Chapter 2
Background on Sampling of Signals

In this chapter, a background on sampling and Fourier analysis of signals is provided. The chapter provides a brief review of the key concepts to establish notation. Some readers may be familiar with this background material. In this case, they can proceed immediately to Chap. 3. Other readers may wish to quickly read this chapter before proceeding.

2.1 Fourier Analysis

The Fourier transform pair for a continuous-time signal $y(t)$ is

Continuous-time Fourier transform

$$\mathcal{F}\big\{f(t)\big\} = F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{2.1}$$

$$\mathcal{F}^{-1}\big\{F(j\omega)\big\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega \tag{2.2}$$

Fourier transforms allow signals to be studied in the *frequency* domain. The Fourier transform will turn out to be a very useful tool when connecting continuous-time signals with the associated sequence of samples.

Detailed issues relating to the existence, uniqueness, and other properties of Fourier transforms will not be discussed here. The reader is referred to the literature cited at the end of the chapter. Table 2.1 gives a summary of commonly used Fourier transform pairs, and Table 2.2 presents the most important properties. Several additional properties of the Fourier transform are summarised below.

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Table 2.1 Fourier transform table

$f(t)\ \forall t \in \mathbb{R}$	$\mathcal{F}\{f(t)\}$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$\mu(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\mu(t) - \mu(t - t_o)$	$\frac{1-e^{-j\omega t_o}}{j\omega}$
$e^{\alpha t}\mu(t),\ \Re\{\alpha\} < 0$	$\frac{1}{j\omega-\alpha}$
$te^{\alpha t}\mu(t),\ \Re\{\alpha\} < 0$	$\frac{1}{(j\omega-\alpha)^2}$
$e^{-\alpha\|t\|},\ \alpha \in \mathbb{R}^+$	$\frac{2\alpha}{\omega^2+\alpha^2}$
$\cos(\omega_o t)$	$\pi(\delta(\omega - \omega_o) + \delta(\omega - \omega_o))$
$\sin(\omega_o t)$	$j\pi(\delta(\omega + \omega_o) - \delta(\omega - \omega_o))$
$\cos(\omega_o t)\mu(t)$	$\pi(\delta(\omega - \omega_o) + \delta(\omega - \omega_o)) + \frac{j\omega}{-\omega^2+\omega_o^2}$
$\sin(\omega_o t)\mu(t)$	$j\pi(\delta(\omega + \omega_o) - \delta(\omega - \omega_o)) + \frac{\omega_o}{-\omega^2+\omega_o^2}$
$e^{-\alpha t}\cos(\omega_o t)\mu(t),\ \alpha \in \mathbb{R}^+$	$\frac{j\omega+\alpha}{(j\omega+\alpha)^2+\omega_o^2}$
$e^{-\alpha t}\sin(\omega_o t)\mu(t),\ \alpha \in \mathbb{R}^+$	$\frac{\omega_o}{(j\omega+\alpha)^2+\omega_o^2}$

Table 2.2 Basic Fourier transform properties

$f(t)$	$\mathcal{F}\{f(t)\}$	Description
$\sum_{i=1}^{l} a_i f_i(t)$	$\sum_{i=1}^{l} a_i F_i(j\omega)$	Linearity
$\frac{dy(t)}{dt}$	$j\omega Y(j\omega)$	Derivative law
$\frac{d^k y(t)}{dt^k}$	$(j\omega)^k Y(j\omega)$	High order derivative
$\int_{-\infty}^{t} y(\tau)\,d\tau$	$\frac{1}{j\omega}Y(j\omega) + \pi Y(0)\delta(\omega)$	Integral law
$y(t-\tau)$	$e^{-j\omega\tau}Y(j\omega)$	Delay
$y(at)$	$\frac{1}{\|a\|}Y(j\frac{\omega}{a})$	Time scaling
$y(-t)$	$Y(-j\omega)$	Time reversal
$\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,d\tau$	$F_1(j\omega)F_2(j\omega)$	Convolution
$y(t)\cos(\omega_o t)$	$\frac{1}{2}\{Y(j\omega - j\omega_o) + Y(j\omega + j\omega_o)\}$	Modulation (cosine)
$y(t)\sin(\omega_o t)$	$\frac{1}{j2}\{Y(j\omega - j\omega_o) - Y(j\omega + j\omega_o)\}$	Modulation (sine)
$F(t)$	$2\pi f(-j\omega)$	Symmetry
$f_1(t) f_2(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\zeta)F_2(j\omega - j\zeta)\,d\zeta$	Time-domain product
$e^{at} f_1(t)$	$F_1(j\omega - a)$	Frequency shift

Lemma 2.1 *The Fourier transform of a constant function $f(t) = 1$ is given by*

$$\mathcal{F}\{1\} = 2\pi\delta(\omega) \tag{2.3}$$

where $\delta(\omega)$ is the Dirac delta function.

Proof The definition of the inverse Fourier transform is used in Eq. (2.2) to yield

$$\mathcal{F}^{-1}\left\{2\pi\delta(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)\,d\omega = 1 \tag{2.4}$$

□

Corollary 2.2 *The Fourier transform of a complex exponential $f(t) = e^{j\omega_o t}$ is given by*

$$\mathcal{F}\left\{e^{j\omega_o t}\right\} = 2\pi\delta(\omega - \omega_o) \tag{2.5}$$

Proof This property follows directly from Lemma 2.1 and the frequency shifting property in Table 2.2. □

The discrete-time Fourier transform (DTFT) and its associated inverse transform are defined as

Discrete-time Fourier transform

$$\mathcal{F}_d\{f_k\} = F_d\left(e^{j\omega\Delta}\right) = \Delta\sum_{k=-\infty}^{\infty} f_k e^{-j\omega k\Delta} \tag{2.6}$$

$$\mathcal{F}_d^{-1}\left\{F_d\left(e^{j\omega\Delta}\right)\right\} = f_k = \frac{1}{2\pi}\int_{-\pi/\Delta}^{\pi/\Delta} F_d\left(e^{j\omega\Delta}\right)e^{j\omega k\Delta}\,d\omega \tag{2.7}$$

Note that the sample period Δ appears explicitly in the above expressions. The usual DTFT sets $\Delta = 1$. In this case, the transform is the same if the sequence $\{y_k\}$ is obtained from samples of a continuous-time signal or is an inherently discrete-time signal. The period Δ has been explicitly included in the definition, as this facilitates the understanding of the connection between discrete-time and continuous-time results, for example, when y_k arises from sampling a continuous signal $y(t)$, i.e., when $y_k = y(k\Delta)$. Indeed, it is obvious that (2.6) is an approximation of the integral in (2.1).

2.2 Sampling of Continuous-Time Signals and Continuous Transforms

The connections between continuous and discrete transforms are next explored in more detail. The following result is first established:

Lemma 2.3 *Defining $s_\Delta(t)$ to be a train of impulses, i.e.,*

$$s_\Delta(t) = \Delta \sum_{k=-\infty}^{\infty} \delta(t - k\Delta) \tag{2.8}$$

the associated Fourier transform $\mathcal{F}\{s_\Delta(t)\} = S_\Delta(j\omega)$ is given by

$$S_\Delta(j\omega) = 2\pi \sum_{\ell=-\infty}^{\infty} \delta(\omega - \omega_\ell) \tag{2.9}$$

where $\omega_\ell = \frac{2\pi}{\Delta}\ell$.

Proof Notice that $s_\Delta(t)$ can also be expressed as

$$s_\Delta(t) = \sum_{\ell=-\infty}^{\infty} e^{-j\omega_\ell t} \tag{2.10}$$

where $\omega_\ell = \frac{2\pi}{\Delta}\ell$. This can be seen, for example, by noting that $s_\Delta(t)$ is a periodic signal with period Δ and hence has a Fourier series

$$s_\Delta(t) = \sum_{\ell=-\infty}^{\infty} C_\ell \, e^{-j\omega_\ell t} \tag{2.11}$$

where

$$C_\ell = \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} s_\Delta(t) \, e^{j\omega_\ell t} \, dt = \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \Delta\delta(t) \, e^{j\omega_\ell t} \, dt = 1 \tag{2.12}$$

for all $\ell \in \mathbb{Z}$.

The associated Fourier transform is then obtained by applying the linearity property and Corollary 2.2 to (2.11), yielding

$$S_\Delta(j\omega) = \mathcal{F}\big\{s_\Delta(t)\big\} = 2\pi \sum_{\ell=-\infty}^{\infty} \delta(\omega - \omega_\ell) \tag{2.13}$$

□

Using Lemma 2.3, the (continuous-time) Fourier transform of a signal $\{y(t)\}$, defined for $t \in (-\infty, \infty)$, can be related to the discrete-time Fourier transform $Y_d(e^{j\omega\Delta})$ of the sequence of samples, $\{y_k\}$, where

$$y_k = y(k\Delta); \quad k \in \mathbb{Z} \tag{2.14}$$

Notice that $\{y_k\}$ defined in (2.14) is a sequence of numbers and, therefore, it does not have a *continuous* Fourier transform. However, it does have a DTFT as defined in (2.6)–(2.7). In order to associate a continuous Fourier transform with the sequence $\{y_k\}$, energy is added to the samples by defining the following *instrumental* signal, where each sample is turned into a scaled impulse:

$$y_\Delta(t) = \Delta \sum_{k=-\infty}^{\infty} y_k \delta(t - k\Delta) \tag{2.15}$$

Notice that

$$y_\Delta(t) = y(t) s_\Delta(t) \tag{2.16}$$

where $s_\Delta(t)$ is defined in (2.8). The signal $y_\Delta(t)$ in (2.15) is not usually considered in pure discrete-time analysis. However, it is useful when connecting continuous- and discrete-time Fourier transforms. The following result can now be established:

Lemma 2.4 *Consider a sequence $\{y_k\}$ arising from sampling a continuous-time signal $\{y_k\}$. Also consider $y_\Delta(t)$ as defined in (2.15). Then*

$$\mathcal{F}\big\{y_\Delta(t)\big\} = \mathcal{F}_d\{y_k\} \tag{2.17}$$

Proof Using the definition of the continuous Fourier transform as in (2.1), it follows that

$$\begin{aligned} \mathcal{F}\big\{y_\Delta(t)\big\} &= \int_{-\infty}^{\infty} \Delta \sum_{k=-\infty}^{\infty} y_k \delta(t - k\Delta) e^{-j\omega t}\, dt \\ &= \Delta \sum_{k=-\infty}^{\infty} y_k \int_{-\infty}^{\infty} \delta(t - k\Delta) e^{-j\omega t}\, dt = \Delta \sum_{k=-\infty}^{\infty} y_k e^{-j\omega k\Delta} \end{aligned} \tag{2.18}$$

which from (2.6) is $\mathcal{F}_d\{y_k\}$. □

In words, the above result establishes that the discrete-time Fourier transform of a sequence is equal to the continuous-time Fourier transform of the associated instrumental signal.

Note that the scaling by the sampling period Δ is important in obtaining this elegant result. If Δ is not specified, then one can arbitrarily set $\Delta = 1$, but this choice will then compact the (discrete) frequency range to $(-\pi, \pi)$.

Lemma 2.4 establishes an equivalence between frequency-domain analysis based on the DTFT of the discrete-time sequence of samples $\{y_k\}$ and frequency-domain analysis based on the Fourier transform of the continuous-time (instrumental) signal $y_\Delta(t)$. The result leads to the following key relationship between the Fourier transform of $y_\Delta(t)$ and the Fourier transform of the original signal $y(t)$:

Lemma 2.5 *Consider a signal $y(t)$, the associated sequence of samples $\{y_k\}$ defined in* (2.14)*, and the instrumental signal $y_\Delta(t)$ defined in* (2.15)*. Then*

$$Y_\Delta(j\omega) = \sum_{\ell=-\infty}^{\infty} Y\left(j\omega - j\frac{2\pi}{\Delta}\ell\right) \tag{2.19}$$

where $Y(j\omega) = \mathcal{F}\{y(t)\}$ and $Y_\Delta(j\omega) = \mathcal{F}\{y_\Delta(t)\} = \mathcal{F}_d\{y_k\}$.

Proof Using (2.16), the Fourier transform of $y_\Delta(t)$ can be obtained using the time-domain product property in Table 2.2. This yields

$$\begin{aligned}
\mathcal{F}\big\{y(t)\,s_\Delta(t)\big\} &= \frac{1}{2\pi} Y(j\omega) * S_\Delta(j\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\zeta) S_\Delta(j\omega - j\zeta)\, d\zeta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\zeta) 2\pi \sum_{\ell=-\infty}^{\infty} \delta\left(\omega - \zeta - \ell\frac{2\pi}{\Delta}\right) d\zeta \\
&= \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} Y(j\zeta)\delta\left(\omega - \zeta - \ell\frac{2\pi}{\Delta}\right) d\zeta \\
&= \sum_{\ell=-\infty}^{\infty} Y\left(j\omega + j\ell\frac{2\pi}{\Delta}\right) \qquad (2.20)
\end{aligned}$$

which establishes (2.19). □

Equation (2.19) describes the *aliasing* effect, which is a cornerstone result in the analysis of sampled signals. Here it is presented in the context of sampling of

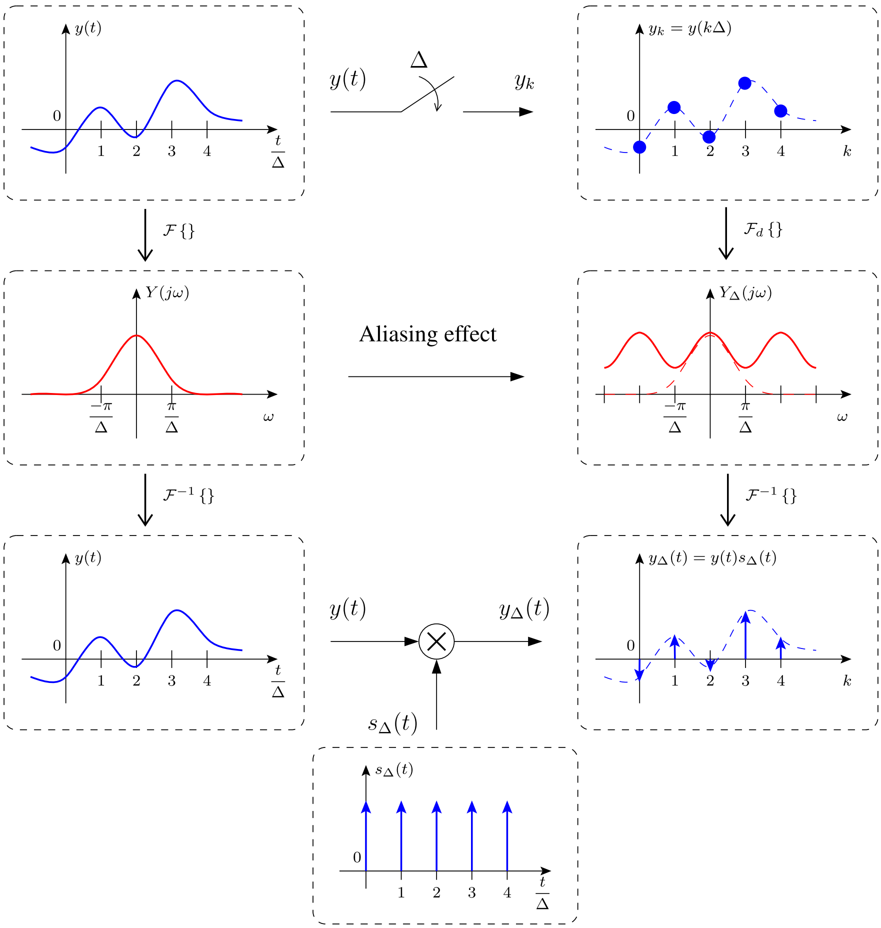


Fig. 2.1 Sampling of a continuous-time signal

deterministic signals. However, it also holds for stochastic signals, as will be shown in Chap. 11.

The aliasing effect is represented in Fig. 2.1. The figure illustrates the fact that the sampled spectrum $Y_\Delta(j\omega)$ can be either obtained as the DTFT of the sequence of samples $\{y_k\}$ or as the Fourier transform of the auxiliary signal $y_\Delta(t)$.

The result Lemma 2.5 is often stated as:

Sampling in the time domain induces folding in the frequency domain.

By virtue of the symmetry inherent in the definition of the Fourier transform and the inverse Fourier transform, the converse is also true. In particular, consider a continuous-time signal, $y(t)$, having Fourier transform $Y(j\omega)$. If the associated Fourier transform, $Y(j\omega)$, is sampled to produce a sequence of equidistant frequency-domain samples $\{Y_\ell = Y(j\ell\frac{2\pi}{\Delta})\}$, at frequency-domain spacing $\omega_s = \frac{2\pi}{\Delta}$, then an associated instrumental form $\{Y_\ell\}$ can be defined via

$$Y_\Delta(j\omega) = 2\pi \sum_{\ell=-\infty}^{\infty} Y_\ell \, \delta\left(\omega - \ell\frac{2\pi}{\Delta}\right) \tag{2.21}$$

The following result is obtained.

Lemma 2.6 *Consider a Fourier transform $Y(j\omega)$, the sequence of (frequency-domain) samples $\{Y_\ell = Y(j\ell\frac{2\pi}{\Delta})\}$, and the instrumental transform defined in (2.21). Then*

$$\mathcal{F}^{-1}\big\{Y_\Delta(j\omega)\big\} = \Delta \sum_{\ell=-\infty}^{\infty} y(t+\ell\Delta) \tag{2.22}$$

where $y(t) = \mathcal{F}^{-1}\{Y(j\omega)\}$.

Proof Notice that

$$Y_\Delta(j\omega) = Y(j\omega)S_\Delta(j\omega) \tag{2.23}$$

where $S_\Delta(j\omega)$ is defined in (2.9). Hence, from the convolution property in Table 2.2, it follows that

$$\begin{aligned}\mathcal{F}^{-1}\big\{Y_\Delta(j\omega)\big\} &= \int_{-\infty}^{\infty} y(t)s_\Delta(t-\tau)\,d\tau \\ &= \Delta \int_{-\infty}^{\infty} y(t) \sum_{\ell=-\infty}^{\infty} \delta(t-\tau-\ell\Delta)\,d\tau = \Delta \sum_{\ell=-\infty}^{\infty} y(t+\ell\Delta)\end{aligned} \tag{2.24}$$

□

The result Lemma 2.6 can be stated as

Sampling in the frequency domain induces folding in the time domain.

2.3 Signal Reconstruction from Samples

In this section, the conditions under which a continuous-time signal can be perfectly reconstructed from the sequence of associated samples are discussed. A special case of interest is where the continuous-time signal has a Fourier transform which is strictly limited to the range $\omega \in (\frac{-\pi}{\Delta}, \frac{\pi}{\Delta})$. Such a signal is said to be a *band-limited signal*.

For this kind of signal, it follows from Lemma 2.5 that the Fourier transforms of $y_\Delta(t)$ and $y(t)$ are identical on the frequency range $(\frac{-\pi}{\Delta}, \frac{\pi}{\Delta})$. It is then a straightforward matter to recover the original signal $y(t)$ from the sequence of samples $\{y_k = y(k\Delta)\}$. Note that the Fourier transform of $\{y_k\}$ is periodic (in the frequency domain). Hence, to recover the original spectrum, it suffices to extract the part of the Fourier transform of $y_\Delta(t)$ lying in the range $(\frac{-\pi}{\Delta}, \frac{\pi}{\Delta})$.

To convert this idea into a reconstruction formula, the following result is first established:

Lemma 2.7 *Consider an ideal low-pass filter defined as*

$$H_{LP}(j\omega) = \begin{cases} 1, & \omega \in (-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}) \\ 0, & \omega \notin (-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}) \end{cases} \tag{2.25}$$

Then, the (*non-causal*) *impulse response of this filter is given by*

$$h_{LP}(t) = \frac{\sin(\frac{\pi}{\Delta}t)}{\pi t} \tag{2.26}$$

Proof The result follows by applying the inverse Fourier transform

$$\begin{aligned} h_{LP}(t) &= \mathcal{F}^{-1}\big\{H_{LP}(j\omega)\big\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{LP}(j\omega)e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{\Delta}}^{\frac{\pi}{\Delta}} e^{j\omega t}\,d\omega = \frac{1}{2\pi}\left[\frac{e^{j\frac{\pi}{\Delta}t} - e^{-j\frac{\pi}{\Delta}t}}{jt}\right] = \frac{\sin(\frac{\pi}{\Delta}t)}{\pi t} \end{aligned} \tag{2.27}$$

□

The ideal low-pass filter impulse response (2.26) can be conveniently expressed using the sinc function as

$$h_{LP}(t) = \frac{1}{\Delta}\operatorname{sinc}\left(\frac{\pi}{\Delta}t\right) \tag{2.28}$$

where $\operatorname{sinc}(x) = \sin(x)/x$. The following result can now be established.

Lemma 2.8 (Nyquist-Shannon Reconstruction Theorem) *Let* $\{y(t)\}$ *be a continuous-time signal which is strictly band-limited to* $(\frac{-\pi}{\Delta}, \frac{\pi}{\Delta})$. *Let* $\{y_k = y(k\Delta)\}$ *be the sequence of samples of* $y(t)$ *with sample period* Δ. *Then, the signal can be reconstructed from its samples as follows*:

$$y(t) = \sum_{k=-\infty}^{\infty} y_k \frac{\sin(\omega_N(t - k\Delta))}{\omega_N(t - k\Delta)} \tag{2.29}$$

where $\omega_N = \frac{\pi}{\Delta}$.

Proof Since $y(t)$ is a band-limited signal, its spectrum is preserved if it is passed through the ideal low-pass filter (2.25). Then

$$Y(j\omega) = H_{LP}(j\omega)Y_d\left(e^{j\omega\Delta}\right) = H_{LP}(j\omega)Y_\Delta(j\omega) \tag{2.30}$$

where $H_{LP}(j\omega)$ is the filter frequency response defined in (2.25). Applying the inverse Fourier transform and using the property that a product in the frequency domain translates to convolution in the time domain leads to:

$$\begin{aligned} y(t) = h_{LP}(t) * y_\Delta(t) &= \int_{-\infty}^{\infty} h_{LP}(t-\sigma)y_\Delta(\sigma)\,d\sigma \\ &= \int_{-\infty}^{\infty} \frac{\sin(\frac{\pi}{\Delta}(t-\sigma))}{\pi(t-\sigma)} \Delta \sum_{k=-\infty}^{\infty} y_k \delta(\sigma - k\Delta)\,d\sigma \\ &= \sum_{k=-\infty}^{\infty} y_k \int_{-\infty}^{\infty} \frac{\sin(\frac{\pi}{\Delta}(t-\sigma))}{\frac{\pi}{\Delta}(t-\sigma)} \delta(\sigma - k\Delta) d\sigma \\ &= \sum_{k=-\infty}^{\infty} y_k \frac{\sin(\frac{\pi}{\Delta}(t-k\Delta))}{\frac{\pi}{\Delta}(t-k\Delta)} \end{aligned} \tag{2.31}$$

□

The frequency $\omega_N = \frac{\pi}{\Delta}$ which appears in Lemma 2.3 is called the Nyquist frequency and corresponds to one half of the sampling frequency $\omega_s = \frac{2\pi}{\Delta}$. Figure 2.2 shows the instantaneous sampling of a band-limited signal. Figure 2.3 illustrates the reconstruction of a signal as the sum of *sinc* functions according to Lemma 2.8.

Lemma 2.8 establishes the *ideal* conditions under which a continuous-time signal can be perfectly reconstructed from its sequence of discrete samples. Note that the result requires the signal to be band-limited and the sampling frequency to be at least twice the highest frequency component of the signal. Additionally, the formula given in (2.29) implies a *non-causal* reconstruction based on *all* the samples of

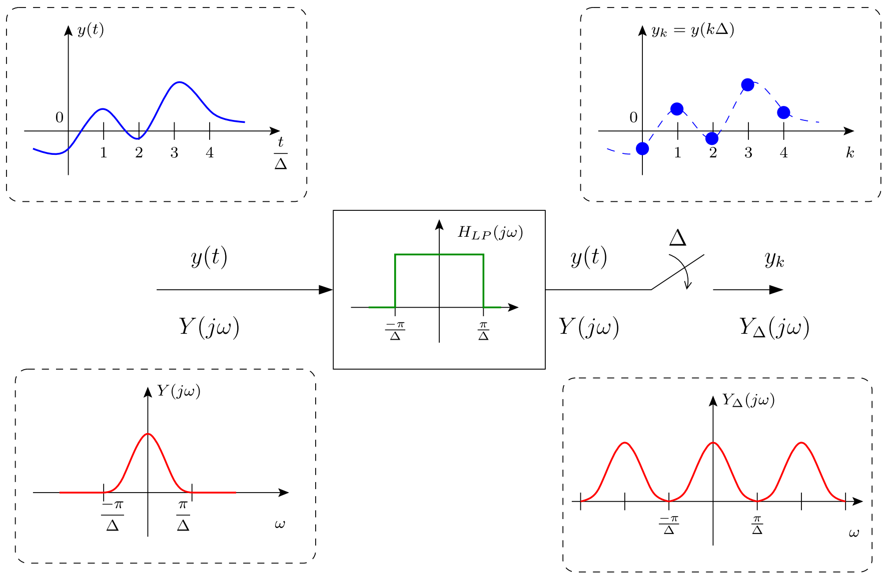
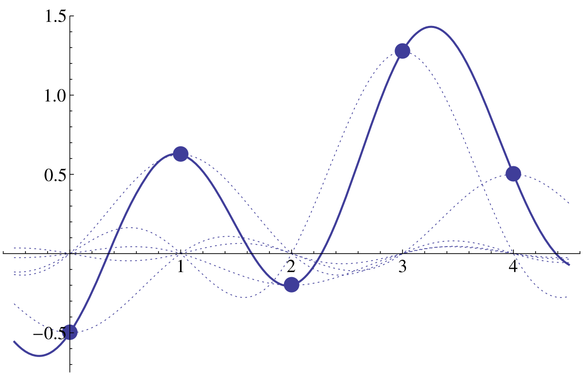


Fig. 2.2 Instantaneous sampling of a band-limited signal

Fig. 2.3 Reconstruction of a signal from samples using the *sinc* function



the signal, since k goes from $-\infty$ to ∞. This kind of reconstruction cannot be performed in practice, but it sets the minimal requirements to perfectly reconstruct the signal. Practical reconstruction strategies use, for example, simple interpolation or extrapolation techniques to approximate $y(t)$ between the samples $\{y_k = y(k\Delta)\}$.

2.4 Anti-aliasing Filters

The material in Sects. 2.2 and 2.3 provides strong motivation to use an anti-aliasing filter prior to taking samples. In particular, say that a continuous signal $y(t)$ is the sum of two other signals, namely $s(t)$ and $n(t)$. Let $s(t)$ be the signal of interest and assume that it is band-limited to the range $[-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}]$. Lemma 2.8 implies that $s(t)$ can be perfectly reconstructed from samples taken at period Δ. On the other hand, the signal $n(t)$ is assumed to be a contamination signal. This signal is assumed to have relatively small spectral content in the range $[-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}]$ but significant spectral components outside of that range. Then considering Lemma 2.5, it follows that the spectral content of the samples $y(t)$, i.e., $Y_\Delta(j\omega)$, will be corrupted in a major way by the *out-of-band* components in $n(t)$. In this case, the samples of $y(t)$ will be a poor representation of the samples of $s(t)$. This leads to the conclusion that it is desirable, in practice, to band-pass filter $y(t)$ prior to taking samples so that the corruption of the samples by out-of-band components in $n(t)$ is mitigated. Such a filter is commonly called an *anti-aliasing filter* (AAF).

The above argument suggests that an ideal band-pass filter should be used. In practice, this is often replaced by a more realistic causal low pass-filter. Note that in Part II of the book, an alternative stochastic interpretation will be given to the above argument.

2.5 Summary

The key points covered in this chapter are:

- The definition of the continuous Fourier transform pair (2.1)–(2.2).
- The definition of the discrete-time Fourier transform pair (2.6)–(2.7).
- The definition of the instrumental signal $y_\Delta(t)$ associated with a sequence $\{y_k\}$, i.e.,

$$y_\Delta(t) = \Delta \sum_{k=-\infty}^{\infty} y_k \delta(t - k\Delta) \tag{2.32}$$

- The fact that the continuous Fourier transform of $y_\Delta(t)$ is *equal* to the discrete-time Fourier transform of $\{y_k\}$.
- The folding or *aliasing* formula for signals, i.e.,

$$Y_\Delta(j\omega) = \sum_{\ell=-\infty}^{\infty} Y\left(j\omega - j\frac{2\pi}{\Delta}\ell\right) \tag{2.33}$$

  where $Y(j\omega) = \mathcal{F}\{y(t)\}$ and $Y_\Delta(j\omega) = \mathcal{F}\{y_\Delta(t)\}$.
- The reconstruction formula for band-limited signals

$$y(t) = \sum_{k=-\infty}^{\infty} y_k \frac{\sin(\omega_N(t - k\Delta))}{\omega_N(t - k\Delta)} \tag{2.34}$$

  where $\omega_N = \frac{\pi}{\Delta}$ is the Nyquist frequency.
- The desirability of including an anti-aliasing filter prior to sampling so that the samples are not dominated by out-of-band contamination.

Further Reading

Further details of Fourier analysis can be found in many textbooks, including:

Gasquet C, Witomski P (1998) Fourier analysis and applications: filtering, numerical computations, wavelets. Springer, Berlin

Lathi BP (2004) Linear systems and signals, 2nd edn. Oxford University Press, Oxford

Oppenheim AV, Schafer RW (1999) Discrete-time signal processing, 2nd edn. Prentice Hall International, New York

Stein E, Shakarchi R (2003) Fourier analysis: an introduction. Princeton University Press, Princeton

Churchill RV, Brown JW (2008) Complex variables and applications, 8th edn. McGraw-Hill, Englewood Cliffs

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