Eq. (2.19) and using the fact that f is bounded,

$$
\lim_{x_n\to+\infty}\frac{u(x)}{x_n} = c + \lim_{x_n\to+\infty}\frac{2}{\sigma_n}\int_{\partial\Omega}\frac{f(z)}{|z-x|^n}\,dz = c.
$$

Suppose now that f is continuous at $z_0 \in \partial\Omega$ and that $|f(z)| \le M$ on $\partial\Omega$. To show that $\lim_{x\to z_0} u(x) = f(z_0)$, it suffices to prove as in Lemma 2.7.4 that $f(z) \le k$ in a neighborhood of z_0 implies that $\limsup_{x\to z_0} u(x) \le k$ for all z, where $k > 0$. Suppose $f(z) \le k$ for all $z \in \partial\Omega \cap B_{z_0,\epsilon}$. Since the term cx_n in Eq. (2.19) has the limit zero as $x \to z_0$, it suffices to consider just the integral term. The integral will be written as a sum of two integrals by splitting the boundary $\partial\Omega$ into the two regions $\partial\Omega \cap \{|z - z_0| < \epsilon\}$ and $\partial\Omega \cap \{|z - z_0| \geq \epsilon\}$. Since $f(z) \leq k$ on $\partial\Omega \cap \{|z - z_0| < \epsilon\}$,

$$\frac{2x_n}{\sigma_n} \int_{\partial\Omega \cap \{|z - z_0| < \epsilon\}} \frac{f(z)}{|z - x|^n}\, dz \leq k\mathbf{PI}(0, 1, \Omega) = k.$$

Letting $\overline{x}$ denote the projection of x onto $\partial\Omega$,

$$\frac{2x_n}{\sigma_n} \int_{\partial\Omega \cap \{|z - z_0| \geq \epsilon\}} \frac{f(z)}{|z - x|^n}\, dz \leq \frac{2Mx_n}{\sigma_n} \int_{\partial\Omega \cap \{|z - \overline{x}|^n \geq \epsilon\}} \frac{1}{|z - \overline{x}|^n}\, dz.$$

Note that for $z \in \partial\Omega \sim B_{z_0,\epsilon}$ and $x \in B_{z_0,\epsilon/2}$, $|z - \overline{x}| \geq \epsilon/2$. Using spherical coordinates relative to $\overline{x}$, for $x \in B_{z_0,\epsilon/2}$

$$\begin{aligned}
\int_{\partial\Omega \cap \{|z - z_0| \geq \epsilon\}} \frac{1}{|z - \overline{x}|^n}\, dz &\leq \int_{\partial\Omega \cap \{|z - \overline{x}| \geq \epsilon/2\}} \frac{1}{|z - \overline{x}|^n}\, dz \\
&\leq \int_{|\theta|=1} \int_{\epsilon/2}^{+\infty} \frac{1}{r^n} r^{n-2}\, dr\, d\theta \\
&= \frac{2\sigma_{n-1}}{\epsilon}.
\end{aligned}$$

Therefore,

$$\limsup_{x \to z_0} \int_{\partial\Omega \cap \{|z - z_0| \geq \epsilon\}} \frac{f(z)}{|z - x|^n}\, dz \leq \limsup_{x \to z_0} \frac{4Mx_n\sigma_{n-1}}{\sigma_n \epsilon} = 0.$$

Hence,

$$\limsup_{x \to z_0} u(x) \leq k + \limsup_{x \to z_0} \int_{\partial\Omega \cap \{|z - \overline{x} \geq \epsilon\}} \frac{f(z)}{|z - \overline{x}|^n}\, dz \leq k.$$

□

The last theorem illustrates the difficulty with uniqueness of the Dirichlet problem for unbounded regions. The solution for a bounded function on the boundary of a half-space is not unique since the choice of c in $\mathbf{PI}(c, f, \Omega)$ is arbitrary.

In some cases, a harmonic function can be extended harmonically across the boundary of a region. This will be shown to be the case when the region is a half-space under appropriate conditions. If $y = (y_1, \ldots, y_n) \in R^n$, y^r will denote the **reflection** of y across the $x_n = 0$ hyperplane; that is, $y^r = (y_1, \ldots, y_{n-1}, -y_n)$. If $\Lambda \subset R^n$, Λ^r will denote the set $\{y^r; y \in \Lambda\}$.

Lemma 2.9.3 *Let $\Omega = \{(x_1, \ldots, x_n) \in R^n; x_n > 0\}$, let Γ be a compact subset of Ω, and let u be a function on Ω^- that is continuous on $\Omega^- \sim \Gamma$, harmonic on $\Omega \sim \Gamma$, and equal to 0 on $\partial\Omega$. Then the function $\tilde{u}$ defined on $R^n \sim (\Gamma \cup \Gamma^r)$ by*

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \sim \Gamma \\ 0 & \text{if } x \in \partial\Omega \\ -u(x^r) & \text{if } x \in (\Omega \sim \Gamma)^r \end{cases}$$

is harmonic on $R^n \sim (\Gamma \cup \Gamma^r)$.

Proof Since the function $\tilde{u}$ is clearly continuous on $R^n \sim (\Gamma \cup \Gamma^r)$, the result follows from Corollary 2.7.10. □

This lemma will be used to prove a less trivial result. In the course of doing so, the fact that the Kelvin transformation is idempotent will be used; that is, if f is a real-valued function on an open set Ω, not containing the reference point of the inversion on which the transformation is based, then the Kelvin transform of f^* on Ω^* is just f on Ω. This property follows immediately from the definition of the transformation.

Theorem 2.9.4 *Let* $0 < \delta < \rho$. *If* u *is continuous on* $B^-_{x,\rho} \sim B^-_{x,\delta}$, *harmonic on* $B_{x,\rho} \sim B^-_{x,\delta}$, *and equal to* 0 *on* $\partial B_{x,\rho}$, *then there is an* $\epsilon > 0$ *such that* u *has a harmonic extension to* $B_{x,\rho+\epsilon} \sim B^-_{x,\delta}$.

Proof Let y be any point of $\partial B_{x,\rho}$ and consider an inversion relative to $\partial B_{y,2\rho}$. Under this inversion, the ball $B_{x,\rho}$ maps onto a half-space Ω, which can be taken to be $\{(x_1, \ldots, x_n); x_n > 0\}$ by choice of coordinates, with the hyperplane $\partial\Omega$ externally tangent to $B_{x,\rho}$, $B = B_{x,\delta}$ maps onto a ball B^* with closure $B^{*-} \subset \Omega$, and $B_{x,\rho} \sim B^-_{x,\delta}$ maps onto $\Omega \sim B^{*-}$. The Kelvin transform u^* of u is continuous on $\Omega^- \sim B^*$, harmonic on $\Omega \sim B^{*-}$, and equal to 0 on $\partial\Omega$. Now restrict u^* to $\Omega^- \sim B^-$. By the preceding lemma, u^* has a harmonic extension to $R^n \sim (B^{*-} \cup (B^{*-})^r)$, where $(B^{*-})^r$ is the reflection of B^{*-} across $\partial\Omega$. By simple geometry, it can be seen that a sufficiently small $\epsilon > 0$ can be chosen so that $B_{x,\rho+\epsilon} \sim B^-_{x,\delta}$ maps under the inversion onto a neighborhood of $\partial\Omega$ which also contains $\Omega \sim B^{*-}$. Restricting u^* to this neighborhood, u^* will be harmonic thereon and its transform will be harmonic on $B_{x,\rho+\epsilon} \sim B^-_{x,\rho}$, thereby extending u harmonically across $\partial B_{x,\rho}$. □

2.9.1 Exercises for Sect. 2.9

1. If $\Omega = B_{0,\rho}$, $\Omega_+ = \Omega \cap R^n_+$, $\Omega_0 = \{x \in \Omega; x_n = 0\}$, the function u is harmonic on Ω_+ and continuous on $\Omega_+ \cup \Omega_0$, and $u = 0$ on Ω_0, show that u has a harmonic extension to Ω.
2. If f is a bounded measurable function on $\partial\Omega$ show that $\mathbf{P}(c, f, \Omega)$ is a bounded harmonic function if and only if $c = 0$.
3. Use the Poisson Integral to find a bounded harmonic function u on the upper half plane $\{(x, y); y > 0\}$ satisfying the condition $u(x, 0) = f(x)$ where $f(x) = -1$ if $x \leq -1$, $f(x) = x$ if $-1 < x < 1$, and $f(x) = 1$ if $x > 1$.

2.10 Neumann Problem for a Disk

Consider a nonempty open set $\Omega \subset R^n$ having compact closure and a smooth boundary. Given a real-valued function g on $\partial\Omega$, the **Neumann problem** is that of finding a harmonic function u on Ω such that $D_{\mathbf{n}}u(x) = g(x), x \in \partial\Omega$. There is an obvious difficulty with uniqueness of the solution if u satisfies the above conditions and c is any constant, then $u+c$ satisfies the same conditions. Also, not every function g can serve as a boundary function for the Neumann problem. If $u \in C^2(\Omega^-)$ solves the Neumann problem for the boundary function g, then by taking $v = 1$ in Green's Identity

$$0 = \int_{\partial\Omega} D_{\mathbf{n}}u(z)\, d\sigma(z) = \int_{\partial\Omega} g(z)\, d\sigma(z);$$

and it follows that a necessary condition for the solvability of the Neumann problem is that the latter integral be zero.

Before getting into the details, the meaning of the statement $D_{\mathbf{n}}u(x) = g(x), x \in \partial\Omega$, should be clarified since the solution u of the Neumann problem may be defined only on Ω. If $\mathbf{n}(x)$ is the outer normal to $\partial\Omega$ at x, by definition $D_{\mathbf{n}}u(x) = \lim_{t\to 1-} D_{\mathbf{n}(x)}u(tx)$, where $D_{\mathbf{n}(x)}u$ denotes the derivative of u in the direction $\mathbf{n}(x)$.

The Poisson integral solved the Dirichlet problem for a disk of any dimension $n \geq 2$. An analogous integral for the Neumann problem is not available for all $n \geq 2$. The $n = 2$ case will be considered in this section and the $n \geq 3$ case in the next section.

Consider the $n = 2$ case, using polar coordinates (r, θ) rather than rectangular coordinates. Let g be a real-valued function on the boundary of the disk $B = B_{y,\rho}$ which satisfies the condition

$$\int_0^{2\pi} g(\rho, \theta)\, d\theta = 0. \tag{2.22}$$

Fourier series will be used to construct a harmonic function u on B satisfying the Neumann condition $D_{\mathbf{n}}u = g$ on ∂B. The use of Fourier series will not only provide a method of approximating the solution, but will also lead to an integral representation analogous to the Poisson integral. Suppose the function $g(\rho, \theta)$ has the Fourier series expansion

$$g(\rho, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta), \tag{2.23}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(\rho,\theta)\cos n\theta\,d\theta \qquad n \ge 0$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} g(\rho,\theta)\sin n\theta\,d\theta \qquad n \ge 1.$$

By Eq. (2.22),

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} g(\rho,\theta)\,d\theta = 0.$$

Noting that the functions $r^n \cos n\theta$, $r^n \sin n\theta$, $n \ge 0$, are harmonic on R^2 (as the real and imaginary parts of z^n), it is conceivable that

$$u(r,\theta) = u_0 + \sum_{n=1}^{\infty}(\alpha_n r^n \cos n\theta + \beta_n r^n \sin n\theta)$$

is a harmonic function. Since $D_{\mathbf{n}}u = \partial u/\partial r$ for a disk,

$$D_{\mathbf{n}}u(\rho,\theta) = \sum_{n=1}^{\infty}(n\alpha_n\rho^{n-1} cos n\theta + n\beta_n\rho^{n-1}\sin n\theta),$$

formally at least. Choosing the α_n and β_n so that

$$\alpha_n = \frac{a_n}{n\rho^{n-1}} \qquad \beta_n = \frac{b_n}{n\rho^{n-1}}, \qquad n \ge 1,$$

it would appear that the Neumann condition $D_{\mathbf{n}}u = g$ on $\partial B_{y,\rho}$ is satisfied. Substituting these values of α_n and β_n in the definition of $u(r,\theta)$,

$$u(r,\theta) = u_0 + \sum_{n=1}^{\infty}\frac{r^n}{n\rho^{n-1}}(a_n \cos n\theta + b_n \sin n\theta). \tag{2.24}$$

Since

$$\begin{aligned} a_n \cos n\theta + b_n \sin n\theta =& \frac{1}{\pi}\int_0^{2\pi}\cos n\phi \cos n\theta\, g(\rho,\phi)\,d\phi \\ &+ \frac{1}{\pi}\int_0^{2\pi}\sin n\phi \sin n\theta\, g(\rho,\phi)\,d\phi \\ =& \frac{1}{\pi}\int_0^{2\pi}\cos n(\phi-\theta) g(\rho,\phi)\,d\phi, \end{aligned}$$

$$
\begin{aligned}
u(r,\theta)&\\
&= u_0 + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{r^n}{n\rho^{n-1}}\int_0^{2\pi}(e^{in(\phi-\theta)}+e^{-in(\phi-\theta)})g(\rho,\phi)\,d\phi\\
&= u_0 + \frac{\rho}{2\pi}\int_0^{2\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{re^{i(\phi-\theta)}}{\rho}\right)^n+\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{re^{-i(\phi-\theta)}}{\rho}\right)^n\right)g(\rho,\phi)\,d\phi
\end{aligned}
$$

for $0 \le r < \rho, 0 \le \theta \le 2\pi$. Using the fact that $\log(1-t) = -\sum_{n=1}^{\infty} t^n/n$ for $|t| < 1$,

$$
u(r,\theta) = u_0 + \frac{\rho}{2\pi}\int_0^{2\pi}\log\left(\frac{\rho^2}{\rho^2-2\rho r\cos(\phi-\theta)+r^2}\right)g(\rho,\phi)\,d\phi.
$$

The ρ^2 in the numerator can be dropped because of Eq. (2.22) resulting in **Dini's formula**

$$
u(r,\theta) = u_0 - \frac{\rho}{2\pi}\int_0^{2\pi}\log(\rho^2+r^2-2\rho r\cos(\phi-\theta))g(\rho,\phi)\,d\phi. \tag{2.25}
$$

Having found a formula for a solution to the Neumann problem for a disk, the Fourier series method could be dispensed and the formula examined independently of its origin. But in light of the fact that the Fourier series method also provides approximate solutions, the method will be retained for the next theorem.

Theorem 2.10.1 *If $g(\rho,\theta)$ is continuous and of bounded variation on $[0,2\pi]$ and $\int_{\partial B_{y,\rho}} g(z)\,d\sigma(z) = 0$, then the function*

$$
u(r,\theta) = u_0 - \frac{\rho}{2\pi}\int_0^{2\pi}\log(\rho^2+r^2-2\rho r\cos(\theta-\phi))g(\rho,\phi)\,d\phi
$$

belongs to $C^0(B_{y,\rho}^-)\cap C^2(B_{y,\rho})$ and solves the Neumann problem for the boundary function g.

Proof Under the conditions on g, the Dirichlet-Jordan test (c.f. [7]) implies that the Fourier series representation Eq. (2.23) of $g(\rho,\theta)$ is valid for each $\theta \in [0,2\pi]$. Define $u(r,\theta)$ for $0 \le r \le \rho$ as in Eq. (2.24). Applying Theorem 1.2.2 twice to the series in (2.24) via the Weierstrass M-test, it is easy to show that $u(r,\theta)$ is harmonic on $B_{y,\rho}$. Since the Fourier coefficients a_n and b_n of g are $\mathcal{O}(\frac{1}{n})$ (c.f. [7]) and $r \le \rho$, the series defining $u(r,\theta)$ in Eq. (2.24) converges uniformly on $B_{y,\rho}^-$, and therefore $u \in C^0(B_{y,\rho}^-)$. Theorem 1.2.2 can be applied again to show that

$$
D_r u(r,\theta) = \sum_{n=1}^{\infty}\frac{r^{n-1}}{\rho^{n-1}}(a_n\cos n\theta + b_n\sin n\theta).
$$

By Abel's limit theorem (c.f. [8]),

$$\lim_{r \to \rho-} D_r u(r, \theta) = g(r, \theta), \qquad 0 \le \theta \le 2\pi. \qquad \square$$

The requirement that g be of bounded variation on $\partial B_{y,\rho}$ in the preceding theorem is unnecessary for the conclusion.

Theorem 2.10.2 *If g is continuous on $\partial B_{y,\rho}$ and $\int_{\partial B_{y,\rho}} g(z)\, d\sigma(z) = 0$, then the function u defined by*

$$u(x) = u_0 - \frac{\rho}{2\pi} \int_{\partial B_{y,\rho}} (\log |x - z|) g(z)\, d\sigma(z) \qquad x \in B_{y,\rho}$$

solves the Neumann problem for the boundary function g.

Proof By Eq. (2.25) and Lemma 2.5.5,

$$\begin{aligned} u_r(r,\theta) &= \frac{\rho}{2\pi} \int_0^{2\pi} \frac{-2r + 2\rho \cos(\phi - \theta)}{\rho^2 + r^2 - 2\rho r \cos(\phi - \theta)} g(\rho, \phi)\, d\phi \\ &= \frac{\rho}{2\pi r} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\phi - \theta)} g(\rho, \phi)\, d\phi - \frac{\rho}{2\pi r} \int_0^{2\pi} g(\rho, \phi)\, d\phi \\ &= \frac{\rho}{r} \mathbf{PI}(g, (r, \theta)). \end{aligned}$$

By Lemma 2.7.6

$$D_{\mathbf{n}} u(\rho, \theta) = \lim_{r \to \rho-} u_r(r, \theta) = g(\rho, \theta). \qquad \square$$

Converting Eq. (2.25) to rectangular coordinates,

$$u(x) = u_0 + \frac{1}{2\pi} \int_{\partial B_{y,\rho}} \log \frac{1}{|z - x|^2} g(z)\, d\sigma(z), \qquad x \in B_{y,\rho}.$$

Putting $x = y$, the log factor becomes a constant and it is clear that the constant u_0 is the value of the solution at the center of the disk.

Remark 2.10.3 Integral representations of solutions to the Neumann problem for half-spaces, quadrants, circular annuli, etc., can be found in [9, 10].

2.10.1 Exercises for Sect. 2.10

1. Verify directly that $r^n \cos n\theta$ and $r^n \sin n\theta$ are harmonic on R^2 without using complex variables.

2. Verify that the Fourier Series representation of a harmonic function on $B_{0,\rho}$ satisfying the Neumann boundary condition

$$D_{\mathbf{n}}u(\rho,\theta) = g(\rho,\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ -1 & \pi < \theta < 2\pi, \end{cases}$$

is

$$u(r,\theta) = u_0 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\rho}{n^2}\left(1-(-1)^n\right)\left(\frac{r}{\rho}\right)^n \sin n\theta.$$

3. Show that any two solutions of the Neumann problem on the disk $\partial B_{y,\rho}$ differ only by a constant. (Hint: Use Eq. 2.2)

2.11 Neumann Problem for the Ball

In constructing the Green function $G_B(x,z)$ and the Poisson integral formula for a ball $B = B_{y,\rho} \subset R^n$, the representation

$$u(x) = \frac{1}{\sigma_n(n-2)}\int_{\partial B}\left(\frac{1}{|z-x|^{n-2}}D_{\mathbf{n}}u - uD_{\mathbf{n}}\frac{1}{|z-x|^{n-2}}\right)d\sigma(z)$$

of a harmonic function was modified by adding an appropriate harmonic function v_x to $1/|z-x|^{n-2}$ in order to eliminate the integral of the first term. It might be possible to eliminate the integral of the the second term by adding a harmonic function v_x to $1/|z-x|^{n-2}$ so that its normal derivative vanishes on ∂B. This goal is not quite achievable, but the procedure can be used to produce a constant normal derivative. The resulting function $K_B(x,z)$ should also be a harmonic function of x for each $z \in \partial B$. The procedure will produce the following representation of a harmonic function u in terms of its boundary normal derivative $D_{\mathbf{n}}u$:

$$u(x) = \frac{1}{\sigma_n(n-2)}\int_{\partial B}K_B(x,z)D_{\mathbf{n}}u(z)\,d\sigma(z) + c\int_{\partial B}u(z)\,d\sigma(z), \qquad x \in B. \tag{2.26}$$

Finding an explicit function v_x accomplishing the above can be done only in the $n = 3$ case. In this case, a function $K_B(x,z)$ and a constant c will be exhibited such that

$$\Delta_{(x)}K_B(x,z) = 0 \qquad \text{for each } z \in \partial B \tag{2.27}$$

$$D_{\mathbf{n}(z)}K_B(x,z) = c \qquad \text{for each } x \in B^-, z \in \partial B, x \neq z, \tag{2.28}$$

where $D_{\mathbf{n}(z)}$ denotes the outer unit normal derivative at $z \in \partial B$. Before reading on, the reader should verify that the function $\log(\pm w + \sqrt{u^2 + v^2 + w^2})$, in the usual calculus notation, is harmonic on its domain.

In order to simplify the notation, it will be assumed that the ball B is centered at the origin; the general case will then follow by a translation. Returning to the discussion at the beginning of this section, a solution $K_B(x, z)$ of Eqs. (2.27) and (2.28) of the form

$$K_B(x, z) = \frac{1}{4\pi r} + k(x, z),$$

will be constructed where $r = |x - z|$. This will be done for a special case first.

Consider a fixed point $x_0 = (0, 0, t)$, $0 < t \le \rho$, and the inverse $x_0^* = (0, 0, \rho^2/t)$ of x_0 relative to $\partial B_{0,\rho}$. The procedure used in deriving the Poisson integral formula will be mimicked by choosing $k(x_0, z)$ to be a multiple of $1/4\pi r_1$ where $r_1 = |x_0^* - z|$. Rather than leaving the multiplier undetermined, for the sake of brevity it will be incorporated here by taking

$$k(x_0, z) = \frac{\rho}{t}\frac{1}{4\pi r_1}.$$

Letting $z = (u, v, w)$ and $r = |x_0 - z|$,

$$\begin{aligned} D_{\mathbf{n}}\left(\frac{1}{r}\right)\Big|_{\partial B} &= \nabla_{(z)}\left(\frac{1}{|x_0 - z|}\right) \cdot \frac{z}{\rho}\Big|_{\partial B} \\ &= \frac{1}{\rho}\left(-\frac{u}{r^3}, -\frac{v}{r^3}, -\frac{w - t}{r^3}\right) \cdot (u, v, w)\Big|_{\partial B} \\ &= \frac{tw - \rho^2}{\rho r^3}\Big|_{\partial B}. \end{aligned}$$

Replacing x_0 by x_0^*,

$$D_{\mathbf{n}}\left(\frac{1}{r_1}\right)\Big|_{\partial B} = \frac{t^* w - \rho^2}{\rho r_1^3}\Big|_{\partial B},$$

where $t^* = \rho^2/t$. By Eq. (2.16), $tr_1 = \rho r$ on ∂B and

$$\begin{aligned} D_{\mathbf{n}}\left(\frac{1}{r_1}\right)\Big|_{\partial B} &= \frac{t^3}{\rho^3}\,\frac{(\rho^2/t)w - \rho^2}{\rho r^3}\Big|_{\partial B} \\ &= \frac{t^2}{\rho^2}\frac{w - t}{r^3}\Big|_{\partial B}, \end{aligned}$$

and so

$$D_{\mathbf{n}}\left(\frac{1}{r}+\frac{\rho}{t}\frac{1}{r_1}\right)\Big|_{\partial B}=\frac{2tw-\rho^2-t^2}{\rho r^3}\Big|_{\partial B}.$$

Since $r^2=|x_0-z|^2=u^2+v^2+(t-w)^2=\rho^2-2tw+t^2$ on ∂B,

$$D_{\mathbf{n}}\left(\frac{1}{4\pi r}+\frac{\rho}{t}\frac{1}{4\pi r_1}\right)\Big|_{\partial B}=-\frac{1}{4\pi\rho r}\Big|_{\partial B}.$$

The addition of $\rho/4\pi t r_1$ to $1/4\pi\rho r$ does not result in a function $K_B(x_0,z)$ satisfying Eq. (2.28). Fortunately, there is another function, namely

$$h(x_0,z)=-\frac{1}{4\pi\rho}\log(t^*-w+r_1),$$

which when added to $k(x_0,z)$ will come closer to satisfying Eq. (2.28). The gradient of $h(x_0,z)$ is given by

$$\nabla_{(z)}h(x_0,z)=-\frac{1}{4\pi\rho r_1(t^*-w+r_1)}(u,v,-r_1+w-t^*).$$

Thus,

$$\begin{aligned}D_{\mathbf{n}}h(x_0,z)|_{\partial B}&=\nabla_{(z)}h(x_0,z)\cdot z\big|_{\partial B}\\&=-\frac{1}{4\pi\rho^2}\frac{(r_1^2-(w-t^*)^2)+w(-r_1+w-t^*)}{r_1(t^*-w+r_1)}\Big|_{\partial B}\\&=-\frac{1}{4\pi\rho^2}\frac{r_1-t^*}{r_1}\Big|_{\partial B}\\&=-\frac{1}{4\pi\rho^2}\left(1-\frac{t^*}{r_1}\right)\Big|_{\partial B}\\&=\frac{1}{4\pi\rho r}-\frac{1}{4\pi\rho^2}\Big|_{\partial B}.\end{aligned}$$

Therefore,

$$D_{\mathbf{n}}\left(\frac{1}{4\pi r}+\frac{\rho}{t}\frac{1}{4\pi r_1}-\frac{1}{4\pi\rho}\log(t^*-w+r_1)\right)\Big|_{\partial B}=-\frac{1}{4\pi\rho^2},$$

a constant. Adding $-(1/4\pi\rho^2)\log t$ to the function just constructed,

$$D_{\mathbf{n}}\left(\frac{1}{4\pi r}+\frac{\rho}{t}\frac{1}{4\pi r_1}-\frac{1}{4\pi\rho}\log(t^*-w+r_1)-\frac{1}{4\pi\rho}\log t\right)\Big|_{\partial B}=-\frac{1}{4\pi\rho^2}.$$

Define

$$K_B(x_0, z) = \frac{1}{4\pi r} + \frac{\rho}{t}\frac{1}{4\pi r_1} - \frac{1}{4\pi\rho}\log(t^* - w + r_1) - \frac{1}{4\pi\rho}\log t.$$

Recall that the above discussion pertains to the special case $x_0 = (0, 0, t)$, $0 < t \leq \rho$. Consider now a fixed point $x = (x_1, x_2, x_3) \in B^-$. Letting $s = \sqrt{x_1^2 + x_2^2}$ and $t = |x|$, the point $x_0 = (0, 0, t)$ will be mapped onto the point $x = (x_1, x_2, x_3)$ by means of the orthogonal transformation

$$A = \begin{bmatrix} \frac{x_2}{s} & \frac{x_1 x_3}{st} & \frac{x_1}{t} \\ -\frac{x_1}{s} & \frac{x_2 x_3}{st} & \frac{x_2}{t} \\ 0 & -\frac{s}{t} & \frac{x_3}{t} \end{bmatrix}.$$

Letting $y = Az$ for $z \in B^-$, $r = |x_0 - z| = |x - y|$, and $r_1 = |x_0^* - z| = |x^* - y|$ since an orthogonal transformation preserves distances. Since $t^* = \rho^2/t = \rho^2/|x|$ and $w = (x \cdot y)/t$, the function constructed above is mapped into the function

$$\begin{aligned} K_B(x, y) &= \frac{1}{4\pi|x - y|} + \frac{\rho}{|x|}\frac{1}{4\pi|x^* - y|} \\ &\quad - \frac{1}{4\pi\rho}\log\left(\frac{\rho^2}{|x|} - \frac{x \cdot y}{|x|} + |x^* - y|\right) - \frac{1}{4\pi\rho}\log|x|. \end{aligned}$$

By Eq. (2.16), $|x||x^* - y| = |y||y^* - x|$ and it follows that the function

$$\log(\rho^2 - x \cdot y + |x||x^* - y|) = \log\left(\frac{\rho^2}{|x|} - \frac{x \cdot y}{|x|} + |x^* - y|\right) + \log|x|$$

is a symmetric function of x and y; since it is a harmonic function of y for each x, it is a harmonic function of x for each y. Likewise, $1/|x||x^* - y|$ is a symmetric function of x and y and is a harmonic function of x for $y \in B^-$. Thus, $\Delta_{(x)} K_B(x, y) = 0$ on B for each $y \in \partial B$. Moreover, by Remark 2.3.4, $D_{\mathbf{n}} K_B(x, y)|_{\partial B} = D_{\mathbf{n}} K_B(x_0, z)|_{\partial B} = -(1/4\pi\rho^2)$ for each $x \in B$. Since $y^* = y$ for $y \in \partial B$,

$$K_B(x, y) = \frac{2}{4\pi|x - y|} - \frac{1}{4\pi\rho}\log(\rho^2 - x \cdot y + \rho|x - y|), \qquad y \in \partial B.$$

Since adding a constant to $K_B(x, y)$ will not affect the validity of Eqs. (2.27) and (2.28), $K_B(x, y)$ can be adjusted so that

$$K_B(x, y) = \frac{1}{2\pi|x - y|} + \frac{1}{4\pi\rho}\log\left(\frac{2\rho^2}{\rho^2 - x \cdot y + \rho|x - y|}\right).$$

The function $K_B(x, z)$ is called the **Green function for the Neumann problem** on the ball B or the **Green function of the second kind** for the ball B. The proof of the following theorem is essentially the same as the proof of Green's representation theorem, Theorem 2.4.2.

Theorem 2.11.1 *If u is harmonic on a neighborhood of B^-, then for each $x \in B$,*

$$u(x) = \frac{1}{4\pi}\int_{\partial B} D_{\mathbf{n}}u(y)\left(\frac{2}{|x-y|} + \frac{1}{\rho}\log\left(\frac{2\rho^2}{\rho^2 - x\cdot y + \rho|x-y|}\right)\right) d\sigma(y) + \frac{1}{4\pi\rho^2}\int_{\partial B} u(y)\, d\sigma(y).$$

The following theorem is stated as an exercise by Kellogg in [11].

Theorem 2.11.2 *If $g \in C^0(\partial B)$ and $\int_{\partial B} g(z)\, d\sigma(z) = 0$, then the function*

$$u(x) = \int_{\partial B} K_B(x, z) g(z)\, d\sigma(z)$$

solves the Neumann problem for the boundary function g.

Proof (Sobolev [12]) If $x \in \partial B$ and $0 < t < 1$, the function $K_B(tx, z)$ is bounded on ∂B and the above integral defining $u(tx)$ is finite. Letting $r = |tx - z|$,

$$\frac{\partial K_B}{\partial x_i}(tx, z) = -\frac{(tx_i - z_i)t}{2\pi r^3} + \frac{1}{4\pi\rho}\frac{trz_i - \rho(tx_i - z_i)t}{r(\rho^2 - t(x\cdot z) + \rho r)}.$$

By Lemma 2.5.5,

$$\frac{\partial u}{\partial x_i}(tx) = \int_{\partial B}\frac{\partial K_B}{\partial x_i}(tx, z) g(z)\, d\sigma(z),$$

and therefore

$$D_{\mathbf{n}(x)}u(tx) = \int_{\partial B} D_{\mathbf{n}(x)}K_B(tx, z) g(z)\, d\sigma(z).$$

Fix $x \in \partial B$. In order to show that $D_{\mathbf{n}(x)}u(x) = \lim_{t\to 1-} D_{\mathbf{n}(x)}u(tx) = g(x)$, it will first be shown that it suffices to prove the result for the special case $g(x) = 0$. Since at least one of the components of $x = (x_1, x_2, x_3)$ is different from zero, it can be assumed that $x_1 \neq 0$, say. Consider the linear function $\ell(u, v, w) = (\rho g(x)/x_1)u$ for which

$$D_{\mathbf{n}(x)}\ell(x) = (\rho g(x)/x_1, 0, 0)\cdot(x_1/\rho, x_2/\rho, x_3/\rho) = g(x).$$

By Theorem 2.11.1 and the definition of u,

$$\ell(y) - u(y) = \int_{\partial B} K_B(y, z)(D_{\mathbf{n}}\ell(z) - g(z))\, d\sigma(z)$$

with $D_{\mathbf{n}}\ell(x) - g(x) = 0$. If it can be shown that $D_{\mathbf{n}(x)}(\ell - u)(x) = 0$, it would follow that $D_{\mathbf{n}(x)}u(x) = D_{\mathbf{n}(x)}\ell(x) = g(x)$. Henceforth, assume that $g(x) = 0$. To show that $D_{\mathbf{n}(x)}u(x) = 0$, the special case $x = (0, 0, \rho)$ will be proven, with the general case following from an orthogonal transformation. Consider $x_0 = (0, 0, t)$ where $\rho/2 < t \le \rho$. Then $D_{\mathbf{n}(x)}u(x) = \lim_{t \to \rho-} D_t u(x_0)$, where

$$D_t u(x_0) = \int_{\partial B} D_t K_B(x_0, z) g(z)\, d\sigma(z).$$

Fix $\epsilon > 0$ and choose $\gamma_0 \in (0, \pi/4)$ such that $|g(z)| < \epsilon$ whenever the angle γ between the line segments joining 0 to x and 0 to z is less than γ_0. Letting $z = (u, v, w)$ and $r = |x_0 - z|$,

$$D_t K_B(x_0, z) = \frac{1}{2\pi}\frac{w - t}{r^3} + \frac{1}{4\pi\rho r}\frac{rw + \rho w - \rho t}{\rho^2 - tw + \rho r}.$$

The above integral over ∂B will be split into a sum of four integrals I_1, I_2, I_3, and I_4 where I_1 is the integral of the first term of this function over $\partial B \cap (\gamma < \gamma_0) \cap (t < w)$, I_2 is the integral of the first term over $\partial B \cap (\gamma < \gamma_0) \cap (w \le t)$, I_3 is the integral of the second term over $\partial B \cap (\gamma < \gamma_0)$, and I_4 is the integral of both terms of the function over $\partial B \cap (\gamma \ge \gamma_0)$. Since $t > \rho/2$,

$$\begin{aligned}
|I_1| &= \left| \int_{\partial B \cap (\gamma < \gamma_0) \cap (t < w)} \frac{1}{2\pi}\frac{w - t}{r^3} g(z)\, d\sigma(z) \right| \\
&\le \frac{\epsilon}{2\pi} \int_{\partial B \cap (\gamma < \gamma_0) \cap (t < w)} \frac{\rho - t}{r^3}\, d\sigma(z) \\
&\le \frac{\epsilon}{2\pi} \int_{\partial B} \frac{\rho^2 - t^2}{r^3} \frac{1}{\rho + t}\, d\sigma(z) \\
&\le 2\epsilon \mathbf{PI}(1 : B)(x_0) \\
&= 2\epsilon.
\end{aligned}$$

Next consider I_2. For $z \in \partial B \cap (\gamma < \gamma_0) \cap (w \le t)$, $r^2 = u^2 + v^2 + (w - t)^2 = (\rho^2 - tw) + t(t - w) \ge t(t - w) \ge (\rho/2)(t - w)$, and therefore $|t - w| \le (2/\rho)r^2$. It follows that

$$\begin{aligned}
|I_2| &\le \int_{\partial B \cap (\gamma < \gamma_0) \cap (w \le t)} \frac{2}{4\pi} \frac{|t - w|}{r^3} |g(z)|\, d\sigma(z) \\
&\le \frac{\epsilon}{\pi\rho} \int_{\partial B \cap (\gamma < \gamma_0) \cap (w \le t)} \frac{1}{r}\, d\sigma(z).
\end{aligned}$$

By Theorem 2.6.1, $(1/4\pi\rho^2)\int_{\partial B}\frac{1}{r}\,d\sigma(z)=\frac{1}{\rho}$ so that

$$|I_2| \le 4\epsilon.$$

Turning to I_3, since $tw \le \rho^2$ and $r=\sqrt{u^2+v^2+(w-t)^2} \ge |w-t|$ on $\partial B \cap (\gamma < \gamma_0)$, another application of Theorem 2.6.1 shows that

$$\begin{aligned}|I_3| &= \left|\frac{1}{4\pi\rho}\int_{\partial B\cap(\gamma<\gamma_0)}\frac{rw+\rho(w-t)}{r(\rho^2-tw+\rho r)}g(z)\,d\sigma(z)\right|\\ &\le \frac{\epsilon}{2\pi\rho}\int_{\partial B\cap(\gamma<\gamma_0)}\frac{1}{r}\,d\sigma(z)\\ &= 2\epsilon.\end{aligned}$$

Lastly, since $\int_{\partial B} g(z)\,d\sigma(z)=0$ and $|\int_{\partial B\cap(\gamma<\gamma_0)} g(z)\,d\sigma(z)| < 4\pi\rho^2\epsilon$,

$$\left|\int_{\partial B\cap(\gamma\ge\gamma_0)} g(z)\,d\sigma(z)\right| < 4\pi\rho^2\epsilon,$$

or

$$\left|\int_{\partial B\cap(\gamma\ge\gamma_0)}\left(-\frac{1}{4\pi\rho^2}g(z)\right)\,d\sigma(z)\right| < \epsilon.$$

Thus,

$$|I_4| \le |g|\int_{\partial B\cap(\gamma\ge\gamma_0)}\left|\frac{2}{4\pi}\frac{w-t}{r^3}+\frac{1}{4\pi\rho}\frac{rw+\rho w-\rho t}{r(\rho^2-tw+\rho r)}-\left(-\frac{1}{4\pi\rho^2}\right)\right|\,d\sigma(z)+\epsilon.$$

Since $r=|x_0-z|=\sqrt{t^2+\rho^2-2\rho t\cos\gamma} \ge \sqrt{t^2+\rho^2-2\rho t\cos\gamma_0} > 0$ for $\gamma \ge \gamma_0$, the integrand on the right tends to zero boundedly in z as $t \to \rho-$. Thus,

$$\limsup_{t\to\rho-}\int_{\partial B} D_t K_B(x_0,z)g(z)\,d\sigma(z) \le 9\epsilon.$$

□

Since $\epsilon > 0$ is arbitrary, $D_{\mathbf{n}(x)}u(x)=\lim_{t\to\rho-} D_t u(x_0)=0$.

2.12 Spherical Harmonics

Solving the Neumann problem for a ball $B=B_{0,\rho}\subset R^n$ is less satisfactory for the $n\ge 4$ case than for the $n=2$ and $n=3$ cases since the Fourier series method is not applicable, nor is there an explicit formula as in the $n=3$ case. A series solution is possible using functions called spherical harmonics. This method of solution will be sketched only briefly. For further details, see [10, 11, 13, 14].

Consider two points $x, y \in B_{0,\rho} \subset R^n, y \neq 0, n \geq 3$, and the fundamental harmonic function $r^{-n+2} = |x-y|^{-n+2}$ with pole y. It is useful to think of y as a parameter in what follows. Letting γ denote the angle between the line segments joining 0 to x and 0 to y,

$$\begin{aligned}\frac{1}{r^{n-2}} &= \frac{1}{(|x|^2+|y|^2-2|x||y|\cos\gamma)^{(n-2)/2}} \\ &= \frac{1}{|y|^{n-2}}\frac{1}{((|x|/|y|)^2+1-2(|x|/|y|)\cos\gamma)^{(n-2)/2}}.\end{aligned} \tag{2.29}$$

Letting $u = \cos\gamma$ and $v = |x|/|y|$,

$$\frac{1}{r^{n-2}} = \frac{1}{|y|^{n-2}}\frac{1}{(1-2uv+v^2)^{(n-2)/2}}.$$

Recalling the binomial coefficients

$$\binom{\alpha}{n} = \alpha(\alpha-1)\times\cdots\times(\alpha-n+1)/n!,$$

defined for real α and $n \geq 1$, and applying the generalized binomial theorem

$$\frac{1}{(1-2uv+v^2)^{(n-2)/2}} = 1 + \sum_{m=1}^{\infty}\binom{-\frac{n-2}{2}}{m}(2uv-v^2)^m$$

provided $|2uv-v^2| < 1$. Suppose $v < \sqrt{2}-1$. Since $|u| \leq 1$, $|2uv-v^2| \leq 2|u|v+v^2 \leq 2v+v^2 = (v+1)^2-1 < 1$ and the above series is absolutely convergent for $v < \sqrt{2}-1$. In fact, absolute convergence holds for $v < 1$ (c.f. [11]). Applying the binomial theorem to the binomials $(2uv-v^2)^m$,

$$\frac{1}{(1-2uv+v^2)^{(n-2)/2}} = 1 + \sum_{m=1}^{\infty}\sum_{k=0}^{m}\binom{-\frac{n-2}{2}}{m}\binom{m}{k}(-1)^k 2^{m-k}u^{m-k}v^{m+k}.$$

Making the substitution $\ell = m+k$ with k fixed,

$$\frac{1}{(1-2uv+v^2)^{(n-2)/2}} = 1 + \sum_{\ell=1}^{\infty}\sum_{k=0}^{[\ell/2]}\binom{-\frac{n-2}{2}}{\ell-k}\binom{\ell-k}{k}(-1)^k 2^{\ell-2k}u^{\ell-2k}v^{\ell}, \tag{2.30}$$

where $[\ell/2]$ is the usual greatest integer function. The coefficient of v^m in this series will be denoted by $P_{n,m}(u)$ and is given by

$$P_{n,m}(u) = \sum_{k=0}^{[m/2]} \binom{-\frac{n-2}{2}}{m-k} \binom{m-k}{k} (-1)^k 2^{m-2k} u^{m-2k},$$

and, in the $n = 3$ case, is called a Legendre polynomial. Consider the factor

$$\begin{aligned} u^{m-2k} v^m &= \left(\frac{x \cdot y}{|x||y|}\right)^{m-2k} \left(\frac{|x|}{|y|}\right)^m, \qquad 0 \le k \le m/2 \\ &= \frac{1}{|y|^{2(m-k)}} (x \cdot y)^{m-2k} (|x|^2)^k \end{aligned}$$

in the general term of the series in Eq. (2.30). It is clear that the latter expression is a polynomial in $x_1, x_2, \ldots, x_n$ of degree m. Recalling that a function $f(x)$ is homogeneous of degree m if $f(\lambda x) = \lambda^m f(x)$, this factor is also homogeneous of degree m. Thus, $1/r^{n-2}$ has the representation

$$\frac{1}{r^{n-2}} = \frac{1}{|y|^{n-2}} \sum_{m=0}^{\infty} \sum_{k_1+\cdots+k_n=m} a_{k_1,\ldots,k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}, \tag{2.31}$$

where the $k_1, \ldots, k_n$ are nonnegative integers. It will be shown now that each term of this series is a harmonic function. Let $\overline{x} = (|x_1|, \ldots, |x_n|)$, $\overline{y} = (|y_1|, \ldots, |y_n|)$, $\overline{u} = (\overline{x} \cdot \overline{y})/|\overline{x}||\overline{y}| = (\overline{x} \cdot \overline{y})/|x||y|$. Note that $\overline{v} = |\overline{x}|/|\overline{y}| = |x|/|y| = v$. As indicated above, $|2\overline{u}v + v^2| \le 2\overline{u}v + v^2 < 1$ for $v < \sqrt{2} - 1$. Thus, the binomial series

$$1 + \sum_{m=1}^{\infty} \binom{-\frac{n-2}{2}}{m} (2\overline{u}v + v^2)^m \tag{2.32}$$

converges absolutely for $v < \sqrt{2} - 1$. Thus, the positive series

$$1 + \sum_{m=1}^{\infty} (-1)^m \binom{-\frac{n-2}{2}}{m} (2\overline{u}v + v^2)^m \tag{2.33}$$

converges for $v < \sqrt{2} - 1$. Expanding the binomials as above, this series becomes the following power series in $|x_1|, \ldots, |x_n|$

$$\sum_{m=0}^{\infty} \sum_{k_1+\cdots+k_n=m} b_{k_1,\ldots,k_n} |x_1|^{k_1} \times \cdots \times |x_n|^{k_n}. \tag{2.34}$$

Since the latter two series are positive series and their partial sums are intertwined, the latter series converges for $v < \sqrt{2} - 1$; that is, the series Eq. (2.31) converges absolutely and can be differentiated term by term for $v < \sqrt{2} - 1$. Thus, for $|x| < (\sqrt{2} - 1)|y|$,

$$0 = \Delta_{(x)} \frac{1}{r^{n-2}} = \frac{1}{|y|^{n-2}} \sum_{m=0}^{\infty} \sum_{k_1+\cdots+k_n=m} a_{k_1,\ldots,k_n} \Delta_{(x)}(x_1^{k_1} \times \cdots \times x_n^{k_n}).$$

Since the series on the right is a power series in $x_1, \ldots, x_n$ and none of the coefficients vanish, each of the homogeneous polynomials in Eq. (2.31) is a harmonic function.

Example 2.12.1 If $n = 3, m = 2, x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$, then $P_{3,2}(u) = (3/2)(u^2 - \frac{1}{3})$ and

$$\begin{aligned}
& P_{3,2}\left(\frac{x \cdot y}{|x||y|}\right)^2 \frac{|x|^2}{|y|^3} \\
&= \frac{3}{2}\left(\frac{(x_1y_1 + x_2y_2 + x_3y_3)^2}{|x|^2|y|^2} - \frac{1}{3}\right)\frac{|x|^2}{|y|^3} \\
&= \frac{1}{2|y|^5}\Big(3(x_1y_1 + x_2y_2 + x_3y_3)^2 - (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2)\Big) \\
&= \frac{1}{2|y|^5}(y_1^2(2x_1^2 - x_2^2 - x_3^2) + y_2^2(2x_2^2 - x_1^2 - x_3^2) + y_3^2(2x_3^2 - x_1^2 - x_2^2) \\
&\quad + 6y_1y_2(x_1x_2) + 6y_1y_3(x_1x_3) + 6y_2y_3(x_2x_3).
\end{aligned}$$

Recalling that $y = (y_1, y_2, y_3)$ is regarded as a parameter, $P_{3,2}\left(\frac{x \cdot y}{|x||y|}\right)\frac{|x|^2}{|y|^3}$ is a linear combination of the following harmonic homogeneous polynomials in x_1, x_2, x_3 of degree 2:

$$2x_1^2 - x_2^2 - x_3^2,\ 2x_2^2 - x_1^2 - x_3^2,\ 2x_3^2 - x_1^2 - x_2^2,\ x_1x_2,\ x_1x_3,\ x_2x_3.$$

Among these six polynomials, five are linearly independent. The first can be expressed as the negative of the sum of the second and third.

Given a harmonic homogeneous polynomial $P_{n,m}(x)$ in $x = (x_1, \ldots, x_n)$ of degree m, putting $x = |x|\theta, \theta \in \partial B_{0,1}$ it can be written

$$P_{n,m}(x) = |x|^m P_{n,m}(\theta) = |x|^m Y_{n,m}(\theta),$$

where $Y_{n,m}(\theta) = P_{n,m}(\theta)$ is a function defined on $\partial B_{0,1}$, called an n-dimensional **spherical harmonic function** of degree m. For each $m \geq 1$, there are $k_{n,m} = (2m+n-2)(n+m-3)!/(n-2)!m!$ linearly independent n-dimensional spherical harmonic functions of degree m, denoted by $Y_{n,m}^{(k)}, k = 1, \ldots, k_{n,m}$ (c.f. [14]). $Y_{n,m}^{(k)}$ is the commonly used notation for the spherical harmonic functions. These functions can be normalized so that the family $\{Y_{n,m}^{(k)}; k = 1, \ldots, k_{n,m}, m \geq 0\}$ is a complete orthonormal system relative to surface area on $\partial B_{0,1}$ with $Y_{n,0}^{(1)}$ equal to a constant.

Now let g be a continuous function on $\partial B_{0,\rho}$ that satisfies the boundary condition

$$\int_{\partial B_{0,\rho}} g(z)\,d\sigma(z) = 0.$$

Then g has the expansion

$$g(\rho\theta) = \sum_{m=1}^{\infty}\sum_{k=1}^{k_{n,m}} b_{n,m,k} Y_{n,m}^{(k)}(\theta),$$

where

$$b_{n,m,k} = \int_{\partial B_{0,1}} g(\rho\theta) Y_{n,m}^{(k)}(\theta)\,d\sigma(\theta),$$

with the term corresponding to $m = 0$ vanishing by the preceding equation and the fact that $Y_{n,0}^{(1)}$ is a constant. The solution u of the Neumann problem for g is then given by

$$u(x) = u_0 + \sum_{m=1}^{\infty}\sum_{k=1}^{k_{n,m}} b_{n,m,k} \frac{|x|^m}{m\rho^{n-1}} Y_{n,m}^{(k)}(\theta),$$

where u_0 is an arbitrary constant (for further details see [10]).

For additional information about spherical harmonics, see the book by Axler, Bourdon, and Ramey [15]. This book contains information on how to access free software for generating spherical harmonic functions.

2.12.1 Exercises for Sect. 2.12

1. Show that $\int_{\partial B_{y,\rho}} g(z)\,d\sigma(z) = 0$ if $u \in C^2(B_{y,\rho}^-)$ solves the Neumann problem $\Delta u = 0$ on $B_{y,\rho}$ and $D_{\mathbf{n}} = g$ on $\partial B_{y,\rho}$.

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