The nature of the representation theory of a finite group $G$ in (finite-dimensional) vector spaces over some field $k$ depends very much on the relation between the order $|G|$ of the group $G$ and the characteristic $\text{char}(k)$ of the field $k$. If $\text{char}(k)$ does not divide $|G|$ then all representations are semisimple, i.e. are direct sums of irreducible representations. The reason for this is the semisimplicity of the group algebra $k[G]$ in this situation. By the modular representation theory of $G$ one means, on the other hand, the case where $\text{char}(k)$ is a divisor of $|G|$ (so that, in particular, $\text{char}(k)$ must be a prime number). The group algebra $k[G]$ now may be far from being semisimple. In the extreme case, for example, where $|G|$ is a power of $\text{char}(k)$, it is a local ring; there is then a single irreducible representation, which is the trivial one, whereas the structure of a general representation will still be very complicated.

As a consequence a whole range of additional tools have to be developed and used in the course of the investigation. To mention some, there is the systematic use of Grothendieck groups (Chap. 2) as well as Green’s direct analysis of indecomposable representations (Chap. 4). There also is the strategy of writing the category of all $k[G]$-modules as the direct product of certain subcategories, the so-called blocks of $G$, by using the action of the primitive idempotents in the center of $k[G]$. Brauer’s approach then establishes correspondences between the blocks of $G$ and blocks of certain subgroups of $G$ (Chap. 5), the philosophy being that one is thereby reduced to a simpler situation. This allows us, in particular, to measure how nonsemisimple a category a block is by the size and structure of its so-called defect group. Beginning in Sect. 4.4 all these concepts are made explicit for the example of the group $G = \text{SL}_2(\mathbb{F}_p)$.

The present book is to be thought of as an introduction to the major tools and strategies of modular representation theory. Its content was taught during a course lasting the full academic year 2010/2011 at Münster. Some basic algebra together with the semisimple case were assumed to be known, although all facts to be used are restated (without proofs) in the text. Otherwise the book is entirely self-contained. The references [1–10] provide a complete list of the sources I have drawn upon. Of course, there already exist several textbooks on the subject. The older ones like [5] and [6] are written in a mostly group theoretic language. The beautiful
book [1] develops the theory entirely from the module theoretic point of view but leaves out completely the comparison with group theoretic concepts. For example, the concept of defect groups can be introduced either purely group theoretically or purely module theoretically. To my knowledge all existing books essentially restrict themselves to a discussion of one of these approaches only. Although my presentation is strongly biased towards the module theoretic point of view, I make an attempt to strike a certain balance by also showing the reader the other aspect. In particular, in the case of defect groups a detailed proof of the equivalence of the two approaches will be given.

This book is not addressed to experts. It does not discuss any very advanced aspects nor any specialized results of the theory. The aim is to familiarize students at the masters level with the basic results, tools, and techniques of a beautiful and important algebraic theory, hopefully enabling them to subsequently pursue their own more specialized problems.

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