Chapter 2
Preliminaries

Classical control design and analysis utilizes the frequency domain tools to specify the system performance. The background of operator theories and single-input and single-output, linear systems is required. In modern control, the time domain approach can be used to deal with multi-input and multi-out cases. Moreover, concepts of linear algebra and matrix-vector operations are used in system analysis and synthesis. Some useful fundamentals will be therefore reviewed in this chapter.

2.1 Linear Algebra and Matrix Theory

This section presents useful and well-known fundamentals of linear algebra and matrix theory, which facilitate the understanding of the subsequent control system concepts and methodology introduced. The stated results can be considered to be purely preliminary in nature, and hence, their proofs are omitted.

2.1.1 Vectors and Matrices

Control systems are, in general, multivariable. That means one deals with more than one variable in input, output, and state. Hence, vectors and matrices are frequently used to represent systems and system interconnections. In engineering and science, one usually has a situation where more than one quantity is closely linked to another. For instance, in specifying the location of a robot on a flat floor, one may use the numbers 2 and 3 to indicate the robot is at 2 units east and 3 units north from where one stands, and following the same logic, one may use −1 and −2 to indicate that the robot is at 1 unit west and 2 units south. Here, (2, 3) and (−1, −2) represent two different locations, and the numbers 2 and 3 are in a fixed order to show that

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particular location, while (3, 2) would represent the position 3 units east and 2 units north. Such a group of numbers in a certain order forms a vector, and the dimensions of a vector correspond to how many numbers there are in the vector. Hence, (2, 3) is a 2-dimensional vector. Conventionally, a vector is defined as a column vector. In the above example, the position vector is thus written as $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$. For any positive integer n, an n-dimensional (usually shortened as n-dim or n-D) vector x is denoted by $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

The transpose of a vector x is denoted as x^{T} and is defined by $x^{\mathrm{T}} = \begin{bmatrix} x_1 \ x_2 \cdots x_n \end{bmatrix}$, a row vector. A group of vectors of the same dimension in a certain order forms a matrix. For example, for $M_i = \begin{bmatrix} m_{i1} \\ \vdots \\ m_{in} \end{bmatrix}$, $1 \le i \le p$, M_i is an n-dim vector and $M = \begin{bmatrix} M_1 \ M_2 \cdots M_P \end{bmatrix}$ is an $n \times p$ matrix. Obviously, a vector is a special case of matrices. $x^{\mathrm{T}} = \begin{bmatrix} x_1 \ x_2 \cdots x_n \end{bmatrix}$ is simply a matrix of $1 \times n$ dimensions. The elements or entries in a matrix can be real numbers or complex numbers. One uses $M \in \mathbb{R}^{n \times p}$ to show the matrix M is of $n \times p$ dimensions and all the elements of M are real numbers; $M \in \mathbb{C}^{n \times p}$ shows a $n \times p$ dimensional matrix M with complex numbers. It is clear that $\mathbb{R}^{n \times p} \subset \mathbb{C}^{n \times p}$. It is also a convention to use capital English letters to show a matrix such as M, whereas lower case letters (sometimes in bold face) are employed to show a vector such as x and lower case letters to show a scalar number. A matrix $M = \begin{bmatrix} m_{11} \cdots m_{1p} \\ \vdots \ \ldots \ \vdots \\ m_{n1} \cdots m_{np} \end{bmatrix}$ of the $n \times p$ dimension can be abbreviated as $M = \{m_{ij}\}_{n \times p}$. Similar to vectors, the transpose of a matrix M is $M^{\mathrm{T}} = \{m_{ji}\}_{p \times n}$. When M is in $\mathbb{C}^{n \times p}$, the complex conjugate transpose of M is defined by $M^* = \{\overline{m}_{ji}\}_{p \times n}$ when $\overline{m}_{ji}$ is the complex conjugate of m_{ji}.

A few manipulations can be defined for vectors and matrices. Two matrices of the same dimensions, e.g., $M = \begin{bmatrix} m_{11} \cdots m_{1p} \\ \vdots \ \ldots \ \vdots \\ m_{n1} \cdots m_{np} \end{bmatrix}$ and $N = \begin{bmatrix} n_{11} \cdots n_{1p} \\ \vdots \ \ldots \ \vdots \\ n_{n1} \cdots n_{np} \end{bmatrix}$, can be added together, i.e., $P = M + N$ where $P = \begin{bmatrix} p_{11} \cdots p_{1p} \\ \vdots \ \ldots \ \vdots \\ p_{n1} \cdots p_{np} \end{bmatrix} = \begin{bmatrix} m_{11} + n_{11} \cdots m_{1p} + n_{1p} \\ \vdots \ \ldots \ \vdots \\ m_{n1} + n_{n1} \cdots m_{np} + n_{np} \end{bmatrix}$. A multiplication is defined for two matrices only

Table 2.1 Classification of normal matrices

Matrix (A)	Definition	Eigenvalues	Diagonal elements	Determinant
Hermitian	$M^* = M$	$\lambda \in \mathbb{R}$	$a_{ii} \in \mathbb{R}$	$\det(M) \in \mathbb{R}$
Positive definite	$x^*Mx > 0, \forall\, x \neq 0$	$\lambda > 0$	$a_{ii} > 0$	$\det(M) > 0$
Positive semi-definite	$x^*Mx \geq 0, \forall\, x$	$\lambda \geq 0$	$a_{ii} \geq 0$	$\det(M) \geq 0$
Unitary $M \in C^{n\times n}$	$M^*M = I$	$\lvert\lambda\rvert = 1$	NA	$\lvert\det(M)\rvert = 1$
Orthogonal $M \in R^{n\times n}$	$M^{\mathrm{T}}M = I$	$\lvert\lambda\rvert = 1$	NA	$\lvert\det(M)\rvert = 1$

when their dimensions are compatible. That is, for $M = \{m_{ij}\}_{n\times p}$, $N = \{n_{kl}\}_{k\times l}$, only when $p = k$, one may have the product $P = MN$, where $P = \{p_{ij}\}_{n\times l}$, with $p_{ij} = \sum_{r=1}^{p(=k)} m_{ir}n_{rj}$. The following paragraph summarizes a few more aspects of vector/matrix manipulations [4].

1. A square matrix M is called nonsingular if a matrix B exists, such that $MB = BM = I$. Define $B = M^{-1}$. The inverse matrix M^{-1} exists if $\det(M) \neq 0$, where $\det(M)$ is the determinant of M. If M^{-1} does not exist, M is said to be singular. If the inverse of M, B, and MB all exist, then $(MB)^{-1} = B^{-1}M^{-1}$.
2. A complex square matrix is called unitary if its inverse is equal to its complex conjugate transpose $M^*M = MM^* = I$, where I denotes the identity matrix of the appropriate dimensions. A square matrix M is called orthogonal if it is real and satisfies $M^{\mathrm{T}}M = MM^{\mathrm{T}} = I$. For an orthogonal matrix, the inverse is its transpose.
3. An $n \times p$ matrix M is of rank m if the maximum number of linearly independent rows (or columns) is m. This equals to the dimension of $\mathrm{img}(M) := \{Mx \mid x \in \mathbb{R}^p\}$.
4. An $n \times p$ matrix M is said to have full row rank if $n \leq p$ and $\mathrm{rank}(M) = n$. It has a full column rank if $n \geq p$ and $\mathrm{rank}(M) = p$.
5. A symmetric matrix M of $n \times n$ dimension is positive definite if $x^{\mathrm{T}}Mx \geq 0$, where x is any n-dimensional (real) vector, and $x^{\mathrm{T}}Mx = 0$, only if $x = 0$. If for any n-dimensional vector x, $x^{\mathrm{T}}Mx \geq 0$ always holds, then M is positive semi-definite. A positive (semi-)definite matrix M may be denoted as $M > 0 (M \geq 0)$. Similarly, negative definite and negative semi-definite matrices may be defined.
6. For a positive definite matrix M, its inverse M^{-1} exists and is also positive definite.
7. All eigenvalues of a positive definite matrix are positive.
8. For two positive definite matrices M_1 and M_2, one has $\alpha M_1 + \beta M_2 > 0$ when α, β are nonnegative and both not zero.
9. A square matrix M is called normal if $MM^* = M^*M$. A normal matrix has the decomposition of $M = U\Lambda U^*$, where $UU^* = I$ and Λ is a diagonal matrix. The following Table 2.1 summarizes the classification of normal matrices.

2.1.2 Linear Spaces

Let $\mathbb{R}$ and $\mathbb{C}$ be real and complex scalar fields, respectively. A linear space V over a field F consists of a set on which two operations are defined. The first one is denoted by "addition (+)"; for each pair of elements x and y in V, there exists a unique element $x+y$ in V. And the second one is a scalar "multiplication (·)"; for each element α in F and each element x in V, there is a unique element αx in V. The following conditions hold with respect to the above two operations.

1. For each element x in V, $1 \cdot x = x$.
2. For all x, y, z in V, $(x+y)+z = x+(y+z)$.
3. For all x, y in V, $x+y = y+x$.
4. For each element x in V, there exists an element y in V, such that $x+y=0$.
5. There exists an element in V denoted by 0, such that $x+0=x$ for each x in V.
6. For each element α in F and each pair of elements x and y in V, $\alpha(x+y) = \alpha x + \alpha y$.
7. For each α, β in F and each element x in V, $(\alpha\beta)x = \alpha(\beta x)$.
8. For each α, β in F and each element x in V, $(\alpha+\beta)x = \alpha x + \beta x$.

Note that one uses the same symbol "0" to denote the element zero and scalar number zero in V and F, respectively. In the following, some basic concepts are reviewed first. These definitions can be easily found in standard linear algebra textbooks, for example see [8].

1. As mentioned in the earlier paragraph, the elements $x+y$ and αx are called the sum of x and y and the product of α and x, respectively, where $x, y \in V, \alpha \in F$.
2. A subset W of a vector space V over a field F is called a subspace of V if W itself is a vector space over F under the operations of addition and scalar multiplication defined on V.
3. Let $x_1, x_2, \ldots, x_k$ be vectors in V, then an element of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$ with $\alpha_i \in F$ is a linear combination over F of $x_1, x_2, \ldots, x_k$.
4. The set of all linear combinations of $x_1, x_2, \ldots, x_k \in V$ is a subspace called the span of $x_1, x_2, \ldots, x_k$, denoted by

$$\text{span}\{x_1, x_2, \ldots, x_k\} = \left\{ x \middle| x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k; \alpha_i \in F \right\}. \quad (2.1)$$

5. Vectors $x_1, x_2, \ldots, x_k$ are said to be linearly dependent if there is at least one x_i that can be expressed as a linear combination of $\{x_j, j = 1, 2, \ldots, k, j \neq i\}$ or there exist constants $c_1, c_2, \ldots, c_k$ which are not all zero, such that $c_1 x_1 + c_2 x_2 + \cdots + c_k x_k = 0$. The vectors $x_1, x_2, \ldots, x_k$ are linearly independent if $c_1 x_1 + c_2 x_2 + \cdots + c_k x_k = 0$ indicates that all $c_1, c_2, \ldots, c_k$ are zero.
6. The vectors $x_1, x_2, \ldots, x_k$ are orthonormal if $x_i{}^* x_j = \delta_{ij} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{otherwise} \end{cases}$, where δ_{ij} is usually called the Kronecker delta.

7. Let W be a subspace of a vector space V, then a set of vectors $\{x_1, x_2, \ldots, x_k\} \in W$ is said to be a basis of W if $x_1, x_2, \ldots, x_k$ are linearly independent and $W = \text{span}\{x_1, x_2, \ldots, x_k\}$. The dimension of a vector subspace W equals to the number of basis vectors.
8. Let W be a subspace of V. The set of all vectors in V that are orthogonal to every vector in W is the orthogonal complement of W and is denoted by $W^{\perp}$. Hence,

$$W^{\perp} = \{y \in V : y^* x = 0, \forall x \in W\}. \tag{2.2}$$

   Each vector x in V can be expressed uniquely in the form $x = x_W + x_{W^{\perp}}$ for $x_W \in W$ and $x_{W^{\perp}} \in W^{\perp}$.
9. A set of vectors $\{u_1, u_2, \ldots, u_k\}$ is said to be an orthonormal basis for a k-dimensional subspace W if the vectors form a basis and are orthonormal. Suppose that the dimension of V is n, it is then possible to find a set of orthonormal basis $\{u_{k+1}, \ldots, u_n\}$ such that

$$W^{\perp} = \text{span}\{u_{k+1}, \ldots, u_n\}. \tag{2.3}$$

10. A collection of subspaces $W_1, W_2, \ldots, W_k$ of V is mutually orthogonal if $x^* y = 0$ whenever $x \in W_i$ and $y \in W_j$ for $i \neq j$.
11. The kernel (or null) space of a matrix $M \in \mathbb{R}^{n \times p}$, which can be viewed as a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^n$, is defined as

$$\ker M = N(M) = \left\{ x \middle| x \in \mathbb{R}^p : Mx = 0 \right\}. \tag{2.4}$$

12. The image (or range) of M is

$$\text{img}(M) = \left\{ y \middle| y \in \mathbb{R}^n : y = Mx, \forall x \in \mathbb{R}^p \right\}. \tag{2.5}$$

13. Let M be an $n \times p$ real, full rank matrix with $n > p$, the orthogonal complement of M is a matrix $M^{\perp}$ of dimension $n \times (n - p)$, such that $\begin{bmatrix} M & M^{\perp} \end{bmatrix}$ is a square, nonsingular matrix with the following property: $M^{\mathrm{T}} M^{\perp} = 0$.
14. The following properties hold:

$$(\ker M)^{\perp} = [\text{img}(M)]^{\mathrm{T}} \quad \text{and} \quad [\text{img}(M)]^{\perp} = \ker M^{\mathrm{T}}. \tag{2.6}$$

2.1.3 Eigenvalues and Eigenvectors

A matrix can be interpreted as a mapping between two linear spaces. For example, a 2×2 matrix $M = \{m_{ij}\}_{2 \times 2}$, $y = Ax$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ are both in $\mathbb{R}^{2\times 1}$ (the two spaces in this case are the same). For most x, the image y would show a rotation of x plus an expansion or reduction in length, which is decided by the matrix M. However, there are some vectors in the space of which the images generated by the mapping M will remain at the same direction as the original vectors. These vectors are the eigenvectors of M, showing somehow the essence (eigen) of the mapping M. The factors of the length change are the eigenvalues of M. Rigorous definitions are given below.

For an $n \times n$ square matrix M, the determinant $\det(\lambda I - M)$ is called the *characteristic polynomial* of M. The *characteristic equation* is given by

$$\det(\lambda I - M) = 0. \tag{2.7}$$

The n roots of the characteristic equation are the *eigenvalues* of M. For an eigenvalue λ of matrix M, there is a nonzero vector ξ such that

$$M\xi = \lambda\xi \tag{2.8}$$

where ξ is called the eigenvector of M corresponding to the eigenvalue λ.

Definition 2.1 The spectral radius of matrix M is defined as

$$\rho(M) = \max_i |\lambda_i(M)| \tag{2.9}$$

where $\{\lambda_i\}$ is the eigenvalue set of M and $|\bullet|$ is the modulus of $\bullet$.

It is easy to show that if M is a Hermitian matrix, i.e., $M = M^*$, then all eigenvalues of M are real. The spectral radius indicates the size of the set which contains all the eigenvalues of M.

Definition 2.2 If M is Hermitian, then there exists a unitary matrix U (i.e., $U^*U = UU^* = I$) and a real diagonal matrix Λ, such that

$$M = U\Lambda U^*. \tag{2.10}$$

In this case, U is the right eigenvector matrix of M.

2.1.4 Matrix Inversion and Pseudoinverse

Matrix inversion is unavoidable and essential in control system manipulation. In this section, the useful formulae of the matrix inversion can be found [4].

Let M be a square $n \times n$ matrix partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{2.11}$$

where $M_{11} : n_1 \times n_1, M_{12} : n_1 \times n_2, M_{21} : n_2 \times n_1, M_{22} : n_2 \times n_2$, and $n_1 + n_2 = n$.

Suppose that M_{11} is nonsingular, then M can be decomposed (block diagonalized) as

$$M = \begin{bmatrix} I & 0 \\ M_{21}M_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} M_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & M_{11}^{-1}M_{12} \\ 0 & I \end{bmatrix} \tag{2.12}$$

where $S = M_{22} - M_{21}M_{11}^{-1}M_{12}$ is the Schur complement of M_{11} in M. Then, if M is nonsingular, it can be derived that

$$M^{-1} = \begin{bmatrix} M_{11}^{-1} + M_{11}^{-1}M_{12}S^{-1}M_{21}M_{11}^{-1} & -M_{11}^{-1}M_{12}S^{-1} \\ -S^{-1}M_{21}M_{11}^{-1} & S^{-1} \end{bmatrix}. \tag{2.13}$$

Dually, if M_{22} and M are nonsingular, then

$$M = \begin{bmatrix} I & M_{12}M_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \widehat{S} & 0 \\ 0 & M_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ M_{22}^{-1}M_{21} & I \end{bmatrix} \tag{2.14}$$

and

$$M^{-1} = \begin{bmatrix} \widehat{S}^{-1} & -\widehat{S}^{-1}M_{12}M_{22}^{-1} \\ -M_{22}^{-1}M_{21}\widehat{S}^{-1} & M_{22}^{-1} + M_{22}^{-1}M_{21}\widehat{S}^{-1}M_{12}M_{22}^{-1} \end{bmatrix} \tag{2.15}$$

where $\widehat{S} = M_{11} - M_{12}M_{22}^{-1}M_{21}$ is called the Schur complement of M_{22} in M. The matrix inversion formulae can be further simplified if M is block triangular as

$$\begin{bmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{bmatrix}^{-1} = \begin{bmatrix} M_{11}^{-1} & 0 \\ -M_{22}^{-1}M_{21}M_{11}^{-1} & M_{22}^{-1} \end{bmatrix}, \tag{2.16}$$

$$\begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}^{-1} = \begin{bmatrix} M_{11}^{-1} & -M_{11}^{-1}M_{12}M_{22}^{-1} \\ 0 & M_{22}^{-1} \end{bmatrix}. \tag{2.17}$$

If both M_{11} and M_{22} are nonsingular, then $\widehat{S}^{-1}$ can be represented by

$$\begin{aligned} \widehat{S}^{-1} &= \left(M_{11} - M_{12}M_{22}^{-1}M_{21}\right)^{-1} \\ &= M_{11}^{-1} + M_{11}^{-1}M_{12}\left(M_{22} - M_{21}M_{11}^{-1}M_{12}\right)^{-1}M_{21}M_{11}^{-1}. \end{aligned} \tag{2.18}$$

The pseudoinverse (also called Moore-Penrose inverse) of a matrix M is denoted as M^{+} which satisfies the following conditions:

$$MM^{+}M = M, \tag{2.19}$$

$$M^{+}MM^{+} = M^{+}, \tag{2.20}$$

$$\left(MM^+\right)^* = MM^+, \tag{2.21}$$

$$\left(M^+M\right)^* = M^+M. \tag{2.22}$$

The pseudoinverse is useful especially when matrix M is either non-square or singular.

2.1.5 Vector Norms and Matrix Norms

Norm is another important concept of vectors and matrices. It can be further developed for functions and systems as well. In this section, definitions of vector norm and matrix norm will be introduced [4]. The concept of norm can be loosely understood as a description of size or volume. A vector norm, denoted by $\|\cdot\|$, of any vector x over the field $\mathbb{C}$, must have the following properties:

1. $\|x\| > 0$, unless $x = 0$, in which case $\|x\| = 0$.
2. $\|cx\| = |c|\|x\|$ where c is any scalar in $\mathbb{C}$.
3. $\|x + y\| \le \|x\| + \|y\|$.

Definition 2.3 Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{C}^n$. The following are norms of $\mathbb{C}^n$.

1. Vector ∞-norm: $\|x\|_\infty = \max\limits_{1\le i\le n} |x_i|$.
2. Vector 1-norm: $\|x\|_1 = \sum\limits_{i=1}^{n} |x_i|$.
3. Vector 2-norm: $\|x\|_2 = \sqrt{x^*x} = \sqrt{\sum\limits_{i=1}^{n} |x_i|^2}$.
4. Vector p-norm (for $1 \le p < \infty$): $\|x\|_p = \left(\sum\limits_{i=1}^{n} |x_i|^p\right)^{1/p}$.

In the case of matrices, a matrix norm satisfies

1. $\|A\| > 0$ unless $A = 0$, in which case $\|A\| = 0$;
2. $\|cA\| = |c|\|A\|$ where c is any scalar in $\mathbb{C}$;
3. $\|A + B\| \le \|A\| + \|B\|$;
4. $\|AB\| \le \|A\|\|B\|$.

Definition 2.4 Let $M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{m1} & m_{m2} & \cdots & m_{mn} \end{bmatrix}$ be a matrix in $\mathbb{C}^{m\times n}$. The following gives a list of different matrix norms, which will be useful for the rest of this book.

1. Matrix 1-norm (column sum): $\|M\|_1 := \max_j \sum_{i=1}^{m} |m_{ij}|$.
2. Matrix 2-norm: $\|M\|_2 := \sqrt{\lambda_{\max}(M^*M)}$.
3. Matrix ∞-norm (row sum): $\|M\|_\infty := \max_i \sum_{j=1}^{n} |m_{ij}|$.
4. Frobenius norm: $\|M\|_F := \sqrt{\text{trace}\,(M^*M)} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}^* m_{ij}}$.

2.1.6 Singular Value Decomposition

The singular values of a matrix M are defined as

$$\sigma_i(M) := \sqrt{\lambda_i(M^*M)}. \tag{2.23}$$

The maximal singular value is denoted as

$$\overline{\sigma}(M) := \max_i (\sigma_i(M)),$$

and the minimal singular value is

$$\underline{\sigma}(M) := \min_i (\sigma_i(M)).$$

It is straightforward from the above definition that the matrix M and its complex conjugate transpose M^* have the same singular values, i.e., $\{\sigma_i(M)\} = \{\sigma_i(M^*)\}$. Let $M \in \mathbb{C}^{m\times n}$; there exist unitary matrices $U = [u_1 \;\; u_2 \cdots u_m] \in \mathbb{C}^{m\times m}$ and $V = [v_1 \; v_2 \cdots v_n] \in \mathbb{C}^{n\times n}$ such that

$$M = U\Sigma V^*, \tag{2.24}$$

where

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.25}$$

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{bmatrix}, \tag{2.26}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $r = \text{rank}(M)$. Equation (2.24) is called the singular value decomposition (SVD) of the matrix M. The matrix admits the decomposition

$$M = \sum_{i=1}^{r} \sigma_i u_i {v_i}^* = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix}^*. \tag{2.27}$$

2.2 Function Spaces and Signals

Controllers or control schemes are as a matter of fact functions in the time domain or the frequency domain. Hence, the synthesis of the required controller, an optimal controller in particular, is a procedure in functional analysis. However, considering that the underlying systems in this book are mainly the linear time-invariant systems and that this book is primarily for practicing control engineers and engineering students, many mathematical definitions and deductions will not be included in order to make it more accessible to the targeted readers. Interested readers are recommended to consult relevant books, for instance [5, 6, 7, 10], for rigorous and in-depth treatment of those mathematical concepts.

2.2.1 Function Spaces

Function spaces useful for the themes introduced in this book are L_2, H_2, L_∞, and H_∞, and their orthogonal complement spaces.

The space L_p (for $1 \leq p < \infty$) consists of all Lebesgue measurable functions $w(t)$ defined in the interval $(-\infty, \infty)$ such that

$$\|w\|_p := \left(\int_{-\infty}^{\infty} |w(t)|^p \mathrm{d}t \right)^{\frac{1}{p}} < \infty. \tag{2.28}$$

The space L_∞ consists of all Lebesgue measurable functions $w(t)$ such that

$$\|w\|_\infty := \operatorname*{ess\,sup}_{t \in R} |w(t)| < \infty. \tag{2.29}$$

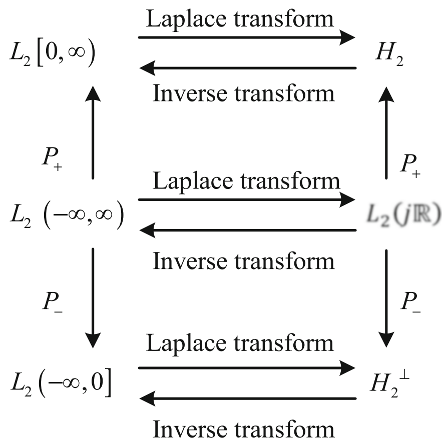

Fig. 2.1 Calculation procedures of function spaces

H_2 is the subspace of L_2 in which every function is analytic in $\mathrm{Re}(s) > 0$ (the real part of $s = \sigma + j\omega \in \mathbb{C}$), the open right-half plane, and H_∞ be a subspace of L_∞ in which every function is analytic and bounded in $\mathrm{Re}(s) > 0$. The space $H_2{}^{\perp}$ is the orthogonal complement of H_2 in L_2. If $G(s)$ is a strictly proper, stable, real, rational transfer function matrix, then $G(s) \in H_2$ implies that $G^{\sim}(s) \in H_2{}^{\perp}$, where $G^{\sim}(s) := G^{\mathrm{T}}(-s)$. The real rational subspace of H_∞ is denoted by RH_∞, which consists of all proper and real, rational, stable transfer function matrices. The relationship between spaces L_2 and H_2 is illustrated in Fig. 2.1 [3, 10].

Definition 2.5 Definitions of L_2, H_2, L_∞, and H_∞ function spaces.

1. L_2 -function space: $G(s) \in L_2$, if

$$\int_{-\infty}^{\infty} \mathrm{trace}\left[G^*(j\omega)\, G(j\omega)\right] \mathrm{d}\omega < \infty. \tag{2.30}$$

   The rational subspace of L_2, denoted by RL_2, consists of all real, rational, strictly proper transfer function matrices with no poles on the imaginary axis jR.
2. H_2-function space: $G(s) \in H_2$, if $G(s)$ is stable and

$$\|G(s)\|_2 := \sqrt{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}\left[G^*(j\omega)\, G(j\omega)\right] \mathrm{d}\omega\right]} < \infty. \tag{2.31}$$

   Hence, the norm for H_2 can be computed just as it is done for L_2. The real rational subspace of H_2, which consists of all strictly proper and real, rational, stable transfer function matrices, is denoted by RH_2.
3. L_∞-function space: $G(s) \in L_\infty$, if

$$\|G(j\omega)\|_\infty := \operatorname*{ess\,sup}_{\omega}\ \overline{\sigma}\left[G(j\omega)\right] < \infty. \tag{2.32}$$

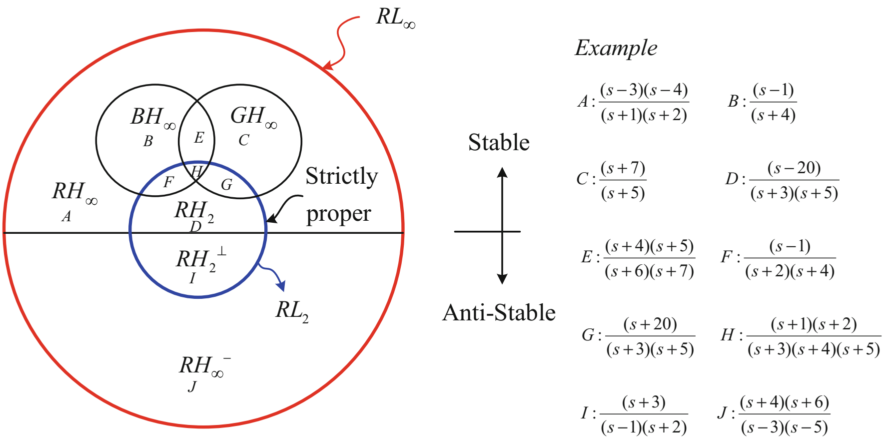

Fig. 2.2 Illustration of the relationship among different function spaces

All proper and real, rational, transfer function matrices with no poles on the imaginary axis form a subspace which is denoted by RL_∞.

4. H_∞ norm, the ∞-norm of Hardy space functions: $G \in H_\infty$, if $G(s)$ is stable and

$$\|G\|_\infty = \sup_{\mathrm{Re}(s)\geq 0} \overline{\sigma}\left[G(s)\right] = \sup_{\omega} \overline{\sigma}\left[G\left(j\omega\right)\right] < \infty. \tag{2.33}$$

H_∞ is a subspace of L_∞ with functions that are analytic and bounded in the open right-half plane. The real, rational subspace of H_∞ is denoted by RH_∞ which consists of all proper and real, rational, stable transfer function matrices.

This book introduces tools and concepts of optimal controller synthesis [3]. Most of the framework is set in the H_∞ function space. For the linear time-variant and causal systems, a given system $G(s) \in RH_\infty$ means the following:

(a) $G(s)$ is stable, and $\lim_{t\to\infty} \Phi(t)$ which is the impulse response of $G(s)$, is bounded.
(b) All poles of $G(s)$ are located in the open left-half plane.
(c) If $G(s)$ has a “minimal” state-space model (A, B, C, D), then the real part of all eigenvalues of the state matrix A is negative.

A state matrix is called *Hurwitz* if the real parts of all its eigenvalues are negative. The following Fig. 2.2 shows the relationship among different function spaces, where $BH_\infty := \{F \in RH_\infty : \|F\|_\infty < 1\}$ is denoted as the set of all stable contractions and GH_∞ is the set of all units of RH_∞, i.e., if $F \in GH_\infty$, then $F \in RH_\infty$ and $F^{-1} \in RH_\infty$.

Example 2.1 Determine the function spaces for each of the following transfer functions: (1) $G_1(s) = \frac{s}{s+1}$; (2) $G_2(s) = \frac{s^2}{s+1}$; and (3) $G_3(s) = \frac{s}{(s-1)(s+2)}$.

1. It is clear that $G_1(s)$ is stable and $\sup_{\omega} |G_1(j\omega)| = \sup_{\omega} \left|\frac{j\omega}{j\omega+1}\right| = \sup_{\omega} \frac{\omega}{\sqrt{1+\omega^2}} = 1 < \infty$ for $\omega > 0$. Hence, $G_1(s) \in RH_\infty$. By decomposition of $G_1(s)$, one has $G_1(s) = \frac{s}{s+1} = 1 - \frac{1}{s+1}$. Thus,

$$\begin{aligned}\|G\|_2 &= \sqrt{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 \mathrm{d}\omega\right]} \\ &= \sqrt{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(1-\frac{1}{j\omega+1}\right)\left(1-\frac{1}{-j\omega+1}\right)\mathrm{d}\omega\right]} \\ &= \sqrt{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(1-\frac{1}{j\omega+1}-\frac{1}{-j\omega+1}+\frac{1}{(j\omega+1)(-j\omega+1)}\right)\mathrm{d}\omega\right]} \\ &= \infty. \end{aligned} \tag{2.34}$$

This implies $G_1(s) \notin RH_2$, which agrees with the fact that $G_1(s)$ is bi-proper.

2. By definition, $G_2(s) \notin RL_\infty$ because of $\sup_{\omega} |G_2(j\omega)| = \sup_{\omega} \left|\frac{(j\omega)^2}{j\omega+1}\right| = \infty$; $G_2(s) \notin RH_2$ because of $\int_{-\infty}^{\infty} |G(j\omega)|^2 \mathrm{d}\omega = \infty$; and $G_2(s) \notin RH_\infty$ because of $\sup_{\mathrm{Re}(s)\geq 0} \left|\frac{s^2}{s+1}\right| = \infty$.
3. Apparently, $G_3(s) \notin RH_\infty$ because $G(s)$ is not analytic at $s=1$; $G_3(s) \in RL_\infty$ because $G(s)$ is analytic on $j\omega$ axis and satisfies $\sup_{\omega} |G(j\omega)| < \infty$; and $G_3(s) \notin RH_2$ because $G(s)$ is not analytic at $s=1$.

2.2.2 Norms for Signals and Systems

Norm symbolizes the size of a system or a function. For the control system analysis and synthesis, norm offers a direct criterion corresponding to design specifications. The detailed treatment of this topic can be found in books, such as [2, 3]. In this book, the following definitions are listed for easy reference. Note that the signal mentioned below is scalar and measurable, and the system is also scalar and linear time-invariant and causal. The vector (matrix) version of these norms can be found in, e.g., the books mentioned above.

Definition 2.6 The 1-norm of a signal $y(t)$ on $(-\infty,\infty)$ is defined as

$$\|y\|_1 := \int_{-\infty}^{\infty} |y(t)|\,\mathrm{d}t. \tag{2.35}$$

Definition 2.7 The 2-norm of a signal $y(t)$ is defined as

$$\|y\|_2 := \sqrt{\int_{-\infty}^{\infty} y^2(t)\mathrm{d}t}. \tag{2.36}$$

Definition 2.8 The ∞-norm of a signal $y(t)$ is defined as

$$\|y\|_\infty := \sup_t |y(t)|. \tag{2.37}$$

Definition 2.9 The 1-norm of a stable system $G(s)$ is defined as

$$\|G\|_1 := \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|\,\mathrm{d}\omega. \tag{2.38}$$

Definition 2.10 The 2-norm of a stable system $G(s)$ is defined as

$$\|G\|_2 := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2\mathrm{d}\omega}. \tag{2.39}$$

For a state-space described system $G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ defined in (2.52) below, the H_2 norm can be determined by

$$\|G\|_2 = \sqrt{\mathrm{trace}\,(B^{\mathrm{T}} P_{\mathrm{o}} B)} \tag{2.40}$$

where P_{o} is the observability gramian, which will be discussed in Chap. 7.

Definition 2.11 The ∞-norm of a stable system $G(s)$ is defined as

$$\|G\|_\infty := \sup_\omega |G(j\omega)|. \tag{2.41}$$

The $\|G\|_\infty$ equals the distance in the complex plane from the origin to the furthest point on the Nyquist plot of $G(s)$. It also appears as the peak value on the Bode magnitude plot of $G(s)$. The Hankel norm is also a representation of function size [3], especially in the design framework of H_∞ loop shaping. The Hankel norm can be exploited to determine the stability margin. Its definition is given below.

Definition 2.12 Hankel norm is used for determining the residual energy of a system before $t = 0$. For a stable system described as $y(t) = Gu(t)$, the Hankel norm is defined as

$$\|G\|_{\mathrm{H}} = \sqrt{\sup_{u \in L_2(-\infty,0)} \frac{\int_0^{\infty} y^{\mathrm{T}}(t) y(t)}{\int_{-\infty}^{0} u^{\mathrm{T}}(t) u(t) \mathrm{d}t}}. \tag{2.42}$$

This can be determined by

$$\|G\|_{\mathrm{H}} = \sqrt{\lambda_{\max}(P_{\mathrm{c}} P_{\mathrm{o}})}, \tag{2.43}$$

where P_{c} and P_{o} are the controllability gramian and observability gramian matrices, respectively, which will be discussed in Chap. 7.

Example 2.2 Given a linear system $G(s)$ as below, determine its H_2 norm and Hankel norm.

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & 2 \end{bmatrix} x. \tag{2.44}$$

The observability gramian P_{o} and controllability gramian P_{c} are

$$P_{\mathrm{o}} = \begin{bmatrix} \frac{1}{2} & -\frac{2}{3} \\ -\frac{2}{3} & 1 \end{bmatrix} \quad \text{and} \tag{2.45}$$

$$P_{\mathrm{c}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}. \tag{2.46}$$

Hence, one can obtain

$$\|G\|_2 = \sqrt{\operatorname{trace}(B^{\mathrm{T}} P_{\mathrm{o}} B)} = \frac{1}{\sqrt{6}}, \tag{2.47}$$

$$\|G\|_H = \sqrt{\lambda_{\max}(P_{\mathrm{c}} P_{\mathrm{o}})} = \frac{1}{6}. \tag{2.48}$$

2.3 Linear System Theory

The aim of this section is to introduce some basic results in linear system theory [1] that are particularly applicable to the work in the following chapters of this book. The descriptions, properties, and algebras of linear systems facilitate the development of optimal and robust control theory. These concepts preliminarily offer tools for system analysis and synthesis, and construct the main scope of modern control theory and control engineering.

2.3.1 Linear Systems

A finite-dimensional LTI dynamic system can be described by the following equations:

$$\begin{aligned} \dot{x} &= Ax + Bu, x(0) = x_0 \\ y &= Cx + Du, \end{aligned} \tag{2.49}$$

where $\forall\, t \geq 0$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, and $y(t) \in \mathbb{R}^p$ is the output vector. The transfer function from u to y is defined as

$$Y(s) = G(s)U(s), \tag{2.50}$$

where $Y(s)$ and $U(s)$ are the Laplace transform of $y(t)$ and $u(t)$, respectively. It can be shown that

$$G(s) = D + C(sI - A)^{-1}B. \tag{2.51}$$

For simplicity, the state-space realization (A,B,C,D) can be written in a compact form as

$$G(s) \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{2.52}$$

The state response in the time domain is

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau, \tag{2.53}$$

and the output response is

$$y(t) = C\mathrm{e}^{At}x_0 + \int_0^t C\mathrm{e}^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t). \tag{2.54}$$

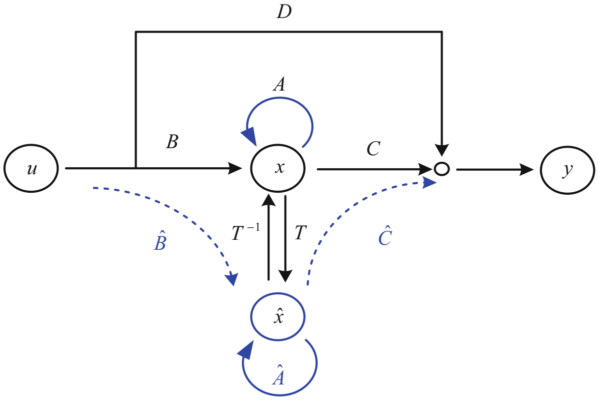

Fig. 2.3 Relationship of state-space similarity transformation

2.3.2 State Similarity Transformation

Different states can be defined to a linear time-invariant system given in (2.49) via $n \times n$ nonsingular matrix T. Let $\breve{x} = Tx$, and then the system can be described by

$$\begin{aligned} \dot{\breve{x}} &= TAT^{-1}\breve{x} + TBu, \breve{x}(0) = \breve{x}_0 = Tx_0 \\ y &= CT^{-1}\breve{x} + Du \end{aligned} \tag{2.55}$$

The transformed system is derived via the state similarity transformation of (T, T^{-1}). One has the same transfer function matrix from the input to the output, though with different state-space model:

$$G(s) \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \overset{s}{=} \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{c|c} \breve{A} & \breve{B} \\ \hline \breve{C} & \breve{D} \end{array}\right], \tag{2.56}$$

where $\breve{A} = TAT^{-1}$, $\breve{B} = TB$, $\breve{C} = CT^{-1}$, and $\breve{D} = D$. The relationship of this transformation is illustrated in Fig. 2.3. The conjugate system $G^{\sim}(s)$ of $G(s)$ is given by

$$G^{\sim}(s) := G^T(-s) \overset{s}{=} \left[\begin{array}{c|c} -A^T & C^T \\ \hline -B^T & D^T \end{array}\right]. \tag{2.57}$$

Finally, if D is invertible, a state-space representation of $G(s)^{-1}$, the inverse of $G(s)$, is given by

$$G(s)^{-1} \overset{s}{=} \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right]. \tag{2.58}$$

2.3.3 *Stability, Controllability, and Observability*

2.3.3.1 Stability

Stability is the most important property of a control system under study. In this section, the concepts of bounded-input-bounded-output (BIBO) stability and asymptotic stability will be discussed.

Definition 2.13 A system is BIBO stable if it generates a bounded output when it is subject to any bounded input.

For a linear system modeled by transfer function $G(s)$, it is called BIBO stable if and only if all the poles of $G(s)$ are in the open left-half plane, i.e., with negative real parts. For instance, given a $G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s-1)(s+2)} \\ 0 & \frac{1}{s+3} \end{bmatrix}$, one can find that the poles of $G(s)$ are $\{-1, 1, -2, -3\}$. Hence, it is not BIBO stable due to that there is a positive real pole $\{1\}$. The following defines the asymptotic stability.

Definition 2.14 A system of (2.49) is called asymptotically stable if, for any given initial state x_0, the state $\|x(t)\| \to 0$, as $t \to \infty$, when $u \equiv 0$.

A necessary and sufficient condition for the system to be asymptotically stable is that the real part of all eigenvalues of A should be negative. The asymptotic stability is also called the *internal stability*, though the term is more often used in a closed-loop system setting. Asymptotic stability implies BIBO stability; however, BIBO stability does not imply the asymptotic stability. That is, asymptotically stable systems must be BIBO stable, but a BIBO stable control system may not be asymptotically stable [1]. The possible discrepancy between BIBO stability and asymptotic stability of a control system arises from whether the underlying system is completely controllable or completely observable. Controllability and observability are introduced next.

2.3.3.2 Controllability

Taking the given system in (2.49), e.g., controllability refers to the ability of the input signal u to transfer the state x from any initial state to any final state in finite time. A system is called *completely controllable* if, for any given initial state x_0 and any final state x_f, there exist a finite time T_f and an input $u(t)$, $0 \le t \le T_f$, which takes $x(0) = x_0$ to $x(T_f) = x_f$. Note that controllability of a system concerns only the matrix pair (A,B), and the state similarity transformation does not affect the controllability.

To verify the controllability and the following observability, the rank test and gramian test are the well-known methods [1]. The following summarizes these schemes.

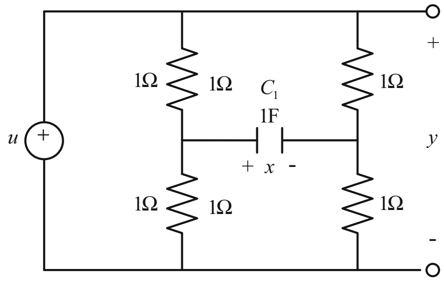

Fig. 2.4 Circuit example on observability

An n _th-order system is completely controllable if any one of the following is true:

1. The controllability matrix $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ is of full rank.
2. The matrix $\begin{bmatrix} \lambda I - A & B \end{bmatrix}$ has full row rank at every eigenvalue of A.
3. The controllability gramian matrix

$$W_c = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau \tag{2.59}$$

   is nonsingular and thus positive definite for every $t > 0$.
4. All the eigenvalues of $A + BF$ can be assigned arbitrarily, where F is an appropriately chosen state feedback matrix and always exists.

A system model in (2.49) is said to be *stabilizable* if there exists a state feedback matrix F such that $A + BF$ is stable (i.e., the state matrix of the feedback system is Hurwitz).

2.3.3.3 Observability

The controllability describes the ability that the input drives the states, of which the dual concept is the observability of a system. Taking the given system in (2.49), e.g., the observability means the extent to which the system state variables are "visible" at the output. A system is called *completely observable* if, by setting the input identical to zero, any initial state $x(0)$ can be uniquely decided by the output $y(t)$, $0 \le t \le T$, for some finite T. For example, Fig. 2.4 shows that the initial voltage across the capacitor, $x(0)$, cannot be determined by the voltage output y. If no input (voltage source) u is applied to the circuit of Fig. 2.4, the initial state (voltage across the capacitor) cannot be deduced from the output y. Note that the observability concerns only the matrix pair (A,C), and the state similarity transformation does not change the observability.

The complete observability of a system can be found by using the rank test or gramian test, which are summarized as follows:

1. The observability matrix $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ is of full rank.
2. The matrix $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ has full column rank at every eigenvalue of A.
3. The observability gramian matrix

$$W_o = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau \tag{2.60}$$

   is nonsingular and thus positive definite for every $t > 0$.
4. All eigenvalues of $A + HC$ can be assigned arbitrarily, where H is an appropriately chosen observer gain matrix and always exists.

A system model in (2.49) is said to be *detectable* if there exists an observer gain matrix F such that $A + HC$ is stable.

2.3.4 Minimal State-Space Realization

For any given LTI system in a state-space model (2.49), an adequately chosen state similarity transfer matrix T can be applied to transform (2.52) into

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{cccc|c} A_{CO} & 0 & A_{13} & 0 & B_{CO} \\ A_{21} & A_{C\bar{O}} & A_{23} & A_{24} & B_{C\bar{O}} \\ 0 & 0 & A_{\bar{C}O} & 0 & 0 \\ 0 & 0 & A_{43} & A_{\bar{C}\bar{O}} & 0 \\ \hline C_{CO} & 0 & C_{\bar{C}O} & 0 & D \end{array}\right]. \tag{2.61}$$

Representation (2.61) is the so-called canonical decomposition form (Kalman canonical decomposition). It can be easily derived that for zero initial states, the transfer function of the system is actually

$$G(s) = D + C(sI - A)^{-1}B = D + C_{CO}(sI - A_{CO})^{-1}B_{CO}, \tag{2.62}$$

which shows that the transfer function only describes the controllable and observable part of the system. Figure 2.5 shows the relation of (2.61) in a block diagram. The dynamics of the uncontrollable, unobservable, or both, if they exist in the system, will not be seen in the input/output relationship (the transfer function). That explains the possible situation of a system being BIBO stable but not asymptotically stable.

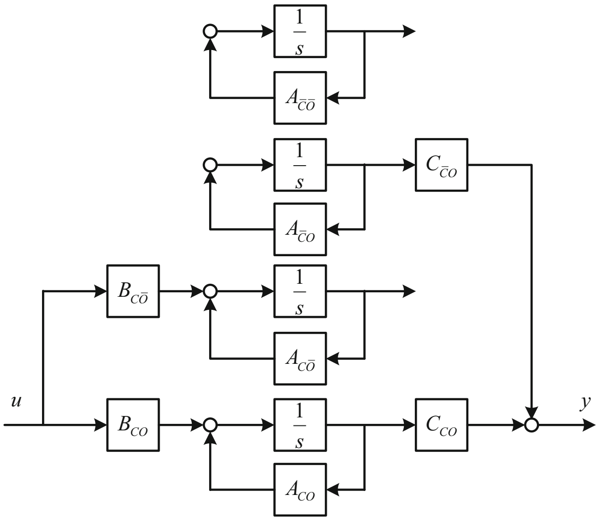

Fig. 2.5 Block diagram of canonical decomposition function

There are many state-space realizations corresponding to the same transfer function. The state-space realization (A,B,C,D) with the least dimensions of the state is called a minimal realization of the transfer function. Minimal realization (A,B,C,D) is always completely controllable and completely observable.

2.3.5 State-Space Algebra

Let state-space realizations of the systems $G_1(s)$ and $G_2(s)$ be given respectively by,

$$\begin{bmatrix} \dot{x}_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} x_1 \\ u_1 \end{bmatrix} \tag{2.63}$$

and

$$\begin{bmatrix} \dot{x}_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} x_2 \\ u_2 \end{bmatrix}. \tag{2.64}$$

Obviously, the system models formed from $G_1(s)$ and $G_2(s)$ could involve the variables from both systems. By augmenting (2.63) and (2.64), one obtains

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I & 0 \\ C_1 & 0 & D_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 \\ \dot{x}_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & A_1 & B_1 \\ 0 & C_1 & D_1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ u_1 \end{bmatrix} \tag{2.65}$$

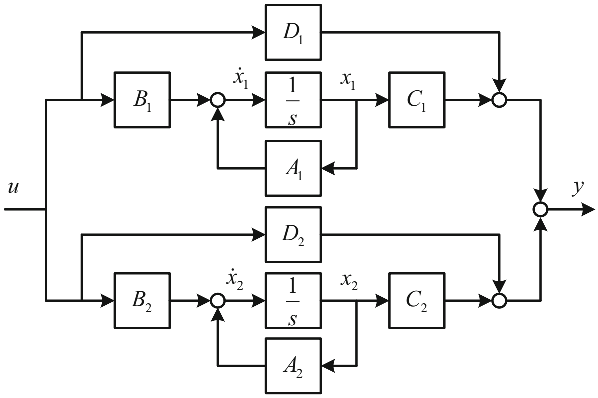

Fig. 2.6 Block diagram of a parallel system

and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & A_2 & B_2 \\ 0 & C_2 & D_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ x_2 \\ u_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_2 & 0 & B_2 \\ 0 & I & 0 \\ C_2 & 0 & D_2 \end{bmatrix} \begin{bmatrix} x_2 \\ \dot{x}_1 \\ u_2 \end{bmatrix}. \tag{2.66}$$

It can be seen in the following that manipulations between two control system models can be realized via the algebra of usual constant matrix operations.

2.3.6 State-Space Formula for Parallel Systems

As shown in Fig. 2.6, let $u_1 = u$ and $u_2 = u$. Since

$$y = y_1 + y_2 = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (D_1 + D_2)\, u, \tag{2.67}$$

a state-space realization of the transfer function from u to $y = y_1 + y_2$ can be found from (2.65) and (2.66) which have the same dimension of the total states as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ y \end{bmatrix} = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D_1 + D_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}, \tag{2.68}$$

i.e.,

$$G_1(s) + G_2(s) \overset{s}{=} \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]. \tag{2.69}$$

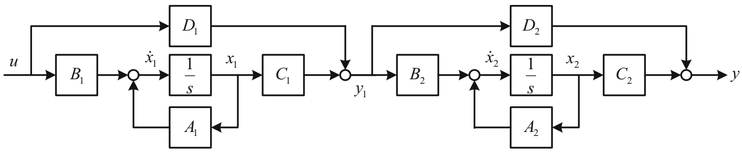

Fig. 2.7 Block diagram of a cascaded system

2.3.7 State-Space Formula for Cascaded Systems

As shown in Fig. 2.7, let $u_1 = u$, $u_2 = y_1$, and $y = y_2$. Then, one has a state-space realization of the transfer function from (2.65) and (2.66) by matrix multiplication as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ y \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & A_2 & B_2 \\ 0 & C_2 & D_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ x_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & A_2 & B_2 \\ 0 & C_2 & D_2 \end{bmatrix} \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I & 0 \\ C_1 & 0 & D_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}$$
$$= \begin{bmatrix} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ D_2C_1 & C_2 & D_2D_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}, \tag{2.70}$$

or equivalently

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \\ y \end{bmatrix} = \begin{bmatrix} A_2 & 0 & B_2 \\ 0 & I & 0 \\ C_2 & 0 & D_2 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & A_1 & B_1 \\ 0 & C_1 & D_1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ u \end{bmatrix} = \begin{bmatrix} A_2 & B_2C_1 & B_2D_1 \\ 0 & A_1 & B_1 \\ C_2 & D_2C_1 & D_2D_1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ u \end{bmatrix}. \tag{2.71}$$

Hence,

$$G_2(s)G_1(s) \overset{s}{=} \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ \hline D_2C_1 & C_2 & D_2D_1 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & B_2C_1 & B_2D_1 \\ 0 & A_1 & B_1 \\ \hline C_2 & D_2C_1 & D_2D_1 \end{array}\right]. \tag{2.72}$$

2.3.8 State-Space Formula for Similarity Transformation

Define a new state variable vector $\breve{x} = Tx$. Then one has

$$\dot{\breve{x}} = T\dot{x}, \tag{2.73}$$

$$x = T^{-1}\breve{x} \tag{2.74}$$

From (2.49), (2.73) and (2.74),

$$\dot{\breve{x}} = T\dot{x} = TAx + TBu = \left(TAT^{-1}\right)\breve{x} + (TB)u \tag{2.75}$$

and

$$\dot{\breve{x}} = y = Cx + Du = \left(CT^{-1}\right)\breve{x} + Du. \tag{2.76}$$

This implies

$$G(s) \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]. \tag{2.77}$$

Consider the specific case that $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, and $T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ $\left(\text{i.e., } T^{-1} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}\right)$ which is helpful to characterize the minimum realization of the state-space solutions later. Then,

$$T\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{bmatrix} = \begin{bmatrix} A_{11} + XA_{21} & A_{12} + XA_{22} & B_1 + XB_2 \\ A_{21} & A_{22} & B_2 \end{bmatrix} \tag{2.78}$$

and

$$\begin{aligned} \begin{bmatrix} TA \\ C \end{bmatrix} T^{-1} &= \begin{bmatrix} A_{11} + XA_{21} & A_{12} + XA_{22} \\ A_{21} & A_{22} \\ C_1 & C_2 \end{bmatrix}\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} A_{11} + XA_{21} & -A_{11}X + XA_{21}X + A_{12} + XA_{22} \\ A_{21} & -A_{21}X + A_{22} \\ C_1 & -C_1X + C_2 \end{bmatrix}. \end{aligned} \tag{2.79}$$

This is equivalent to the matrix manipulations of

$$\begin{aligned} & X \circlearrowleft \overset{\curvearrowright\, -X}{\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]} \\ &= \left[\begin{array}{cc|c} A_{11} + XA_{21} & -A_{11}X + XA_{21}X + A_{12} + XA_{22} & B_1 + XB_2 \\ A_{21} & -A_{21}X + A_{22} & B_2 \\ \hline C_1 & -C_1X + C_2 & D \end{array}\right] = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]. \end{aligned} \tag{2.80}$$

2.4 Linear Fractional Transformations and Chain Scattering-Matrix Description

Consider a general feedback control framework shown in Fig. 2.8, where P denotes the interconnection system of the controlled plant, namely, the standard control (or compensation) configuration (SCC) [10]. The closed-loop transfer function from w to z in Fig. 2.8 is given by

$$\mathrm{LFT_l}\,(P, K) = \mathrm{LFT_l}\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, K\right) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}, \tag{2.81}$$

where LFT stands for the linear fractional transformation and the subscript “l” stands for “lower.” Different from the LFT, the chain scattering-matrix description (CSD) developed in the network circuits provides a straightforward interconnection in a cascaded way. The CSD transforms a LFT into a two-port network connection. Thus, many known theories which have been developed for a two-port network can then be used. The definition of CSD is briefly introduced below, while the details on background, properties, and use of CSD will be described in Chaps. 3, 4, and 5. Figure 2.9 shows the right and left CSD representations.

Define right and left CSD transformations with G and K denoted by $\mathrm{CSD_r}(G,K)$ and by $\mathrm{CSD_l}\left(\tilde{G}, K\right)$, respectively [9], as

$$\mathrm{CSD_r}\,(G, K) = \mathrm{CSD_r}\left(\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, K\right) := (G_{12} + G_{11}K)\,(G_{22} + G_{21}K)^{-1} \tag{2.82}$$

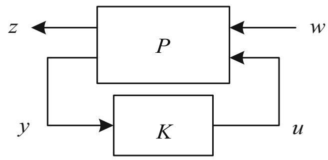

Fig. 2.8 Linear fractional transformation

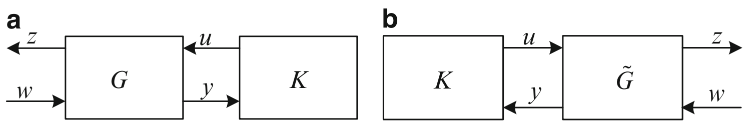

Fig. 2.9 Right and left CSD (**a**) Right CSD (**b**) Left CSD

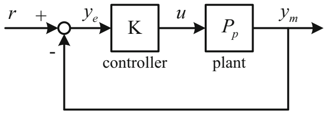

Fig. 2.10 Unity feedback control system

and

$$\mathrm{CSD_l}\left(\tilde{G}, K\right) = \mathrm{CSD_l}\left(\begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{bmatrix}, K\right) := -\left(\tilde{G}_{11} - K\tilde{G}_{21}\right)^{-1}\left(\tilde{G}_{12} - K\tilde{G}_{22}\right), \tag{2.83}$$

where G_{22} and $\tilde{G}_{11}$ are square and invertible. Note that, if P_{21} is invertible, the SCC matrix P can be transformed to a right CSD as

$$G = \begin{bmatrix} P_{12} - P_{11}P_{21}{}^{-1}P_{22} & P_{11}P_{21}{}^{-1} \\ -P_{21}{}^{-1}P_{22} & P_{21}{}^{-1} \end{bmatrix}. \tag{2.84}$$

Also, if P_{12} is invertible, the SCC matrix P can be transformed to a left CSD as

$$\tilde{G} = \begin{bmatrix} P_{12}{}^{-1} & P_{12}{}^{-1}P_{11} \\ P_{22}P_{12}{}^{-1} & P_{21} - P_{22}P_{12}{}^{-1}P_{11} \end{bmatrix}. \tag{2.85}$$

Example 2.3 Consider the unity feedback control system in Fig. 2.10, where P_p is a SISO-controlled plant. Find its corresponding $\mathrm{LFT_l}$ and CSD representations.

Let $z = \begin{pmatrix} y_e \\ u \end{pmatrix}$, $w = r$, and $y = y_e$.

From the unity feedback control system, by definition, as $u = 0$, one has $y_m = 0$; hence, $r = y_e$ from $r - P_p u = y_e$ so that $P_{11} = \dfrac{\begin{pmatrix} y_e \\ u \end{pmatrix}}{r}\Big|_{u=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $P_{21} = \frac{y_e}{r}\big|_{u=0} = 1$. Similarly, as $r = 0$, one can also obtain

$$P_{12} = \frac{\begin{pmatrix} y_e \\ u \end{pmatrix}}{u}\Big|_{r=0} = \begin{pmatrix} -P_p \\ 1 \end{pmatrix} \text{ and } P_{22} = \frac{y_e}{u}\Big|_{r=0} = -P_p.$$

The closed-loop transfer function from r to $z = \begin{pmatrix} y_e \\ u \end{pmatrix}$ is presented as below (Fig. 2.11).

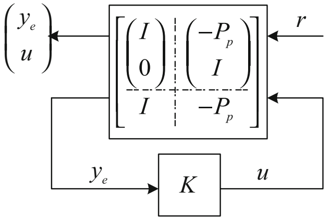

Fig. 2.11 LFT of closed-loop transfer function form

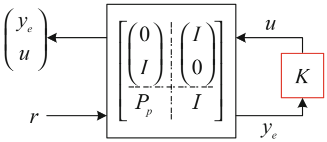

Fig. 2.12 Right CSD of closed-loop transfer function form

From

$$z = \mathrm{LFT}_l\,(P, K)\,w = \left[P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\right]w,$$

one has

$$\begin{pmatrix} y_e \\ u \end{pmatrix} = \begin{pmatrix} 1 \\ K \end{pmatrix}(1 + P_P K)^{-1} r.$$

From the control block diagram, as $y_e = 0$, one has, by the definition of the right CSD, $G_{11} = \dfrac{\begin{pmatrix} y_e \\ u \end{pmatrix}}{u}\Big|_{y_e=0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $G_{21} = \frac{r}{u}\big|_{y_e=0} = P_p$, since $r - P_p u = y_e = 0$. As $u = 0$, one then has

$$G_{12} = \frac{\begin{pmatrix} y_e \\ u \end{pmatrix}}{y_e}\Big|_{u=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and from } r - P_p u = y_e.$$

Equivalently, the closed-loop transfer function from r to $z = \begin{pmatrix} y_e \\ u \end{pmatrix}$ can be represented by Fig. 2.12.

From $z = \mathrm{CSD_r}(G,K)\,w = (G_{11}K + G_{12})(G_{21}K + G_{22})^{-1}w$, one has

$$\begin{pmatrix} y_e \\ u \end{pmatrix} = \begin{pmatrix} 1 \\ K \end{pmatrix}(1 + P_p K)^{-1} r.$$

This concludes that

$$z = \mathrm{LFT_l}\,(P, K)\,w = \mathrm{CSD_r}\,(G, K)\,w.$$

Exercises

1. Prove that all the eigenvalues $\lambda(H)$ of a Hamiltonian matrix H are symmetric to the $j\omega$-axis.
2. Determine the rank of $A = \begin{bmatrix} 1 & 2 & 5 & 1 \\ 2 & 4 & -1 & 2 \\ 1 & 2 & 1 & 9 \end{bmatrix}$.
3. Let $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Utilize the least square approach to solve $Ax = b$ where $A = QR$.
4. Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}.$$

   Find the response of $x_1(t)$ and $x_2(t)$.
5. Sketch the state trajectories of the following system in the (x_1, x_2, x_3) plane for $x_0 = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}$ and input $u(t) = 0$. Determine the controllability of the above system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 5 & 1 \end{bmatrix} x.$$

6. The transfer function of a linear system is given by

$$\frac{Y(s)}{U(s)} = \frac{s + a}{s^3 + 7s^2 + 14s + 8}.$$

   (a) Determine the values of a when the system is not completely controllable or not completely observable.
   (b) Define the state variables and derive the state-space model of which one of the states is unobservable.
   (c) Define the state variables and derive the state-space model of which one of the states is uncontrollable.

7. The state-space model of a third-order system is shown below:

$$\dot{x}_1 = 2x_1 + 3x_2 + 3x_3 + u$$
$$\dot{x}_2 = -2x_1 - 3x_2 - 2u$$
$$\dot{x}_3 = -2x_1 - 2x_2 - 5x_3 + 2u$$
$$y = 7x_1 + 6x_2 + 4x_3.$$

Use state similarity transformation to decouple the state-space model and explain the observability and controllability for each of the subsystems.

8. Consider the following systems and decide in which function space they belong to.

(a) $\dfrac{(s+1)}{(s+2)(s+4)}$

(b) $\dfrac{2s-1}{(s+1)(s+3)}$

(c) $\dfrac{1}{s-2}$

9. Consider the linear system below. Determine its H_2 norm, H_∞ norm, and Hankel norm.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

References

1. Chen CT (2009) Linear system theory and design. Oxford University Press, New York
2. Doyle JC, Francis B, Tannenbaum A (1992) Feedback control theory. Macmillan Publishing Company, New York
3. Francis BA (1987) A course in H_∞ control theory, vol 88, Lecture notes in control and information sciences. Springer, Berlin
4. Golub GH, Van Loan CF (1989) Matrix computations. The Johns Hopkins University Press, London
5. Green M, Limebeer DJN (1995) Linear robust control. Prentice Hall, Englewood Cliffs
6. Helton JW, Merino O (1998) Classical control using H_∞ methods. SIAM, Philadelphia
7. Rudin W (1973) Functional analysis. McGraw-Hill, New York
8. Strang G (2004) Linear algebra and its applications, 4th edn. Academic, New York
9. Tsai MC, Tsai CS (1993) A chain scattering matrix description approach to H_∞ control. IEEE Trans Autom Control 38:1416–1421
10. Zhou K, Doyle JC, Glover K (1996) Robust and optimal control. Prentice Hall, Upper Saddle River

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