Chapter 2
Linear Parameter-Varying Modeling and Control Synthesis Methods

This chapter is split into the following two main parts: modeling of LPV systems and control synthesis methods for LPV systems.

2.1 Modeling of LPV Systems

Throughout this book, the control synthesis methods used rely on the existence of an LPV model with polytopic parameter dependence. Unfortunately, this is not the most intuitive form that an LPV model can take. Many physical systems have parameter variations that can be easily represented with LFTs. For this reason, we will demonstrate how to convert an LPV model with LFT parameter dependency into an LPV model with polytopic parameter dependence.

Consider the following open-loop, discrete-time LPV system with LFT parameter dependency:

$$\begin{bmatrix} x(k+1) \\ l(k) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_p & B_w & B_u \\ C_l & D_{lp} & D_{lw} & D_{lu} \\ C_z & D_{zp} & D_{zw} & D_{zu} \\ C_y & D_{yp} & D_{yw} & D_{yu} \end{bmatrix} \begin{bmatrix} x(k) \\ p(k) \\ w(k) \\ u(k) \end{bmatrix} \tag{2.1}$$
$$p(k) = \Theta(k) l(k)$$

where $x(k)$ is the state at time k, $w(k)$ is the exogenous input, and $u(k)$ is the control input. The vectors $z(k)$ and $y(k)$ are the performance output and the measurement for control. Also, $p(k)$ and $l(k)$ are the pseudo-input and pseudo-output connected by $\Theta(k)$. The time-varying parameter $\Theta(k)$ follows the structure

$$\Theta(k) \in \Theta = \left\{ \mathrm{diag}(\theta_1 I_{n_1}, \theta_2 I_{n_2}, \cdots, \theta_N I_{n_N}) \right\}. \tag{2.2}$$

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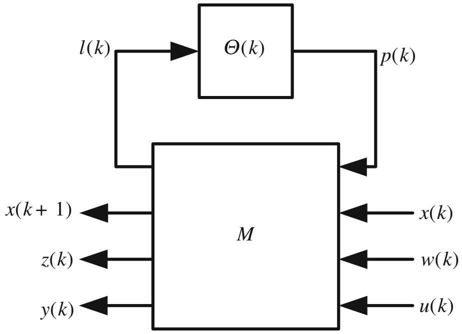

Fig. 2.1 Diagram of the upper LFT of the state space matrices

To emphasize the fact that there exists an LFT with respect to the time-varying parameter matrix $\Theta(k)$, the state-space matrices can be rearranged into the following upper LFT (Fig. 2.1):

$$\begin{bmatrix} l(k) \\ x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \underbrace{\left[\begin{array}{c|ccc} D_{lp} & C_l & D_{lw} & D_{lu} \\ \hline B_p & A & B_w & B_u \\ D_{zp} & C_z & D_{zw} & D_{zu} \\ D_{yp} & C_y & D_{yw} & D_{yu} \end{array}\right]}_{=:M} \begin{bmatrix} p(k) \\ x(k) \\ w(k) \\ u(k) \end{bmatrix} \tag{2.3}$$

$$p(k) = \Theta(k) l(k).$$

The time-varying matrix $\Theta(k)$ can be absorbed back into the state space matrices such that the state space matrices would be given by

$$\begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \underbrace{\begin{bmatrix} A(\Theta(k)) & B_w(\Theta(k)) & B_u(\Theta(k)) \\ C_z(\Theta(k)) & D_{zw}(\Theta(k)) & D_{zu}(\Theta(k)) \\ C_y(\Theta(k)) & D_{yw}(\Theta(k)) & D_{yu}(\Theta(k)) \end{bmatrix}}_{=:H(\Theta)} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix} \tag{2.4}$$

where

$$\begin{aligned} H(\Theta) &:= \mathscr{F}_u(M, \Theta) \\ &= \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix} + \begin{bmatrix} B_p \\ D_{zp} \\ D_{yp} \end{bmatrix} \Theta(k) \left(I - D_{lp}\Theta(k)\right)^{-1} \left[C_l \ D_{lw} \ D_{lu} \right]. \end{aligned} \tag{2.5}$$

It is clear from (2.5) that when the matrix D_{lp} is non-zero, then the system matrices are not affine functions, i.e., a linear combination of the time-varying parameters plus a constant translation. Since, as previously mentioned, all control synthesis methods covered in this book rely on an LPV model with a polytopic parameter dependence, the system matrices must be affine functions of the time-varying parameters. If the matrix D_{lp} is non-zero, then some approximation must be made. If the parameter variation is “small”, then a first-order Taylor series approximation can be performed.

2.1.1 First-Order Taylor Series Approximation of LPV Systems

Using the first-order Taylor series expansion at $\Theta = \bar{\Theta}$, the LPV system can be approximated as

$$\hat{H}(\Theta(k)) = H(\bar{\Theta}) + \sum_{i=1}^{N} \left[\nabla H(\bar{\Theta})\right]_i \, (\theta_i(k) - \bar{\theta}_i) \tag{2.6}$$

where $\theta_i(k)$, for $i = 1, \ldots, N$ are the individual parameters in $\Theta(k)$, and $\left[\nabla H(\bar{\Theta})\right]_i$ is the partial derivative of the LFT system $H(\Theta)$ with respect to θ_i solved at $\bar{\Theta}$. The i-th partial derivative of the upper LFT system $H(\Theta)$ is computed by [39]

$$[\nabla H(\Theta)]_i = M_{21}[I - \Theta M_{11}]^{-1} E_i [I - M_{11}\Theta]^{-1} M_{12}, \tag{2.7}$$

where

$$M_{11} = D_{lp}, \quad M_{12} = \begin{bmatrix} C_l & D_{lw} & D_{lu} \end{bmatrix}, \quad M_{21} = \begin{bmatrix} B_p \\ D_{zp} \\ D_{yp} \end{bmatrix}, \tag{2.8}$$

and the matrices E_i are defined such that

$$\Theta(k) = \sum_{i=1}^{N} \theta_i(k) E_i. \tag{2.9}$$

After performing this first-order Taylor series approximation, then the approximated system $\hat{H}(\Theta(k))$ will have affine parameter dependence with respect to $\Theta(k)$. As shown in the next section, a polytopic LPV model can be obtained from an LPV system with affine parameter dependence.

2.1.2 Polytopic Linear Time-Varying System with Barycentric Coordinates

The LPV system with affine parameter dependence can be represented by the following polytopic linear time-varying (LTV) system

$$\begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \underbrace{\begin{bmatrix} A(\lambda(k)) & B_w(\lambda(k)) & B_u(\lambda(k)) \\ C_z(\lambda(k)) & D_{zw}(\lambda(k)) & D_{zu}(\lambda(k)) \\ C_y(\lambda(k)) & D_{yw}(\lambda(k)) & D_{yu}(\lambda(k)) \end{bmatrix}}_{=:H(\lambda)} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix} \tag{2.10}$$

where the system matrices $A(\lambda(k))$, $B_w(\lambda(k))$, $B_u(\lambda(k))$, $C_z(\lambda(k))$, $C_y(\lambda(k))$, $D_{zw}(\lambda(k))$, $D_{zu}(\lambda(k))$, $D_{yw}(\lambda(k))$, and $D_{yu}(\lambda(k))$ belong to the polytope

$$\begin{aligned} \mathfrak{D} = \Big\{ & (A, B_w, B_u, C_z, C_y, D_{zw}, D_{zu}, D_{yw}, D_{yu})(\lambda_k) : \\ & (A, B_w, B_u, C_z, C_y, D_{zw}, D_{zu}, D_{yw}, D_{yu})(\lambda_k) \\ & = \sum_{i=1}^{N} \lambda_i(k)(A, B_w, B_u, C_z, C_y, D_{zw}, D_{zu}, D_{yw}, D_{yu})_i, \lambda_k \in \Lambda_N \Big\}, \end{aligned} \tag{2.11}$$

with $(A, B_w, B_u, C_z, C_y, D_{zw}, D_{zu}, D_{yw}, D_{yu})_i$ the vertices of the polytope and $\lambda_k = \lambda(k) \in \mathbb{R}^N$ the vector of time-varying barycentric coordinates lying in the unit simplex

$$\Lambda_N = \left\{ \zeta \in \mathbb{R}^N : \sum_{i=1}^{N} \zeta_i = 1, \zeta_i \geq 0,\ i = 1, \cdots, N \right\}. \tag{2.12}$$

The vertices of the polytope $\mathfrak{D}$ are obtained by solving the system matrices of $\hat{H}(\Theta)$ at each of the vertices $\mathscr{V}_i$ for $i = 1, \ldots, N$ (See examples in Fig. 2.2). Then each of the state space matrices in $H(\lambda_k)$ are computed as the convex combination of the vertice systems of this polytope, such that, for example, the state matrix would be computed by

$$A(\lambda_k) = \sum_{i=1}^{N} \lambda_i(k) A_i. \tag{2.13}$$

Each of the other matrices in $H(\lambda_k)$ are computed in the same way. The convex combination coefficients $\{\lambda_i(\Theta)\}$ for a given Θ and set of vertices $\{\mathscr{V}_i\}$ are also known as the barycentric coordinates. The barycentric coordinate function is defined in [60] as

$$\lambda_i(\Theta) = \frac{\Upsilon_i(\Theta)}{\sum_i \Upsilon_i(\Theta)}, \tag{2.14}$$

where $\Upsilon_i(\Theta)$ is the weight function of vertex i for a point Θ inside of the convex polytope. The weight function is

$$\Upsilon_i(\Theta) = \frac{\mathrm{vol}(\mathscr{V}_i)}{\Pi_{j \in \mathrm{ind}(\mathscr{V}_i)}(n_j \cdot (\mathscr{V}_i - \Theta))}, \tag{2.15}$$

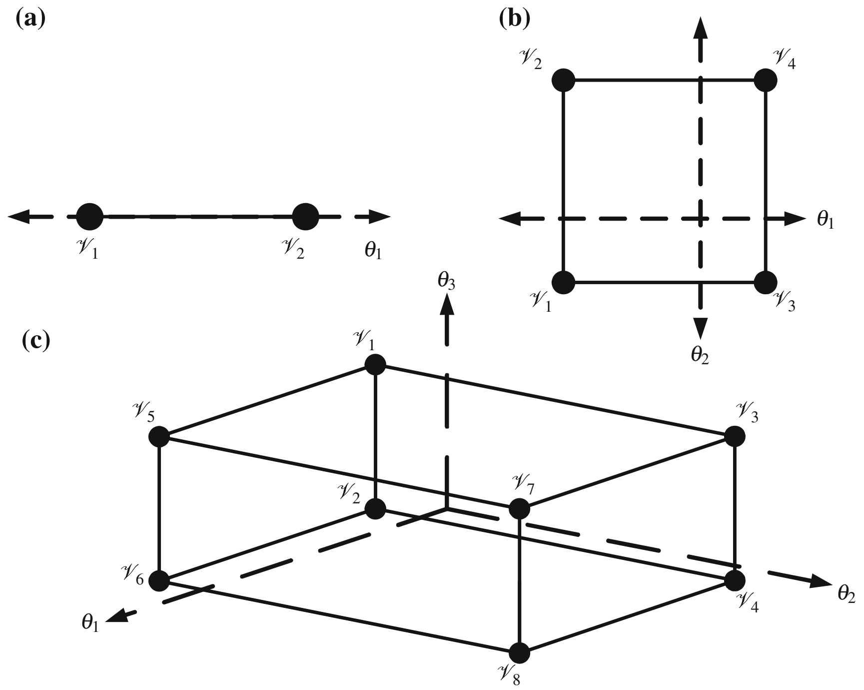

Fig. 2.2 Examples of possible parameter space polytopes

where $\text{vol}(\mathscr{V}_i)$ is the volume of the parallelepiped span by the normals to the facets incident on vertex i, i.e., $\mathscr{V}_i$, $\{n_j\}$ is the collection of normal vectors to the facets incident on vertex i, and $\text{ind}(\mathscr{V}_i)$ denotes the set of indices j such that the facet normal to n_j contains the vertex $\mathscr{V}_i$. The volume of a parallelepiped can be found as

$$\text{vol}(\mathscr{V}_i) = |\det(n_{\text{ind}})| \,. \tag{2.16}$$

where n_{ind} is a matrix whose rows are the vectors n_j where $j \in \text{ind}(\mathscr{V}_i)$.

Since the polytopic LTV system has been defined, we will now focus our attention in the next section on the performance specifications for the polytopic LTV system.

2.2 Performance of Discrete-Time Polytopic LPV Systems

Consider the $\mathscr{H}_2$ or $\mathscr{H}_\infty$ weighted closed-loop discrete-time LPV system

$$H := \begin{cases} x(k+1) = \mathscr{A}(\lambda_k)x(k) + \mathscr{B}_w(\lambda_k)w(k), \quad x(0) = 0 \\ z(k) = \mathscr{C}_z(\lambda_k)x(k) + \mathscr{D}_w(\lambda_k)w(k) \end{cases} \tag{2.17}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^r$ is the exogenous input, and $z(k) \in \mathbb{R}^p$ is the performance output. The system matrices $\mathscr{A}(\lambda_k)$, $\mathscr{B}_w(\lambda_k)$, $\mathscr{C}_z(\lambda_k)$, and $\mathscr{D}_w(\lambda_k)$ belong to a polytope similar to $\mathfrak{D}$ in (2.11).

The $\mathscr{H}_\infty$ performance of the system (2.17) from $w(k)$ to $z(k)$ is defined by the quantity

$$\|H\|_\infty = \sup_{\|w(k)\|_2 \neq 0} \frac{\|z(k)\|_2}{\|w(k)\|_2} \tag{2.18}$$

with $w(k) \in \ell_2$ and $z(k) \in \ell_2$. In robust control, the $\mathscr{H}_\infty$ norm has proved to be extremely useful and has various interpretations [54]. For example, in the frequency domain, the $\mathscr{H}_\infty$ norm is the peak value of the transfer function magnitude, for a single-input single-output (SISO) system. On the other hand, in the time domain the $\mathscr{H}_\infty$ norm can be thought of as the worst-case gain for sinusoidal inputs at any frequency. The $\mathscr{H}_\infty$ norm has been extremely useful in robust and LPV control because it is convenient for representing unstructured model uncertainties, and can therefore be useful in gaging the robustness of a system. By using the bounded real lemma, an upper bound for the $\mathscr{H}_\infty$ performance is characterized by the following lemma [13].

Lemma 2.1 *($\mathscr{H}_\infty$ **Performance**) Consider the system H given by (2.17). If there exist bounded matrices $G(\lambda_k)$ and $P(\lambda_k) = P^T(\lambda_k) > 0$ for all $\lambda_k \in \Lambda_N$ such that*

$$\begin{bmatrix} P(\lambda_{k+1}) & \mathscr{A}(\lambda_k)G(\lambda_k) & \mathscr{B}_w(\lambda_k) & 0 \\ G^T(\lambda_k)\mathscr{A}^T(\lambda_k) & G(\lambda_k) + G^T(\lambda_k) - P(\lambda_k) & 0 & G^T(\lambda_k)\mathscr{C}_z^T(\lambda_k) \\ \mathscr{B}_w^T(\lambda_k) & 0 & \eta I & \mathscr{D}_w^T(\lambda_k) \\ 0 & \mathscr{C}_z(\lambda_k)G(\lambda_k) & \mathscr{D}_w(\lambda_k) & \eta I \end{bmatrix} > 0 \tag{2.19}$$

then the system H is exponentially stable and

$$\|H\|_\infty \leq \inf_{P(\lambda_k), G(\lambda_k), \eta} \eta.$$

This lemma is an extension of a standard result provided by [18, 19].

The $\mathscr{H}_2$ norm has two main interpretations: deterministic and stochastic, depending on what type of input signal is considered. For the deterministic interpretation, the input signal has bounded energy (ℓ_2 norm) and the $\mathscr{H}_2$ norm is the peak magnitude (or ℓ_∞ norm) of the performance output divided by the energy of the input. For the stochastic interpretation, the input signal is assumed to be white noise with unit intensity and the $\mathscr{H}_2$ norm is then the energy of the performance output (ℓ_2 norm) [54]. For discrete-time LTI systems, there are three main definitions that are usually used to define the $\mathscr{H}_2$ norm [12, 13, 55]. They are as follows:

1. If $H(q)$ represents the transfer function matrix of a system $H(q)$, then its $\mathscr{H}_2$ norm can be defined as

$$\|H(q)\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} \text{trace}\left\{H^T(e^{j\omega})H(e^{j\omega})\right\} d\omega. \tag{2.20}$$

2. If $\{e_1, \ldots, e_r\}$ is a basis for the input space and $z_i(k)$ is the output of the system H when an impulse $\delta(k)e_i$ is applied, then its $\mathscr{H}_2$ norm can be defined as

$$\|H\|_2^2 = \sum_{i=1}^{r} \|z_i\|_2^2. \tag{2.21}$$

3. If $z(k)$ is the output of an LTI system when a zero-mean white noise Gaussian process $w(k)$ with identity covariance matrix is applied, then its $\mathscr{H}_2$ norm can be defined as

$$\|H\|_2^2 = \lim_{\mathrm{m}\to\infty} \sup \mathscr{E}\left\{\frac{1}{\mathrm{m}} \sum_{k=0}^{\mathrm{m}} z^T(k)z(k)\right\} \tag{2.22}$$

where $\mathscr{E}$ denotes the expectation operator and the positive integer m denotes the time horizon.

Since the idea of a transfer function is not well defined for time-varying systems, the first definition has not been extended to LTV systems. The second and third definitions can be extended to LTV systems. However, since the computation of the norm with the second definition can depend on the selection of the basis for the input space, the third definition has received more attention [8, 12, 13, 27, 55]. An upper bound for the $\mathscr{H}_2$ performance given by the third definition is characterized by the following lemma [13].

Lemma 2.2 *(*$\mathscr{H}_2$ ***Performance****) Consider the system* H *given by (2.17). If there exists bounded matrices* $G(\lambda_k)$*,* $P(\lambda_k) = P^T(\lambda_k) > 0$*, and* $W(\lambda_k) = W^T(\lambda_k)$ *for all* $\lambda_k \in \Lambda_N$ *such that*

$$\begin{bmatrix} P(\lambda_{k+1}) & \mathscr{A}(\lambda_k)G(\lambda_k) & \mathscr{B}_w(\lambda_k) \\ G^T(\lambda_k)\mathscr{A}^T(\lambda_k) & G(\lambda_k) + G^T(\lambda_k) - P(\lambda_k) & 0 \\ \mathscr{B}_w^T(\lambda_k) & 0 & I \end{bmatrix} > 0 \tag{2.23}$$

and

$$\begin{bmatrix} W(\lambda_k) - \mathscr{D}_w(\lambda_k)\mathscr{D}_w^T(\lambda_k) & \mathscr{C}_z(\lambda_k)G(\lambda_k) \\ G^T(\lambda_k)\mathscr{C}_z^T(\lambda_k) & G(\lambda_k) + G^T(\lambda_k) - P(\lambda_k) \end{bmatrix} > 0 \tag{2.24}$$

then the system H *is exponentially stable and its* $\mathscr{H}_2$ *performance is bounded by* ν *given by*

$$\nu^2 = \inf_{P(\lambda_k), G(\lambda_k), W(\lambda_k)} \sup_{\lambda_k \in \Lambda_N} \text{trace}\{W(\lambda_k)\}$$

such that $\|H\|_2 \leq \nu$*.*

The proof for this lemma can be found in [13].

Note that the parameter-dependent LMI conditions in Lemmas 2.1 and 2.2 must be evaluated for all λ_k in the unit simplex Λ_N. This leads to an infinite dimensional problem. By imposing an affine parameter-dependent structure on the Lyapunov matrix $P(\lambda_k)$, such that

$$P(\lambda_k) = \sum_{i=1}^{N} \lambda_i(k) P_i, \quad i = 1, \ldots, N, \tag{2.25}$$

a finite set of LMIs in terms of the vertices of the polytope $\mathfrak{D}$ can be obtained.

To reduce conservatism, the parameter variation rate

$$\Delta\lambda_i(k) = \lambda_i(k+1) - \lambda_i(k), \quad i = 1, \ldots, N \tag{2.26}$$

is assumed to be limited. Two limits have been considered in the literature. The first rate limit considered in the literature [11, 12, 40] is given by

$$-b\lambda_i(k) \le \Delta\lambda_i(k) \le b\left(1 - \lambda_i(k)\right), \quad i = 1, \ldots, N, \tag{2.27}$$

with $b \in [0, 1]$. With this parameter variation rate bound and the affine parameter-dependent structure in (2.25), the $\mathscr{H}_\infty$ performance criteria in Lemma 2.1 can be transformed into a finite number of LMIs, as shown in the next Lemma [11].

Lemma 2.3 *(**Finite $\mathscr{H}_\infty$ Performance with rate limit** (2.27)) The system H (2.17) has an $\mathscr{H}_\infty$ performance bounded by η if there exist matrices $G_i \in \mathbb{R}^{n\times n}$ and symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} (1-b)P_i + bP_\ell & \star & \star & \star \\ G_i^T \mathscr{A}_i^T & G_i + G_i^T - P_i & \star & \star \\ \mathscr{B}_{w,i}^T & 0 & \eta I & \star \\ 0 & \mathscr{C}_{z,i} G_i & \mathscr{D}_{w,i} & \eta I \end{bmatrix} > 0 \tag{2.28}$$

holds for $i = 1, \ldots, N$ and $\ell = 1, \ldots, N$ and

$$\begin{bmatrix} (1-b)P_i + (1-b)P_j + 2bP_\ell & \star & \star & \star \\ G_j^T \mathscr{A}_i^T + G_i^T \mathscr{A}_j^T & G_i + G_i^T + G_j + G_j^T - P_i - P_j & \star & \\ \mathscr{B}_{w,i}^T + \mathscr{B}_{w,j}^T & 0 & 2\eta I & \star \\ 0 & \mathscr{C}_{z,i} G_j + \mathscr{C}_{z,j} G_i & \mathscr{D}_{w,i} + \mathscr{D}_{w,j} & 2\eta I \end{bmatrix} > 0 \tag{2.29}$$

holds for $\ell = 1, \ldots, N$, $i = 1, \ldots, N-1$, and $j = i+1, \ldots, N$.

A proof for this lemma can be found in [11].

Likewise, the $\mathscr{H}_2$ performance criteria in Lemma 2.2 can also be transformed into a finite number of LMIs as shown in [12] for the case when $\mathscr{D}_w = 0$.

Lemma 2.4 *(**Finite $\mathscr{H}_2$ Performance with rate limit** (2.27)) Consider the system H (2.17), with $\mathscr{D}_w = 0$. If there exist matrices $G_i \in \mathbb{R}^{n\times n}$ and symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ and $W_i \in \mathbb{R}^{p\times p}$ such that*

$$\begin{bmatrix} (1-b)P_i + bP_\ell & \star & \star \\ G_i^T \mathscr{A}_i^T & G_i + G_i^T - P_i & \star \\ \mathscr{B}_{w,i}^T & 0 & I \end{bmatrix} > 0, \tag{2.30}$$

for $i = 1, \ldots, N$ and $\ell = 1, \ldots, N$,

$$\begin{bmatrix} (1-b)\left(P_i + P_j\right) + 2bP_\ell & \star & \star \\ G_j^T \mathscr{A}_i^T + G_i^T \mathscr{A}_j^T & G_i + G_i^T + G_j + G_j^T - P_i - P_j & \star \\ \mathscr{B}_{w,i}^T + \mathscr{B}_{w,j}^T & 0 & 2I \end{bmatrix} > 0 \tag{2.31}$$

for $\ell = 1, \ldots, N$, $i = 1, \ldots, N-1$, and $j = i+1, \ldots, N$,

$$\begin{bmatrix} W_i & \star \\ G_i^T \mathscr{C}_{z,i}^T & G_i + G_i^T - P_i \end{bmatrix} > 0 \tag{2.32}$$

for $i = 1, \ldots, N$, and

$$\begin{bmatrix} W_i + W_j & \star \\ G_j^T \mathscr{C}_{z,i}^T + G_i^T \mathscr{C}_{z,j}^T & G_i + G_i^T + G_j + G_j^T - P_i - P_j \end{bmatrix} > 0 \tag{2.33}$$

for $i = 1, \ldots, N-1$ and $j = i+1, \ldots, N$, then the system H, with $\mathscr{D}_w = 0$, is exponentially stable and has an $\mathscr{H}_2$ performance bounded by ν given by

$$\nu^2 = \min_{G_i, P_i, W_i} \max_i \operatorname{trace}\{W_i\}. \tag{2.34}$$

A proof for this lemma can be found in [12].

While the rate limit (2.27) can be useful, it may or may not be very realistic. To see this, one may consider the example parameter variation with $N = 2$ and $b = 0.5$ as displayed in Fig. 2.3. In this example, the time-varying parameter starts at one extreme and moves the other extreme as quickly as the parameter variation rate limit (2.27) allows. It is clear that the maximum parameter variation rate is dependent on the current value of the parameters with the rate limit given by (2.27).

A more realistic parameter variation limit that is not dependent on the current value of the time-varying parameter is considered in [13, 42]. This limit is given by

$$-b \leq \Delta\lambda_i(k) \leq b, \quad i = 1, \ldots, N, \tag{2.35}$$

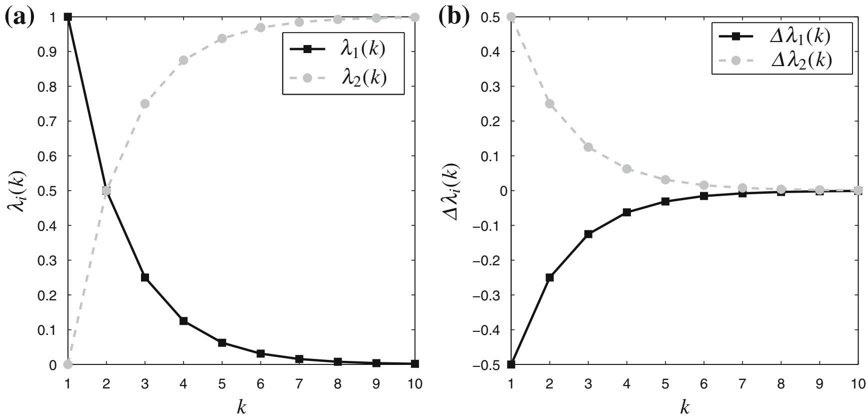

Fig. 2.3 Fastest possible parameter transition between the extreme conditions with $N = 2$ and $b = 0.5$ when the rate limit given by (2.27) is in effect

with $b \in [0, 1]$. When using this parameter variation rate, the uncertainty domain, where the vector $(\lambda(k), \Delta\lambda(k))^T \in \mathbb{R}^{2N}$ takes values, may be modeled by the compact set

$$\Gamma_b = \Bigg\{\delta \in \mathbb{R}^{2N} : \delta \in \text{co}\{g^1, \ldots, g^M\}, g^j = \begin{pmatrix} f^j \\ h^j \end{pmatrix}, \quad f^j \in \mathbb{R}^N, \quad h^j \in \mathbb{R}^N,$$
$$\sum\nolimits_{i=1}^N f_i^j = 1 \text{ with } f_i^j \geq 0, i = 1, \ldots, N, \sum\nolimits_{i=1}^N h_i^j = 0, \quad j = 1, \ldots, M\Bigg\} \tag{2.36}$$

defined as the convex combination of the vectors g^j, for $j = 1, \ldots, M$, given a priori. This definition of Γ_b ensures that $\lambda(k) \in \Lambda_N$ and that

$$\sum_{i=1}^N \Delta\lambda_i(k) = 0 \tag{2.37}$$

holds for all $k \geq 0$. For a given bound b, the columns of Γ_b can be generated as the columns of a matrix V as follows [13] (in MATLAB code)

```
V = zeros(2*N,N^2+(N-1)^2+(N-1));
for i = 1:1:N
    V(i,(i-1)*N+1) = 1;
    ind = 1;
    for j = 1:1:N
        if j ISNOT i
            V(i,(i-1)*N+ind+1) = 1;
```

```
                    V(N+i,(i-1)*N+ind+1) = -b;
                    V(N+j,(i-1)*N+ind+1) = b;

                    V(i,N^2+(i-1)*(N-1)+ind) = b;
                    V(j,N^2+(i-1)*(N-1)+ind) = 1-b;
                    V(N+i,N^2+(i-1)*(N-1)+ind) = -b;
                    V(N+j,N^2+(i-1)*(N-1)+ind) = b;

                    ind = ind + 1;
                end
            end
        end
        f = V(1:N,:);
        h = V(N+1:2*N,:);
```

With the uncertainty set Γ_b, each $\lambda_i(k)$ and $\Delta\lambda_i(k)$ for $i = 1, 2, \ldots, N$ are given by

$$\lambda_i(k) = \sum_{j=1}^{M} f_i^j \gamma_j(k) \quad \text{and} \quad \Delta\lambda_i(k) = \sum_{j=1}^{M} h_i^j \gamma_j(k) \tag{2.38}$$

such that the affine representation of $P(\lambda_k)$ is given by

$$\begin{aligned} P(\lambda_k) &= \sum_{i=1}^{N} \lambda_i(k) P_i = \sum_{i=1}^{N} \left(\sum_{j=1}^{M} f_i^j \gamma_j(k) \right) P_i \\ &= \sum_{j=1}^{M} \gamma_j(k) \left(\sum_{i=1}^{N} f_i^j P_i \right) = \sum_{j=1}^{M} \gamma_j(k) \tilde{P}_j = \tilde{P}(\gamma(k)) \end{aligned} \tag{2.39}$$

with $\tilde{P}_j = \sum_{i=1}^{N} f_i^j P_i$ as shown in [13]. Using the same structure for λ_k, the system matrices in H (2.17) are also converted to the new representation in terms of $\gamma(k) \in \Lambda_M$, such that

$$\mathscr{A}(\lambda_k) = \tilde{\mathscr{A}}(\gamma(k)) = \sum_{j=1}^{M} \gamma_j(k) \tilde{\mathscr{A}}_j \tag{2.40}$$

with $\tilde{\mathscr{A}}_j = \sum_{i=1}^{N} f_i^j \mathscr{A}_i$. All other matrices in H are converted in the same way. It is also shown in [13], that by combining (2.38) with the fact that $\lambda_{k+1} = \lambda_k + \Delta\lambda_k$,

$$P(\lambda_{k+1}) = \sum_{i=1}^{N}(\lambda_i(k) + \Delta\lambda_i(k))P_i = \sum_{i=1}^{N}\left(\sum_{j=1}^{M}\left(f_i^j + h_i^j\right)\gamma_j(k)\right)P_i$$
$$= \sum_{j=1}^{M}\gamma_j(k)\left(\sum_{i=1}^{N}\left(f_i^j + h_i^j\right)P_i\right) = \sum_{j=1}^{M}\gamma_j(k)\hat{P}_j = \hat{P}(\gamma(k)) \quad (2.41)$$

with $\hat{P}_j = \sum_{i=1}^{N}\left(f_i^j + h_i^j\right)P_i$. The authors of [13] note that due to these representations of $P(\lambda_k)$ and $P(\lambda_{k+1})$, the LMIs of Lemma 2.1 and Lemma 2.2 can be rewritten with a dependency on $\gamma(k)$. They also note that a convenient parameterization of the slack variable $G(\lambda_k)$ is given by

$$G(\lambda_k) = G(\gamma(k)) = \sum_{j=1}^{M}\gamma_j(k)G_j, \quad \gamma(k) \in \Lambda_M. \quad (2.42)$$

Using these parameterizations, the next two lemmas present a finite-dimensional set of LMIs that guarantee the LMI conditions of Lemmas 2.1 and 2.2 [13].

Lemma 2.5 ***(Finite $\mathscr{H}_\infty$ Performance with rate limit (2.35))*** *Consider the system H given by (2.17). Assume that the vectors f^j and h^j of Γ_b are given. If there exist, for $j = 1, \ldots, M$, matrices $G_j \in \mathbb{R}^{n\times n}$ and, for $i = 1, \ldots, N$, symmetric positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} \sum_{i=1}^{N}\left(f_i^j + h_i^j\right)P_i & \star & \star & \star \\ G_j^T\tilde{\mathscr{A}}_j^T & G_j + G_j^T - \sum_{i=1}^{N} f_i^j P_i & \star & \star \\ \tilde{\mathscr{B}}_{w,j}^T & 0 & \eta I & \star \\ 0 & \tilde{\mathscr{C}}_{z,j}G_j & \tilde{\mathscr{D}}_{w,j} & \eta I \end{bmatrix} > 0 \quad (2.43)$$

for $j = 1, \ldots, M$ and

$$\begin{bmatrix} \sum_{i=1}^{N}\left(f_i^j + f_i^\ell + h_i^j + h_i^\ell\right)P_i & \star & \star & \star \\ G_j^T\tilde{\mathscr{A}}_\ell^T + G_\ell^T\tilde{\mathscr{A}}_j^T & \Theta_{22,j\ell} & \star & \star \\ \tilde{\mathscr{B}}_{w,j}^T + \tilde{\mathscr{B}}_{w,\ell}^T & 0 & 2\eta I & \star \\ 0 & \tilde{\mathscr{C}}_{z,j}G_\ell + \tilde{\mathscr{C}}_{z,\ell}G_j & \tilde{\mathscr{D}}_{w,j} + \tilde{\mathscr{D}}_{w,\ell} & 2\eta I \end{bmatrix} > 0 \quad (2.44)$$

with

$$\Theta_{22,j\ell} = G_j + G_j^T + G_\ell + G_\ell^T - \sum_{i=1}^{N}\left(f_i^j + f_i^\ell\right)P_i$$

for $j = 1, \ldots, M-1$ and $\ell = j+1, \ldots, M$, then the system H is exponentially stable and

$$\|H\|_\infty \leq \min_{P_i, G_j, \eta} \eta.$$

The proof for this lemma can be found in [13].

Lemma 2.6 ***(Finite $\mathscr{H}_2$ Performance with rate limit** (2.35)) *Consider the system H given by (2.17). Assume that the vectors f^j and h^j of Γ_b are given. If there exist, for $j = 1, \ldots, M$, matrices $G_j \in \mathbb{R}^{n\times n}$ and, for $i = 1, \ldots, N$, symmetric positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$ and $W_i \in \mathbb{R}^{p\times p}$ such that*

$$\begin{bmatrix} \sum_{i=1}^{N} \left(f_i^j + h_i^j\right) P_i & \star & \star \\ G_j^T \tilde{\mathscr{A}}_j^T & G_j + G_j^T - \sum_{i=1}^{N} f_i^j P_i & \star \\ \tilde{\mathscr{B}}_{w,j}^T & 0 & I \end{bmatrix} > 0 \tag{2.45}$$

for $j = 1, \ldots, M$,

$$\begin{bmatrix} \sum_{i=1}^{N} \left(f_i^j + f_i^\ell + h_i^j + h_i^\ell\right) P_i & \star & \star \\ G_j^T \tilde{\mathscr{A}}_\ell^T + G_\ell^T \tilde{\mathscr{A}}_j^T & G_j + G_j^T + G_\ell + G_\ell^T - \sum_{i=1}^{N} \left(f_i^j + f_i^\ell\right) P_i & \star \\ \tilde{\mathscr{B}}_{w,j}^T + \tilde{\mathscr{B}}_{w,\ell}^T & 0 & 2I \end{bmatrix} > 0 \tag{2.46}$$

for $j = 1, \ldots, M-1$, and $\ell = j+1, \ldots, M$,

$$\begin{bmatrix} \sum_{i=1}^{N} f_i^j W_i - \tilde{\mathscr{D}}_{w,j} \tilde{\mathscr{D}}_{w,j}^T & \star \\ G_j^T \tilde{\mathscr{C}}_{z,j}^T & G_j + G_j^T - \sum_{i=1}^{N} f_i^j P_i \end{bmatrix} > 0 \tag{2.47}$$

for $j = 1, \ldots, M$,

$$\begin{bmatrix} \sum_{i=1}^{N} \left(f_i^j + f_i^\ell\right) W_i - \tilde{\mathscr{D}}_{w,j} \tilde{\mathscr{D}}_{w,\ell}^T + \tilde{\mathscr{D}}_{w,\ell} \tilde{\mathscr{D}}_{w,j}^T & \star \\ G_j^T \tilde{\mathscr{C}}_{z,\ell}^T + G_\ell^T \tilde{\mathscr{C}}_{z,j}^T & G_j + G_j^T + G_\ell + G_\ell^T - \sum_{i=1}^{N} \left(f_i^j + f_i^\ell\right) P_i \end{bmatrix} > 0 \tag{2.48}$$

for $j = 1, \ldots, M-1$, and $\ell = j+1, \ldots, M$, then the system H is exponentially stable and its $\mathscr{H}_2$ performance is bound by ν given by

$$\nu^2 = \min_{P_i, G_j, W_i} \max_i \operatorname{trace}\{W_i\}.$$

The proof for this lemma can be found in [13].

2.3 Control Synthesis Methods for LPV Systems

In this section, the gain-scheduled static output feedback controller synthesis results from [11–13] are reviewed.

Consider the following $\mathscr{H}_\infty$ and $\mathscr{H}_2$ weighted, discrete-time polytopic time-varying systems H^∞ and H^2:

$$H^\infty := \begin{cases} x(k+1) = A(\lambda_k)x(k) + B_{\infty w}(\lambda_k)w_\infty(k) + B_u(\lambda_k)u(k) \\ z_\infty(k) = C_{\infty z}(\lambda_k)x(k) + D_{\infty w}(\lambda_k)w_\infty(k) + D_{\infty u}(\lambda_k)u(k) \\ y(k) = C_y x(k), \quad C_y = \begin{bmatrix} I_q, & 0 \end{bmatrix} \end{cases} \tag{2.49}$$

$$H^2 := \begin{cases} x(k+1) = A(\lambda_k)x(k) + B_{2w}(\lambda_k)w_2(k) + B_u(\lambda_k)u(k) \\ z_2(k) = C_{2z}(\lambda_k)x(k) + D_{2w}(\lambda_k)w_2(k) + D_{2u}(\lambda_k)u(k) \\ y(k) = C_y x(k), \quad C_y = \begin{bmatrix} I_q, & 0 \end{bmatrix} \end{cases} \tag{2.50}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w_\infty(k) \in \mathbb{R}^{r_\infty}$ and $w_2(k) \in \mathbb{R}^{r_2}$ are the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ exogenous inputs, $z_\infty(k) \in \mathbb{R}^{p_\infty}$ and $z_2(k) \in \mathbb{R}^{p_2}$ are the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance outputs, and $y \in \mathbb{R}^q$ is the measurement for control. The system matrices of H^∞ and H^2 belong to a polytope similar to the one given in (2.11).

2.3.1 $\mathscr{H}_\infty$ Control Synthesis

In [11], a finite set of LMIs is presented which can be used to synthesize a stabilizing, static output feedback LPV controller for the system H^∞ with a guaranteed $\mathscr{H}_\infty$ performance bound. The rate of variation of the parameters (2.26) is assumed to be limited by (2.27).

Extending the analysis result presented in Lemma 2.3, the authors of [11] characterize a finite set of LMI conditions for the synthesis of a gain scheduled $\mathscr{H}_\infty$ static output feedback controller for the system (2.49).

Lemma 2.7 *Consider the system H^∞ given by (2.49). If there exist matrices $G_{i,1} \in \mathbb{R}^{q\times q}$, $G_{i,2} \in \mathbb{R}^{n-q,q}$, $G_{i,3} \in \mathbb{R}^{n-q\times n-q}$, $Z_i \in \mathbb{R}^{m\times q}$, and symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} (1-b)P_i + bP_\ell & \star & \star & \star \\ G_i^T A_i^T + Z_i^T B_{u,i}^T & G_i + G_i^T - P_i & \star & \star \\ B_{\infty w,i}^T & 0 & \eta I & \star \\ 0 & C_{\infty z,i}G_i + D_{\infty u,i}Z_i & D_{\infty w,i} & \eta I \end{bmatrix} > 0 \tag{2.51}$$

hold for $i = 1, \ldots, N$ and $\ell = 1, \ldots, N$ and

$$\begin{bmatrix} (1-b)P_i + (1-b)P_j + 2bP_\ell & \star & \star & \star \\ \Theta_{21,ij} & G_i + G_i^T + G_j + G_j^T - P_i - P_j & \star & \star \\ B_{\infty w,i}^T + B_{\infty w,j}^T & 0 & 2\eta I & \star \\ 0 & \Theta_{42,ij} & D_{\infty w,i} + D_{\infty w,j} & 2\eta I \end{bmatrix} > 0 \tag{2.52}$$

with

$$\Theta_{21,ij} = G_j^T A_i^T + G_i^T A_j^T + Z_j^T B_{u,i}^T + Z_i^T B_{u,j}^T$$
$$\Theta_{42,ij} = C_{\infty z,i} G_j + C_{\infty z,j} G_i + D_{\infty u,i} Z_j + D_{\infty u,j} Z_i$$

hold for $\ell = 1, \dots, N$, $i = 1, \dots, N-1$, *and* $j = i+1, \dots, N$, *with*

$$G_i = \begin{bmatrix} G_{i,1} & 0 \\ G_{i,2} & G_{i,3} \end{bmatrix} \quad \text{and} \quad Z_i = \begin{bmatrix} Z_{i,1} & 0 \end{bmatrix},$$

then the parameter-dependent static output feedback gain

$$K(\lambda_k) = \hat{Z}(\lambda_k)\hat{G}(\lambda_k)^{-1}, \tag{2.53}$$

with

$$\hat{Z}(\lambda(k)) = \sum_{i=1}^{N} \lambda_i(k) Z_{i,1} \quad \text{and} \quad \hat{G}(\lambda(k)) = \sum_{i=1}^{N} \lambda_i(k) G_{i,1}$$

stabilizes the system (2.49) with a guaranteed $\mathscr{H}_\infty$ *performance bounded by* η *for all* $\lambda \in \Lambda_N$ *and* $\Delta\lambda$ *that satisfies (2.27).*

A proof for this lemma can be found in [11].

2.3.2 $\mathscr{H}_2$ Control Synthesis

In [12], a finite set of LMIs is presented which can be used to synthesize a stabilizing, static output feedback LPV controller for the system H^2 with a guaranteed $\mathscr{H}_2$ performance bound. The rate of variation of the parameters (2.26) is assumed to be limited by (2.27).

Extending the analysis result presented in Lemma 2.4, the authors of [12] characterize a finite set of LMI conditions for the synthesis of a gain scheduled $\mathscr{H}_2$ static output feedback controller for the system (2.50) with $D_{2w} = 0$.

Lemma 2.8 *Consider the system* H^2 *given by (2.49) with* $D_{2w} = 0$. *If there exist matrices* $G_{i,1} \in \mathbb{R}^{q\times q}$, $G_{i,2} \in \mathbb{R}^{n-q,q}$, $G_{i,3} \in \mathbb{R}^{n-q\times n-q}$, $Z_i \in \mathbb{R}^{m\times q}$, *and symmetric matrices* $P_i \in \mathbb{R}^{n\times n}$ *and* $W_i \in \mathbb{R}^{p_2\times p_2}$ *such that*

$$\begin{bmatrix} (1-b)P_i + bP_\ell & \star & \star \\ G_i^T A_i^T + Z_i^T B_{u,i}^T & G_i + G_i^T - P_i & \star \\ B_{2w,i}^T & 0 & I \end{bmatrix} > 0 \tag{2.54}$$

for $i = 1, \ldots, N$ *and* $\ell = 1, \ldots, N$,

$$\begin{bmatrix} (1-b)P_i + (1-b)P_j + 2bP_\ell & \star & \star \\ G_j^T A_i^T + G_i^T A_j^T + Z_j^T B_{u,i}^T + Z_i^T B_{u,j}^T & G_i + G_i^T + G_j + G_j^T - P_i - P_j & \star \\ B_{2w,i}^T + B_{2w,j}^T & 0 & 2I \end{bmatrix} > 0 \tag{2.55}$$

for $\ell = 1, \ldots, N$, $i = 1, \ldots, N-1$, *and* $j = i+1, \ldots, N$,

$$\begin{bmatrix} W_i & \star \\ G_i^T C_{2z,i}^T + Z_i^T D_{2u,i}^T & G_i + G_i^T - P_i \end{bmatrix} > 0 \tag{2.56}$$

for $i = 1, \ldots, N$, *and*

$$\begin{bmatrix} W_i + W_j & \star \\ G_j^T C_{2z,i}^T + G_i^T C_{2z,j}^T + Z_j^T D_{2u,i}^T + Z_i^T D_{2u,j}^T & G_i + G_i^T + G_j + G_j^T - P_i - P_j \end{bmatrix} > 0 \tag{2.57}$$

for $i = 1, \ldots, N-1$ *and* $j = i+1, \ldots, N$, *with*

$$G_i = \begin{bmatrix} G_{i,1} & 0 \\ G_{i,2} & G_{i,3} \end{bmatrix} \quad \text{and} \quad Z_i = \begin{bmatrix} Z_{i,1} & 0 \end{bmatrix},$$

then the parameter-dependent static output feedback gain

$$K(\lambda_k) = \hat{Z}(\lambda_k)\hat{G}(\lambda_k)^{-1}, \tag{2.58}$$

with

$$\hat{Z}(\lambda(k)) = \sum_{i=1}^{N} \lambda_i(k) Z_{i,1} \quad \text{and} \quad \hat{G}(\lambda(k)) = \sum_{i=1}^{N} \lambda_i(k) G_{i,1}$$

stabilizes the system (2.50) with a guaranteed $\mathscr{H}_2$ *performance bounded by* ν *given by*

$$\nu^2 = \min_{G_i, Z_i, P_i, W_i} \max_i \operatorname{trace}\{W_i\}. \tag{2.59}$$

for all $\lambda \in \Lambda_N$ *and* $\Delta\lambda$ *that satisfies (2.27).*

A proof for this lemma can be found in [12].

2.3.3 Mixed $\mathscr{H}_2/\mathscr{H}_\infty$ Control Synthesis

In [13], gain-scheduled static output feedback synthesis LMIs that stabilize the systems H^∞ and H^2 with $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance bounds are presented. The rate of variation of the parameters (2.26) is assumed to be limited by a priori known bound b, given by (2.35).

The authors of [13] extend the analysis results of Lemmas 2.5 and 2.6 to characterize a finite set of LMI conditions for the synthesis of a gain scheduled mixed $\mathscr{H}_2/\mathscr{H}_\infty$ static output feedback controller for the systems H^2 (2.50) and H^∞ (2.49).

Lemma 2.9 *Consider the systems H^∞ (2.49) and H^2 (2.50). Assume that the vectors f^j and h^j of Γ_b are given. Additionally, assume that a prescribed $\mathscr{H}_\infty$ performance bound η is given. If there exist, for $i = 1, \ldots, N$, matrices $G_{i,1} \in \mathbb{R}^{n\times n}$, $Z_{i,1} \in \mathbb{R}^{m\times q}$ and symmetric positive-definite matrices $P_{\infty,i} \in \mathbb{R}^{n\times n}$, $P_{2,i} \in \mathbb{R}^{n\times n}$, and $W_i \in \mathbb{R}^{p\times p}$, and, for $j = 1, \ldots, M$, matrices $G_{\infty j,2} \in \mathbb{R}^{(n-q)\times q}$, $G_{2j,2} \in \mathbb{R}^{(n-q)\times q}$, $G_{\infty j,3} \in \mathbb{R}^{(n-q)\times(n-q)}$, and $G_{2j,3} \in \mathbb{R}^{(n-q)\times(n-q)}$ such that*

$$\begin{bmatrix} \sum_{i=1}^N \left(f_i^j + h_i^j\right) P_{\infty,i} & \star & \star & \star \\ G_{\infty,j}^T \tilde{A}_j^T + Z_j^T \tilde{B}_{u,j}^T & G_{\infty,j} + G_{\infty,j} - \sum_{i=1}^N f_i^j P_{\infty,i} & \star & \star \\ \tilde{B}_{\infty w,j}^T & 0 & \eta I & \star \\ 0 & \tilde{C}_{\infty z,j} G_{\infty,j} + \tilde{D}_{\infty u,j} Z_j & \tilde{D}_{\infty w,j} & \eta I \end{bmatrix} = \Theta_j > 0 \tag{2.60}$$

for $j = 1, \ldots, M$ and

$$\begin{bmatrix} \sum_{i=1}^N \left(f_i^j + f_i^\ell + h_i^j + h_i^\ell\right) P_{\infty,i} & \star & \star & \star \\ \Theta_{21,j\ell} & \Theta_{22,j\ell} & \star & \star \\ \tilde{B}_{\infty w,j}^T + \tilde{B}_{\infty w,\ell}^T & 0 & 2\eta I & \star \\ 0 & \Theta_{42,j\ell} & \tilde{D}_{\infty w,j} + \tilde{D}_{\infty w,\ell} & 2\eta I \end{bmatrix} = \Theta_{jl} > 0 \tag{2.61}$$

with

$$\Theta_{21,j\ell} = G_{\infty,j}^T \tilde{A}_\ell^T + G_{\infty,\ell}^T \tilde{A}_j^T + Z_j^T \tilde{B}_{u,\ell}^T + Z_\ell^T \tilde{B}_{u,j}^T$$

$$\Theta_{22,j\ell} = G_{\infty,j} + G_{\infty,j}^T + G_{\infty,\ell} + G_{\infty,\ell}^T - \sum_{i=1}^N \left(f_i^j + h_i^j\right) P_{\infty,i}$$

$$\Theta_{42,j\ell} = \tilde{C}_{\infty z,j} G_{\infty,\ell} + \tilde{C}_{\infty z,\ell} G_{\infty,j} + \tilde{D}_{u,j} Z_\ell + \tilde{D}_{u,\ell} Z_j$$

for $j = 1, \ldots, M-1$ and $\ell = j+1, \ldots, M$, and

$$\begin{bmatrix} \sum_{i=1}^N \left(f_i^j + h_i^j\right) P_{2,i} & \star & \star \\ G_{2,j}^T \tilde{A}_j^T + Z_j^T \tilde{B}_{u,j}^T & G_{2,j} + G_{2,j}^T - \sum_{i=1}^N f_i^j P_{2,i} & \star \\ \tilde{B}_{w2,j}^T & 0 & I \end{bmatrix} = \Phi_j > 0 \tag{2.62}$$

for $j = 1, \ldots, M$, *and*

$$\begin{bmatrix} \sum_{i=1}^{N} \left(f_i^j + f_i^\ell + h_i^j + h_i^\ell \right) P_{2,i} & \star & \star \\ \Phi_{21,j\ell} & \Phi_{22,j\ell} & \star \\ \tilde{B}_{w2,j}^T + \tilde{B}_{w2,\ell}^T & 0 & 2I \end{bmatrix} = \Phi_{j\ell} > 0 \tag{2.63}$$

with

$$\Phi_{21,j\ell} = G_{2,j}^T \tilde{A}_\ell^T + G_{2,\ell}^T \tilde{A}_j^T + Z_j^T \tilde{B}_{u,\ell}^T + Z_\ell^T \tilde{B}_{u,j}^T$$
$$\Phi_{22,j\ell} = G_{2,j} + G_{2,j}^T + G_{2,\ell} + G_{2,\ell}^T - \sum_{i=1}^{N} \left(f_i^j + f_i^\ell \right) P_{2,i}$$

for $j = 1, \ldots, M-1$ *and* $\ell = j+1, \ldots, M$, *and*

$$\begin{bmatrix} \sum_{i=1}^{N} f_i^j W_i - \tilde{D}_{2w,j} \tilde{D}_{2w,j}^T & \star \\ G_{2,j}^T \tilde{C}_{2z,j}^T + Z_j^T \tilde{D}_{2u,j}^T & G_{2,j} + G_{2,j}^T - \sum_{i=1}^{N} f_i^j P_{2,i} \end{bmatrix} = \Psi_j > 0 \tag{2.64}$$

for $j = 1, \ldots, M$, *and*

$$\begin{bmatrix} \sum_{i=1}^{N} \left(f_i^j + f_i^\ell \right) W_i - \tilde{D}_{2w,j} \tilde{D}_{2w,\ell}^T + \tilde{D}_{2w,\ell} \tilde{D}_{2w,j}^T & \star \\ G_{2,j}^T \tilde{C}_{2z,\ell}^T + G_{2,\ell}^T \tilde{C}_{2z,j}^T + Z_j^T \tilde{D}_{2u,\ell}^T + Z_\ell^T \tilde{D}_{2u,j}^T & \Psi_{22,j\ell} \end{bmatrix} = \Psi_{j\ell} > 0 \tag{2.65}$$

with

$$\Psi_{22,j\ell} = G_{2,j} + G_{2,j}^T + G_{2,\ell} + G_{2,\ell}^T - \sum_{i=1}^{N} \left(f_i^j + f_i^\ell \right) P_{2,i}$$

for $j = 1, \ldots, M-1$ *and* $\ell = j+1, \ldots, M$ *where*

$$G_{\infty j} = \begin{bmatrix} \sum_{i=1}^{N} f_i^j G_{i,1} & 0 \\ G_{\infty j,2} & G_{\infty j,3} \end{bmatrix}, \quad G_{2j} = \begin{bmatrix} \sum_{i=1}^{N} f_i^j G_{i,1} & 0 \\ G_{2j,2} & G_{2j,3} \end{bmatrix}, \quad \textit{and}$$
$$Z_j = \begin{bmatrix} \sum_{i=1}^{N} f_i^j Z_{i,1} & 0 \end{bmatrix}, \tag{2.66}$$

then the parameter-dependent static output feedback gain

$$K(\lambda_k) = \hat{Z}(\lambda_k) \hat{G}(\lambda_k)^{-1} \tag{2.67}$$

with

$$\hat{Z}(\lambda_k) = \sum_{i=1}^{N} \lambda_i(k) Z_{i,1} \quad \textit{and} \quad \hat{G}(\lambda_k) = \sum_{i=1}^{N} \lambda_i(k) G_{i,1} \tag{2.68}$$

stabilizes the system H^{∞} with a guaranteed $\mathscr{H}_\infty$ performance bounded by η and the system H^2 with a guaranteed $\mathscr{H}_2$ performance bounded by ν given by

$$\nu^2 = \min_{P_{\infty,i}, P_{2,i}, G_{i,1}, G_{\infty j,2}, G_{2j,2}, G_{\infty j,3}, G_{2j,3}, Z_{i,1}, W_i} \max_i \text{trace}\{W_i\}. \tag{2.69}$$

The proof for Lemma 2.9 is provided by [13].

Note that, as with any multi-objective controller synthesis, the mixed $\mathscr{H}_2/\mathscr{H}_\infty$ controller synthesis LMIs in Lemma 2.9 can be solved in a few different ways depending on the needs of the control designer. For instance, a controller with the best possible $\mathscr{H}_2$ performance is found with respect to a fixed, predetermined $\mathscr{H}_\infty$ performance η by solving the LMIs to minimize

$$\sum_{i=1}^{N} \text{trace}\{W_i\},$$

while ensuring that $\|H^{\infty}\|_\infty < \eta$. Likewise, a controller with the best possible $\mathscr{H}_\infty$ performance is found with respect to a fixed, predetermined $\mathscr{H}_2$ performance by first adding the following LMIs to the controller synthesis:

$$\overline{W} - W_i > 0, \quad i = 1, \ldots, N, \tag{2.70}$$

where $\overline{W}$ is selected to provide the desired $\mathscr{H}_2$ performance, and then minimizing η in the $\mathscr{H}_\infty$ LMIs.

Suppose that a control design problem or application had certain system outputs that were required to maintain hard constraints instead of just minimizing a weighted $\mathscr{H}_2$ or $\mathscr{H}_\infty$ performance. This would require that the closed-loop system have a guaranteed ℓ_2 to ℓ_∞ gain, which will be covered in the next chapter.

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