

# Chapter 2

## Preliminaries

In this chapter, we provide some mathematical preliminaries, useful technical lemmas, and properties of the ocean disturbance, which will be extensively used throughout this book. The chapter is organized as follows. Firstly, the Hamilton principle is introduced in Sect. 2.1. Then, a brief introduction of the ocean disturbance on marine flexible structures is given in Sect. 2.2. Subsequently, the function approximation using NNs is presented in Sect. 2.3, followed by Sect. 2.4 about some useful technical lemmas for completeness.

### 2.1 The Hamilton Principle

As opposed to lumped mechanical systems, flexible mechanical systems have an infinite number of degrees of freedom, and the model of the system is described by using continuous functions of space and time. The Hamilton principle permits the derivation of equations of motion from energy quantities in a variational form and generates the motion equations of the flexible mechanical systems. The Hamilton principle [193, 194] is represented by

$$\int_{t_1}^{t_2} \delta(E_k - E_p + W) dt = 0, \tag{2.1}$$

where  $t_1$  and  $t_2$  are two time instants,  $t_1 < t < t_2$  is the operating interval,  $\delta$  denotes the variational operator,  $E_k$  and  $E_p$  are the kinetic and potential energies of the system, respectively, and  $W$  denotes the work done by the nonconservative forces acting on the system, including internal tension, transverse load, linear structural damping, and external disturbance. The principle states that the variation of the kinetic and potential energies plus the variation of work done by loads during any time interval  $[t_1, t_2]$  must equal zero.

There are some advantages using the Hamilton principle to derive the mathematical model of the flexible mechanical systems. Firstly, this approach is independent of the coordinates, and the boundary conditions can be automatically generated by

this approach [88]. In addition, the kinetic energy, the potential energy, and the work done by the nonconservative forces in the Hamilton principle can be directly used to design the Lyapunov function of the closed-loop system.

## 2.2 The Ocean Disturbance on Marine Mechanical Structures

Vortex-induced vibration (VIV) is a direct consequence of lift and drag oscillations due to the vortex shedding formation behind bluff bodies [195]. The marine flexible structures used in offshore production system may get out of control when the structural natural frequency of the risers and cables equals frequency of vortex shedding. The effects of a time-varying ocean current,  $U(x, t)$ , on a riser or a cable can be modeled as a vortex excitation force [196, 197]. The current profile  $U(x, t)$  is a function that relates the depth to the ocean surface current velocity  $U(t)$ . The distributed load on a marine flexible structure,  $f(x, t)$ , can be expressed as a combination of the inline drag force,  $f_D(x, t)$ , consisting of a mean drag and an oscillating drag about the mean modeled as

$$f_D(x, t) = \frac{1}{2}\rho_s C_D(x, t)U(x, t)^2 D + A_D \cos(4\pi f_v(x, t)t + \theta), \quad (2.2)$$

and an oscillating lift force  $f_L(x, t)$ , perpendicular to  $f_D(x, t)$ , about a mean deflected profile,

$$f_L(x, t) = \frac{1}{2}\rho_s C_L(x, t)U(x, t)^2 D \cos(2\pi f_v(x, t)t + \vartheta), \quad (2.3)$$

where  $\rho_s$  is the sea water density,  $C_D(x, t)$  and  $C_L(x, t)$  are the time and spatially varying drag and lift coefficients, respectively,  $D$  is the outer diameter of the flexible structures,  $f_v(x, t)$  is the shedding frequency,  $\theta$  and  $\vartheta$  are the phase angles, and  $A_D$  is the amplitude of the oscillatory part of the drag force, typically 20 % of the first term in  $f_D(x, t)$  [197]. The nondimensional vortex shedding frequency [4] can be expressed as

$$f_v(x, t) = \frac{S_t U(x, t)}{D}, \quad (2.4)$$

where  $S_t$  is the Strouhal number.

In this book, we consider the deflection of the marine flexible structures in transverse and longitudinal directions. Hence, the distributed load can be expressed as

$$f(x, t) = f_D(x, t) \frac{1}{2}\rho_s C_D(x, t)U(x, t)^2 D + A_D \cos(4\pi f_v(x, t)t + \theta). \quad (2.5)$$

The transverse vortex-induced vibration (VIV) from the lift component is not considered in this book, but the proposed method can be similarly applied without any loss of generality if only the lift component is considered.

## 2.3 Function Approximation

In this book, a class of linearly parameterized NNs with radial basis functions (RBF) is used to approximate the continuous function  $f_j(Z) : \mathbb{R}^q \rightarrow \mathbb{R}$ ,

$$f_{nn,j}(Z) = W_j^T S_j(Z), \quad (2.6)$$

where the input vector  $Z = [Z_1, Z_2, \dots, Z_q]^T \in \Omega_Z \subset \mathbb{R}^q$ , the weight vector  $W_j \in \mathbb{R}^l$ , the NN node number  $l > 1$ , and  $S_j(Z) = [s_1, s_2, \dots, s_l]^T \in \mathbb{R}^l$ . Universal approximation results indicate that, if  $l$  is chosen sufficiently large,  $W_j^T S_j(Z)$  can approximate any continuous function,  $f_j(Z)$ , over a compact set  $\Omega_Z \subset \mathbb{R}^q$  to any desired accuracy. This is achieved as

$$f_j(Z) = W_j^{*T} S_j(Z) + \epsilon_j(Z) \quad \forall Z \in \Omega_Z \in \mathbb{R}^q, \quad (2.7)$$

where  $W_j^*$  is the ideal constant weight vector, and  $\epsilon_j(Z)$  is the approximation error, which is bounded over the compact set, i.e.,  $|\epsilon_j(Z)| \leq \epsilon_j^*$  for all  $Z \in \Omega_Z$  with  $\epsilon_j^* > 0$  is an unknown constant. The ideal weight vector  $W_j^*$  is an ‘‘artificial’’ quantity required for analytical purposes.  $W_j^*$  is defined as the value of  $W_j$  that minimizes  $|\epsilon_j|$  for all  $Z \in \Omega_Z \subset \mathbb{R}^q$ , i.e.,

$$W_j^* = \arg \min_{W_j \in \mathbb{R}^l} \left\{ \sup_{Z \in \Omega_Z} |f_j(Z) - W_j^T S_j(Z)| \right\}. \quad (2.8)$$

Typical choices for  $s_k(Z)$  include the sigmoid function, hyperbolic tangent function, and RBF. The RBF NN is a particular network architecture that uses  $l$  Gaussian functions of the form

$$s_k(Z) = \exp \left[ \frac{-(Z - \mu_k)^T (Z - \mu_k)}{\eta_k^2} \right], \quad k = 1, 2, \dots, l, \quad (2.9)$$

where  $\mu_k = [\mu_{k1}, \mu_{k2}, \dots, \mu_{kq}]^T$  is the center of the receptive field, and  $\eta_k$  is the width of the Gaussian function [198].

## 2.4 Lemmas

**Lemma 2.1** [199] *Let  $\phi_1(x, t) \in \mathbb{R}$  and  $\phi_2(x, t) \in \mathbb{R}$  be functions defined for  $x \in [0, L]$  and  $t \in [0, \infty)$ . The Cauchy–Schwarz inequality is*

$$\int_0^L \phi_1 \phi_2 dx \leq \left( \int_0^L \phi_1^2 dx \right)^{\frac{1}{2}} \left( \int_0^L \phi_2^2 dx \right)^{\frac{1}{2}}. \quad (2.10)$$

**Lemma 2.2** [88] *The following inequalities hold:*

$$\phi_1 \phi_2 \leq |\phi_1 \phi_2| \leq \phi_1^2 + \phi_2^2 \quad \forall \phi_1, \phi_2 \in \mathbb{R}. \quad (2.11)$$

**Lemma 2.3** [88] *The following inequalities hold:*

$$|\phi_1\phi_2| = \left| \left( \frac{1}{\sqrt{\delta}}\phi_1 \right) (\sqrt{\delta}\phi_2) \right| \leq \frac{1}{\delta}\phi_1^2 + \delta\phi_2^2 \quad \forall \phi_1, \phi_2 \in \mathbb{R} \text{ and } \delta > 0. \quad (2.12)$$

**Lemma 2.4** [200] *Let  $\phi(x, t) \in \mathbb{R}$  be a function defined for  $x \in [0, L]$  and  $t \in [0, \infty)$  and satisfying the boundary condition*

$$\phi(0, t) = 0 \quad \forall t \in [0, \infty). \quad (2.13)$$

*Then the following inequalities hold:*

$$\int_0^L \phi^2 dx \leq L^2 \int_0^L [\phi']^2 dx, \quad (2.14)$$

$$\phi^2 \leq L \int_0^L [\phi']^2 dx. \quad (2.15)$$

*If in addition to Eq. (2.13), the function  $\phi(x, t)$  satisfies the boundary condition*

$$\phi'(0, t) = 0 \quad \forall t \in [0, \infty), \quad (2.16)$$

*then the following inequality also holds:*

$$[\phi']^2 \leq L \int_0^L [\phi'']^2 dx. \quad (2.17)$$

*Proof* Define the inner product

$$(\phi_1, \phi_2) = \frac{1}{2} \int_0^L \phi_1\phi_2 dx + \frac{1}{2\alpha}\phi_1(L)\phi_2(L)$$

and the operator  $\mathcal{A}_0\phi = [-\phi'', \alpha\phi'(L, t)]^T$ ,  $\alpha > 0$ . Then we have

$$\begin{aligned} (\mathcal{A}_0\phi, \phi) &= \frac{1}{2} \int_0^L -\phi''\phi dx + \frac{1}{2}\phi'(L)\phi(L) \\ &= \frac{1}{2} \int_0^L [\phi']^2 dx. \end{aligned}$$

Since the operator  $\mathcal{A}_0$  is positive and symmetric, we have

$$\begin{aligned} (\mathcal{A}_0\phi, \phi) &\geq \lambda_{\min}(\mathcal{A}_0)\|\phi\|^2 \\ &= \lambda_{\min}(\mathcal{A}_0) \left( \frac{1}{2} \int_0^L [\phi']^2 dx + \frac{1}{2\alpha}[\phi(L)]^2 \right), \end{aligned}$$

where  $\lambda_{\min}(\mathcal{A}_0)$  is the minimum eigenvalue of  $\mathcal{A}_0$  given by the solutions of  $(\lambda I - \mathcal{A}_0)\phi = 0$ . Therefore, we obtain

$$\begin{aligned} \int_0^L [\phi']^2 dx &\geq \lambda_{\min}(\mathcal{A}_0) \left( \int_0^L [\phi]^2 dx + \frac{1}{\alpha} [\phi(L)]^2 \right) \\ &\geq \lambda_{\min}(\mathcal{A}_0) \int_0^L [\phi]^2 dx. \end{aligned}$$

The eigenfunctions of  $\mathcal{A}_0$  have the following form:

$$W(x) = \sin \beta x,$$

where the pinned boundary condition at  $x = 0$  has been used, and  $\lambda = \beta^2$ . The frequency equation is

$$\beta \sin \beta L - \alpha \cos \beta L = 0.$$

We are free to choose  $\alpha = \frac{\sin(1)}{L \cos(1)}$ , so the minimum solution of the above equation is  $\beta = \frac{1}{L}$  and  $\lambda_{\min}(\mathcal{A}_0) = \frac{1}{L^2}$ , and we obtain

$$L^2 \int_0^L [\phi']^2 dx \geq \int_0^L [\phi]^2 dx.$$

Define  $\phi_1(x, t) = \phi'(x, t)$  and  $\phi_2(x, t) = \chi(s - x) = \begin{cases} 1, & x \leq s, \\ 0, & x > s, \end{cases}$  where  $s \in [0, L]$  is a constant. Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \int_0^L \phi_1 \phi_2 dx &= \int_0^L \phi'(x, t) \chi(s - x) dx = \phi(s, t) \leq s^{\frac{1}{2}} \left( \int_0^L [\phi']^2 dx \right)^{\frac{1}{2}} \\ &\leq L^{\frac{1}{2}} \left( \int_0^L [\phi']^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, we have

$$\phi^2 \leq L \int_0^L [\phi']^2 dx.$$

Similarly, we have

$$[\phi']^2 \leq L \int_0^L [\phi'']^2 dx. \quad \square$$

**Lemma 2.5** *Let  $\phi(x, t) \in \mathbb{R}$  be a function defined for  $x \in [0, L]$  and  $t \in [0, \infty)$  and satisfying the boundary condition*

$$\phi(0, t) = C \quad \forall t \in [0, \infty), \quad (2.18)$$

where  $C$  is a constant. Then the following inequality holds:

$$(\phi - C)^2 \leq L \int_0^L [\phi']^2 dx \quad \forall (x, t) \in [0, L] \times [0, \infty). \quad (2.19)$$

*Proof* Define  $\phi_1(x, t) = \phi'(x, t)$  and  $\phi_2(x, t) = \chi(s - x) = \begin{cases} 1, & x \leq s, \\ 0, & x > s, \end{cases}$  where  $s \in [0, L]$  is a constant. Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \int_0^L \phi_1 \phi_2 dx &= \int_0^L \phi'(x, t) \chi(s - x) dx \\ &= \phi(s, t) - C \\ &\leq s^{\frac{1}{2}} \left( \int_0^L [\phi']^2 dx \right)^{\frac{1}{2}} \\ &\leq L^{\frac{1}{2}} \left( \int_0^L [\phi']^2 dx \right)^{\frac{1}{2}}. \end{aligned} \quad (2.20)$$

Therefore, we have

$$(\phi - C)^2 \leq L \int_0^L [\phi']^2 dx, \quad \forall (x, t) \in [0, L] \times [0, \infty). \quad \square \quad (2.21)$$

**Lemma 2.6** [201] *Rayleigh–Ritz theorem: Let  $A \in \mathbb{R}^{n \times n}$  be a real, symmetric, positive-definite matrix; therefore, all the eigenvalues of  $A$  are real and positive. Let  $\lambda_{\min}$  and  $\lambda_{\max}$  denote the minimum and maximum eigenvalues of  $A$ , respectively. Then for all  $x \in \mathbb{R}^n$ , we have*

$$\lambda_{\min} \|x\|^2 \leq x^T A x \leq \lambda_{\max} \|x\|^2, \quad (2.22)$$

where  $\|\cdot\|$  denotes the standard Euclidean norm.

**Lemma 2.7** [202, 203] *For bounded initial conditions,  $\forall x$  and  $\forall t \geq 0$ , if there exists a  $C^1$  continuous and positive-definite Lyapunov function  $V(x, t) : \mathfrak{R}^n \times \mathfrak{R}_+ \rightarrow \mathfrak{R}$  satisfying  $\kappa_1(\|x\|) \leq V(x, t) \leq \kappa_2(\|x\|)$  and such that  $\dot{V}(x, t) \leq -\lambda V(x, t) + c$ , where  $\kappa_1, \kappa_2 : \mathbb{R}^n \rightarrow \mathbb{R}$  are class  $K$  functions, and  $c$  is a positive constant, then the equilibrium point  $x = 0$  of the system  $\dot{x} = f(x, t)$  is uniformly bounded.*

**Lemma 2.8** [204] *For any real-valued continuous function  $f(x, y, z)$ ,  $x \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^n$ ,  $z \in \mathbb{R}^p$ , there are smooth scalar functions  $\alpha(x, y) \geq 0$  and  $\beta(z) \geq 0$  such that*

$$|f(x, y, z)| \leq \alpha(x, y) + \beta(z). \quad (2.23)$$

**Lemma 2.9** [205] *Consider the basis functions of Gaussian RBF NN (2.9) with  $\hat{Z}$  being the input vector. If  $\hat{Z} = Z - \epsilon \bar{\psi}$ , where  $\bar{\psi}$  is a bounded vector, and  $\epsilon > 0$  is a*

constant, then we have

$$s_i(\hat{Z}) = \exp\left[\frac{-(\hat{Z} - \mu_j)^T(\hat{Z} - \mu_j)}{\eta_j^2}\right], \quad j = 1, 2, \dots, l, \quad (2.24)$$

$$S(\hat{Z}) = S(Z) + \epsilon S_t,$$

where  $S_t$  is a bounded vector function.

**Lemma 2.10** [206] *Suppose that a system output  $y(t)$  and its first  $n$  derivatives are bounded and such that  $|y^{(k)}| < Y_K$  with positive constants  $Y_K$ . Consider the following linear system:*

$$\begin{aligned} \epsilon \dot{\pi}_i &= \pi_{i+1}, \quad i = 1, \dots, n-1, \\ \epsilon \dot{\pi}_n &= -\bar{\lambda}_1 \pi_n - \bar{\lambda}_2 \pi_{n-1} - \dots - \bar{\lambda}_{n-1} \pi_2 - \pi_1 + \eta(t), \end{aligned} \quad (2.25)$$

where  $\epsilon$  is any small positive constant, and the parameters  $\bar{\lambda}_1$  to  $\bar{\lambda}_{n-1}$  are chosen such that the polynomial  $s^n + \bar{\lambda}_1 s^{n-1} + \dots + \bar{\lambda}_{n-1} s + 1$  is Hurwitz. Then, the follow property holds:

$$\xi_k = \frac{\pi_k}{\epsilon^{k-1}} - \eta^{(k-1)} = -\epsilon \psi^{(k)}, \quad k = 1, \dots, n-1, \quad (2.26)$$

where  $\psi = \pi_n + \bar{\lambda}_1 \pi_{n-1} + \dots + \bar{\lambda}_{n-1} \pi_1$  with  $\psi^{(k)}$  denoting the  $k$ th derivative of  $\psi$ . Also, there exist positive constants  $t^*$  and  $h_k$  such that for all  $t > t^*$ , we have  $\|\xi_k\| \leq \epsilon h_k$ ,  $k = 1, 2, 3, \dots, n$ .

**Lemma 2.11** [207] *For any positive constants  $k_b$ , let  $\mathcal{Z}_1 := \{z_3 \in \mathbb{R} : -k_b < z_3 < k_b\} \subset \mathbb{R}$  and  $\mathcal{N} := \mathcal{R}^l \times \mathcal{Z}_1 \subset \mathbb{R}^{l+1}$  be open sets. Consider the system*

$$\dot{\eta} = h(t, \eta), \quad (2.27)$$

where  $\eta = [w, z_1]^T \in \mathcal{N}$ , and  $h : \mathbb{R}_+ \times \mathcal{N} \rightarrow \mathbb{R}^{l+1}$  is piecewise continuous in  $t$  and locally Lipschitz in  $z$ , uniformly in  $t$ , on  $\mathbb{R}_+ \times \mathcal{N}$ . Suppose that there exist functions  $U : \mathbb{R}^l \rightarrow \mathbb{R}_+$  and  $V_3 : \mathcal{Z}_1 \rightarrow \mathbb{R}_+$ , continuously differentiable and positive definite in their respective domains, such that

$$V_3(z_3) \rightarrow \infty \quad \text{as } z_3 \rightarrow -k_b \text{ or } z_3 \rightarrow k_b, \quad (2.28)$$

$$\gamma_1(\|w\|) \leq U(w) \leq \gamma_2(\|w\|) \quad (2.29)$$

where  $\gamma_1$  and  $\gamma_2$  are class  $K_\infty$  functions. Let  $V(\eta) := V_1(z_3) + U(w)$ , and let  $z_3(0)$  belong to the set  $z_3 \in (-k_b, k_b)$ . If

$$\dot{V} = \frac{\partial V}{\partial \eta} h \leq 0, \quad (2.30)$$

then  $z_3(t)$  remains in the open set  $z_3 \in (-k_b, k_b)$  for all  $t \in [0, \infty)$ .

**Definition 2.12** (Barrier Lyapunov Function [207]) A BLF is a scalar function  $V(x)$ , defined with respect to the system  $\dot{x} = f(x)$  on an open region  $\mathcal{D}$  containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of  $\mathcal{D}$ , has the property  $V(x) \rightarrow \infty$  as  $x$  approaches the boundary of  $\mathcal{D}$ , and satisfies  $V(x(t)) \leq b$ ,  $t \geq 0$ , along the solution of  $\dot{x} = f(x)$  for  $x(0) \in \mathcal{D}$  and some constant  $b$ .

As discussed in [207], there are many functions  $V_1(z_1)$  satisfying Definition 2.12, which may be symmetric or asymmetric. Asymmetric barrier functions are more general than their counterparts and thus can offer more flexibility for control design to obtain better performance. However, they are considerably more difficult to construct analytically and to employ for control design. For clarity, the following symmetric *BLF* candidate considered in [207] is used in this book:

$$V_1 = \frac{1}{2} \log \frac{k_b^2}{k_b^2 - z_1^3}, \quad (2.31)$$

where  $\log(\cdot)$  denotes the natural logarithm of  $(\cdot)$ , and  $k_b$  is the constraint on  $z_1$ . The *BLF* escapes to infinity at  $z_1 = k_b$ . It can be shown that  $V_1$  is positive definite and  $C^1$  continuous in the set  $z_1 < k_b$ . The control design and results in this book can be extended to the asymmetric *BLF* case.

**Definition 2.13** (SGUUB [65]) The solution  $X(t)$  of a system is semi-globally uniformly ultimately bounded (SGUUB) if, for any compact set  $\Omega_0$  and all  $X(t_0) \in \Omega_0$ , there exist  $\mu > 0$  and  $T(\mu, X(t_0))$  such that  $\|X(t)\| \leq \mu$  for all  $t \geq t_0 + T$ .





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