2
Linear differential equations

Systems of linear differential equations form the focus of our first line of investigation. In particular, we will develop a *theory of existence and uniqueness* of solutions of homogeneous initial-value problems of the form $\dot{x}(t) = A(t)x(t)$, $x(\tau) = \xi$, under the assumption that A is piecewise continuous. The special case of constant A forms an important sub-class for which, as we shall see, the solution x of the initial-value problem is given in terms of the matrix exponential function by $x(t) = \exp(A(t-\tau))\xi$ for all $t \in \mathbb{R}$. Then, we extend the existence and uniqueness theory to inhomogeneous initial-value problems of the form $\dot{x}(t) = A(t)x(t) + b(t)$, $x(\tau) = \xi$, where b is a piecewise continuous extraneous input or forcing function. In certain circumstances, the function b is open to choice, and may be chosen so as to ensure that the unique solution of the initial-value problem has some desirable properties: questions relating to the extent to which solutions may be influenced through the choice of input form the basis of *linear control theory* - fundamentals of which form the focus of Chapter 3.

For a periodic function A (that is, a function A with the property that, for some $p > 0$, $A(t+p) = A(t)$ for all $t \in \mathbb{R}$), it is intuitively reasonable to surmise the existence of periodic solutions of the homogeneous differential equation $\dot{x}(t) = A(t)x(t)$: we investigate this and related issues pertaining to such periodic differential equations, within the framework of what is traditionally referred to as *Floquet theory*[1].

In this chapter, we make free use of the material presented in Appendices

[1] Gaston Floquet (1847-1920), French.

H. Logemann and E. P. Ryan, *Ordinary Differential Equations*,
Springer Undergraduate Mathematics Series,
DOI: 10.1007/978-1-4471-6398-5_2,

A.1-A.3, including generalized eigenspaces, matrix norms, the concepts of piecewise continuous and piecewise continuously differentiable functions, and the triangle inequality for integrals.

2.1 Homogeneous linear systems

Whilst we are primarily interested in linear differential equations over the real field $\mathbb{R}$, the ensuing analysis applies equally to differential equations over the complex field $\mathbb{C}$. On occasions, it will prove notationally and analytically convenient to consider the complex case. For this reason, we develop the theory in the context of a field $\mathbb{F}$ which is either $\mathbb{R}$ or $\mathbb{C}$ (precisely which of these being largely immaterial).

Let J be an interval and let $A\colon J \to \mathbb{F}^{N\times N}$ be a piecewise continuous function (see Appendix A.3) from J to the space $\mathbb{F}^{N\times N}$ of $N \times N$ matrices with entries in $\mathbb{F}$ and equipped with the norm induced by the 2-norm on $\mathbb{F}^N$:

$$\|L\| := \sup_{z\neq 0} \frac{\|Lz\|}{\|z\|}$$

(see Appendix A.2).

First, we will consider the issue of existence and uniqueness of solutions of the linear homogeneous initial-value problem

$$\dot{x}(t) = A(t)x(t), \quad x(\tau) = \xi \tag{2.1}$$

for initial data $(\tau, \xi) \in J \times \mathbb{F}^N$. Since A is not continuous, but only piecewise continuous, it would be unreasonable to expect that there exists a continuously differentiable function $x\colon J \to \mathbb{F}^N$ satisfying the initial-value problem (2.1).

Exercise 2.1

Let $N = 1$, $J = [-1, 1]$ and $\tau = 0$. Provide an example of a piecewise continuous function $A : J \to \mathbb{R}$ and $\xi \in \mathbb{R}$ with the property that there does not exist a continuously differentiable function $x : J \to \mathbb{R}$ such that $x(0) = \xi$ and $\dot{x}(t) = A(t)x(t)$ for all $t \in [-1, 1]$

By a *solution* of (2.1) we mean a continuous function $x\colon J_x \to \mathbb{F}^N$ satisfying

$$x(t) = \xi + \int_\tau^t A(\sigma)x(\sigma)\,\mathrm{d}\sigma \quad \forall\, t \in J_x,$$

where $J_x \subset J$ is an interval such that $\tau \in J_x$. Note that, by Theorems A.30 and A.31 (generalized fundamental theorems of calculus), $x\colon J_x \to \mathbb{F}^N$ is a solution of (2.1) if, and only if, x is piecewise continuously differentiable (Appendix A.3), with $x(\tau) = \xi$ and

$$\dot{x}(t) = A(t)x(t) \quad \forall\, t \in J_x \backslash E,$$

where E is the set of points in J at which A fails to be continuous. Since A is piecewise continuous, the set E is "small" in the sense, that, for all $t_1, t_2 \in J$ with $t_1 < t_2$, the intersection $E \cap [t_1, t_2]$ has at most finitely many elements. Note that not every point in E is necessarily a point of discontinuity of a solution of (2.1) (for example, if $\xi = 0$, then the zero function is a solution).

Exercise 2.2

Provide an example of discontinuous A and $\xi \neq 0$ with the property that there exists a solution $x\colon J \to \mathbb{F}^N$ of (2.1) and a point $\sigma \in E$ such that x is continuously differentiable in an open interval containing σ.

If A is continuous on J, then every solution $x : J_x \to \mathbb{F}^N$ is continuously differentiable and (2.1) is satisfied for all $t \in J_x$.

In certain contexts, the initial condition in (2.1) is not relevant, in which case we say that a continuous function $x\colon J_x \to \mathbb{F}^N$, where $J_x \subset J$ is an interval, is a solution of the differential equation $\dot{x}(t) = A(t)x(t)$ if there exists $\tau \in J_x$ such that

$$x(t) = x(\tau) + \int_\tau^t A(\sigma)x(\sigma)\,\mathrm{d}\sigma \quad \forall\, t \in J_x. \tag{2.2}$$

Note that, by Theorems A.30 and A.31, $x\colon J_x \to \mathbb{F}^N$ is a solution of the differential equation in this sense if, and only if, x is piecewise continuously differentiable and the differential equation $\dot{x}(t) = A(t)x(t)$ is satisfied for every $t \in J_x$ which is not a point of discontinuity of A. The next exercise asserts that, if (2.2) holds for some $\tau \in J_x$, then (2.2) holds for all $\tau \in J_x$.

Exercise 2.3

Let $x\colon J_x \to \mathbb{F}^N$ be a solution of the differential equation $\dot{x}(t) = A(t)x(t)$. Show that

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} A(\sigma)x(\sigma)\,\mathrm{d}\sigma \quad \forall\, t_1, t_2 \in J_x.$$

Our goals are to show that, for each $(\tau, \xi) \in J \times \mathbb{F}^N$, (2.1) admits precisely one solution defined on J and to characterize that solution explicitly in terms of A, τ and ξ. In particular, we will establish the existence of a map $\Phi\colon J \times J \to \mathbb{F}^{N\times N}$ – referred to as the *transition matrix function* – such that $J \to \mathbb{F}^N$, $t \mapsto \Phi(t,\tau)\xi$ is the unique solution on J of (2.1).

2.1.1 Transition matrix function

To make progress, a number of preliminary technicalities are required.

Lemma 2.1

Define the sequence (M_n) of continuous matrix-valued functions $M_n\colon J \times J \to \mathbb{F}^{N\times N}$ by the recursion:

$$M_1(t,s) := I, \quad M_{n+1}(t,s) := I + \int_s^t A(\sigma)M_n(\sigma,s)\,\mathrm{d}\sigma \;\; \forall (t,s) \in J\times J,\ \forall\, n \in \mathbb{N}.$$

For each closed and bounded interval $[a,b] \subset J$, the sequence (M_n) is uniformly convergent on $[a,b] \times [a,b]$.

Proof

First note that

$$M_{n+1}(t,s) - M_n(t,s) = \int_s^t A(\sigma_1)\int_s^{\sigma_1} A(\sigma_2)\cdots\int_s^{\sigma_{n-1}} A(\sigma_n)\,\mathrm{d}\sigma_n\cdots\mathrm{d}\sigma_2\,\mathrm{d}\sigma_1$$
$$\forall\,(t,s) \in J\times J,\ \forall\, n\in\mathbb{N} \quad (2.3)$$

and

$$\int_s^t\int_s^{\sigma_1}\cdots\int_s^{\sigma_{n-1}}\mathrm{d}\sigma_n\cdots\mathrm{d}\sigma_2\mathrm{d}\sigma_1 = \frac{(t-s)^n}{n!} \quad \forall\,(t,s)\in J\times J,\ \forall\, n\in\mathbb{N}, \quad (2.4)$$

as can be easily verified (see Exercise 2.4). Let $a, b \in J$, with $a < b$, be arbitrary and write $X := [a,b]\times[a,b]$. Since A is piecewise continuous, there exists $K > 0$ such that

$$\|A(t)\| \le K \quad \forall\, t \in [a,b],$$

which, in conjunction with (2.3), (2.4) and the triangle inequality for integrals (see Proposition A.28), yields

$$\begin{aligned}\|M_{n+1}(t,s) - M_n(t,s)\| &\le K^n\left|\int_s^t\int_s^{\sigma_1}\cdots\int_s^{\sigma_{n-1}}\mathrm{d}\sigma_n\cdots\mathrm{d}\sigma_2\mathrm{d}\sigma_1\right| \\ &= \frac{K^n|t-s|^n}{n!} \le \frac{K^n(b-a)^n}{n!} \quad \forall\,(t,s)\in X,\ \forall\, n\in\mathbb{N}.\end{aligned}$$

Define the real sequence (m_n) by

$$m_1 := 1, \quad m_{n+1} := \frac{K^n(b-a)^n}{n!} \quad \forall\, n\in\mathbb{N},$$

and note that the series $\sum_{n=1}^{\infty} m_n$ is convergent, with limit $\exp(K(b-a))$. Let (f_n) be the sequence of functions $f_n \in C(X, \mathbb{F}^{N\times N})$ given by

$$\begin{aligned} f_1(t,s) &:= M_1(t,s) = I, \quad \forall\,(t,s)\in X \\ f_{n+1}(t,s) &:= M_{n+1}(t,s) - M_n(t,s) \ \ \forall\,(t,s)\in X, \quad \forall\, n\in\mathbb{N}. \end{aligned}$$

Then,

$$\|f_n(t,s)\| \le m_n \ \ \forall\,(t,s)\in X,\ \forall\, n\in\mathbb{N}.$$

By the Weierstrass[2] criterion (Corollary A.23), the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent. Equivalently, the sequence (S_n) of its partial sums $S_n := \sum_{k=1}^{n} f_k$ is uniformly convergent on X. Noting that $S_n(t,s) = M_n(t,s)$ for all $(t,s)\in X$, we may conclude that the sequence $\big(M_n\big)$ is uniformly convergent on $X = [a,b]\times[a,b]$. □

Exercise 2.4

Prove that (2.3) and (2.4) hold.

In view of Lemma 2.1 and since $[a,b]\subset J$ is arbitrary, we may define a function $\Phi\colon J\times J \to \mathbb{F}^{N\times N}$ by setting

$$\Phi(t,s) := \lim_{n\to\infty} M_n(t,s) \quad \forall (t,s)\in J\times J. \tag{2.5}$$

Since each M_n is continuous and, by Lemma 2.1, the sequence (M_n) converges uniformly on $X = [a,b]\times[a,b]$ for all $a,b\in J$ with $a<b$, it follows that Φ is continuous (see Proposition A.22). Moreover, for $n\ge 2$,

$$M_n(t,s) = M_1(t,s) + \sum_{k=1}^{n-1}\big(M_{k+1}(t,s) - M_k(t,s)\big) \ \ \forall\,(t,s)\in J\times J,$$

and thus, invoking (2.3), we have

$$\begin{aligned} \Phi(t,s) = I &+ \int_s^t A(\sigma_1)\mathrm{d}\sigma_1 + \int_s^t A(\sigma_1)\int_s^{\sigma_1} A(\sigma_2)\mathrm{d}\sigma_2\,\mathrm{d}\sigma_1 \\ &+ \int_s^t A(\sigma_1)\int_s^{\sigma_1} A(\sigma_2)\int_s^{\sigma_2} A(\sigma_3)\mathrm{d}\sigma_3\,\mathrm{d}\sigma_2\,\mathrm{d}\sigma_1 + \cdots \\ &\forall\,(t,s)\in J\times J. \end{aligned} \tag{2.6}$$

Note that $\Phi(t,t) = I$ for all $t\in J$. The function Φ is referred to as the *transition matrix function*; A is said to be its *generator* or, alternatively, we say that *Φ is generated by A*. The series representation of Φ given in (2.6) is the *Peano-Baker*[3] *series*. It converges to Φ uniformly on $[a,b]\times[a,b]$ for every interval $[a,b]\subset J$.

[2] Karl Theodor Wilhelm Weierstrass (1815-1897), German.
[3] Giuseppe Peano (1858-1932), Italian; Henry Frederick Baker (1866-1956), British.

Example 2.2

Let $\mathbb{F} = \mathbb{R}$ and let $A\colon \mathbb{R} \to \mathbb{R}^{2\times 2}$ be given by $A(t) = \begin{pmatrix} 0 & 2t \\ 0 & 0 \end{pmatrix}$.

Noting that $A(t)A(s) = 0$ for all $t, s \in \mathbb{R}$, we see that the Peano-Baker series terminates after two terms to give

$$\Phi(t,s) = I + \int_s^t A(\sigma)\,\mathrm{d}\sigma = \begin{pmatrix} 1 & t^2 - s^2 \\ 0 & 1 \end{pmatrix} \quad \forall\,(t,s) \in \mathbb{R}\times\mathbb{R}.$$

△

If $J = \mathbb{R}$ and A is constant, then the Peano-Baker series gives

$$\begin{aligned} \Phi(t,\tau) &= I + (t-\tau)A + \frac{(t-\tau)^2 A^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!} A^k \\ &= \exp(A(t-\tau)) \quad \forall\, t,\tau \in \mathbb{R} \end{aligned} \tag{2.7}$$

and so we identify Φ with the matrix exponential function: in particular,

$$\begin{aligned} \Phi(t,0) &= \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \exp(At) \quad \forall\, t \in \mathbb{R}, \\ \Phi(t,\tau) &= \Phi(t-\tau, 0) \quad \forall\, t,\tau \in \mathbb{R}. \end{aligned}$$

For further details on the matrix exponential, see Proposition A.27.

Exercise 2.5

Assume that $A\colon \mathbb{R} \to \mathbb{F}^{N\times N}$ is such that, for all $t, s \in \mathbb{R}$, the matrices $A(t)$ and $A(s)$ commute. Show that the transition matrix function Φ is given by

$$\Phi(t,\tau) = \exp\left(\int_\tau^t A(\sigma)\,\mathrm{d}\sigma\right).$$

We proceed to establish basic properties of the transition matrix function.

Corollary 2.3

The transition matrix function Φ satisfies

$$\Phi(t,s) = I + \int_s^t A(\sigma)\Phi(\sigma,s)\,\mathrm{d}\sigma \quad \forall\,(t,s) \in J \times J. \tag{2.8}$$

Moreover, for each $s \in J$, the function $t \mapsto \Phi(t,s)$ is piecewise continuously differentiable with derivative

$$\partial_1 \Phi(t,s) = A(t)\Phi(t,s) \quad \forall\, t \in J\backslash E$$

where $E \subset J$ is the set of points at which A fails to be continuous.

Proof

The identity (2.8) follows from (2.5), the defining equation for Φ, in conjunction with Lemma 2.1 and Theorem A.32. The remaining claims are an immediate consequence of (2.8) and Theorem A.30. $\square$

The next result (the so-called Gronwall[4] lemma) is a basic tool in differential and integral equations. It will not only be used in this chapter, but it will also be invoked, in Chapter 4, in the context of nonlinear differential equations.

Lemma 2.4 (Gronwall's lemma)

Let $I \subset \mathbb{R}$ be an interval, let $\tau \in I$, and let $g, h : I \to [0, \infty)$ be continuous. If, for some constant $c \geq 0$,

$$g(t) \leq c + \left| \int_\tau^t h(\sigma) g(\sigma) \, \mathrm{d}\sigma \right| \quad \forall\, t \in I\,, \tag{2.9}$$

then

$$g(t) \leq c \exp\left(\left| \int_\tau^t h(\sigma) \, \mathrm{d}\sigma \right| \right) \quad \forall\, t \in I\,. \tag{2.10}$$

Note that whilst (2.9) (the hypothesis in Lemma 2.4) is an inequality in g (involving c and h), the inequality (2.10) (the conclusion in Lemma 2.4) provides a bound for g in terms of c and h.

Proof of Lemma 2.4

Define $G, H \colon I \to [0, \infty)$ by setting

$$G(t) := c + \left| \int_\tau^t h(\sigma) g(\sigma) \, \mathrm{d}\sigma \right| \quad \text{and} \quad H(t) = \left| \int_\tau^t h(\sigma) \, \mathrm{d}\sigma \right| \quad \forall\, t \in I\,.$$

By hypothesis, $0 \leq g(t) \leq G(t)$ for all $t \in I$. Let $t \in I$ be arbitrary. We consider two cases: $t \geq \tau$ and $t < \tau$.

Case 1. Assume that $t \geq \tau$. The inequality in (2.10) evidently holds for $t = \tau$. Hence, without loss of generality we may assume that $t > \tau$. Then

$$G(s) = c + \int_\tau^s h(\sigma) g(\sigma) \, \mathrm{d}\sigma \quad \text{and} \quad H(s) = \int_\tau^s h(\sigma) \, \mathrm{d}\sigma \quad \forall\, s \in [\tau, t]\,.$$

Differentiation yields

$$G'(s) = h(s) g(s) \leq h(s) G(s) = H'(s) G(s) \quad \forall\, s \in [\tau, t]\,.$$

[4] Thomas Hakon Grönwall (1877-1932), Swedish.

Therefore,

$$\big(G(s)\exp\big(-H(s)\big)\big)' = (G'(s) - H'(s)G(s))\exp\big(-H(s)\big) \leq 0 \quad \forall\, s \in [\tau, t]$$

which, on integration, gives

$$G(t)\exp\big(-H(t)\big) \leq G(\tau) = c.$$

Hence, we arrive at the requisite inequality

$$g(t) \leq G(t) \leq c\exp\big(H(t)\big) = c\exp\left(\left|\int_\tau^t h(s)\mathrm{d}s\right|\right).$$

Case 2. Assume that $t < \tau$. In this case,

$$G(s) = c + \int_s^\tau h(\sigma)g(\sigma)\,\mathrm{d}\sigma \quad \text{and} \quad H(s) = \int_s^\tau h(\sigma)\,\mathrm{d}\sigma \quad \forall\, s \in [t, \tau]\,,$$

and differentiation yields

$$G'(s) = -h(s)g(s) \geq -h(s)G(s) = H'(s)G(s) \quad \forall\, \sigma \in [t, \tau]\,.$$

An argument analogous to that used in Case 1 gives the desired inequality. □

Exercise 2.6

In the above proof, complete Case 2 by providing an argument similar to that of Case 1.

We are now in a position to state and prove the existence and uniqueness result which asserts that the initial-value problem (2.1) has precisely one solution defined on J.

Theorem 2.5

Let $(\tau, \xi) \in J \times \mathbb{F}^N$. The function

$$x\colon J \to \mathbb{F}^N, \quad t \mapsto x(t) := \varPhi(t, \tau)\xi. \tag{2.11}$$

is a solution of the initial-value problem (2.1). Moreover, if $y\colon J_y \to \mathbb{F}^N$ is also a solution of (2.1), then $y(t) = x(t)$ for all $t \in J_y$.

Proof

Let $(\tau, \xi) \in J \times \mathbb{F}^N$ be arbitrary. It is immediate that the function x given by (2.11) is a solution of (2.1), since, by Corollary 2.3,

$$x(t) = \varPhi(t, \tau)\xi = \xi + \int_\tau^t A(\sigma)\varPhi(\sigma, \tau)\xi\,\mathrm{d}\sigma = \xi + \int_\tau^t A(\sigma)x(\sigma)\,\mathrm{d}\sigma \quad \forall\, t \in J.$$

Let $y\colon J_y \to \mathbb{F}^N$ be another solution of (2.1). Then

$$e(t) := x(t) - y(t) = \int_\tau^t A(\sigma)\big(x(\sigma) - y(\sigma)\big)\mathrm{d}\sigma = \int_\tau^t A(\sigma)e(\sigma)\,\mathrm{d}\sigma \quad \forall\, t \in J_y.$$

Invoking the triangle inequality for integrals (Proposition A.28), we conclude

$$\|e(t)\| \le \left| \int_\tau^t \|A(\sigma)\|\|e(\sigma)\|\,\mathrm{d}\sigma \right| \quad \forall\, t \in J_y.$$

By Gronwall's lemma (Lemma 2.4), it follows that $e(t) = 0$ for all $t \in J_y$, showing that $y(t) = x(t)$ for all $t \in J_y$. □

Further properties of the transition matrix function readily follow.

Corollary 2.6

For all $t, \sigma, \tau \in J$,

$$\Phi(\tau,\tau) = I, \quad \Phi(t,\tau) = \Phi(t,\sigma)\Phi(\sigma,\tau) \;\text{ and }\; \Phi^{-1}(t,\tau) = \Phi(\tau,t).$$

Proof

Let $\sigma, \tau \in J$ and $\xi \in \mathbb{F}^N$ be arbitrary. The first identity follows immediately from (2.5), the defining equation for Φ, and the definition of M_n (see Lemma 2.1). To prove the second identity, set $\zeta := \Phi(\sigma,\tau)\xi$ and define the functions $y, z : J \to \mathbb{F}^N$ by $y(t) := \Phi(t,\tau)\xi$ and $z(t) = \Phi(t,\sigma)\zeta$. By Theorem 2.5, y is the unique solution of the initial-value problem $\dot{x}(t) = A(t)x(t)$, $x(\tau) = \xi$, and z is the unique solution of the initial-value problem

$$\dot{x}(t) = A(t)x(t), \quad x(\sigma) = \zeta. \tag{2.12}$$

Noting that $y(\sigma) = \Phi(\sigma,\tau)\xi = \zeta$, we see that y also solves the initial-value problem (2.12). Hence, by Theorem 2.5, $y(t) = z(t)$ for all $t \in J$, and thus, in particular,

$$\Phi(t,\sigma)\Phi(\sigma,\tau)\xi = \Phi(t,\sigma)\zeta = z(t) = y(t) = \Phi(t,\tau)\xi$$

Since $\xi \in \mathbb{F}^N$ is arbitrary, we have $\Phi(t,\sigma)\Phi(\sigma,\tau) = \Phi(t,\tau)$. Finally, as an immediate consequence of this identity, we have

$$\Phi(\tau,t)\Phi(t,\tau) = \Phi(\tau,\tau) = I,$$

and so $\Phi(t,\tau)$ is invertible with inverse $\Phi^{-1}(t,\tau) = \Phi(\tau,t)$. □

Exercise 2.7

Let $\varPhi$ be the transition matrix function generated by $A\colon J \to \mathbb{F}^{N\times N}$. Define $\tilde{A}$ by $\tilde{A}(t) = -A^*(t)$ for all $t \in J$. Prove that the transition matrix function $\tilde{\varPhi}$ generated by $\tilde{A}$ is given by

$$\tilde{\varPhi}(t,s) = \varPhi^*(s,t) \quad \forall\, (t,s) \in J \times J.$$

Here M^* denotes the Hermitian transposition of a matrix M (see also Appendix A.1). (*Hint.* Prove that, if $x\colon J \to \mathbb{F}^N$ is a solution of $\dot{x}(t) = A(t)x(t)$ and $y\colon J \to \mathbb{F}^N$ is a solution of $\dot{y}(t) = -A^*(t)y(t)$, then, for some scalar c, we have $\langle x(t), y(t)\rangle = c$ for all $t \in J$.)

2.1.2 Solution space

Let $\mathcal{S}_{\mathrm{hom}}$ denote the set of all solutions $x\colon J \to \mathbb{F}^N$ of the homogeneous differential equation $\dot{x}(t) = A(t)x(t)$, that is, the set of functions $x\colon J \to \mathbb{F}^N$ that solve the initial-value problem (2.1) for some $(\tau, \xi) \in J \times \mathbb{F}^N$. It is easy to show that the set $\mathcal{S}_{\mathrm{hom}}$ forms a vector space, a subspace of $C(J, \mathbb{F}^N)$, the so-called *solution space* of the homogeneous differential equation. If $y_1, \ldots, y_N \in \mathcal{S}_{\mathrm{hom}}$, then $w(t) := \det(y_1(t), \ldots, y_N(t))$ is called the *Wronskian*[5] associated with the solutions $y_1, \ldots, y_N$. Next, we establish some some properties of the solution space and the Wronskian. Recall that the *trace* of a square matrix $M = (m_{ij}) \in \mathbb{F}^{N\times N}$ is defined by $\operatorname{tr} M := \sum_{j=1}^N m_{jj}$, the sum if its diagonal elements.

Proposition 2.7

(1) Let $b_1, \ldots, b_N$ be a basis of $\mathbb{F}^N$ and let $\tau \in J$. Then the functions $y_j\colon J \to \mathbb{F}^N$ defined by $y_j(t) := \varPhi(t,\tau)b_j$, $j = 1, 2, \ldots, N$, form a basis of the solution space $\mathcal{S}_{\mathrm{hom}}$. In particular, $\mathcal{S}_{\mathrm{hom}}$ is N-dimensional and, for every solution $x\colon J \to \mathbb{F}^N$, there exist scalars $\gamma_1, \ldots, \gamma_N$ such that $x(t) = \sum_{j=1}^N \gamma_j y_j(t)$ for all $t \in J$.

(2) Let $y_1, \ldots, y_N$ be in $\mathcal{S}_{\mathrm{hom}}$ and let w be the associated Wronskian. Then

$$w(t) = w(\tau) \det \varPhi(t,\tau) \quad \forall\, (t,\tau) \in J \times J, \tag{2.13}$$

and moreover, $\dot{w}(t) = (\operatorname{tr} A(t))w(t)$ for all $t \in J$ which are not points of discontinuity of A, and so

$$w(t) = w(\tau) \exp\left(\int_\tau^t \operatorname{tr} A(s)\, \mathrm{d}s\right) \quad \forall\, t \in J\,. \tag{2.14}$$

[5] Josef-Maria Hoëné de Wronski (1778-1853), Polish.

In particular, if $w(\tau) = 0$ for some $\tau \in J$, then $w(t) = 0$ for all $t \in J$, or, equivalently, if $w(\tau) \neq 0$ for some $\tau \in J$, then $w(t) \neq 0$ for all $t \in J$.

(3) Elements $y_1, \ldots, y_n$ of $\mathcal{S}_{\rm hom}$, where $n \leq N$, are linearly independent (as elements in the vector space $C(J, \mathbb{F}^N)$) if, and only if, for every $t \in J$, the vectors $y_1(t), \ldots, y_n(t)$ are linearly independent (as elements of $\mathbb{F}^N$).

Proof

(1) Theorem 2.5 ensures that $y_1, \ldots, y_N$ are solutions and so are in $\mathcal{S}_{\rm hom}$. Moreover, these solutions are linearly independent (in the vector space $C(J, \mathbb{F}^N)$). Indeed, if, for $\alpha_1, \ldots, \alpha_N \in \mathbb{F}$, we have $\sum_{j=1}^N \alpha_j y_j(t) = 0$ for all $t \in J$, then $\sum_{j=1}^N \alpha_j y_j(\tau) = \sum_{j=1}^N \alpha_j b_j = 0$, and so, by linear independence of $b_1, \ldots, b_N$ (in $\mathbb{F}^N$), it follows that $\alpha_1 = \ldots = \alpha_N = 0$. Next, we show that $y_1, \ldots, y_N$ form a basis of $\mathcal{S}_{\rm hom}$. Let x be an arbitrary element of $\mathcal{S}_{\rm hom}$. Then, by Theorem 2.5, $x(t) = \Phi(t, \tau) x(\tau)$ for all $t \in J$. Since $b_1, \ldots, b_N$ form a basis of $\mathbb{F}^N$, there exist scalars $\gamma_1, \ldots, \gamma_N$ such that $x(\tau) = \sum_{j=1}^N \gamma_j b_j$. Consequently, $x(t) = \sum_{j=1}^N \gamma_j \Phi(t, \tau) b_j = \sum_{j=1}^N \gamma_j y_j(t)$ for all $t \in J$, showing that $y_1, \ldots, y_N$ span $\mathcal{S}_{\rm hom}$.

(2) Let $\tau \in J$ be fixed, but arbitrary. Since $y_j \in \mathcal{S}_{\rm hom}$ for $j = 1, \ldots, N$, it follows from Theorem 2.5 that $y_j(t) = \Phi(t, \tau) y_j(\tau)$ for all $t \in J$ and all $j = 1, \ldots, N$. Hence, for all $t \in J$,

$$w(t) = \det \Phi(t, \tau) \det(y_1(\tau), \ldots, y_N(\tau)) = w(\tau) \det \Phi(t, \tau),$$

establishing (2.13). Moreover, writing $\Phi(t, \tau) = (\varphi_1(t, \tau), \ldots, \varphi_N(t, \tau))$, where $\varphi_j(t, \tau)$ denotes the j-th column of $\Phi(t, \tau)$, it follows, from the definition of the determinant (see (A.8) in Appendix A.1) and the product rule for differentiation, that, for all $t \in J$ which are not points of discontinuity of A,

$$\big(\partial_1 \det \Phi\big)(t, \tau) = \sum_{j=1}^N \det \big(\varphi_1(t, \tau), \ldots, \varphi_{j-1}(t, \tau), \partial_1 \varphi_j(t, \tau), \ldots, \varphi_N(t, \tau)\big),$$

where ∂_1 denotes the derivative with respect to the first argument. In the following we assume that τ is not a point of discontinuity of A. Then, since $\Phi(\tau, \tau) = I$, the above identity yields for $t = \tau$,

$$\big(\partial_1 \det \Phi\big)(\tau, \tau) = \sum_{j=1}^N \det \big(e_1, \ldots, e_{j-1}, A(\tau) e_j, e_{j+1}, \ldots, e_N\big),$$

where $e_1, \ldots, e_N$ denotes the canonical basis of $\mathbb{F}^N$. Denoting the entries of

$A(t)$ by $a_{ij}(t)$ it follows that

$$\big(\partial_1 \det \Phi\big)(\tau,\tau) = \sum_{j=1}^{N} a_{jj}(\tau) = \operatorname{tr} A(\tau).$$

Therefore, differentiation of (2.13) with respect to t at $t = \tau$ yields

$$\dot{w}(\tau) = w(\tau)\operatorname{tr} A(\tau). \tag{2.15}$$

The argument leading to (2.15) applies to any $\tau \in J$ which is not a point of discontinuity of A and therefore $\dot{w}(t) = (\operatorname{tr} A(t))w(t)$ for every $t \in J$ which is not a point of discontinuity of A. Furthermore, (2.14) now follows from Exercise 2.5 and Theorem 2.5.

(3) Let $y_1, \ldots, y_n$ be in $\mathcal{S}_{\mathrm{hom}}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$.
Sufficiency. Assume that $y_1(t), \ldots, y_n(t)$ are linearly independent vectors in $\mathbb{F}^N$ for all $t \in J$. It immediately follows that

$$\alpha_1 y_1 + \cdots + \alpha_n y_n = 0 \implies \alpha_k = 0,\ k = 1, \ldots, n$$

and so $y_1, \ldots, y_n$ are linearly independent in $\mathcal{S}_{\mathrm{hom}}$.
Necessity. Let $y_1, \ldots, y_n$ be linearly independent in $\mathcal{S}_{\mathrm{hom}}$. Let $\tau \in J$ be arbitrary. Assume that $\alpha_1 y_1(\tau) + \cdots + \alpha_n y_n(\tau) = 0$. Then, $y := \alpha_1 y_1 + \cdots + \alpha_n y_n$ solves the initial-value problem $\dot{x}(t) = A(t)x(t)$, $x(\tau) = 0$, which we know has unique solution 0. Therefore, $y = 0$ and so, by linear independence of the functions $y_1, \ldots, y_n$, we have $\alpha_k = 0$, $k = 1, \ldots, n$. This establishes linear independence of $y_1(\tau), \ldots, y_n(\tau)$ and, as $\tau \in J$ is arbitrary, the result follows. □

Statement (3) of Proposition (2.7) says that linear independence of $y_1, \ldots, y_n \in \mathcal{S}_{\mathrm{hom}}$ as functions is equivalent to linear independence of $y_1(t), \ldots, y_n(t)$ (as vectors in $\mathbb{F}^N$) for every $t \in J$. The following exercise shows that if $y_1, \ldots, y_n \in C(J, \mathbb{F}^N)$ are not required to be solutions of $\dot{x}(t) = A(t)x(t)$, then this equivalence does not hold.

Exercise 2.8

Show, by counterexample, that linear independence of $y_1, \ldots, y_n \in C(J, \mathbb{F}^N)$ does not imply linear independence of $y_1(t), \ldots, y_n(t) \in \mathbb{F}^N$ for all $t \in J$.

A *fundamental system* for the homogeneous differential equation $\dot{x}(t) = A(t)x(t)$ is a set of N linearly independent solutions, or, equivalently, a basis of $\mathcal{S}_{\mathrm{hom}}$. If $\{\psi_1, \ldots, \psi_N\}$ is a fundamental system, then the matrix-valued function $\Psi \colon J \to \mathbb{F}^{N\times N}$ defined by

$$\Psi(t) := \big(\psi_1(t), \ldots, \psi_n(t)\big) \quad \forall\, t \in J$$

is said to be a *fundamental matrix* for the differential equation $\dot{x}(t) = A(t)x(t)$.

Proposition 2.8

Let Ψ be a fundamental matrix. Then $\Psi(t)$ is invertible for every $t \in J$ and

$$\Phi(t,\tau) = \Psi(t)\Psi^{-1}(\tau) \quad \forall\, t,\tau \in J.$$

Proof

By part (3) of Proposition 2.7 we may infer that $\Psi(t)$ is invertible for all $t \in J$. Let $\xi \in \mathbb{F}^N$ be arbitrary and define $x\colon J \to \mathbb{F}^N$ by setting $x(t) := \Psi(t)\Psi^{-1}(\tau)\xi$. Obviously, x is a linear combination of the columns of Ψ and consequently, x is a solution. Moreover, $x(\tau) = \xi$, so that x solves the initial-value problem (2.1). By Theorem 2.5, the function $t \mapsto \Phi(t,\tau)\xi$ is the unique solution of (2.1). Hence, $x(t) = \Phi(t,\tau)\xi$ for all $t \in J$, showing that

$$\Phi(t,\tau)\xi = \Psi(t)\Psi^{-1}(\tau)\xi \quad \forall\, t \in J.$$

Since ξ was arbitrary, we obtain that $\Phi(t,\tau) = \Psi(t)\Psi^{-1}(\tau)$ for all $t \in J$. $\square$

Exercise 2.9

Let $\mathbb{F} = \mathbb{R}$ and let $A\colon \mathbb{R} \to \mathbb{R}^{2\times 2}$ be given by

$$A(t) := \begin{pmatrix} 0 & 1 \\ 0 & 2t \end{pmatrix}.$$

Find two linearly independent solutions of $\dot{x}(t) = A(t)x(t)$ and hence determine the transition matrix function Φ.

2.1.3 Autonomous systems

Let us now turn attention to the case of constant A, that is, we consider the *autonomous* homogeneous initial-value problem, with $J = \mathbb{R}$,

$$\dot{x}(t) = Ax(t), \quad x(\tau) = \xi \in \mathbb{F}^N, \quad \text{where } A \in \mathbb{F}^{N\times N}. \tag{2.16}$$

Recall that, for $M \in \mathbb{F}^{N\times N}$, $\exp(M) := \sum_{k=0}^{\infty}(1/k!)M^k$ and that, by (2.7),

$$\Phi(t,\tau) = \exp(A(t-\tau)) \;\; \forall\, t,\tau \in \mathbb{R}.$$

In particular,

$$\Phi(t,0) = \exp(At) \;\; \forall\, t \in \mathbb{R} \;\text{ and }\; \Phi(t,\tau) = \Phi(t-\tau,0) \;\; \forall\, t,\tau \in \mathbb{R}. \tag{2.17}$$

The following is an immediate corollary of Theorem 2.5.

Corollary 2.9

The function $\mathbb{R} \to \mathbb{F}^N$, $t \mapsto \exp(A(t-\tau))\xi$ is the unique solution of the autonomous homogeneous initial-value problem (2.16).

In view of (2.17), we see that the unique solution (on $\mathbb{R}$) $t \mapsto \Phi(t,\tau)\xi = \Phi(t-\tau,0)\xi$ of the initial-value problem (2.16) is simply a translation of the solution $t \mapsto \Phi(t,0)\xi$ (on $\mathbb{R}$) of the initial-value problem $\dot{x}(t) = Ax(t)$, $x(0) = \xi$. Consequently, we may assume without loss of generality that $\tau = 0$ in (2.16).

We briefly digress to record following some important properties of the matrix exponential function.

Lemma 2.10

Let $P, Q \in \mathbb{F}^{N\times N}$.

(1) If P is diagonal, that is, $P = \mathrm{diag}(p_1, \ldots, p_n)$, then

$$\exp(P) = \mathrm{diag}\big(\exp(p_1), \ldots, \exp(p_N)\big).$$

(2) $\exp(P^*) = (\exp(P))^*$.

(3) For all $t \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(Pt) = P\exp(Pt) = \exp(Pt)P.$$

(4) If $PQ = QP$, then $\exp(P)Q = Q\exp(P)$ and

$$\exp(P+Q) = \exp(P)\exp(Q). \tag{2.18}$$

(5) $\exp(-P)\exp(P) = \exp(P)\exp(-P) = I$, that is, $\exp(P)$ is invertible with inverse $\exp(-P)$.

Proof

The proofs of parts (1)-(3) are straightforward (see Exercise 2.10). To prove part (4), assume that P and Q commute, that is, $PQ = QP$. Then,

$$\exp(P)Q = \sum_{k=0}^{\infty}\frac{1}{k!}P^kQ = \sum_{k=0}^{\infty}\frac{1}{k!}QP^k = Q\sum_{k=0}^{\infty}\frac{1}{k!}P^k = Q\exp(P).$$

Let $z \in \mathbb{F}^N$ and define $y\colon \mathbb{R} \to \mathbb{C}^N$ by setting $y(t) = \exp(Pt)\exp(Qt)z$. Using part (3) and the product rule, differentiation of y leads to

$$\dot{y}(t) = P\exp(Pt)\exp(Qt)z + \exp(Pt)Q\exp(Qt)z = (P+Q)y(t) \quad \forall\, t \in \mathbb{R},$$

where we have used the fact that $\exp(Pt)Q = Q\exp(Pt)$. Moreover, $y(0) = z$. The unique solution of the initial-value problem $\dot{x} = (P+Q)x$, $x(0) = z$, is the function $t \mapsto \exp((P+Q)t)z$, and so

$$\exp((P+Q)t)z = y(t) = \exp(Pt)\exp(Qt)z \quad \forall\, t \in \mathbb{R}.$$

As $z \in \mathbb{F}^N$ is arbitrary, we have $\exp((P+Q)t) = \exp(Pt)\exp(Qt)$ for all $t \in \mathbb{R}$.

Finally, statement (5) is an immediate consequence of statement (4) (on setting $Q = -P$). This completes the proof. □

Exercise 2.10

Prove assertions (1)-(3) of Lemma 2.10.

Exercise 2.11

Let $P, Q \in \mathbb{F}^{N\times N}$. Show that, if P and Q do not commute, then (2.18) does not hold in general.

In the following, we will show how, in principle, N linearly independent solutions (or, equivalently, a fundamental matrix) of the autonomous differential equation (2.16), over the complex field $\mathbb{F} = \mathbb{C}$, can be computed. In this context, a pivotal role is played by the concepts of generalized eigenspaces and algebraic/geometric multiplicities of eigenvalues, the definitions of which (together with key results) can be found in Appendix A.1. For $A \in \mathbb{C}^{N\times N}$ it is convenient to define

$$\sigma(A) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } A\}.$$

The set $\sigma(A)$ (the set of all eigenvalues of A) is called the *spectrum* of $\sigma(A)$.

Theorem 2.11

Let $A \in \mathbb{C}^{N\times N}$. For $\lambda \in \sigma(A)$, let $m(\lambda)$ denote the algebraic multiplicity of λ, denote its associated generalized eigenspace by $E(\lambda) := \ker(A - \lambda I)^{m(\lambda)}$, and, for $z \in \mathbb{C}^N$, define $x_z \colon \mathbb{R} \to \mathbb{C}^N$ by $x_z(t) := \exp(At)z$.

(1) For $\lambda \in \sigma(A)$ and $z \in E(\lambda)$,

$$x_z(t) = e^{\lambda t} \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!}(A - \lambda I)^k z \quad \forall\, t \in \mathbb{R}. \tag{2.19}$$

(2) Let $B(\lambda)$ be a basis of $E(\lambda)$ and write

$$\mathcal{B} := \cup_{\lambda\in\sigma(A)} B(\lambda),$$

The set of functions $\{x_z \colon z \in \mathcal{B}\}$ is a basis of the solution space of (2.16) (with $\mathbb{F} = \mathbb{C}$).

Proof

Let $\lambda \in \sigma(A)$ and $z \in E(\lambda)$. Then, $(A - \lambda I)^k z = 0$ for all $k \geq m(\lambda)$ and so

$$x_z(t) = \exp(At)z = e^{\lambda t}\big(\exp(A - \lambda I)t\big)z = e^{\lambda t}\sum_{k=0}^{m(\lambda)-1}\frac{t^k}{k!}(A - \lambda I)^k z \quad \forall\, t \in \mathbb{R},$$

establishing statement (1).

By the generalized eigenspace decomposition theorem (see Theorem A.8), $\mathcal{B}$ is a basis of $\mathbb{C}^N$ and so, by Proposition 2.7, $\{x_z \colon z \in \mathcal{B}\}$ is a basis of the solution space of (2.16) (with $\mathbb{F} = \mathbb{C}$), proving statement (2). □

Theorem 2.11 shows that, by computing the eigenvalues of A and computing $m(\lambda)$ linearly independent generalized eigenvectors associated with λ for each $\lambda \in \sigma(A)$, N linearly independent solutions of (2.16) (with $\mathbb{F} = \mathbb{C}$) can be obtained by using formula (2.19).

The next result, a consequence of Theorem 2.11, says, roughly speaking, that the growth of $\|\exp(At)\|$ as $t \to \infty$ is determined by the spectrum of A.

Theorem 2.12

Let $A \in \mathbb{C}^{N\times N}$, set $\mu_A := \max\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ and

$$\varGamma_A := \{\gamma \in \mathbb{R} : \text{there exists } M_\gamma \geq 1 \text{ such that } \|\exp(At)\| \leq M_\gamma e^{\gamma t} \ \ \forall\, t \geq 0\}.$$

(1) $(\mu_A, \infty) \subset \varGamma_A$ and $\inf \varGamma_A = \mu_A$.

(2) $\mu_A \in \varGamma_A$ if, and only if, every $\lambda \in \sigma(A)$ satisfying $\operatorname{Re}\lambda = \mu_A$ is semisimple.

(3) Let $\gamma \in \mathbb{R}$. If, for all $\xi \in \mathbb{C}^N$, $\lim_{t\to\infty}\exp((A - \gamma I)t)\xi = 0$, then $\mu_A < \gamma$.

Before we prove Theorem 2.12, we state an immediate corollary.

Corollary 2.13

Let $A \in \mathbb{C}^{N\times N}$ and define μ_A as in Theorem 2.12.

(1) $\mu_A < 0$ if, and only if, $\|\exp(At)\|$ decays exponentially fast as $t \to \infty$.

(2) $\mu_A < 0$ if, and only if, $\lim_{t\to\infty}\exp(At)\xi = 0$ for every $\xi \in \mathbb{C}^N$.

(3) If $\mu_A = 0$, then $\sup_{t\geq 0}\|\exp(At)\| < \infty$ if, and only if, all purely imaginary eigenvalues of A are semisimple.

Proof of Theorem 2.12

Let $\lambda \in \sigma(A)$ and let $z \in \mathbb{C}^N$ be an associated generalized eigenvector. Then, by statement (1) of Theorem 2.11,

$$\exp(At)z = e^{\lambda t} \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!}(A - \lambda I)^k z \quad \forall\, t \in \mathbb{R}, \tag{2.20}$$

where $m(\lambda)$ denotes the algebraic multiplicity of λ. Let $z_1, \ldots, z_N$ be a basis of $\mathbb{C}^N$ consisting of generalized eigenvectors of A (such a basis exists by Theorem A.8) and define the invertible matrix $Z := (z_1, \ldots, z_n) \in \mathbb{C}^{N \times N}$.

(1) Let $\gamma \in (\mu_A, \infty)$ be arbitrary. We will show that $\gamma \in \Gamma_A$ (and so $(\mu_A, \infty) \subset \Gamma_A$). Noting that $\gamma > \operatorname{Re}\lambda$ for all $\lambda \in \sigma(A)$ and invoking (2.20), we may infer the existence of $L \geq 1$ such that

$$\|\exp(At)z_i\| \leq L e^{\gamma t}\|z_i\| \quad \forall\, t \geq 0,\ i = 1, \ldots, N. \tag{2.21}$$

Let $\xi \in \mathbb{C}^N$ be arbitrary and write $\eta := Z^{-1}\xi$. Then $\xi = \sum_{i=1}^N \eta_i z_i$, where η_i, $i = 1, \ldots, N$, are the components of η and so

$$\|\exp(At)\xi\| \leq \sum_{i=1}^N |\eta_i| \|\exp(At)z_i\| \leq \|Z^{-1}\|\|\xi\| \sum_{i=1}^N \|\exp(At)z_i\|,$$

which, in conjunction with (2.21) and writing $M_\gamma := L\|Z^{-1}\| \sum_{i=1}^N \|z_i\|$, gives

$$\|\exp(At)\xi\| \leq M_\gamma e^{\gamma t}\|\xi\| \quad \forall\, \xi \in \mathbb{C}^N,\ \forall\, t \geq 0.$$

Since ξ is arbitrary, it follows that $\|\exp(At)\| \leq M_\gamma e^{\gamma t}$ for all $t \geq 0$. Therefore, $\gamma \in \Gamma_A$, showing that $(\mu_A, \infty) \subset \Gamma_A$. As an immediate consequence of the latter inclusion, we obtain $\inf \Gamma_A \leq \mu_A$. On the other hand, by (2.20), $\inf \Gamma_A \geq \mu_A$. Therefore, $\mu_A = \inf \Gamma_A$, completing the proof of statement (1).

(2) We proceed to prove statement (2). We will use the fact that an eigenvalue λ of A is semisimple if, and only if, the generalized eigenspace $E(\lambda)$ coincides with the eigenspace $\ker(A - \lambda I)$ (see Proposition A.10 in Appendix A.1). If all $\lambda \in \sigma(A)$ satisfying $\operatorname{Re}\lambda = \mu_A$ are semisimple, then, invoking (2.20), it is clear that for every generalized eigenvector z of A, there exists $L_z \geq 1$ such that $\|\exp(At)z\| \leq L_z e^{\mu_A t}\|z\|$ for all $t \geq 0$. By an argument identical to that used in the proof of the inclusion $(\mu_A, \infty) \subset \Gamma_A$, it follows that there exists $M \geq 1$ such that $\|\exp(At)\| \leq M e^{\mu_A t}$ for all $t \geq 0$, implying that $\mu_A \in \Gamma_A$. Conversely, assume that $\mu_A \in \Gamma_A$. Let $\lambda \in \sigma(A)$ be such that $\operatorname{Re}\lambda = \mu_A$ and let $z \in E(\lambda)$. Then, by (2.20), for all $t \in \mathbb{R}$,

$$\Big\| \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!}(A - \lambda I)^k z \Big\| = \|e^{-\lambda t}\exp(At)\| = e^{-\mu_A t}\|\exp(At)\|$$

By hypothesis, $\sup_{t\geq 0} e^{-\mu_A t}\|\exp(At)\| < \infty$, and hence, $(A-\lambda I)z = 0$. This holds for every $z \in E(\lambda)$ and consequently, λ is semisimple.

(3) Finally, to prove statement (3), let $\lambda \in \sigma(A)$ and let $v \in \mathbb{C}^N$ be an associated eigenvector. By hypothesis,

$$e^{(\lambda-\gamma)t}v = e^{-\gamma t}\exp(At)v = \exp((A-\gamma I)t)v \to 0 \quad \text{as } t \to \infty$$

and so $\operatorname{Re}\lambda < \gamma$. Since $\lambda \in \sigma(A)$ was arbitrary, we conclude that $\mu_A < \gamma$. $\square$

Next, we turn attention to the special case of (2.16) over the real field $\mathbb{F} = \mathbb{R}$. In particular, we consider the initial-value problem

$$\dot{x}(t) = Ax(t), \quad x(0) = \xi \in \mathbb{R}^N, \quad A \in \mathbb{R}^{N\times\mathbb{N}} \tag{2.22}$$

and will show how to compute N linearly independent *real* solutions. As a prelude, we set the following.

Exercise 2.12

Let $V \subset \mathbb{C}^N$ be a subspace that is closed under complex conjugation (that is, if $v \in V$, then $\bar{v} \in V$). Show that V has a real basis.

If A is a *real* $N \times N$ matrix and $\lambda \in \sigma(A)$ is a *real* eigenvalue of algebraic multiplicity $m(\lambda)$, then the associated generalized eigenspace $\ker(A-\lambda I)^{m(\lambda)}$ is closed under complex conjugation and so, by Exercise 2.12, has a *real basis.* This fact is used implicitly in the following theorem. Furthermore, for $z \in \mathbb{C}^N$, the real and imaginary parts of z, denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively, should be interpreted in the natural componentwise manner.

Theorem 2.14

Let $A \in \mathbb{R}^{N\times N}$. For $\lambda \in \sigma(A)$, let $m(\lambda)$ denote the algebraic multiplicity of λ, denote its associated generalized eigenspace by $E(\lambda) := \ker(A-\lambda I)^{m(\lambda)}$, and let $B(\lambda)$ be a basis thereof, chosen to be a real basis whenever λ is real. For all $z \in \mathbb{C}^N$, define real solutions $x_z, y_z : \mathbb{R} \to \mathbb{R}^N$ of (2.22) by $x_z(t) := \exp(At)\operatorname{Re} z$ and $y_z(t) := \exp(At)\operatorname{Im} z$.

(1) Let B_0 (respectively, B_+) denote the union of all $B(\lambda)$ with $\lambda \in \sigma(A)$ and $\operatorname{Im}\lambda = 0$ (respectively, $\operatorname{Im}\lambda > 0$). The set of functions $\mathbb{R} \to \mathbb{R}^N$ given by

$$\{x_z\colon\ z \in B_0 \cup B_+\} \cup \{y_z\colon\ z \in B_+\},$$

forms a basis of the solution space of (2.22).

(2) If λ is a real eigenvalue of A, then, for every $z \in E(\lambda)$, the function x_z can be expressed in the form

$$x_z(t) = e^{\lambda t} \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!}(A-\lambda I)^k \mathrm{Re}\, z\,. \tag{2.23}$$

(3) If $\lambda = \alpha + i\beta$, with $\beta \neq 0$, is an eigenvalue of A, then, for every $z \in E(\lambda)$, the functions x_z and y_z can be expressed as follows

$$x_z(t) = e^{\alpha t} \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!}\big[\cos(\beta t)\mathrm{Re}\left((A-\lambda I)^k z\right) - \sin(\beta t)\mathrm{Im}\left((A-\lambda I)^k z\right)\big]\,. \tag{2.24}$$

and

$$y_z(t) = e^{\alpha t} \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!}\big[\cos(\beta t)\mathrm{Re}\left((A-\lambda I)^k z\right) + \sin(\beta t)\mathrm{Im}\left((A-\lambda I)^k z\right)\big]\,. \tag{2.25}$$

Theorem 2.14 shows that, by computing the eigenvalues of A and computing $m(\lambda)$ linearly independent generalized eigenvectors associated with λ for each $\lambda \in \sigma(A)$, N linearly independent real solutions of (2.22) can be obtained by using formulas (2.23)-(2.25).

Proof of Theorem 2.14

Let $\lambda \in \sigma(A)$ and $z \in E(\lambda)$. By Theorem 2.11,

$$\exp(At)z = e^{\lambda t}\big(\exp(A-\lambda I)t\big)z = e^{\lambda t} \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!}(A-\lambda I)^k z \quad \forall\, t \in \mathbb{R}.$$

Therefore, for all $t \in \mathbb{R}$,

$$x_z(t) := \exp(At)\mathrm{Re}\, z = \mathrm{Re}\left(\exp(At)z\right) = \mathrm{Re}\left(e^{\lambda t} \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!}(A-\lambda I)^k z\right).$$

Statement (2) follows immediately.

Now assume $\lambda = \alpha + i\beta$, with $\beta \neq 0$. Then,

$$x_z(t) = e^{\alpha t}\mathrm{Re}\left(e^{i\beta t} \sum_{k=0}^{m(\lambda)-1} \frac{t^k}{k!}(A-\lambda I)^k z\right) \quad \forall\, t \in \mathbb{R},$$

from which (2.24) follows. Since $y_z(t) = \exp(At)\mathrm{Im}\, z = \mathrm{Im}\,(\exp(At)z)$ for all $t \in \mathbb{R}$, an analogous calculation yields (2.25). This establishes statement(3).

It remains to prove statement (1). To this end, observe that B_0 is either empty or is a set of real vectors in $\mathbb{R}^N$. Noting that complex eigenvalues of A occur in conjugate pairs, it is readily seen that, if $B(\lambda) = \{v_1, \ldots, v_p\}$ is a basis of $E(\lambda)$, then $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is a basis of $E(\bar{\lambda})$. Writing $B_- := \{\bar{v} \colon v \in B_+\}$, it follows, by the generalized eigenspace decomposition theorem (see Theorem A.8), that $B_0 \cup B_+ \cup B_-$ is a basis of $\mathbb{C}^N$. If B_+ is non-empty, then writing $B_+ = \{v_1, \ldots, v_q\}$, we have

$$\mathrm{span}(B_+ \cup B_-) = \mathrm{span}\{v_1, \ldots, v_q, \bar{v}_1, \ldots, \bar{v}_q\} = \mathrm{span}\, B_1,$$

where

$$B_1 := \{\mathrm{Re}\, v_1, \ldots, \mathrm{Re}\, v_q, \mathrm{Im}\, v_1, \ldots, \mathrm{Im}\, v_q\}.$$

If $B_+ = \emptyset$, then $B_1 := \emptyset$. We may now conclude that $\mathcal{B} = B_0 \cup B_1$ is a real basis of $\mathbb{C}^N$. Moreover,

$$\mathcal{B} = \{x_z(0) \colon z \in B_0 \cup B_+\} \cup \{y_z(0) \colon z \in B_+\}$$

showing that the N functions $\mathbb{R} \to \mathbb{R}^N$ in the set $\{x_z \colon z \in B_0 \cup B_+\} \cup \{y_z \colon z \in B_+\}$ are linearly independent solutions of (2.22). This completes the proof. □

2.2 Inhomogeneous linear systems

In the following, let $A \colon J \to \mathbb{F}^{N \times N}$ and $b \colon J \to \mathbb{F}^N$ be piecewise continuous and let Φ be the transition matrix function generated by A. We will consider the issue of existence and uniqueness of solutions of the linear inhomogeneous initial-value problem

$$\dot{x}(t) = A(t)x(t) + b(t), \quad x(\tau) = \xi, \quad (\tau, \xi) \in J \times \mathbb{F}^N. \tag{2.26}$$

A *solution* of (2.26) is a continuous function $x \colon J_x \to \mathbb{F}^N$ satisfying

$$x(t) = \xi + \int_\tau^t \big(A(\sigma)x(\sigma) + b(\sigma)\big)\mathrm{d}\sigma \quad \forall\, t \in J_x.$$

where $J_x \subset J$ is an interval such that $\tau \in J_x$. By Theorems A.30 and A.31, $x : J_x \to \mathbb{R}^N$ is a solution of (2.26) if, and only if, x is piecewise continuously differentiable, $x(\tau) = \xi$ and

$$\dot{x}(t) = A(t)x(t) + b(t) \quad \forall\, t \in J_x \backslash E,$$

where E is the set of points in J at which A or b fail to be continuous. Piecewise continuity of A and b implies that the set E is "small" in the sense that, for all $t_1, t_2 \in J$ with $t_1 < t_2$, the intersection $E \cap [t_1, t_2]$ has at most finitely many elements. If A and b are continuous on J, then x is continuously differentiable and the differential equation in (2.26) holds for all $t \in J$.

Theorem 2.15

Let $(\tau, \xi) \in J \times \mathbb{F}^N$. The function

$$x\colon J \to \mathbb{F}^N, \quad t \mapsto \Phi(t,\tau)\xi + \int_\tau^t \Phi(t,\sigma)b(\sigma)\,\mathrm{d}\sigma. \tag{2.27}$$

is a solution of the initial-value problem (2.26). Moreover, if $y\colon J_y \to \mathbb{F}^N$ is another solution of (2.26), then $y(t) = x(t)$ for all $t \in J_y$.

Proof

Let $(\tau, \xi) \in J \times \mathbb{F}^N$ be arbitrary. We first show that x, given by (2.27), is a solution. Invoking Corollary 2.3, we have

$$\begin{aligned} x(t) = \left(I + \int_\tau^t A(\sigma)\Phi(\sigma,\tau)\mathrm{d}\sigma\right)\xi \\ + \int_\tau^t \left(I + \int_\sigma^t A(\eta)\Phi(\eta,\sigma)\mathrm{d}\eta\right) b(\sigma)\,\mathrm{d}\sigma \quad \forall\, t \in J. \end{aligned}$$

Changing the order of integration and then relabelling the variables of integration, we find

$$\begin{aligned} \int_\tau^t \int_\sigma^t A(\eta)\Phi(\eta,\sigma)b(\sigma)\,\mathrm{d}\eta\mathrm{d}\sigma &= \int_\tau^t \int_\tau^\eta A(\eta)\Phi(\eta,\sigma)b(\sigma)\,\mathrm{d}\sigma\mathrm{d}\eta \\ &= \int_\tau^t A(\sigma)\int_\tau^\sigma \Phi(\sigma,\eta)b(\eta)\,\mathrm{d}\eta\mathrm{d}\sigma\,. \end{aligned}$$

Therefore,

$$\begin{aligned} x(t) &= \xi + \int_\tau^t \left(A(\sigma)\left(\Phi(\sigma,\tau)\xi + \int_\tau^\sigma \Phi(\sigma,\eta)b(\eta)\,\mathrm{d}\eta\right) + b(\sigma)\right)\mathrm{d}\sigma \\ &= \xi + \int_\tau^t \big(A(\sigma)x(\sigma) + b(\sigma)\big)\,\mathrm{d}\sigma \quad \forall\, t \in J \end{aligned}$$

and so x is a solution of (2.26).

Finally, let $y\colon J_y \to \mathbb{R}^N$ be another solution of (2.26). Then

$$e(t) := x(t) - y(t) = \int_\tau^t A(\sigma)\big(x(\sigma) - y(\sigma)\big)\mathrm{d}\sigma = \int_\tau^t A(\sigma)e(\sigma)\,\mathrm{d}\sigma \quad \forall\, t \in J_y.$$

Therefore, e solves the initial-value problem $\dot{e}(t) = A(t)e(t)$, $e(\tau) = 0$, and so, by Theorem 2.5, e must be the zero function. Hence, $y(t) = x(t)$ for all $t \in J_y$. □

The formula (2.27) for the (unique) solution of the inhomogeneous initial-value problem (2.26) is frequently referred to as *the variation of parameters formula.*

In certain contexts, the initial condition in (2.26) is not relevant, in which case we say that a continuous function $x\colon J_x \to \mathbb{F}^N$, where $J_x \subset J$ is an interval, is a solution of the differential equation $\dot{x}(t) = A(t)x(t) + b(t)$ if there exists $\tau \in J_x$ such that

$$x(t) = x(\tau) + \int_\tau^t \big(A(\sigma)x(\sigma) + b(\sigma)\big)\mathrm{d}\sigma \quad \forall\, t \in J_x. \tag{2.28}$$

Note that, by Theorems A.30 and A.31, $x\colon J_x \to \mathbb{F}^N$ is a solution of the differential equation in this sense if, and only if, x is piecewise continuously differentiable and the differential equation $\dot{x}(t) = A(t)x(t) + b(t)$ is satisfied for every $t \in J_x$ which is not a point of discontinuity of A or b. The next exercise asserts that, if (2.28) holds for some $\tau \in J_x$, then (2.28) holds for all $\tau \in J_x$.

Exercise 2.13

Let $x\colon J_x \to \mathbb{F}^N$ be a solution of the differential equation $\dot{x}(t) = A(t)x(t) + b(t)$. Show that

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} \big(A(\sigma)x(\sigma) + b(\sigma)\big)\mathrm{d}\sigma \quad \forall\, t_1, t_2 \in J_x.$$

Let $\mathcal{S}_{\mathrm{ih}}$ denote the set of all solutions $x : J \to \mathbb{F}^N$ of the inhomogeneous differential equation $\dot{x}(t) = A(t)x(t) + b(t)$. The following result contains information on the structure of $\mathcal{S}_{\mathrm{ih}}$.

Corollary 2.16

Let $y \in \mathcal{S}_{\mathrm{ih}}$. Then

$$\mathcal{S}_{\mathrm{ih}} = y + \mathcal{S}_{\mathrm{hom}} = \{y + x : x \in \mathcal{S}_{\mathrm{hom}}\},$$

where $\mathcal{S}_{\mathrm{hom}}$ is the solution space of the homogeneous system $\dot{x}(t) = A(t)x(t)$.

Exercise 2.14

Prove Corollary 2.16.

Corollary 2.16 says that $\mathcal{S}_{\mathrm{ih}}$ is an affine linear space: the sum of an arbitrary solution y of the inhomogeneous problem (sometimes also called a *particular solution*) and the (linear) solution space of the associated homogeneous problem.

Finally, we consider the inhomogeneous initial-value problem with constant A, namely

$$\dot{x}(t) = Ax(t) + b(t), \quad x(\tau) = \xi \in \mathbb{F}^N, \tag{2.29}$$

where $A \in \mathbb{F}^{N\times N}$, $b\colon J \to \mathbb{F}^N$ is piecewise continuous and $\tau \in J$. By Theorem 2.15, we may immediately conclude the following.

Corollary 2.17

The function

$$x\colon J \to \mathbb{F}^N,\ t \mapsto \exp(A(t-\tau))\xi + \int_\tau^t \exp(A(t-\sigma))b(\sigma)\,\mathrm{d}\sigma$$

is a solution of the inhomogeneous initial-value problem (2.29). Moreover, if $y\colon J_y \to \mathbb{F}^N$ is also a solution of (2.29), then $y(t) = x(t)$ for all $t \in J_y$.

2.3 Systems with periodic coefficients: Floquet theory

Periodic phenomena feature prominently in the sciences and engineering: rotation of the Earth around its axis, heart beat, alternating electric current, to mention just a few examples. Correspondingly, the study of systems with periodic coefficients is a classical theme in differential equations. Here, we turn attention to linear homogeneous systems with $J = \mathbb{R}$ and a piecewise continuous periodic function $A\colon \mathbb{R} \to \mathbb{F}^{N\times N}$ with period $p > 0$:

$$\dot{x}(t) = A(t)x(t), \quad A(t+p) = A(t) \ \ \forall\, t \in \mathbb{R}. \tag{2.30}$$

It is natural to ask if there exist periodic solutions of the homogeneous system (2.30). By a periodic solution, we mean a solution x with the property that, for some $q > 0$, $x(t) = x(t+q)$ for all $t \in \mathbb{R}$. Observe that a constant solution qualifies as a periodic solution and, since $x = 0$ is a solution of (2.30), one might argue that there always exists a periodic solution. Disregarding the zero or trivial solution, our primary concern is the existence or otherwise of *non-zero* periodic solutions and, more generally, the qualitative behaviour of solutions of (2.30).

The following example illustrates the fact that non-zero periodic solutions of (2.30) need not necessarily exist.

Example 2.18

Consider the scalar initial-value problem with $\mathbb{F} = \mathbb{R}$

$$\dot{x}(t) = (1 + \sin t)x(t), \quad x(0) = \xi.$$

Here, $A\colon t \mapsto 1 + \sin t$ is periodic with period $p = 2\pi$. The unique solution of the initial-value problem is $x\colon t \mapsto \xi e^{(1+t-\cos t)}$ which fails to be periodic for all $\xi \neq 0$. $\triangle$

We briefly digress to state a result - the *spectral mapping theorem* - which will play a key role in our investigations.

Theorem 2.19 (Spectral mapping theorem)

Let $a_n \in \mathbb{C}$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, assume that the series $\sum_{n=0}^{\infty} a_n z^n =: f(z)$ converges for all $z \in \mathbb{C}$ and let $M \in \mathbb{C}^{N\times N}$. Then the series $f(M) := \sum_{n=0}^{\infty} a_n M^n$ converges in $\mathbb{C}^{N\times N}$ and $f(M)$ has the following properties.

(1) $\sigma(f(M)) = \big\{f(\lambda)\colon\ \lambda \in \sigma(M)\big\}$.

(2) If f is injective on $\sigma(M)$, then, for each $\lambda \in \sigma(M)$, the algebraic multiplicities of $f(\lambda) \in \sigma(f(M))$ and λ coincide.

(3) If f is injective on $\sigma(M)$ and $f'(\lambda) \neq 0$ whenever $\lambda \in \sigma(M)$ is not semisimple, then, for each $\lambda \in \sigma(M)$, the $f(\lambda)$-eigenspace $\ker(f(M) - f(\lambda)I)$ coincides with the λ-eigenspace $\ker(M - \lambda I)$ (and so, *a fortiori*, the geometric multiplicities of $f(\lambda)$ and λ coincide).

In order to avoid disrupting the presentation of our main concern, namely, the investigation of qualitative features of solutions of (2.30), we relegate the proof of the spectral mapping theorem to the end of the current chapter (see Section 2.4) and embark on our first task of identifying conditions under which (2.30) has a periodic solution.

Let Φ be the transition matrix function generated by A (a p-periodic function $\mathbb{R} \to \mathbb{F}^{N\times N}$). Let $\xi \in \mathbb{F}^N$, $\tau \in \mathbb{R}$ and set $y(t) := \Phi(t + p, \tau + p)\xi$ for all

$t \in \mathbb{R}$. Then, by Corollary 2.3,

$$y(t) - \xi = \int_{\tau+p}^{t+p} A(\sigma)\Phi(\sigma, \tau + p)\xi \, \mathrm{d}\sigma$$

whence

$$y(t) - \xi = \int_{\tau}^{t} A(\sigma + p)\Phi(\sigma + p, \tau + p)\xi \, \mathrm{d}\sigma = \int_{\tau}^{t} A(\sigma)y(\sigma) \, \mathrm{d}\sigma \quad \forall \, t \in \mathbb{R}$$

and so y is the unique solution of the initial-value problem $\dot{x}(t) = A(t)x(t)$, $x(\tau) = \xi$. Therefore, $\Phi(t + p, \tau + p)\xi = \Phi(t, \tau)\xi$ for all $t \in \mathbb{R}$. Since $\tau \in \mathbb{R}$ and $\xi \in \mathbb{F}^N$ are arbitrary, we may deduce the following property of Φ:

$$\Phi(t + p, \tau + p) = \Phi(t, \tau) \quad \forall \, (t, \tau) \in \mathbb{R} \times \mathbb{R}. \tag{2.31}$$

Therefore, for all $(t, \tau) \in \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} \Phi(t + p, \tau) &= \Phi(t + p, \tau + p)\Phi(\tau + p, \tau) = \Phi(t, \tau)\Phi(\tau, \tau - p) \\ &= \Phi(t, \tau)\Phi(\tau, 0)\Phi(0, \tau - p) = \Phi(t, 0)\Phi(p, \tau) \\ &= \Phi(t, 0)\Phi(p, 0)\Phi(0, \tau). \end{aligned}$$

We may now infer by induction that, for all $n \in \mathbb{N}$,

$$\Phi(t + np, \tau) = \Phi(t, 0)\Phi^n(p, 0)\Phi(0, \tau) \quad \forall \, (t, \tau) \in \mathbb{R} \times \mathbb{R}. \tag{2.32}$$

Exercise 2.15

Prove, by induction, that (2.32) holds for all $n \in \mathbb{N}$.

The following result gives a necessary and sufficient condition for the existence of a non-zero periodic solution of period np, where $n \in \mathbb{N}$.

Proposition 2.20

Let $n \in \mathbb{N}$. System (2.30) has a non-zero periodic solution x of period np if, and only if, $\Phi(p, 0)$ has an eigenvalue λ such that $\lambda^n = 1$.

Proof

To prove sufficiency, assume that λ is an eigenvalue of $\Phi(p, 0)$ and $\lambda^n = 1$. Let $v \in \mathbb{C}^N$ be an associated eigenvector. Then $v \neq 0$ and $\Phi^n(p, 0)v = \lambda^n v = v$. The unique solution $x \colon \mathbb{R} \to \mathbb{F}^N$ of the initial-value problem

$$\dot{x}(t) = A(t)x(t), \quad x(0) = v,$$

is given by $x(t) = \Phi(t,0)v$. Invoking (2.32), with $\tau = 0$, gives

$$x(t+np) = \Phi(t+np,0)v = \Phi(t,0)\Phi^n(p,0)v = \Phi(t,0)v = x(t)\ \forall\, t \in \mathbb{R},$$

and so x is a non-zero periodic solution of period np.

We proceed to prove necessity. To this end, assume that x is a non-zero periodic solution of (2.30), with period np. Then $v := x(0) \neq 0$ (because the zero function is the unique solution of the initial-value problem $\dot{y}(t) = A(t)y(t)$, $y(0) = 0$). Invoking (2.32), with $\tau = 0$, we have

$$\Phi(t,0)v = x(t) = x(t+np) = \Phi(t+np,0)v = \Phi(t,0)\Phi^n(p,0)v,$$

and thus, $\Phi(t,0)\big(I - \Phi^n(p,0)\big)v = 0$. Consequently $\big(I - \Phi^n(p,0)\big)v = 0$ and so 1 is an eigenvalue of $\Phi^n(p,0)$. By Theorem 2.19 (with $f(z) = z^n$),

$$\sigma(\Phi^n(p,0)) = \big\{\lambda^n \colon \lambda \in \sigma(\Phi(p,0))\big\}.$$

Therefore, $\Phi(p,0)$ has an eigenvalue λ with the property that $\lambda^n = 1$. □

Example 2.21

For $\mathbb{F} = \mathbb{R}$ and $N = 3$, consider (2.30) with $A\colon \mathbb{R} \to \mathbb{R}^{3\times 3}$ (period $p = 2\pi$) given by

$$A(t) := \begin{pmatrix} 0 & 1 & \sin t \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, the Peano-Baker series terminates and the state transition function Φ is given by

$$\begin{aligned}\Phi(t,\tau) &= I + \int_\tau^t A(s_1)\mathrm{d}s_1 + \int_\tau^t A(s_1)\int_\tau^{s_1} A(s_2)\mathrm{d}s_2\mathrm{d}s_1 \\ &= \begin{pmatrix} 1 & t-\tau & \cos\tau - \cos t + (t-\tau)^2/2 \\ 0 & 1 & t-\tau \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

Therefore,

$$\Phi(p,0) = \Phi(2\pi,0) = \begin{pmatrix} 1 & 2\pi & 2\pi^2 \\ 0 & 1 & 2\pi \\ 0 & 0 & 1 \end{pmatrix}$$

which evidently has eigenvalue $\lambda = 1$. By Proposition 2.20, it follows that (2.30) has a non-zero periodic solution of period 2π. Inspection of the form of Φ reveals that (2.30) can have no non-constant periodic solutions. Indeed, for every $\xi = (\xi_1,\xi_2,\xi_3)^* \in \mathbb{R}^3$ and every $\tau \in \mathbb{R}$, the function x defined by $x(t) := \Phi(t,\tau)\xi$ is unbounded (and hence not periodic) if $(\xi_2,\xi_3) \neq (0,0)$ and is constant if $(\xi_2,\xi_3) = (0,0)$. We therefore conclude that all non-zero periodic solutions are constant and are of the form $x(t) = (\xi_1,0,0)^*$ for all $t \in \mathbb{R}$. △

Exercise 2.16

Let $n \in \mathbb{N}$. Assume that $\Phi(p,0)$ has an eigenvalue λ such that $\lambda^n = 1$ and that the function $\mathbb{C} \to \mathbb{C}, z \mapsto z^n$ is injective on $\sigma(\Phi(p,0))$ (the latter condition holds trivially for $n = 1$).

(a) Show that $x : \mathbb{R} \to \mathbb{F}^N$ is a np-periodic solution of (2.30) if, and only if, $x(0) \in \ker(\Phi(p,0) - \lambda I)$. (*Hint.* Inspect the proof of Proposition 2.20. Make use of Theorem 2.19.)

(b) Let $\mathcal{S}_{np}$ denote the set of all np-periodic solutions of (2.30). Show that $\mathcal{S}_{np}$ is a vector space and that $\dim \mathcal{S}_{np} = \dim \ker(\Phi(p,0) - \lambda I)$.

Exercise 2.17

Let $n \in \mathbb{N}$ and $\mu \in \mathbb{C}$. Show that system (2.30) has a non-zero solution $x\colon \mathbb{R} \to \mathbb{C}^N$ with the property

$$x(t + np) = \mu x(t) \quad \forall\, t \in \mathbb{R}$$

if, and only if, $\Phi(p,0)$ has an eigenvalue λ such that $\lambda^n = \mu$.

(*Hint.* Note that the claim is a generalization of Proposition 2.20 (which corresponds to the special case of $\mu = 1$). Inspect the the proof of Proposition 2.20 and modify it in a suitable way.)

Next we present a variant of Proposition 2.20 which provides sufficient conditions for (2.30) to have a non-constant periodic solution.

Proposition 2.22

Let $n \in \mathbb{N}$ with $n \geq 2$. If the function $\mathbb{C} \to \mathbb{C}, z \mapsto z^n$ is injective on $\sigma(\Phi(p,0))$ and if $\Phi(p,0)$ has an eigenvalue λ such that $\lambda^n = 1$ and $\lambda^k \neq 1$, $k = 1, \ldots, n-1$, then, for each non-zero $\xi \in \ker(\Phi(p,0) - \lambda I)$, the solution of (2.30), with initial data $x(0) = \xi$, is non-constant and periodic.

Example 2.21 shows that the above proposition does not hold in the case of $n = 1$.

Proof of Proposition 2.22

Assume that $n \in \mathbb{N}$, $n \geq 2$, λ is an eigenvalue of $\Phi(p,0)$ with $\lambda^n = 1$ and $\lambda^k \neq 1$, $1 \leq k \leq n-1$. By hypothesis, the function $f\colon z \mapsto z^n$ is injective on $\sigma(\Phi(p,0))$. Moreover, since $\Phi(p,0)$ is invertible, $0 \notin \sigma(\Phi(p,0))$ and so $f'(\lambda) \neq 0$ for all $\lambda \in \sigma(\Phi(p,0))$. Therefore, by the spectral mapping theorem (Theorem 2.19), $\ker(\Phi^n(p,0) - I) = \ker(\Phi(p,0) - \lambda I)$. Let $\xi \in \ker(\Phi^n(p,0) - I) = \ker(\Phi(p,0) -$

$\lambda I)$. With initial data $x(0) = \xi$, (2.30) has unique solution $x\colon \mathbb{R} \to \mathbb{F}^N$ given by $x(t) = \Phi(t,0)\xi$. Invoking (2.32), we obtain

$$x(t+np) - x(t) = \big(\Phi(t+np,0) - \Phi(t,0)\big)\xi = \Phi(t,0)\big(\Phi^n(p,0) - I\big)\xi = 0 \;\forall\, t \in \mathbb{R},$$

and so x is np-periodic. It remains to show that x is not constant if $\xi \neq 0$. Seeking a contradiction, suppose that $x(t) = \Phi(t,0)\xi$ is constant for some non-zero ξ in $\ker(\Phi(p,0) - \lambda I)$. Fixing k, $1 \leq k \leq n-1$, we have $\Phi^k(p,0)\xi = \lambda^k\xi \neq \xi$. Since x is constant, x is kp-periodic, whence the contradiction

$$\xi = x(0) = x(kp) = \Phi(kp,0)\xi = \Phi^k(p,0)\xi = \lambda^k\xi \neq \xi,$$

where we have used once again (2.32). Therefore, x is non-constant, completing the proof. ☐

The next exercise shows that, whilst Proposition 2.22 provides sufficient conditions for the existence of a non-constant periodic solution of (2.30), this solution may have a period smaller than p.

Exercise 2.18

Let $\mathbb{F} = \mathbb{R}$, $N = 3$, $p = 2\pi$ and let $A\colon \mathbb{R} \to \mathbb{R}^{3\times 3}$ be the continuous, p-periodic function given by

$$A(t) := \begin{pmatrix} 0 & 1/2 & 0 \\ -1/2 & 0 & 0 \\ 0 & 0 & 1+\sin t \end{pmatrix}.$$

Show that the transition matrix function Φ generated by A is such that

$$\Phi(t,0) = \begin{pmatrix} \cos(t/2) & \sin(t/2) & 0 \\ -\sin(t/2) & \cos(t/2) & 0 \\ 0 & 0 & \exp(1-\cos t + t) \end{pmatrix}.$$

Verify that $\sigma(\Phi(p,0) = \{-1, e^p\}$, and so the hypotheses of Proposition 2.22 hold. Show that, for each non-zero $\xi \in \ker(\Phi(p,0)+I)$, the solution of (2.30), with initial data $x(0) = \xi$, is non-constant and periodic with period $\pi < 2\pi = p$.

Proposition 2.20 (and its generalization in Exercise 2.17) and Proposition 2.22 serve to illustrate the fact that the eigenvalues of the matrix $\Phi(p,0)$ play a crucial role in the analysis of solutions of the system (2.30): these eigenvalues are known as *Floquet multipliers* and are all non-zero, because $\Phi(p,0)$ is non-singular.

We now consider inhomogeneous systems with piecewise continuous periodic $A : \mathbb{R} \to \mathbb{F}^{N\times N}$ and $b : \mathbb{R} \to \mathbb{F}^N$, each with period $p > 0$:

$$\dot{x}(t) = A(t)x(t) + b(t), \quad A(t+p) = A(t),\; b(t+p) = b(t) \;\; \forall\, t \in \mathbb{R}. \tag{2.33}$$

Exercise 2.19

Set $\eta := \int_0^p \Phi(p,s)b(s)\mathrm{d}s$. Show that (2.33) has a p-periodic solution if, and only if, $\eta \in \operatorname{im}\big(I - \Phi(p,0)\big)$.

We proceed to investigate further the existence of p-periodic solutions of (2.33). In the following, let $\mathcal{S}_p$ denote the set of all p-periodic solutions of the homogeneous equation (2.30). It is easy to show that $\mathcal{S}_p$ is a vector space (a subspace of $\mathcal{S}_{\mathrm{hom}}$), see Exercise 2.16. The homogeneous equation

$$\dot{y}(t) = \tilde{A}(t)y(t), \quad \text{where } \tilde{A}(t) := -A^*(t) \text{ for all } t \in \mathbb{R}, \tag{2.34}$$

is said to be the *adjoint* equation of (2.30). The transition matrix $\tilde{\Phi}$ generated by $\tilde{A}$ is given by $\tilde{\Phi}(t,s) = \Phi^*(s,t)$ for all $s,t \in \mathbb{R}$, see Exercise 2.7. The space of all p-periodic solutions of the adjoint equation (2.34) is denoted by $\tilde{\mathcal{S}}_p$. For later purposes, we state and prove the following result which shows that the dimensions of $\mathcal{S}_p$ and $\tilde{\mathcal{S}}_p$ coincide.

Lemma 2.23

$\dim \mathcal{S}_p = \dim \tilde{\mathcal{S}}_p = \dim \ker(\Phi(p,0) - I)$.

Proof

Invoking Proposition 2.7, Exercise 2.16, and Proposition 2.20 shows that

$$\dim \mathcal{S}_p = \dim \ker(\Phi(p,0) - I) \quad \text{and} \quad \dim \tilde{\mathcal{S}}_p = \dim \ker(\tilde{\Phi}(p,0) - I).$$

Therefore, it only remains to prove that

$$\dim \ker(\Phi(p,0) - I) = \dim \ker(\tilde{\Phi}(p,0) - I). \tag{2.35}$$

Since $\tilde{\Phi}(t,s) = \Phi^*(s,t)$, it follows that

$$(\tilde{\Phi}(p,0) - I)^* = \Phi(0,p) - I = \Phi(0,p)(I - \Phi(p,0)).$$

Consequently, since $\Phi(p,0)$ is invertible,

$$\operatorname{rk}(\tilde{\Phi}(p,0) - I) = \operatorname{rk}(\tilde{\Phi}(p,0) - I)^* = \operatorname{rk}\big(\Phi(0,p)(I - \Phi(p,0))\big) = \operatorname{rk}(\Phi(p,0) - I).$$

Finally, by the dimension formula (see (A.5) in Appendix A.1),

$$\operatorname{rk}(\Phi(p,0)-I) + \dim\ker(\Phi(p,0)-I) = N = \operatorname{rk}(\tilde{\Phi}(p,0)-I) + \dim\ker(\tilde{\Phi}(p,0)-I),$$

and (2.35) follows. □

The following theorem provides a necessary and sufficient condition for the existence of p-periodic solutions of the inhomogeneous equation (2.33).

Theorem 2.24

(1) There exists a p-periodic solution of the inhomogeneous equation (2.33) if, and only if,

$$\int_0^p \langle y(s), b(s)\rangle \,\mathrm{d}s = 0 \quad \forall\, y \in \tilde{\mathcal{S}}_p, \tag{2.36}$$

where $\langle\, , \rangle$ denotes the standard inner product in $\mathbb{F}^N$ (see Appendix A.1).

(2) If (2.36) does not hold, then every solution $x : \mathbb{R} \to \mathbb{F}^N$ of (2.33) is unbounded (and, *a fortiori*, non-periodic).

Proof

Set $\eta := \int_0^p \Phi(p,s)b(s)\mathrm{d}s$. By Exercise 2.19, (2.33) has a p-periodic solution if, and only if, $\eta \in \operatorname{im}\big(I - \Phi(p,0)\big)$. By Theorem A.1,

$$\operatorname{im}(I - \Phi(p,0)) = \big(\ker(I - \Phi^*(p,0))\big)^{\perp}.$$

Moreover, since $I - \Phi^*(p,0) = \Phi^*(p,0)\big(\Phi^*(0,p) - I\big)$ and $\Phi(p,0)$ is invertible, we have $\ker\big(I - \Phi^*(p,0)\big) = \ker\big(\Phi^*(0,p) - I\big)$. We may now infer that (2.33) has a p-periodic solution if, and only if,

$$\langle z, \eta\rangle = 0 \quad \forall\, z \in \ker\big(\Phi^*(0,p) - I\big). \tag{2.37}$$

Therefore, to establish statement (1), it suffices to prove that (2.36) and (2.37) are equivalent. With this in mind, observe that, by part (a) of Exercise 2.16 applied in the context of the adjoint equation (2.34),

$$\tilde{\mathcal{S}}_p = \big\{\tilde{\Phi}(\cdot,0)z\colon z \in \ker\big(\tilde{\Phi}(p,0) - I\big)\big\} = \big\{\tilde{\Phi}(\cdot,0)z\colon z \in \ker\big(\Phi^*(0,p) - I\big)\big\}.$$

Therefore, (2.36) is equivalent to

$$\int_0^p \langle\tilde{\Phi}(s,0)z, b(s)\rangle \mathrm{d}s = 0 \quad \forall\, z \in \ker\big(\Phi^*(0,p) - I\big)$$

and, noting that

$$\begin{aligned}
\int_0^p \langle\tilde{\Phi}(s,0)z, b(s)\rangle \mathrm{d}s &= \int_0^p \langle\tilde{\Phi}(s,p)\tilde{\Phi}(p,0)z, b(s)\rangle \mathrm{d}s \\
&= \int_0^p \langle\Phi^*(0,p)z, \Phi(p,s)b(s)\rangle \mathrm{d}s \\
&= \langle\Phi^*(0,p)z, \eta\rangle = \langle z, \eta\rangle \quad \forall\, z \in \ker\big(\Phi^*(0,p) - I\big),
\end{aligned}$$

we may conclude that (2.36) holds if, and only if, (2.37) holds, completing the proof of statement (1).

To prove statement (2), let $x : \mathbb{R} \to \mathbb{F}^N$ be an arbitrary solution of (2.33). Let $k \in \mathbb{N}_0$ and define $x_k \colon \mathbb{R} \to \mathbb{F}^N$ by $x_k(t) := x(t + kp)$ for all $t \in \mathbb{R}$. It is straightforward to show that x_k is a solution of (2.33). Therefore,

$$x_k(t) = \Phi(t,0)x_k(0) + \int_0^t \Phi(t,s)b(s)\mathrm{d}s \quad \forall\, t \in \mathbb{R}.$$

Hence, $x_k(p) = \Phi(p,0)x_k(0) + \eta$, and thus,

$$x((k+1)p) = \Phi(p,0)x(kp) + \eta \quad \forall\, k \in \mathbb{N}_0.$$

By induction on k, we obtain

$$x(kp) = \Phi^k(p,0)x(0) + \sum_{j=0}^{k-1} \Phi^j(p,0)\eta \quad \forall\, k \in \mathbb{N}. \tag{2.38}$$

By hypothesis, (2.36) does not hold. Since (2.36) is equivalent to (2.37), it follows that there exists $\zeta \in \ker(\Phi^*(0,p) - I)$ such that $\langle \zeta, \eta \rangle \neq 0$. Now $\Phi^*(0,p)\zeta = \zeta$, whence $\zeta = \Phi^*(p,0)\zeta$ and so $\langle \zeta, z \rangle = \langle \zeta, \Phi^n(p,0)z \rangle$ for all $z \in \mathbb{F}^N$ and all $n \in \mathbb{N}$. Invoking (2.38) leads to

$$\langle \zeta, x(kp) \rangle = \langle \zeta, x(0) \rangle + k\langle \zeta, \eta \rangle \quad \forall\, k \in \mathbb{N}. \tag{2.39}$$

Since $\langle \zeta, \eta \rangle \neq 0$, the right-hand side of (2.39) is unbounded and, as a consequence, the sequence $(x(kp))$ is unbounded. This shows that x is unbounded. □

We record consequences of Theorem 2.24 in two corollaries, the first of which is immediate and does not require a proof.

Corollary 2.25

The inhomogeneous equation (2.33) has a p-periodic solution if, and only if, it has a bounded solution $\mathbb{R} \to \mathbb{F}^N$.

Corollary 2.26

There exists a p-periodic solution of the inhomogeneous equation (2.33) for every piecewise continuous p-periodic forcing function b if, and only if, there does not exist a non-zero p-periodic solution of the homogeneous equation (2.30) (that is, 1 is not a Floquet multiplier).

Proof

To prove sufficiency, assume that the homogeneous equation (2.30) does not have a non-zero p-periodic solution. Then $\mathcal{S}_p = \{0\}$, and thus, by Lemma 2.23, $\tilde{\mathcal{S}}_p = \{0\}$. It now follows from Theorem 2.24 that the inhomogeneous equation (2.33) has a p-periodic solution for every piecewise continuous p-periodic b. Conversely, to prove necessity, assume that (2.33) has a p-periodic solution for every piecewise continuous p-periodic b. Let $y \in \tilde{\mathcal{S}}_p$. It then follows that (2.33) has a p-periodic solution for $b = y$. Consequently, by Theorem 2.24, $\int_0^p \|y(s)\|^2 \mathrm{d}s = 0$, implying that $y = 0$. Since $y \in \tilde{\mathcal{S}}_p$ was arbitrary, we conclude that $\tilde{\mathcal{S}}_p = \{0\}$, and hence, by Lemma 2.23, $\mathcal{S}_p = \{0\}$, completing the proof. $\square$

Example 2.27

Consider the harmonic oscillator with 2π-periodic forcing

$$\ddot{y}(t) + \omega^2 y(t) = \cos t, \quad \omega \in \mathbb{R}$$

which may be expressed in the form (2.33) with constant A and 2π-periodic b given by

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad b(t) = \begin{pmatrix} 0 \\ \cos t \end{pmatrix}.$$

By Corollary 2.26, we may conclude the existence of a 2π-periodic solution if, and only if, $\omega^2 \neq 1$. $\triangle$

We proceed with a deeper investigation into connections between Floquet multipliers and qualitative behaviour of solutions of the homogeneous equation (2.30). In order to do so, we require the concept of matrix logarithm: for matrices G and H in $\mathbb{C}^{N\times N}$, we say that G is a *logarithm* of H if $\exp(G) = H$. If G is a logarithm of H, then, by Theorem 2.19,

$$\sigma(H) = \{e^\lambda : \lambda \in \sigma(G)\}.$$

Thus, every eigenvalue of G is a logarithm of some eigenvalue of H and, conversely, every eigenvalue of H has a logarithm which is an eigenvalue of G. We say that G is a *principal logarithm* of H if G is a logarithm of H and

$$\sigma(G) = \{\operatorname{Log} \lambda \colon \lambda \in \sigma(H)\}, \tag{2.40}$$

where $\operatorname{Log} : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ denotes the (scalar) principal logarithm, that is, for every nonzero $z \in \mathbb{C}$, $\operatorname{Log} z$ is the unique complex number with the properties that $z = e^{\operatorname{Log} z}$ and $\operatorname{Im}(\operatorname{Log} z) \in [0, 2\pi)$.

Corollary 2.28

Let $G \in \mathbb{C}^{N\times N}$ be a principal logarithm of $H \in \mathbb{C}^{N\times N}$. Then the algebraic and geometric multiplicities of each $\lambda \in \sigma(H)$ coincide with those of $\operatorname{Log}\lambda \in \sigma(G)$.

Proof

By hypothesis, $H = \exp(G)$ and (2.40) holds. Since, for all $z_1, z_2 \in \sigma(G)$, we have that $z_1 - z_2 \neq 2k\pi i$ for every $k \in \mathbb{Z}\backslash\{0\}$, it follows that the exponential function exp is injective on $\sigma(G)$. Furthermore, $\exp'(z) = \exp(z) \neq 0$ for all $z \in \sigma(G)$. Consequently, the claim follows from Theorem 2.19 (with $f = \exp$). □

Exercise 2.20

Find a matrix H which has a logarithm G with the property that there exists $\lambda \in \sigma(G)$ such that the algebraic and geometric multiplicities of λ do not coincide with those of $e^\lambda \in \sigma(H)$.

The question of existence of principal matrix logarithms is settled by the next result.

Proposition 2.29

If $H \in \mathbb{C}^{N\times N}$ is invertible, then there exists a principal logarithm of H.

In order to avoid disrupting the investigation of qualitative features of solutions of (2.30), we relegate the proof of Proposition 2.29 to the end of the current chapter (see Section 2.4).

Returning to the context of system (2.30), we now establish the following (Floquet) representation for $\Phi(\cdot, 0)$.

Theorem 2.30

Let $G \in \mathbb{C}^{N\times N}$ be a logarithm of $\Phi(p, 0)$. There exists a piecewise continuously differentiable p-periodic function $\Theta\colon \mathbb{R} \to \mathbb{C}^{N\times N}$, with $\Theta(0) = I$ and $\Theta(t)$ non-singular for all t, such that

$$\Phi(t, 0) = \Theta(t)\exp(tp^{-1}G) \quad \forall\, t \in \mathbb{R}.$$

Proof

Invoking (2.32) with $n = 1$ and $\tau = 0$, we have

$$\Phi(t + p, 0) = \Phi(t, 0)\Phi(p, 0). \tag{2.41}$$

Set $F := p^{-1}G$ and define the continuous function $\Theta\colon \mathbb{R} \to \mathbb{C}^{N\times N}$ by

$$\Theta(t) := \Phi(t,0)\exp(-tF).$$

Then $\Theta(0) = I$, $\Theta(t)$ is nonsingular for all t, and $\Phi(t,0) = \Theta(t)\exp(tF)$ for all $t \in \mathbb{R}$. Since $\Phi(\,\cdot\,,0)$ is piecewise continuously differentiable, it follows that Θ is also piecewise continuously differentiable. Moreover, for all $t \in \mathbb{R}$,

$$\Theta(t+p) = \Phi(t+p,0)\exp(-(t+p)F) = \Phi(t+p,0)\exp(-G)\exp(-tF).$$

Since $\Phi(p,0) = \exp(G)$, we have $\Phi(0,p) = \exp(-G)$ and so, for all $t \in \mathbb{R}$,

$$\Theta(t+p) = \Phi(t+p,0)\Phi(0,p)\exp(-tF) = \Phi(t,0)\Phi(p,0)\Phi(0,p)\exp(-tF),$$

where we have used (2.41) to obtain the second equation. Consequently, we have $\Theta(t+p) = \Phi(t,0)\exp(-tF) = \Theta(t)$ for all $t \in \mathbb{R}$ and so Θ is p-periodic. □

Equipped with Theorem 2.30, we are now in a position to make further connections between Floquet multipliers (eigenvalues of $\Phi(p,0)$) and qualitative properties of solutions of (2.30). A Floquet multiplier is said to be *semisimple* if its algebraic and geometric multiplicities (as an eigenvalue of $\Phi(p,0)$) coincide.

Theorem 2.31

(1) Every solution of (2.30) is bounded on $\mathbb{R}_+$ if, and only if, the modulus of each Floquet multiplier is not greater than 1 and any Floquet multiplier with modulus equal to 1 is semisimple.

(2) Every solution of (2.30) tends to zero at $t \to \infty$ if, and only if, the modulus of each Floquet multiplier is less than 1.

Proof

Let $(\tau,\xi) \in \mathbb{R} \times \mathbb{F}^N$ be arbitrary. The solution $x\colon \mathbb{R} \to \mathbb{F}^N$ of the initial-value problem

$$\dot{x}(t) = A(t)x(t), \quad x(\tau) = \xi,$$

is given by $x(t) = \Phi(t,\tau)\xi = \Phi(t,0)\Phi(0,\tau)\xi = \Phi(t,0)\zeta$, where $\zeta := \Phi(0,\tau)\xi$. Proposition 2.29 guarantees the existence of a principal logarithm G of $\Phi(p,0)$ and, moreover, by Corollary 2.28, the algebraic and geometric multiplicities of each $\lambda \in \sigma(\Phi(p,0))$ coincide with those of $\operatorname{Log}\lambda \in \sigma(G)$. Writing $F := p^{-1}G$ and invoking Theorem 2.30, we have

$$x(t) = \Theta(t)\exp(tF)\zeta \quad \forall\, t \in \mathbb{R},$$

where $\Theta\colon \mathbb{R} \to \mathbb{C}^{N\times N}$ is piecewise continuously differentiable (and hence continuous) and p-periodic, with $\Theta(0) = I$ and $\Theta(t)$ invertible for all t. Therefore, we may infer the existence of $M > 0$ such that $\|\Theta(t)\| \leq M$ and $\|\Theta^{-1}(t)\| \leq M$ for all $t \in \mathbb{R}$. Since $pF = G$, we have

$$\sigma(\Phi(p,0)) = \{e^{\mu p}\colon\ \mu \in \sigma(F)\}$$

and, moreover, the algebraic and geometric multiplicities of each $\mu \in \sigma(F)$ coincide with those of $e^{\mu p} \in \sigma(\Phi(p,0))$. We record three particular consequences.

(a) Every eigenvalue of F has non-positive real part if, and only if, every eigenvalue of $\Phi(p,0)$ has modulus not greater than 1.

(b) Every eigenvalue of F with zero real part is semisimple if, and only if, every eigenvalue of $\Phi(p,0)$ with modulus equal to 1 is semisimple.

(c) Every eigenvalue of F has negative real part if, and only if, every eigenvalue of $\Phi(p,0)$ has modulus less than 1.

Now, define $y\colon \mathbb{R} \to \mathbb{C}^N$ by $y(t) := \Theta^{-1}(t)x(t)$. Then, $\|y(t)\| \leq M\|x(t)\| \leq M^2\|y(t)\|$ and, in particular, x is bounded on $\mathbb{R}_+$ if, and only if, y is bounded on $\mathbb{R}_+$. Furthermore,

$$y(t) = \exp(tF)\zeta \quad \forall\, t \in \mathbb{R}.$$

Thus, Θ determines a one-to-one correspondence between the solutions of the nonautonomous system (2.30) and the solutions of the autonomous system

$$\dot{y} = Fy. \tag{2.42}$$

Therefore, we may conclude the following.

(d) Every solution of (2.30) is bounded on $\mathbb{R}_+$ if, and only if, every solution of (2.42) is bounded on $\mathbb{R}_+$.

(e) Every solution of (2.30) tends to zero as $t \to \infty$ if, and only if, every solution of (2.42) tends to zero as $t \to \infty$.

The conjunction of Corollary 2.13 and equivalences (a), (b) and (d) above now give statement (1). Similarly, the conjunction of Corollary 2.13 and equivalences (c) and (e) yield statement (2). □

Example 2.32

In this example, we consider Hill's equation[6]

$$\ddot{y}(t) + a(t)y(t) = 0, \quad a(t+p) = a(t) \ \ \forall\, t \in \mathbb{R}, \tag{2.43}$$

[6] George William Hill (1838-1914), US American.

where a is piecewise continuous and $p > 0$. Hill's equation describes an undamped oscillation with restoring force at time t equal to $-a(t)y(t)$. The two-dimensional first-order system associated with (2.43) is given by

$$\dot{x}(t) = A(t)x(t), \quad A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix} \quad \forall\, t \in \mathbb{R}. \tag{2.44}$$

Let Φ be the transition matrix function generated by A. Our intention is to apply Theorem 2.31 in the context of (2.44). To this end, we calculate the Floquet multipliers. Now,

$$\det(\lambda I - \Phi(p,0)) = \lambda^2 - \lambda \operatorname{tr} \Phi(p,0) + \det \Phi(p,0),$$

and, by statement (2) of Proposition 2.7,

$$\det \Phi(p,0) = \exp\left(\int_0^p \operatorname{tr} A(s)\mathrm{d}s\right) = 1.$$

Moreover, noting that $\Phi(t,0)$ is of the form

$$\Phi(t,0) = \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \dot{\varphi}_1(t) & \dot{\varphi}_2(t) \end{pmatrix} \quad \forall\, t \in \mathbb{R},$$

where φ_1 and φ_2 are the unique solutions of (2.43) satisfying $\varphi_1(0) = 1 = \dot{\varphi}_2(0)$ and $\dot{\varphi}_1(0) = 0 = \varphi_2(0)$, respectively, it follows that

$$\operatorname{tr} \Phi(p,0) = \varphi_1(p) + \dot{\varphi}_2(p).$$

Consequently,

$$\det(\lambda I - \Phi(p,0)) = \lambda^2 - 2\gamma\lambda + 1, \quad \text{where} \quad \gamma := \tfrac{1}{2}(\varphi_1(p) + \dot{\varphi}_2(p)), \tag{2.45}$$

and the Floquet multipliers are given by

$$\lambda_\pm = \gamma \pm \sqrt{\gamma^2 - 1}.$$

Invoking Theorem 2.31, we draw the following conclusions.

Case 1: $|\gamma| > 1$. Then $\lambda_+ > 1$ (if $\gamma > 1$) or $\lambda_- < -1$ (if $\gamma < -1$), and hence, at least one solution of (2.44) is unbounded on $\mathbb{R}_+$.

Case 2: $|\gamma| < 1$. Then $\lambda_\pm = \gamma \pm i\delta$ with $\delta > 0$. Since $\lambda_+\lambda_- = 1$, it follows that $|\lambda_+| = |\lambda_-| = 1$. Moreover, λ_+ and λ_- are simple (and *a fortiori* semisimple) and hence all solutions of (2.44) are bounded on $\mathbb{R}_+$.

Case 3: $|\gamma| = 1$. Then $\gamma = \pm 1$ and $\lambda_+ = \lambda_- = \gamma$. All solutions of (2.44) are bounded on $\mathbb{R}_+$ if, and only if, γ is semisimple. Since the algebraic multiplicity of γ is two, γ is semisimple if, and only if, $\ker(\gamma I - \Phi(p,0)) = \mathbb{C}^2$. Consequently,

γ is semisimple if, and only if, $\Phi(p,0) = \gamma I$, that is, $\varphi_1(p) = \dot{\varphi}_2(p) = \gamma$ and $\dot{\varphi}_1(p) = \varphi_2(p) = 0$.

Irrespective of semisimplicity of γ, by Proposition 2.20, there exists at least one non-zero periodic solution of period p if $\gamma = 1$ and of period $2p$ if $\gamma = -1$. Furthermore, we claim that, in the case of γ being semisimple, every solution is p-periodic (if $\gamma = 1$) or $2p$-periodic (if $\gamma = -1$). To see this, assume that γ is semisimple. Then the matrix

$$G := \begin{pmatrix} \log\gamma & 0 \\ 0 & \log\gamma \end{pmatrix}.$$

is a logarithm of $\Phi(p,0) = \gamma I$. By Theorem 2.30, there exists a piecewise continuously differentiable p-periodic function $\Theta : \mathbb{R} \to \mathbb{C}^{2\times 2}$ such that

$$\Phi(t,0) = \Theta(t)\exp(tp^{-1}G) \quad \forall\, t \in \mathbb{R}.$$

If $\gamma = 1$, then $G = 0$, and hence $\Phi(t,0) = \Theta(t)$ for all $t \in \mathbb{R}$, showing that $\Phi(t+p,0) = \Phi(t,0)$ for all $t \in \mathbb{R}$. Every solution x of (2.44) is of the form $x(t) = \Phi(t,0)x(0)$ and is therefore p-periodic. If $\gamma = -1$, then

$$G = \begin{pmatrix} i\pi & 0 \\ 0 & i\pi \end{pmatrix},$$

whence

$$\Phi(t,0) = \Theta(t)\begin{pmatrix} e^{(i\pi/p)t} & 0 \\ 0 & e^{(i\pi/p)t} \end{pmatrix} \quad \forall\, t \in \mathbb{R}.$$

Therefore, $\Phi(t+2p,0) = \Phi(t,0)$ for all $t \in \mathbb{R}$, showing that every solution x of (2.44) is $2p$-periodic.

Finally, we analyse a specific example. Assume that the function a is given by

$$a(t) = \begin{cases} \omega^2, & m \le t < m+\tau \\ 0, & m+\tau \le t < m+1, \quad \text{where } m \in \mathbb{Z}. \end{cases} \tag{2.46}$$

Here $\omega > 0$ and $\tau \in (0,1)$, Obviously, a is a piecewise continuous periodic function with period equal to 1. With this choice of a, Hill's equation (2.43) describes an undamped oscillator, the restoring force of which is switched off on the intervals $[m+\tau, m+1)$, $m \in \mathbb{Z}$. Since a is piecewise constant, $\Phi(1,0)$ can easily be determined analytically. A routine calculation yields

$$\Phi(1,0) = \begin{pmatrix} \cos(\omega\tau) - \omega(1-\tau)\sin(\omega\tau) & \omega^{-1}\sin(\omega\tau) + (1-\tau)\cos(\omega\tau) \\ -\omega\sin(\omega\tau) & \cos\omega\tau \end{pmatrix}.$$

In particular,

$$\gamma = \frac{1}{2}\big(2\cos(\omega\tau) - \omega(1-\tau)\sin(\omega\tau)\big)$$

We consider two "extreme" scenarios.

Scenario 1: τ is close to 1. In this scenario, the restoring force is switched on "most" of the time and so one might expect the behaviour of the solutions to be similar to those of the harmonic oscillator $\ddot{y} + \omega^2 y = 0$ (for which every solution is periodic, of period $2\pi/\omega$, and so *a fortiori* is bounded). However, we show that this is not the case. To this end, let $\omega = \pi/\tau$ and note that

$$\Phi(1,0) = \begin{pmatrix} -1 & \tau - 1 \\ 0 & -1 \end{pmatrix}$$

and $\gamma = -1$. Clearly, $\lambda = -1$ is an eigenvalue of $\Phi(1,0)$ and so, by Proposition 2.20, there exists a non-zero periodic solution of period 2. Since the eigenvalue $\lambda = -1$ is not semisimple, it follows from Case 3 above that there exists at least one solution which is unbounded on $\mathbb{R}_+$. A more detailed analysis (see Exercise 2.21) reveals that φ_1 is periodic of period 2 and φ_2 is unbounded on $\mathbb{R}_+$. Consequently, denoting the components of $\xi \in \mathbb{R}^2$ by ξ_1 and ξ_2 and setting $x(t) := \Phi(t,0)\xi$ for all $t \in \mathbb{R}$, the solution x is periodic of period 2 if, and only if, $\xi_2 = 0$ and, furthermore, x is unbounded on $\mathbb{R}_+$ if, and only if, $\xi_2 \neq 0$. These observations are valid for all $\tau \in (0,1)$: in particular, they hold when τ is close to 1, in which case we have $\omega \approx \pi$ and so the 2-periodic solutions do indeed mimic the behaviour of the harmonic oscillator $\ddot{y} + \omega^2 y = 0$; however, all other non-zero solutions are unbounded and so the behaviour of the system differs markedly from that of the harmonic oscillator.

Scenario 2: τ is close to 0. In this scenario, the restoring force is switched off "most" of the time and one might expect that the behaviour of the solutions is similar to those of the "double integrator" $\ddot{y} = 0$ (which has unbounded solutions, for example, $y(t) = t$). However, this is not the case. For every $\omega > 0$, we have $0 < \gamma < 1$ for all sufficiently small $\tau \in (0,1)$. Consequently, by Case 2 above, for all $\tau > 0$ sufficiently small, all solutions of (2.44) are bounded on $\mathbb{R}_+$. This behaviour differs markedly from that of the double integrator. △

Exercise 2.21

Assume that, in Example 2.32, the periodic function a is given by (2.46) with $\tau \in (0,1)$ and $\omega = \pi/\tau$. Show that φ_1 is periodic of period 2 and φ_2 is unbounded with $\varphi_2(n) = (-1)^n n(1-\tau)$ for all $n \in \mathbb{N}$.

Exercise 2.22

Assume that in Example 2.32 the periodic function a is even. Show that in this case $\gamma = \varphi_1(p) = \dot{\varphi}_2(p)$.

The following corollary of Theorem 2.31 provides a criterion for the existence of at least one solution of (2.30) which is unbounded on $\mathbb{R}_+$.

Corollary 2.33

If $\int_0^p \operatorname{tr} A(s)\,\mathrm{d}s$ has positive real part, then (2.30) has a solution x with $\limsup_{t\to\infty} \|x(t)\| = \infty$.

Proof

By statement (2) of Proposition 2.7, we have

$$\det \Phi(p,0) = \exp\left(\int_0^p \operatorname{tr} A(s)\,\mathrm{d}s\right).$$

Let λ_j, $j = 1, \dots, d$, be the distinct eigenvalues of $\Phi(p,0)$, with algebraic multiplicities m_j, $j = 1, \dots, d$. Then $\det\big(\Phi(p,0) - \lambda I\big) = \prod_{j=1}^d (\lambda_j - \lambda)^{m_j}$, which, upon evaluation at $\lambda = 0$, shows that $\det \Phi(p,0) = \prod_{j=1}^d \lambda_j^{m_j}$. Hence,

$$\prod_{j=1}^d \lambda_j^{m_j} = \exp\left(\int_0^p \operatorname{tr} A(s)\,\mathrm{d}s\right).$$

Therefore, invoking the hypothesis,

$$\prod_{j=1}^d |\lambda_j|^{m_j} = \exp\left(\operatorname{Re} \int_0^p \operatorname{tr} A(s)\,\mathrm{d}s\right) > 1.$$

Consequently, there exists $j \in \{1, \dots, d\}$ such that $|\lambda_j| > 1$ and so, by Theorem 2.31, there must exist a solution x which is unbounded on $\mathbb{R}_+$. □

Exercise 2.23

Consider (2.30) with $N = 2$, $\mathbb{F} = \mathbb{R}$ and

$$A(t) = \begin{pmatrix} 1 + \sin t & a \\ b & 1 - \cos t \end{pmatrix},$$

where $a, b \in \mathbb{R}$ are arbitrary constants. Show that there exists at least one solution which is unbounded on $\mathbb{R}_+$.

The converse of Corollary 2.33 does not hold. Specifically, if $\int_0^p \operatorname{tr} A(s)\,\mathrm{d}s$ has negative real part, then we cannot conclude that every solution x of (2.30) is bounded on $\mathbb{R}_+$, as the following exercise shows.

Exercise 2.24

Consider (2.30) with $N = 2$, $\mathbb{F} = \mathbb{R}$ and

$$A(t) = \frac{1}{2}\begin{pmatrix} -2 + 3\cos^2 t & 2 - 3\sin t \cos t \\ -2 - 3\sin t \cos t & -2 + 3\sin^2 t \end{pmatrix}.$$

In this case, A is π-periodic and $\int_0^\pi \operatorname{tr} A(s)\,\mathrm{d}s = -\pi/2 < 0$. Show that

$$t \mapsto x(t) := e^{t/2}\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

is a solution of (2.30), and is such that $\|x(t)\| \to \infty$ as $t \to \infty$.

2.4 Proof of Theorem 2.19 and Proposition 2.29

We conclude this chapter with proofs of Theorem 2.19 (the spectral mapping theorem) and Proposition 2.29.

Proof of Theorem 2.19

Let $M \in \mathbb{C}^{N\times N}$ and let $a_n \in \mathbb{C}$, with $n \in \mathbb{N}_0$, be such that the series $f(z) = \sum_{n=0}^\infty a_n z^n$ converges for all $z \in \mathbb{C}$. By Proposition A.27, $f(M) := \sum_{n=0}^\infty a_n M^n$ is a well-defined element of $\mathbb{C}^{N\times N}$. Let λ_j, $j = 1, \ldots, d$, be the distinct eigenvalues of M with associated algebraic multiplicities m_j, $j = 1, \ldots, d$. Noting that, if T is invertible, then

$$f(T^{-1}MT) = \sum_{n=0}^\infty a_n \big(T^{-1}MT\big)^n = T^{-1}\left(\sum_{n=0}^\infty M^n\right) T = T^{-1}f(M)T$$

and so, without loss of generality, we may assume that M is in Jordan[7] canonical form (see Theorem A.9) which we express as $M = \operatorname{diag}\big(J_1, \ldots, J_\ell\big)$. The generic block $J \in \{J_1, \ldots, J_\ell\}$ takes the form

$$J = \lambda I + K \quad \text{for some } \lambda \in \{\lambda_1, \ldots, \lambda_d\},$$

where, for some $r \in \mathbb{N}$, I is the $r\times r$ identity matrix and $K \in \mathbb{R}^{r\times r}$ is a matrix with every superdiagonal entry equal to 1, all other entries being 0, that is,

$$K = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

(if $r = 1$, then $K = 0$). Note that $K^n = 0$ for all $n \geq r$,

$$J^n = (\lambda I + K)^n = \sum_{k=0}^n \binom{n}{k} \lambda^{n-k} K^k \quad \forall\, n \in \mathbb{N}_0.$$

[7] Marie Ennemond Camille Jordan (1838-1922), French.

Furthermore, term-by-term differentiation of the power series yields

$$\frac{f^{(k)}(\lambda)}{k!} = \sum_{n=k}^{\infty} a_n \binom{n}{k} \lambda^{n-k} \quad \forall\, k \in \mathbb{N}_0,$$

where $f^{(k)}$ denotes the k-th derivative of f (with $f^{(0)} := f$). Therefore,

$$f(J) = \sum_{n=0}^{\infty} a_n J^n = \sum_{k=0}^{r-1} \sum_{n=k}^{\infty} a_n \binom{n}{k} \lambda^{n-k} K^k = \sum_{k=0}^{r-1} \frac{f^{(k)}(\lambda)}{k!} K^k.$$

In particular, $f(J)$ has the following upper triangular structure

$$f(J) = \begin{pmatrix} f(\lambda) & f'(\lambda) & * & \cdots & * & * \\ 0 & f(\lambda) & f'(\lambda) & \cdots & * & * \\ 0 & 0 & f(\lambda) & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & f(\lambda) & f'(\lambda) \\ 0 & 0 & 0 & \cdots & 0 & f(\lambda) \end{pmatrix} \tag{2.47}$$

and so $\sigma(f(J)) = \{f(\lambda)\}$. Of course, in the case of $r = 1$, (2.47) should be interpreted as the scalar $f(J) = f(\lambda)$. Since $f(M) = \operatorname{diag}\big(f(J_1), \ldots, f(J_\ell)\big)$, it now follows that

$$\sigma(f(M)) = \{f(\lambda) \colon \lambda \in \sigma(M)\},$$

completing the proof of statement (1).

We proceed to prove statement (2). To this end note that the above argument also shows that, for each $\lambda \in \sigma(M)$ and every Jordan block J associated with λ, the algebraic multiplicity of λ as an eigenvalue of J coincides with the algebraic multiplicity of $f(\lambda)$ as an eigenvalue of $f(J)$. From this, we may infer that the algebraic multiplicity of $f(\lambda)$ as an eigenvalue of $f(M)$ cannot be less than the algebraic multiplicity of λ as an eigenvalue of M. Moreover, since f is injective on $\sigma(M)$, the number of distinct eigenvalues of $f(M)$ coincides with the number d of distinct eigenvalues of M. Since the algebraic multiplicities of the eigenvalues sum to N in each case, it follows that the algebraic multiplicity of each $\lambda \in \sigma(M)$ coincides with that of $f(\lambda) \in \sigma(f(M))$.

To prove statement (3), let J be any Jordan block in $M = \operatorname{diag}\big(J_1, \ldots, J_\ell\big)$ associated with $\lambda \in \sigma(M)$. If J is scalar, then trivially we have $\ker(f(J) - f(\lambda)I) = \ker(J - \lambda I) = \mathbb{C}$. If J is not scalar, then λ is not a semisimple, and so, by hypothesis, the additional property $f'(\lambda) \neq 0$ holds. By (2.47), we then have

$$\ker\big(f(J) - f(\lambda)I\big) = \operatorname{span}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \ker\big(J - \lambda I\big).$$

Consequently, defining $\Lambda := \{k\colon J_k \text{ is associated with } \lambda\} \subset \{1,\dots,\ell\}$, it follows that

$$\dim\ker(f(J_k) - f(\lambda)I) = \dim\ker(J_k - \lambda I) = 1 \quad \forall\, k \in \Lambda.$$

Moreover, by injectivity of f on $\sigma(M)$, $f(\lambda) \notin \sigma(f(J_k)$ for all $k \in \{1,\dots,\ell\}\backslash\Lambda$, and thus

$$\dim\ker(f(J_k) - f(\lambda)I) = \dim\ker(J_k - \lambda I) = 0 \quad \forall\, k \in \{1,\dots,\ell\}\backslash\Lambda.$$

Therefore, we may conclude that

$$\dim\ker(f(M) - f(\lambda)I) = \dim\ker(M - \lambda I) = \#\Lambda,$$

where $\#\Lambda$ denotes the number of elements of Λ. Finally, let $v \in \ker(M - \lambda I)$. Then $M^n v = \lambda^n v$ for all $n \in \mathbb{N}_0$ and so $f(M)v = f(\lambda)v$. Therefore, $\ker(M - \lambda I) \subset \ker(f(M) - f(\lambda)I)$ and since these subspaces have the same dimension, they must coincide. This completes the proof. $\square$

Proof of Proposition 2.29

Let λ_j, $j = 1,\dots,d$, be the distinct eigenvalues of H with associated algebraic multiplicities m_j, $j = 1,\dots,d$. Note that, if G is a logarithm of H and T is invertible, then $\exp(T^{-1}GT) = T^{-1}\exp(G)T = T^{-1}HT$ and so $T^{-1}GT$ is a logarithm of $T^{-1}HT$. Therefore, without loss of generality, we may assume that H is in Jordan canonical form (see Theorem A.9) which can be expressed as

$$H = \operatorname{diag}\big(J_1,\dots,J_\ell\big), \quad \text{where } \ell \geq d.$$

The generic block $J \in \{J_1,\dots,J_\ell\}$ takes the form

$$J = \lambda I + K \quad \text{for some } \lambda \in \{\lambda_1,\dots,\lambda_d\},$$

where, for some $r \in \mathbb{N}$, I is the $r \times r$ identity matrix and $K \in \mathbb{R}^{r\times r}$ is a matrix with every superdiagonal entry equal to 1, all other entries being 0, that is,

$$K = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

(of course, if $r = 1$, then $K = 0$). We record that $\sigma(J) = \{\lambda\}$ and

$$\ker(J - \lambda I) = \ker(K) = \operatorname{span}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} =: V. \tag{2.48}$$

Choose $\delta > 0$ sufficiently small so that $e^\delta - 1 < 3\delta/2 < 1$. Furthermore, choose $\varepsilon > 0$ sufficiently small so that $M := (\varepsilon/\lambda)K$ has norm $\|M\| \le \delta/2$. Next, we will invoke the contraction mapping theorem (Theorem A.25) to prove the existence of a logarithm of $I + M$. To this end, set $\Omega := \{X \in \mathbb{C}^{r\times r}\colon\ \|X\| \le \delta\}$ and define $F\colon \Omega \to \mathbb{C}^{r\times r}$ by

$$F(X) := X + I + M - \exp(X) = M - \sum_{k=2}^{\infty} \frac{1}{k!} X^k.$$

Then,

$$\|F(X)\| \le \|M\| + \sum_{k=2}^{\infty} \frac{\delta^k}{k!} \le \frac{\delta}{2} + e^\delta - 1 - \delta = e^\delta - 1 - \frac{\delta}{2} \le \delta \quad \forall\, X \in \Omega$$

and so $F(\Omega) \subset \Omega$. Observing that

$$X^k - Y^k = \sum_{j=1}^{k} X^{k-j}(X - Y)Y^{j-1} \quad \forall\, X, Y \in \Omega,\ \forall\, k \in \mathbb{N} \tag{2.49}$$

(see Exercise 2.25 below for details), we obtain

$$\begin{aligned}\|F(X) - F(Y)\| &= \left\| \sum_{k=2}^{\infty} \frac{1}{k!} \big(Y^k - X^k\big) \right\| \\ &\le \sum_{k=2}^{\infty} \frac{\delta^{k-1}}{(k-1)!} \|X - Y\| = \big(e^\delta - 1\big)\|X - Y\| \quad \forall\, X, Y \in \Omega.\end{aligned}$$

Recalling that $e^\delta - 1 < 1$, it follows that F is a contraction on Ω and so, by the contraction mapping theorem (Theorem A.25), has a fixed point $Z \in \Omega$. Therefore, $I + M = \exp(Z)$, that is, Z is a logarithm of $I + M$.

Next, we will use Z to construct a principal logarithm of J. To this end, define $Q := \operatorname{diag}\big(\varepsilon, \varepsilon^2, \dots, \varepsilon^r\big)$. Then $Q^{-1}KQ = \varepsilon K$ and so

$$Q^{-1}JQ = \lambda I + \varepsilon K = \lambda\big(I + M\big) = \lambda \exp(Z).$$

Let $\nu \in \mathbb{C}$ be a logarithm of λ, and so $e^\nu = \lambda$ (such a logarithm exists since $\lambda \ne 0$ by invertibility of H). Therefore,

$$J = \lambda Q \exp(Z) Q^{-1} = \lambda \exp(QZQ^{-1}) = \exp(\nu I + QZQ^{-1}),$$

showing that $P := \nu I + QZQ^{-1}$ is a logarithm of J. By Theorem 2.19, $e^\mu = \lambda$ for every $\mu \in \sigma(P)$. Hence, for $\mu \in \sigma(P)$ and $v \in \ker(P - \mu I)$,

$$(J - \lambda I)v = (\exp(P) - \lambda I)v = e^\mu v - \lambda v = 0,$$

and thus, by (2.48), $\ker(P-\mu I) = V$. Consequently, $\sigma(P) = \{\mu\}$ is a singleton. Setting $L := P + (2k\pi i)I$, where $k \in \mathbb{Z}$ is such that $\operatorname{Log}\lambda = \mu + 2k\pi i$, we obtain

$$\exp(L) = J \quad \text{and} \quad \sigma(L) = \{\operatorname{Log}\lambda\},$$

that is, L is a principal logarithm of J.

We have now shown that, for each $j = 1, \ldots, \ell$, there exists a principle logarithm L_j of J_j and so $G := \operatorname{diag}(L_1, \ldots, L_\ell)$ is a principal logarithm of $H = \operatorname{diag}(J_1, \ldots, J_\ell)$, completing the proof. □

Exercise 2.25

Prove (2.49) by induction on k.

(*Hint.* Note that $X^{k+1} - Y^{k+1} = (X+Y)(X^k - Y^k) + XY^k - YX^k$.)

Springer
http://www.springer.com/978-1-4471-6397-8

Ordinary Differential Equations
Analysis, Qualitative Theory and Control
Logemann, H.; Ryan, E.P.
2014, XIII, 333 p. 40 illus., Softcover
ISBN: 978-1-4471-6397-8