Let $G$ be a finite group. Over any commutative ring $R$ we have the group ring
\[ R[G] = \left\{ \sum_{g \in G} a_g g : a_g \in R \right\} \]
with addition
\[ \left( \sum_{g \in G} a_g g \right) + \left( \sum_{g \in G} b_g g \right) = \sum_{g \in G} (a_g + b_g) g \]
and multiplication
\[ \left( \sum_{g \in G} a_g g \right) \left( \sum_{g \in G} b_g g \right) = \sum_{g \in G} \left( \sum_{h \in G} a_h b_{h^{-1} g} \right) g. \]

We fix an algebraically closed field $k$ of characteristic $p > 0$. The representation theory of $G$ over $k$ is the module theory of the group ring $k[G]$. This is our primary object of study in the following.

### 2.1 The Setting

The main technical tool of our investigation will be a $(0, p)$-ring $R$ for $k$ which is a complete local commutative integral domain $R$ such that

- the maximal ideal $m_R \subseteq R$ is principal,
- $R/m_R = k$, and
- the field of fractions of $R$ has characteristic zero.

**Exercise** The only ideals of $R$ are $m_R^j$ for $j \geq 0$ and $\{0\}$. 
We note that there must exist an integer $e \geq 1$—the ramification index of $R$—such that $Rp = m_R^e$. There is, in fact, a canonical $(0, p)$-ring $W(k)$ for $k$—its ring of Witt vectors—with the additional property that $m_{W(k)} = W(k)p$. Let $K/K_0$ be any finite extension of the field of fractions $K_0$ of $W(k)$. Then

$$R := \{ a \in K : \text{Norm}_{K/K_0}(a) \in W(k) \}$$

is a $(0, p)$-ring for $k$ with ramification index equal to $[K : K_0]$. Proofs for all of this can be found in §3–6 of [9].

In the following we fix a $(0, p)$-ring $R$ for $k$. We denote by $K$ the field of fractions of $R$ and by $\pi_R$ a choice of generator of $m_R$, i.e. $m_R = R\pi_R$. The following three group rings are now at our disposal:

$$\begin{array}{ccc}
K[G] & \leq & K[G] \\
\uparrow & \uparrow & \uparrow \\
R[G] & \xrightarrow{pr} & k[G]
\end{array}$$

such that


As explained before Proposition 1.7.4 there are the corresponding homomorphisms between Grothendieck groups

$$\begin{array}{ccc}
K_0(K[G]) & \xrightarrow{\kappa} & K_0(k[G]) \\
K_0(R[G]) & \xrightarrow{\rho} & K_0(k[G]).
\end{array}$$

For the vertical arrow observe that, quite generally for any $R[G]$-module $M$, we have

$$K[G] \otimes_{R[G]} M = K \otimes_R R[G] \otimes_{R[G]} M = K \otimes_R M.$$

We put

$$R_K(G) := R(K[G]) \quad \text{and} \quad R_k(G) := R(k[G]).$$

Since $K$ has characteristic zero the group ring $K[G]$ is semisimple, and we have

$$R_K(G) = K_0(K[G])$$

by Remark 1.7.3.iii. On the other hand, as a finite-dimensional $k$-vector space the group ring $k[G]$ of course is left and right artinian. In particular we have the Cartan
homomorphism

\[ c_G : K_0(\mathbb{k}[G]) \longrightarrow R_k(G) \]

\[ [P] \longmapsto [P]. \]

Hence, so far, there is the diagram of homomorphisms

\[
\begin{array}{ccc}
R_K(G) & \xrightarrow{\kappa} & R_k(G) \\
\downarrow & & \downarrow \rho \\
K_0(R[G]) & \xrightarrow{c_G} & K_0(k[G]).
\end{array}
\]

Clearly, \( R[G] \) is an \( R \)-algebra which is finitely generated as an \( R \)-module. Let us collect some of what we know in this situation.

- (Proposition 1.3.6) \( R[G] \) is left and right noetherian, and any finitely generated \( R[G] \)-module is complete as well as \( R[G] \pi_R \)-adically complete.
- (Proposition 1.5.5) \( 1 \in R[G] \) can be written as a sum of pairwise orthogonal primitive idempotents; the set of all central idempotents in \( R[G] \) is finite; \( 1 \) is equal to the sum of all primitive idempotents in \( Z(R[G]) \); any \( R[G] \)-module has a block decomposition.
- (Proposition 1.5.7) For any idempotent \( \varepsilon \in k[G] \) there is an idempotent \( e \in R[G] \) such that \( \varepsilon = e + R[G] \pi_R \).
- (Proposition 1.5.12) The projection map \( R[G] \longrightarrow k[G] \) restricts to a bijection between the set of all central idempotents in \( R[G] \) and the set of all central idempotents in \( k[G] \); in particular, the block decomposition of an \( R[G] \)-module \( M \) reduces modulo \( R[G] \pi_R \) to the block decomposition of the \( k[G] \)-module \( M/\pi_R M \).
- (Proposition 1.6.10) Any finitely generated \( R[G] \)-module \( M \) has a projective cover \( P \longrightarrow M \) such that \( P/\text{Jac}(R[G]) \longrightarrow M/\text{Jac}(R[G])/\pi_R \). We emphasize that \( R[G] \pi_R \subseteq \text{Jac}(R[G]) \) by Lemma 1.3.5.iii. Moreover, the ideal \( \text{Jac}(k[G]) = \text{Jac}(R[G])/R[G] \pi_R \) in the left artinian ring \( k[G] \) is nilpotent by Proposition 1.2.1.vi. We apply Proposition 1.7.4 to the rings \( R[G] \) and \( k[G] \) and we see that the maps \( \{P\} \longmapsto \{P/\pi_R P\} \longmapsto \{P/\text{Jac}(R[G]) P\} \) induce the commutative diagram of bijections between finite sets

\[
\begin{array}{ccc}
\widehat{R[G]} & \simeq & \widehat{k[G]} \\
\downarrow \simeq & & \downarrow \simeq \\
\widehat{R[G]} = \widehat{k[G]} & & \widehat{k[G]}
\end{array}
\]
as well as the commutative diagram of isomorphisms

\[
\begin{array}{ccc}
\mathbb{Z}[\tilde{R}[G]] & \xrightarrow{\cong} & \mathbb{Z}[\tilde{k}[G]] \\
\cong & & \cong \\
K_0(R[G]) & \xrightarrow{\rho} & K_0(k[G]).
\end{array}
\]

For purposes of reference we state the last fact as a proposition.

**Proposition 2.1.1** The map \( \rho : K_0(R[G]) \xrightarrow{\cong} K_0(k[G]) \) is an isomorphism; its inverse is given by sending \([M]\) to the class of a projective cover of \(M\) as an \(R[G]\)-module.

We define the composed map

\[
e_G : K_0(k[G]) \xrightarrow{\rho^{-1}} K_0(R[G]) \xrightarrow{\kappa} K_0(K[G]) = R_K(G).
\]

**Remark 2.1.2** Any finitely generated projective \(R\)-module is free.

**Proof** Since \(R\) is an integral domain \(1\) is the only idempotent in \(R\). Hence the free \(R\)-module \(R\) is indecomposable by Corollary 1.5.2. On the other hand, according to Proposition 1.7.4.i the map

\[
\tilde{R} \xrightarrow{\cong} \hat{k}
\]

\[
\{P\} \mapsto \{P/\pi_R P\}
\]

is bijective. Obviously, \(k\) is up to isomorphism the only simple \(k\)-module. Hence \(R\) is up to isomorphism the only finitely generated indecomposable projective \(R\)-module. An arbitrary finitely generated projective \(R\)-module \(P\), by Lemma 1.1.6.i, is a finite direct sum of indecomposable ones. It follows that \(P\) must be isomorphic to some \(R^n\).

\[\square\]

## 2.2 The Triangle

We already have the two sides

\[
\begin{array}{ccc}
R_K(G) & K_0(k[G]) & R_k(G) \\
\downarrow{e_G} & & \downarrow{e_G} \\
R_k(G) & & R_K(G)
\end{array}
\]

of the triangle. To construct the third side we first introduce the following notion.
Definition Let $V$ be a finite-dimensional $K$-vector space; a lattice $L$ in $V$ is an $R$-submodule $L \subseteq V$ for which there exists a $K$-basis $e_1, \ldots, e_d$ of $V$ such that

$$L = Re_1 + \cdots + Re_d.$$ 

Obviously, any lattice is free as an $R$-module. Furthermore, with $L$ also $aL$, for any $a \in K^\times$, is a lattice in $V$.

Lemma 2.2.1

i. Let $L$ be an $R$-submodule of a $K$-vector space $V$; if $L$ is finitely generated then $L$ is free.

ii. Let $L \subseteq V$ be an $R$-submodule of a finite-dimensional $K$-vector space $V$; if $L$ is finitely generated as an $R$-module and $L$ generates $V$ as a $K$-vector space then $L$ is a lattice in $V$.

iii. For any two lattices $L$ and $L'$ in $V$ there is an integer $m \geq 0$ such that $\pi_R^m L \subseteq L'$.

Proof i. and ii. Let $d \geq 0$ be the smallest integer such that the $R$-module $L$ has $d$ generators $e_1, \ldots, e_d$. The $R$-module homomorphism

$$R^d \longrightarrow L$$

$$(a_1, \ldots, a_d) \longmapsto a_1 e_1 + \cdots + a_d e_d$$

is surjective. Suppose that $(a_1, \ldots, a_d) \neq 0$ is an element in its kernel. Since at least one $a_i$ is nonzero the integer

$$\ell := \max\{ j \geq 0 : a_1, \ldots, a_d \in m^j \}$$

is defined. Then $a_i = \pi_R^\ell b_i$ with $b_i \in R$, and $b_{i_0} \in R^\times$ for at least one index $1 \leq i_0 \leq d$. Computing in the vector space $V$ we have

$$0 = a_1 e_1 + \cdots + a_d e_d = \pi_R^\ell (b_1 e_1 + \cdots + b_d e_d)$$

and hence

$$b_1 e_1 + \cdots + b_d e_d = 0.$$ 

But the latter equation implies $e_{i_0} = -\sum_{i \neq i_0} b_{i_0}^{-1} b_i e_i \in \sum_{i \neq i_0} Re_i$, which is a contradiction to the minimality of $d$. It follows that the above map is an isomorphism. This proves i. and, in particular, that

$$L = Re_1 + \cdots + Re_d.$$ 

For ii. it therefore suffices to show that $e_1, \ldots, e_d$, under the additional assumption that $L$ generates $V$, is a $K$-basis of $V$. This assumption immediately guarantees
that the $e_1, \ldots, e_d$ generate the $K$-vector space $V$. To show that they are $K$-linearly independent let

$$c_1 e_1 + \cdots + c_d e_d = 0 \quad \text{with } c_1, \ldots, c_d \in K.$$ 

We find a sufficiently large $j \geq 0$ such that $a_i := \pi^j_R c_i \in R$ for any $1 \leq i \leq d$. Then

$$0 = \pi^j_R \cdot 0 = \pi^j_R (c_1 e_1 + \cdots + c_d e_d) = a_1 e_1 + \cdots + a_d e_d.$$

By what we have shown above we must have $a_i = 0$ and hence $c_i = 0$ for any $1 \leq i \leq d$.

iii. Let $e_1, \ldots, e_d$ and $f_1, \ldots, f_d$ be $K$-bases of $V$ such that

$$L = Re_1 + \cdots + Re_d \quad \text{and} \quad L' = Rf_1 + \cdots + Rf_d.$$

We write

$$e_j = c_{1j} f_1 + \cdots + c_{dj} f_d \quad \text{with } c_{ij} \in K,$$

and we choose an integer $m \geq 0$ such that

$$\pi^m_R c_{ij} \in R \quad \text{for any } 1 \leq i, j \leq d.$$ 

It follows that $\pi^m_R e_j \in Rf_1 + \cdots + Rf_d = L'$ for any $1 \leq j \leq d$ and hence that $\pi^m_R L \subseteq L'$.

Suppose that $V$ is a finitely generated $K[G]$-module. Then $V$ is finite-dimensional as a $K$-vector space. A lattice $L$ in $V$ is called $G$-invariant if we have $gL \subseteq L$ for any $g \in G$. In particular, $L$ is a finitely generated $R[G]$-submodule of $V$, and $L/\pi_R L$ is a $k[G]$-module of finite length.

**Lemma 2.2.2** Any finitely generated $K[G]$-module $V$ contains a lattice which is $G$-invariant.

**Proof** We choose a basis $e_1, \ldots, e_d$ of the $K$-vector space $V$. Then $L' := Re_1 + \cdots + Re_d$ is a lattice in $V$. We define the $R[G]$-submodule

$$L := \sum_{g \in G} g L'$$

of $V$. With $L'$ also $L$ generates $V$ as a $K$-vector space. Moreover, $L$ is finitely generated by the set $\{ge_i : 1 \leq i \leq d, g \in G\}$ as an $R$-module. Therefore, by Lemma 2.2.1.ii, $L$ is a $G$-invariant lattice in $V$. 

Whereas the $K[G]$-module $V$ always is projective by Remark 1.7.3 a $G$-invariant lattice $L$ in $V$ need not to be projective as an $R[G]$-module. We will encounter an example of this later on.
**Theorem 2.2.3** Let $L$ and $L'$ be two $G$-invariant lattices in the finitely generated $K[G]$-module $V$; we then have

$$[L/π_R L] = [L'/π_R L'] \quad \text{in } R_k(G).$$

**Proof** We begin by observing that, for any $a \in K^\times$, the map

$$L/π_R L \xrightarrow{a} (aL)/π_R (aL)$$

$$x + π_R L \mapsto ax + π_R (aL)$$

is an isomorphism of $k[G]$-modules, and hence

$$[L/π_R L] = [(aL)/π_R (aL)] \quad \text{in } R_k(G).$$

By applying Lemma 2.2.1.iii (and replacing $L$ by $π^n_R L$ for some sufficiently large $m \geq 0$) we therefore may assume that $L \subseteq L'$. By applying Lemma 2.2.1.iii again to $L$ and $L'$ (while interchanging their roles) we find an integer $n \geq 0$ such that

$$π^n_R L' \subseteq L \subseteq L'.$$

We now proceed by induction with respect to $n$. If $n = 1$ we have the two exact sequences of $k[G]$-modules

$$0 \longrightarrow L/π_R L' \longrightarrow L'/π_R L' \longrightarrow L'/L \longrightarrow 0$$

and

$$0 \longrightarrow π_R L'/π_R L \longrightarrow L/π_R L \longrightarrow L/π_R L' \longrightarrow 0.$$  

It follows that

$$[L'/π_R L'] = [L/π_R L'] + [L'/L] = [L/π_R L'] + [π_R L'/π_R L]$$

$$= [L/π_R L'] + [L/π_R L] - [L/π_R L']$$

$$= [L/π_R L]$$

in $R_k(G)$. For $n \geq 2$ we consider the $R[G]$-submodule

$$M := π_R^{n-1} L' + L.$$

It is a $G$-invariant lattice in $V$ by Lemma 2.2.1.ii and satisfies

$$π_R^{n-1} L' \subseteq M \subseteq L' \quad \text{and} \quad π_R M \subseteq L \subseteq M.$$

The above lemma and theorem imply that

\[ Z[M_K[G]] \rightarrow R_K(G) \]

\[ \{V\} \mapsto [L/\pi_R L], \]

where \( L \) is any \( G \)-invariant lattice in \( V \), is a well-defined homomorphism. If \( V_1 \) and \( V_2 \) are two finitely generated \( K[G] \)-modules and \( L_1 \subseteq V_1 \) and \( L_2 \subseteq V_2 \) are \( G \)-invariant lattices then \( L_1 \oplus L_2 \) is a \( G \)-invariant lattice in \( V_1 \oplus V_2 \) and

\[ [(L_1 \oplus L_2)/\pi_R (L_1 \oplus L_2)] = [L_1/\pi_R L_1 \oplus L_2/\pi_R L_2] = [L_1/\pi_R L_1] + [L_2/\pi_R L_2]. \]

It follows that the subgroup \( \text{Rel} \subseteq Z[M_K[G]] \) lies in the kernel of the above map so that we obtain the homomorphism

\[ d_G: \ R_K(G) \rightarrow R_k(G) \]

\[ \{V\} \mapsto [L/\pi_R L]. \]

It is called the decomposition homomorphism of \( G \). The Cartan–Brauer triangle is the diagram

\[ R_K(G) \xrightarrow{d_G} R_k(G) \]

\[ \xrightarrow{c_G} K_0(k[G]). \]

**Lemma 2.2.4** The Cartan–Brauer triangle is commutative.

**Proof** Let \( P \) be a finitely generated projective \( R[G] \)-module. We have to show that

\[ d_G(\kappa([P])) = c_G(\rho([P])) \]

holds true. By definition the right-hand side is equal to \([P/\pi_R P] \in R_K(G)\). Moreover, \( \kappa([P]) = [K \otimes_R P] \in R_K(G) \). According to Proposition 1.6.4 the \( R(G) \)-module \( P \) is a direct summand of a free \( R[G] \)-module. But \( R[G] \) and hence any free \( R[G] \)-module also is free as an \( R \)-module. We see that \( P \) as an \( R \)-module is finitely generated projective and hence free by Remark 2.1.2. We conclude that \( P \cong R^d \) is a \( G \)-invariant lattice in the \( K[G] \)-module \( K \otimes_R P \cong K \otimes_R R^d = (K \otimes_R R)^d = K^d \), and we obtain \( d_G(\kappa([P])) = d_G(\rho([K \otimes_R P])) = [P/\pi_R P]. \)

Let us consider two “extreme” situations where the maps in the Cartan–Brauer triangle can be determined completely. First we look at the case where \( p \) does not divide the order \( |G| \) of the group \( G \). Then \( k[G] \) is semisimple. Hence we have \( k[G] = k[G] \) and \( K_0(k[G]) = R_k(G) \) by Remark 1.7.3. The map \( c_G \), in particular, is the identity.
Proposition 2.2.5 If \( p \nmid |G| \) then any \( R[G] \)-module \( M \) which is projective as an \( R \)-module also is projective as an \( R[G] \)-module.

Proof We consider any “test diagram” of \( R[G] \)-modules

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & N \\
\downarrow{\beta} & & \downarrow{\gamma} \\
L & \rightarrow & 0
\end{array}
\]

Viewing this as a “test diagram” of \( R \)-modules our second assumption ensures the existence of an \( R \)-module homomorphism \( \alpha_0 : M \rightarrow L \) such that \( \beta \circ \alpha_0 = \alpha \). Since \( |G| \) is a unit in \( R \) by our first assumption, we may define a new \( R \)-module homomorphism \( \tilde{\alpha} : M \rightarrow L \) by

\[
\tilde{\alpha}(x) := |G|^{-1} \sum_{g \in G} g \alpha_0(g^{-1} x)
\]

for any \( x \in M \).

One easily checks that \( \tilde{\alpha} \) satisfies

\[
\tilde{\alpha}(hx) = h \tilde{\alpha}(x)
\]

for any \( h \in G \) and any \( x \in M \).

This means that \( \tilde{\alpha} \) is, in fact, an \( R[G] \)-module homomorphism. Moreover, we compute

\[
\begin{align*}
\beta(\tilde{\alpha}(x)) &= |G|^{-1} \sum_{g \in G} g \beta(\alpha_0(g^{-1} x)) \\
&= |G|^{-1} \sum_{g \in G} g \alpha(g^{-1} x) \\
&= |G|^{-1} \sum_{g \in G} gg^{-1} \alpha(x) = \alpha(x).
\end{align*}
\]

Corollary 2.2.6 If \( p \nmid |G| \) then all three maps in the Cartan–Brauer triangle are isomorphisms; more precisely, we have the triangle of bijections

\[
\begin{array}{ccc}
\hat{K}[G] & \xrightarrow{\sim} & \hat{k}[G] \\
\uparrow{\sim} & & \uparrow{\sim} \\
\{P\} \mapsto \{K \otimes_R P\} & \xrightarrow{\sim} & \{P/\pi_R P\} \\
\otimes & & \\
\hat{R}[G].
\end{array}
\]

Proof We already have remarked that \( c_G \) is the identity. Hence it suffices to show that the map \( \kappa : K_0(R[G]) \rightarrow R_K(G) \) is surjective. Let \( V \) be any finitely generated \( K[G] \)-module. By Lemma 2.2.2 we find a \( G \)-invariant lattice \( L \) in \( V \). It satisfies \( V = K \otimes_R L \) by definition. Proposition 2.2.5 implies that \( L \) is a finitely generated
projective $R[G]$-module. We conclude that $[L] \in K_0(R[G])$ with $\kappa([L]) = [V]$. This argument in fact shows that the map

$$M_{R[G]}/ \cong \to M_{K[G]}/ \cong$$

$$[P] \mapsto [K \otimes_R P]$$

is surjective. Let $P$ and $Q$ be two finitely generated projective $R[G]$-modules such that $K \otimes_R P \cong K \otimes_R Q$ as $K[G]$-modules. The commutativity of the Cartan–Brauer triangle then implies that

$$[P/\pi_R P] = d_G([K \otimes_R P]) = d_G([K \otimes_R Q]) = [Q/\pi_R Q].$$

Let $P = P_1 \oplus \cdots \oplus P_s$ and $Q = Q_1 \oplus \cdots \oplus Q_t$ be decompositions into indecomposable submodules. Then

$$P/\pi_R P = \bigoplus_{i=1}^s P_i/\pi_R P_i \quad \text{and} \quad Q/\pi_R Q = \bigoplus_{j=1}^t Q_j/\pi_R Q_j$$

are decompositions into simple submodules. Using Proposition 1.7.1 the identity

$$\sum_{i=1}^s [P_i/\pi_R P_i] = [P/\pi_R P] = [Q/\pi_R Q] = \sum_{j=1}^t [Q_j/\pi_R Q_j]$$

implies that $s = t$ and that there is a permutation $\sigma$ of \{1, \ldots, s\} such that

$$Q_j/\pi_R Q_j \cong P_{\sigma(j)}/\pi_R P_{\sigma(j)} \quad \text{for any } 1 \leq j \leq s$$

as $k[G]$-modules. Applying Proposition 1.7.4.i we obtain

$$Q_j \cong P_{\sigma(j)} \quad \text{for any } 1 \leq j \leq s$$

as $R[G]$-modules. It follows that $P \cong Q$ as $R[G]$-modules. Hence the above map between sets of isomorphism classes is bijective. Obviously, if $K \otimes_R P$ is indecomposable (i.e. simple) then $P$ was indecomposable. Vice versa, if $P$ is indecomposable then the above reasoning says that $P/\pi_R P$ is simple. Because of $[P/\pi_R P] = d_G([K \otimes_R P])$ it follows that $K \otimes_R P$ must be indecomposable. □

The second case is where $G$ is a $p$-group. For a general group $G$ we have the ring homomorphism

$$k[G] \to k$$

$$\sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g$$
2.2 The Triangle

which is called the augmentation of \( k[G] \). It makes \( k \) into a simple \( k[G] \)-module which is called the trivial \( k[G] \)-module. Its kernel is the augmentation ideal

\[
I_k[G] := \left\{ \sum_{g \in G} a_g g \in k[G] : \sum_{g \in G} a_g = 0 \right\}.
\]

**Proposition 2.2.7** If \( G \) is a \( p \)-group then we have \( \text{Jac}(k[G]) = I_k[G] \); in particular, \( k[G] \) is a local ring and the trivial \( k[G] \)-module is, up to isomorphism, the only simple \( k[G] \)-module.

**Proof** We will prove by induction with respect to the order \( |G| = p^n \) of \( G \) that the trivial module, up to isomorphism, is the only simple \( k[G] \)-module. There is nothing to prove if \( n = 0 \). We therefore suppose that \( n \geq 1 \). Note that we have

\[
g^{p^n} = 1 \quad \text{and hence} \quad (g - 1)^{p^n} = g^{p^n} - 1 = 1 - 1 = 0
\]

for any \( g \in G \). The center of a nontrivial \( p \)-group is nontrivial. Let \( g_0 \neq 1 \) be a central element in \( G \). We now consider any simple \( k[G] \)-module \( M \) and we denote by \( \pi : k[G] \longrightarrow \text{End}_k(M) \) the corresponding ring homomorphism. Then

\[
\left( \pi(g_0) - \text{id}_M \right)^{p^n} = \pi(g_0 - 1)^{p^n} = \pi((g_0 - 1)^{p^n}) = \pi(0) = 0.
\]

Since \( g_0 \) is central \( (\pi(g_0) - \text{id}_M)(M) \) is a \( k[G] \)-submodule of \( M \). But \( M \) is simple. Hence \( (\pi(g_0) - \text{id}_M)(M) = \{0\} \) or \( = M \). The latter would inductively imply that \( (\pi(g_0) - \text{id}_M)^{p^n}(M) = M \neq \{0\} \) which is contradiction. We obtain \( \pi(g_0) = \text{id}_M \) which means that the cyclic subgroup \( \langle g_0 \rangle \) is contained in the kernel of the group homomorphism \( \pi : G \longrightarrow \text{Aut}_k(M) \). Hence we have a commutative diagram of group homomorphisms

\[
\begin{array}{ccc}
G & \longrightarrow & \text{Aut}_k(M) \\
\downarrow \pi & & \downarrow \pi \\
G/\langle g_0 \rangle & \longrightarrow & \end{array}
\]

We conclude that \( M \) already is a simple \( k[G/\langle g_0 \rangle] \)-module and therefore is the trivial module by the induction hypothesis.

The identity \( \text{Jac}(k[G]) = I_k[G] \) now follows from the definition of the Jacobson radical, and Proposition 1.4.1 implies that \( k[G] \) is a local ring. \( \square \)

Suppose that \( G \) is a \( p \)-group. Then Propositions 2.2.7 and 1.7.1 together imply that the map

\[
\mathbb{Z} \overset{\approx}{\longrightarrow} R_k(G) \\
m \longmapsto m[k]
\]
is an isomorphism. To compute the inverse map let $M$ be a $k[G]$-module of finite length, and let $\{0\} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$ be a composition series. We must have $M_i / M_{i-1} \cong k$ for any $1 \leq i \leq t$. It follows that

$$[M] = \sum_{i=1}^{t} [M_i / M_{i-1}] = t \cdot [k] \quad \text{and} \quad \dim_k M = \sum_{i=1}^{t} \dim_k M_i / M_{i-1} = t.$$ 

Hence the inverse map is given by

$$[M] \mapsto \dim_k M.$$

Furthermore, from Proposition 1.7.4 we obtain that

$$\mathbb{Z} \xrightarrow{\cong} K_0(R[G])$$

$$m \mapsto m[R[G]]$$

is an isomorphism. Because of $\dim_k k[G] = |G|$ the Cartan homomorphism, under these identifications, becomes the map

$$c_G: \mathbb{Z} \rightarrow \mathbb{Z}$$

$$m \mapsto m \cdot |G|.$$ 

For any finitely generated $K[G]$-module $V$ and any $G$-invariant lattice $L$ in $V$ we have $\dim_K V = \dim_k L / \pi_R L$. Hence the decomposition homomorphism becomes

$$d_G: R_K(G) \rightarrow \mathbb{Z}$$

$$[V] \mapsto \dim_K V.$$ 

Altogether the Cartan–Brauer triangle of a $p$-group $G$ is of the form

$$R_K(G) \xrightarrow{[V] \mapsto \dim_K V} \mathbb{Z}$$

$$1 \mapsto [K[G]]$$

$$\mathbb{Z} \cdot |G|.$$ 

We also see that the trivial $K[G]$-module $K$ has the $G$-invariant lattice $R$ which cannot be projective as an $R[G]$-module if $G \neq \{1\}$.

Before we can establish the finer properties of the Cartan–Brauer triangle we need to develop the theory of induction.

### 2.3 The Ring Structure of $R_F(G)$, and Induction

In this section we let $F$ be an arbitrary field, and we consider the group ring $F[G]$ and its Grothendieck group $R_F(G) := R(F[G])$. 
Let $V$ and $W$ be two (finitely generated) $F[G]$-modules. The group $G$ acts on the tensor product $V \otimes_F W$ by

$$g(v \otimes w) := gv \otimes gw \quad \text{for } v \in V \text{ and } w \in W.$$  

In this way $V \otimes_F W$ becomes a (finitely generated) $F[G]$-module, and we obtain the multiplication map

$$\mathbb{Z}[\mathcal{M}_{F[G]}] \times \mathbb{Z}[\mathcal{M}_{F[G]}] \rightarrow \mathbb{Z}[\mathcal{M}_{F[G]}]$$

$$(\{V\}, \{W\}) \mapsto \{V \otimes_F W\}.$$  

Since the tensor product, up to isomorphism, is associative and commutative this multiplication makes $\mathbb{Z}[\mathcal{M}_{F[G]}]$ into a commutative ring. Its unit element is the isomorphism class $\{F\}$ of the trivial $F[G]$-module.

**Remark 2.3.1** The subgroup Rel is an ideal in the ring $\mathbb{Z}[\mathcal{M}_{F[G]}]$.

**Proof** Let $V$ be a (finitely generated) $F[G]$-module and let

$$0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$$

be a short exact sequence of $F[G]$-modules. We claim that the sequence

$$0 \rightarrow V \otimes_F L \xrightarrow{id_V \otimes \alpha} V \otimes_F M \xrightarrow{id_V \otimes \beta} V \otimes_F N \rightarrow 0$$

is exact as well. This shows that the subgroup Rel is preserved under multiplication by $\{V\}$. The exactness in question is purely a problem about $F$-vector spaces. But as vector spaces we have $M \cong L \oplus N$ and hence $V \otimes_F M \cong (V \otimes_F L) \oplus (V \otimes_F N)$. \hfill $\square$

It follows that $R_F(G)$ naturally is a commutative ring with unit element $[F]$ such that

$$[V] \cdot [W] = [V \otimes_F W].$$

Let $H \subseteq G$ be a subgroup. Then $F[H] \subseteq F[G]$ is a subring (with the same unit element). Any $F[G]$-module $V$, by restriction of scalars, can be viewed as an $F[H]$-module. If $V$ is finitely generated as an $F[G]$-module then $V$ is a finite-dimensional $F$-vector space and, in particular, is finitely generated as an $F[H]$-module. Hence we have the ring homomorphism

$$\text{res}_H^G : R_F(G) \rightarrow R_F(H)$$

$$[V] \mapsto [V].$$

On the other hand, for any $F[H]$-module $W$ we have, by base extension, the $F[G]$-module $F[G] \otimes_{F[H]} W$. Obviously, the latter is finitely generated over $F[G]$ if the
former was finitely generated over \( F[H] \). The first Frobenius reciprocity says that

\[
\text{Hom}_{F[G]}(F[G] \otimes_{F[H]} W, V) \xrightarrow{\cong} \text{Hom}_{F[H]}(W, V) \\
\alpha \mapsto [w \mapsto \alpha(1 \otimes w)]
\]

is an \( F \)-linear isomorphism for any \( F[H] \)-module \( W \) and any \( F[G] \)-module \( V \).

**Remark 2.3.2** For any short exact sequence

\[
0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0
\]

of \( F[H] \)-modules the sequence of \( F[G] \)-modules

\[
0 \rightarrow F[G] \otimes_{F[H]} L \xrightarrow{\text{id}_{F[G]} \otimes \alpha} F[G] \otimes_{F[H]} M \xrightarrow{\text{id}_{F[G]} \otimes \beta} F[G] \otimes_{F[H]} N \rightarrow 0
\]

is exact as well.

**Proof** Let \( g_1, \ldots, g_r \in G \) be a set of representatives for the left cosets of \( H \) in \( G \). Then \( g_1, \ldots, g_r \) also is a basis of the free right \( F[H] \)-module \( F[G] \). It follows that, for any \( F[H] \)-module \( W \), the map

\[
W' \xrightarrow{\cong} F[G] \otimes_{F[H]} W \\
(w_1, \ldots, w_r) \mapsto g_1 \otimes w_1 + \cdots + g_r \otimes w_r
\]

is an \( F \)-linear isomorphism. We see that, as a sequence of \( F \)-vector spaces, the sequence in question is just the \( r \)-fold direct sum of the original exact sequence with itself. \( \square \)

**Remark 2.3.3** For any \( F[H] \)-module \( W \), if the \( F[G] \)-module \( F[G] \otimes_{F[H]} W \) is simple then \( W \) is a simple \( F[H] \)-module.

**Proof** Let \( W' \subseteq W \) be any \( F[H] \)-submodule. Then \( F[G] \otimes_{F[H]} W' \) is an \( F[G] \)-submodule of \( F[G] \otimes_{F[H]} W \) by Remark 2.3.2. Since the latter is simple we must have \( F[G] \otimes_{F[H]} W' = \{0\} \) or \( = F[G] \otimes_{F[H]} W \). Comparing dimensions using the argument in the proof of Remark 2.3.2 we obtain

\[
[G : H] \cdot \dim_F W' = \dim_F F[G] \otimes_{F[H]} W' = 0 \quad \text{or} \quad \dim_F F[G] \otimes_{F[H]} W = [G : H] \cdot \dim_F W.
\]

We see that \( \dim_F W' = 0 \) or \( = \dim_F W \) and therefore that \( W' = \{0\} \) or \( = W \). This proves that \( W \) is a simple \( F[H] \)-module. \( \square \)

It follows that the map

\[
\mathbb{Z}[\mathfrak{m}_{F[H]}] \longrightarrow \mathbb{Z}[\mathfrak{m}_{F[G]}] \\
\{W\} \longmapsto \{F[G] \otimes_{F[H]} W\}
\]
preserves the subgroups Rel in both sides and therefore induces an additive homomorphism

\[
\text{ind}_H^G : \ R_F(H) \longrightarrow R_F(G)
\]

\[
[W] \longmapsto [F[G] \otimes_{F[H]} W]
\]

(which is not multiplicative!).

**Proposition 2.3.4** We have

\[
\text{ind}_H^G(y) \cdot x = \text{ind}_H^G(y \cdot \text{res}_H^G(x)) \quad \text{for any } x \in R_F(G) \text{ and } y \in R_F(H).
\]

**Proof** It suffices to show that, for any \(F[G]\)-module \(V\) and any \(F[H]\)-module \(W\), we have an isomorphism of \(F[G]\)-modules

\[
(F[G] \otimes_{F[H]} W) \otimes_F V \cong F[G] \otimes_{F[H]} (W \otimes_F V).
\]

One checks (exercise!) that such an isomorphism is given by

\[
(g \otimes w) \otimes v \longmapsto g \otimes (w \otimes g^{-1}v).
\]

\[\square\]

**Corollary 2.3.5** The image of \(\text{ind}_H^G : R_F(H) \longrightarrow R_F(G)\) is an ideal in \(R_F(G)\).

We also mention the obvious transitivity relations

\[
\text{res}_{H'}^H \circ \text{res}_H^G = \text{res}_{H'}^G \quad \text{and} \quad \text{ind}_H^G \circ \text{ind}_{H'}^G = \text{ind}_{H'}^G
\]

for any chain of subgroups \(H' \subseteq H \subseteq G\).

An alternative way to look at induction is the following. Let \(W\) be any \(F[H]\)-module. Then

\[
\text{Ind}_H^G(W) := \{ \phi : G \longrightarrow W : \phi(gh) = h^{-1}\phi(g) \text{ for any } g \in G, h \in H \}
\]

equipped with the *left translation action* of \(G\) given by

\[
s \phi(g') := \phi(g^{-1}g')
\]

is an \(F[G]\)-module called the *module induced* from \(W\). But, in fact, the map

\[
F[G] \otimes_{F[H]} W \xrightarrow{\cong} \text{Ind}_H^G(W)
\]

\[
\left( \sum_{g \in G} a_g g \right) \otimes w \longmapsto \phi(g') := \sum_{h \in H} a_{g'h} hw
\]

is an isomorphism of \(F[G]\)-modules. This leads to the *second Frobenius reciprocity* isomorphism

\[
\text{Hom}_{F[G]}(V, \text{Ind}_H^G(W)) \xrightarrow{\cong} \text{Hom}_{F[H]}(V, W)
\]

\[
\alpha \longmapsto [v \longmapsto \alpha(v)(1)].
\]
We also need to recall the character theory of $G$ in the semisimple case. For this we assume for the rest of this section that the order of $G$ is prime to the characteristic of the field $F$. Any finitely generated $F[G]$-module $V$ is a finite-dimensional $F$-vector space. Hence we may introduce the function

$$\chi_V: \ G \longrightarrow F$$

$$g \longmapsto \text{tr}(g; V) = \text{trace of } V \underset{g}{\rightarrow} V$$

which is called the character of $V$. It depends only on the isomorphism class of $V$. Characters are class functions on $G$, i.e. they are constant on each conjugacy class of $G$. For any two finitely generated $F[G]$-modules $V_1$ and $V_2$ we have

$$\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2} \quad \text{and} \quad \chi_{V_1 \otimes_F V_2} = \chi_{V_1} \cdot \chi_{V_2}.$$ 

Let $\text{Cl}(G, F)$ denote the $F$-vector space of all class functions $G \longrightarrow F$. By pointwise multiplication of functions it is a commutative $F$-algebra. The above identities imply that the map

$$\text{Tr}: \ R_F(G) \longrightarrow \text{Cl}(G, F)$$

$$[V] \longmapsto \chi_V$$

is a ring homomorphism.

**Definition** The field $F$ is called a splitting field for $G$ if, for any simple $F[G]$-module $V$, we have $\text{End}_{F[G]}(V) = F$.

If $F$ is algebraically closed then it is a splitting field for $G$.

**Theorem 2.3.6**

i. If the field $F$ has characteristic zero then the characters $\{\chi_V : \{V \in \hat{F[G]}\} \}$ are $F$-linearly independent.

ii. If $F$ is a splitting field for $G$ then the characters $\{\chi_V : \{V \in \hat{F[G]}\} \}$ form a basis of the $F$-vector space $\text{Cl}(G, F)$.

iii. If $F$ has characteristic zero then two finitely generated $F[G]$-modules $V_1$ and $V_2$ are isomorphic if and only if $\chi_{V_1} = \chi_{V_2}$ holds true.

**Corollary 2.3.7**

i. If $F$ has characteristic zero then the map $\text{Tr}$ is injective.

ii. If $F$ is a splitting field for $G$ then the map $\text{Tr}$ induces an isomorphism of $F$-algebras

$$F \otimes_\mathbb{Z} R_F(G) \overset{\simeq}{\longrightarrow} \text{Cl}(G, F).$$
iii. If $F$ has characteristic zero then the map 
\[ \mathfrak{M}_{F[G]} / \cong \longrightarrow \text{Cl}(G, F) \]
\[ \{V\} \longmapsto \chi_V \]
is injective.

**Proof** For i. and ii. use Proposition 1.7.1. \qed

## 2.4 The Burnside Ring

A $G$-set $X$ is a set equipped with a $G$-action
\[ G \times X \longrightarrow X \]
\[ (g, x) \longmapsto g x \]
such that
\[ 1 x = x \quad \text{and} \quad g(hx) = (gh)x \quad \text{for any } g, h \in G \text{ and any } x \in X. \]

Let $X$ and $Y$ be two $G$-sets. Their disjoint union $X \cup Y$ is a $G$-set in an obvious way. But also their cartesian product $X \times Y$ is a $G$-set with respect to
\[ g(x, y) := (gx, gy) \quad \text{for } (x, y) \in X \times Y. \]

We will call $X$ and $Y$ isomorphic if there is a bijective map $\alpha : X \longrightarrow Y$ such that $\alpha(gx) = g\alpha(x)$ for any $g \in G$ and $x \in X$.

Let $S_G$ denote the set of all isomorphism classes $\{X\}$ of finite $G$-sets $X$. In the free abelian group $\mathbb{Z}[S_G]$ we consider the subgroup Rel generated by all elements of the form
\[ \{X \cup Y\} - \{X\} - \{Y\} \quad \text{for any two finite } G \text{-sets } X \text{ and } Y. \]

We define the factor group
\[ B(G) := \mathbb{Z}[S_G] / \text{Rel}, \]
and we let $[X] \in B(G)$ denote the image of the isomorphism class $\{X\}$. In fact, the map
\[ \mathbb{Z}[S_G] \times \mathbb{Z}[S_G] \longrightarrow \mathbb{Z}[S_G] \]
\[ ([X], [Y]) \longmapsto \{X \times Y\} \]
makes $\mathbb{Z}[S_G]$ into a commutative ring in which the unit element is the isomorphism class of the $G$-set with one point. Because of $(X_1 \cup X_2) \times Y = (X_1 \times Y) \cup (X_2 \times Y)$
the subgroup Rel is an ideal in \( \mathbb{Z}[S_G] \). We see that \( B(G) \) is a commutative ring. It is called the Burnside ring of \( G \).

Two elements \( x, y \) in a \( G \)-set \( X \) are called equivalent if there is a \( g \in G \) such that \( y = gx \). This defines an equivalence relation on \( X \). The equivalence classes are called \( G \)-orbits. They are of the form \( Gx = \{ gx : g \in G \} \) for some \( x \in X \). A nonempty \( G \)-set which consists of a single \( G \)-orbit is called simple (or transitive or a principal homogeneous space). The decomposition of an arbitrary \( G \)-set \( X \) into its \( G \)-orbits is the unique decomposition of \( X \) into simple \( G \)-sets. In particular, the only \( G \)-subsets of a simple \( G \)-set \( Y \) are \( Y \) and \( \emptyset \). We let \( S_G \) denote the set of isomorphism classes of simple \( G \)-sets.

**Lemma 2.4.1** \( \mathbb{Z}[S_G] \overset{\sim}{\longrightarrow} B(G) \).

**Proof** If \( X = Y_1 \cup \cdots \cup Y_n \) is the decomposition of \( X \) into its \( G \)-orbits then we put

\[
\pi(\{X\}) := \{Y_1\} + \cdots + \{Y_n\}.
\]

This defines an endomorphism \( \pi \) of \( \mathbb{Z}[S_G] \) which is idempotent with \( \text{im}(\pi) = \mathbb{Z}[S_G] \). It is rather clear that \( \text{Rel} \subseteq \text{ker}(\pi) \). Moreover, using the identities

\[
[Y_1] + [Y_2] = [Y_1 \cup Y_2], \quad [Y_1 \cup Y_2] + [Y_3] = [Y_1 \cup Y_2 \cup Y_3], \quad \ldots,
\]

we see that

\[
[X] = [Y_1] + \cdots + [Y_n]
\]

and hence that \( \{X\} - \pi(\{X\}) \in \text{Rel} \). As in the proof of Proposition 1.7.1 these three facts together imply

\[
\mathbb{Z}[S_G] = \mathbb{Z}[S_G] \oplus \text{Rel}.
\]

For any subgroup \( H \subseteq G \) the coset space \( G/H \) is a simple \( G \)-set with respect to

\[
G \times G/H \longrightarrow G/H \quad (g, g'H) \mapsto gg'H.
\]

Simple \( G \)-sets of this form are called standard \( G \)-sets.

**Remark 2.4.2** Each simple \( G \)-set \( X \) is isomorphic to some standard \( G \)-set \( G/H \).

**Proof** We fix a point \( x \in X \). Let \( G_x := \{ g \in G : gx = x \} \) be the stabilizer of \( x \) in \( G \). Then

\[
G/G_x \overset{\sim}{\longrightarrow} X \quad gG_x \mapsto gx
\]

is an isomorphism.
It follows that the set \( S_G \) is finite.

**Lemma 2.4.3** Two standard \( G \)-sets \( G/H_1 \) and \( G/H_2 \) are isomorphic if and only if there is a \( g_0 \in G \) such that \( g_0^{-1}H_1g_0 = H_2 \).

*Proof* If \( g_0^{-1}H_1g_0 = H_2 \) then
\[
\begin{align*}
G/H_1 & \xrightarrow{\sim} G/H_2 \\
gH_1 & \longmapsto g g_0 H_2
\end{align*}
\]
is an isomorphism of \( G \)-sets. Vice versa, let
\[
\alpha : G/H_1 \xrightarrow{\sim} G/H_2
\]
be an isomorphism of \( G \)-sets. We have \( \alpha(1H_1) = g_0H_2 \) for some \( g_0 \in G \) and then
\[
g_0H_2 = \alpha(1H_1) = \alpha(h_1H_1) = h_1\alpha(1H_1) = h_1g_0H_2
\]
for any \( h_1 \in H_1 \). This implies \( g_0^{-1}H_1g_0 \subseteq H_2 \). On the other hand
\[
g_0^{-1}H_1 = \alpha^{-1}(1H_2) = \alpha^{-1}(h_2H_2) = h_2\alpha^{-1}(1H_2) = h_2g_0^{-1}H_1
\]
for any \( h_2 \in H_2 \) which implies \( g_0H_2g_0^{-1} \subseteq H_1 \). \( \square \)

**Exercise 2.4.4** Let \( G/H_1 \) and \( G/H_2 \) be two standard \( G \)-sets; we then have
\[
[G/H_1] \cdot [G/H_2] = \sum_{g \in H_1 \setminus G/H_2} [G/H_1 \cap gH_2g^{-1}] \quad \text{in } B(G)
\]
where \( H_1 \setminus G/H_2 \) denotes the space of double cosets \( H_1gH_2 \) in \( G \).

Let \( F \) again be an arbitrary field. For any finite set \( X \) we have the finite-dimensional \( F \)-vector space
\[
F[X] := \left\{ \sum_{x \in X} a_x x : a_x \in F \right\}
\]
“with basis \( X \)”. Suppose that \( X \) is a finite \( G \)-set. Then the group \( G \) acts on \( F[X] \) by
\[
g \left( \sum_{x \in X} a_x x \right) := \sum_{x \in X} a_x gx = \sum_{x \in X} a_{g^{-1}x} x.
\]
In this way \( F[X] \) becomes a finitely generated \( F[G] \)-module (called a *permutation module*). If \( \alpha : X \xrightarrow{\sim} Y \) is an isomorphism of finite \( G \)-sets then
\[
\tilde{\alpha} : F[X] \xrightarrow{\sim} F[Y]
\]
\[
\sum_{x \in X} a_x x \longmapsto \sum_{x \in X} a_x \alpha(x) = \sum_{y \in Y} a_{\alpha^{-1}(y)} y
\]
is an isomorphism of $F[G]$-modules. It follows that the map

$$S_G \longrightarrow \mathcal{M}_{F[G]}/ \cong$$

$$\{X\} \longmapsto \{F[X]\}$$

is well defined. We obviously have

$$F[X_1 \cup X_2] = F[X_1] \oplus F[X_2]$$

for any two finite $G$-sets $X_1$ and $X_2$. Hence the above map respects the subgroups Rel in both sides and induces a group homomorphism

$$b: \quad B(G) \longrightarrow R_F(G)$$

$$[X] \longmapsto [F[X]].$$

**Remark** There is a third interesting Grothendieck group for the ring $F[G]$ which is the factor group

$$A_F(G) := \mathbb{Z}[\mathcal{M}_{F[G]}]/\text{Rel}_\oplus$$

with respect to the subgroup Rel$_\oplus$ generated by all elements of the form

$$\{M \oplus N\} - \{M\} - \{N\}$$

where $M$ and $N$ are arbitrary finitely generated $F[G]$-modules. We note that Rel$_\oplus \subseteq$ Rel. The above map $b$ is the composite of the maps

$$B(G) \longrightarrow A_F(G) \xrightarrow{\text{pr}} R_F(G)$$

$$[X] \longmapsto [F[X]] \longmapsto [F[X]].$$

**Remark 2.4.5** For any two finite $G$-sets $X_1$ and $X_2$ we have

$$F[X_1 \times X_2] \cong F[X_1] \otimes_F F[X_2]$$


**Proof** The vectors $(x_1, x_2)$, resp. $x_1 \otimes x_2$, for $x_1 \in X_1$ and $x_2 \in X_2$, form an $F$-basis of the left-, resp. right-, hand side. Hence there is a unique $F$-linear isomorphism mapping $(x_1, x_2)$ to $x_1 \otimes x_2$. Because of

$$g(x_1, x_2) = (gx_1, gx_2) \longmapsto gx_1 \otimes gx_2 = g(x_1 \otimes x_2),$$

for any $g \in G$, this map is an $F[G]$-module isomorphism. □

It follows that the map

$$b: \quad B(G) \longrightarrow R_F(G)$$
is a ring homomorphism. Note that the unit element \([G/G]\) in \(B(G)\) is mapped to the class \([F]\) of the trivial module \(F\) which is the unit element in \(R_F(G)\).

**Lemma 2.4.6**

i. For any standard \(G\)-set \(G/H\) we have

\[
b([G/H]) = \text{ind}^G_H(1)
\]

where 1 on the right-hand side denotes the unit element of \(R_F(H)\).

ii. For any finite \(G\)-set \(X\) we have

\[
\text{tr}(g; F[X]) = |\{x \in X : gx = x\}| \in F \quad \text{for any } g \in G.
\]

**Proof** i. Let \(F\) be the trivial \(F[H]\)-module. It suffices to establish an isomorphism

\[
F[G/H] \cong F[G] \otimes_{F[H]} F.
\]

For this purpose we consider the \(F\)-bilinear map

\[
\beta : F[G] \times F \to F[G/H]
\]

\[
\left( \sum_{g \in G} a_g g, a \right) \mapsto \sum_{g \in G} a_g g H.
\]

Because of

\[
\beta(gh, a) = ghH = gH = \beta(g, a) = \beta(g, ha)
\]

for any \(h \in H\) the map \(\beta\) is \(F[H]\)-balanced and therefore induces a well-defined \(F\)-linear map

\[
\tilde{\beta} : F[G] \otimes_{F[H]} F \to F[G/H]
\]

\[
\left( \sum_{g \in G} a_g g \right) \otimes a \mapsto \sum_{g \in G} a_g g H.
\]

As discussed in the proof of Remark 2.3.2 we have \(F[G] \otimes_{F[H]} F \cong F^{[G:H]}\) as \(F\)-vector spaces. Hence \(\tilde{\beta}\) is a map between \(F\)-vector spaces of the same dimension. It obviously is surjective and therefore bijective. Finally the identity

\[
\tilde{\beta}\left( g' \left( \sum_{g \in G} a_g g \right) \otimes a \right) = \tilde{\beta}\left( \sum_{g \in G} a_g g' g \right) = \sum_{g \in G} a_g g' g H
\]

\[
= g' \left( \sum_{g \in G} a_g g H \right) = g' \tilde{\beta}\left( \sum_{g \in G} a_g g \right) \otimes a
\]

for any \(g' \in G\) shows that \(\tilde{\beta}\) is an isomorphism of \(F[G]\)-modules.
ii. The matrix \((a_{x,y})_{x,y}\) of the \(F\)-linear map \(F[X] \xrightarrow{g} F[X]\) with respect to the basis \(X\) is given by the equations

\[
gy = \sum_{x \in X} a_{x,y} x.
\]

But \(gy \in X\) and hence

\[
a_{x,y} = \begin{cases} 
1 & \text{if } x = gy \\
0 & \text{otherwise.}
\end{cases}
\]

It follows that

\[
\text{tr}(g; F[X]) = \sum_{x \in X} a_{x,x} = \sum_{gx = x} 1 = \left| \{x \in X : gx = x\} \right| \in F. \quad \square
\]

**Remark**

1. The map \(b\) rarely is injective. Let \(G = S_3\) be the symmetric group on three letters. It has four conjugacy classes of subgroups. Using Lemma 2.4.1, Remark 2.4.2, and Lemma 2.4.3 it therefore follows that \(B(S_3) \cong \mathbb{Z}^4\). On the other hand, \(S_3\) has only three conjugacy classes of elements. Hence Proposition 1.7.1 and Theorem 2.3.6.ii imply that \(R_C(S_3) \cong \mathbb{Z}^3\).

2. In general the map \(b\) is not surjective either. But there are many structural results about its cokernel. For example, the Artin induction theorem implies that

\[
|G| \cdot R_Q(G) \subseteq \text{im}(b).
\]

We therefore introduce the subring

\[
P_F(G) := \text{im}(b) \subseteq R_F(G).
\]

Let \(\mathcal{H}\) be a family of subgroups of \(G\) with the property that if \(H' \subseteq H\) is a subgroup of some \(H \in \mathcal{H}\) then also \(H' \in \mathcal{H}\). We introduce the subgroup \(B(G, \mathcal{H}) \subseteq B(G)\) generated by all \([G/H]\) for \(H \in \mathcal{H}\) as well as its image \(P_F(G, \mathcal{H}) \subseteq P_F(G)\) under the map \(b\).

**Lemma 2.4.7** \(B(G, \mathcal{H})\) is an ideal in \(B(G)\), and hence \(P_F(G, \mathcal{H})\) is an ideal in \(P_F(G)\).

**Proof** We have to show that, for any \(H_1 \in \mathcal{H}\) and any subgroup \(H_2 \subseteq G\), the element \([G/H_1] \cdot [G/H_2]\) lies in \(B(G, \mathcal{H})\). This is immediately clear from Exercise 2.4.4. But a less detailed argument suffices. Obviously \([G/H_1] \cdot [G/H_2]\) is the sum of the classes of the \(G\)-orbits in \(G/H_1 \times G/H_2\). Let \(G(g_1 H_1, g_2 H_2) = G(H_1, g_1^{-1} g_2 H_2)\) be such a \(G\)-orbit. The stabilizer \(H' \subseteq G\) of the element \((H_1, g_1^{-1} g_2 H_2)\) is contained in \(H_1\) and therefore belongs to \(\mathcal{H}\). It follows that

\[
[G(g_1 H_1, g_2 H_2)] = [G/H'] \in B(G, \mathcal{H}). \quad \square
\]
Definition

i. Let $\ell$ be a prime number. A finite group $H$ is called $\ell$-hyper-elementary if it contains a cyclic normal subgroup $C$ such that $\ell \nmid |C|$ and $H/C$ is an $\ell$-group.

ii. A finite group is called hyper-elementary if it is $\ell$-hyper-elementary for some prime number $\ell$.

Exercise Let $H$ be an $\ell$-hyper-elementary group. Then:

i. Any subgroup of $H$ is $\ell$-hyper-elementary;

ii. Let $C \subseteq H$ be a cyclic normal subgroup as in the definition, and let $L \subseteq H$ be any $\ell$-Sylow subgroup; then the map $C \times L \xrightarrow{\sim} H$ sending $(c, g)$ to $cg$ is a bijection of sets.

Let $\mathcal{H}_{he}$ denote the family of hyper-elementary subgroups of $G$. By the exercise Lemma 2.4.7 is applicable to $\mathcal{H}_{he}$.

Theorem 2.4.8 (Solomon) Suppose that $F$ has characteristic zero; then

$$P_F(G, \mathcal{H}_{he}) = P_F(G).$$

Proof Because of Lemma 2.4.7 it suffices to show that the unit element $1 \in P_F(G)$ already lies in $P_F(G, \mathcal{H}_{he})$. According to Lemma 2.4.6.ii the characters

$$\chi_{F[X]}(g) = \left|\{x \in X : gx = x\}\right| \in \mathbb{Z},$$

for any $g \in G$ and any finite $G$-set $X$, have integral values. Hence we have the well-defined ring homomorphisms

$$t_g : P_F(G) \longrightarrow \mathbb{Z}$$

$$z \longmapsto \text{Tr}(z)(g)$$

for $g \in G$. On the one hand, by Corollary 2.3.7.i, they satisfy

$$\bigcap_{g \in G} \ker(t_g) = \ker(\text{Tr}|P_F(G)) = \{0\}. \quad (2.4.1)$$

On the other hand, we claim that

$$t_g(P_F(G, \mathcal{H}_{he})) = \mathbb{Z} \quad (2.4.2)$$

holds true for any $g \in G$. We fix a $g_0 \in G$ in the following. Since the image $t_{g_0}(P_F(G, \mathcal{H}_{he}))$ is an additive subgroup of $\mathbb{Z}$ and hence is of the form $n\mathbb{Z}$ for some $n \geq 0$ it suffices to find, for any prime number $\ell$, an $\ell$-hyper-elementary subgroup $H \subseteq G$ such that the integer
\[ t_{g_0}(\left[F[G/H]\right]) = \chi_{F[G/H]}(g_0) = \left| \{x \in G/H : g_0x = x\} \right| = \left| \{gH \in G/H : g^{-1}g_0g \in H\} \right| \]

is not contained in \( \ell \mathbb{Z} \). We also fix \( \ell \).

The wanted \( \ell \)-hyper-elementary subgroup \( H \) will be found in a chain of subgroups

\[ C \subseteq \langle g_0 \rangle \subseteq H \subseteq N \]

with \( C \) being normal in \( N \) which is constructed as follows. Let \( n \geq 1 \) be the order of \( g_0 \), and write \( n = \ell^s m \) with \( \ell \nmid m \). The cyclic subgroup \( \langle g_0 \rangle \subseteq G \) generated by \( g_0 \) then is the direct product

\[ \langle g_0 \rangle = \langle g_0^{\ell^s} \rangle \times \langle g_0^m \rangle \]

where \( \langle g_0^m \rangle \) is an \( \ell \)-group and \( C := \langle g_0^{\ell^s} \rangle \) is a cyclic group of order prime to \( \ell \).

We define \( N := \{g \in G : gCg^{-1} = C\} \) to be the normalizer of \( C \) in \( G \). It contains \( \langle g_0 \rangle \), of course. Finally, we choose \( H \subseteq N \) in such a way that \( H/C \) is an \( \ell \)-Sylow subgroup of \( N/C \) which contains the \( \ell \)-subgroup \( \langle g_0 \rangle/C \). By construction \( H \) is \( \ell \)-hyper-elementary.

In the next step we study the cardinality of the set

\[ \{gH \in G/H : g_0gH = gH\} = \{gH \in G/H : g^{-1}g_0g \in H\}. \]

Suppose that \( g^{-1}g_0g \in H \). Then \( g^{-1}Cg \subseteq g^{-1}\langle g_0 \rangle g = \langle g^{-1}g_0g \rangle \subseteq H \). But, the two sides having coprime orders, the projection map \( g^{-1}Cg \rightarrow H/C \) has to be the trivial map. It follows that \( g^{-1}Cg = C \) which means that \( g \in N \). This shows that

\[ \{gH \in G/H : g_0gH = gH\} = \{gH \in N/H : g_0gH = gH\}. \]

The cardinality of the right-hand side is the number of \( \langle g_0 \rangle \)-orbits in \( N/H \) which consist of one point only. We note that the subgroup \( C \), being normal in \( N \) and contained in \( H \), acts trivially on \( N/H \). Hence the \( \langle g_0 \rangle \)-orbits coincide with the orbits of the \( \ell \)-group \( \langle g_0 \rangle/C \). But, quite generally, the cardinality of an orbit, being the index of the stabilizer of any point in the orbit, divides the order of the acting group. It follows that the cardinality of any \( \langle g_0 \rangle \)-orbit in \( N/H \) is a power of \( \ell \). We conclude that

\[ \left| \{gH \in N/H : g_0gH = gH\} \right| \equiv |N/H| = [N : H] \mod \ell. \]

But by the choice of \( H \) we have \( \ell \nmid [N : H] \). This establishes our claim.

By (2.4.2) we now may choose an element \( z_g \in P_\ell(G, H_{he}) \), for any \( g \in G \), such that \( t_g(z_g) = 1 \). We then have

\[ t_g \left( \prod_{g' \in G} (z_{g'} - 1) \right) = 0 \quad \text{for any } g \in G, \]
and (2.4.1) implies that
\[
\prod_{g' \in G} (z_{g'} - 1) = 0.
\]
Multiplying out the left-hand side and using that \( P_F(G, \mathcal{H}_{he}) \) is additively and multiplicatively closed easily shows that \( 1 \in P_F(G, \mathcal{H}_{he}) \).

\[\square\]

2.5 Clifford Theory

As before \( F \) is an arbitrary field. We fix a normal subgroup \( N \) in our finite group \( G \). Let \( W \) be an \( F[N] \)-module. It is given by a homomorphism of \( F \)-algebras
\[
\pi: \quad F[N] \longrightarrow \text{End}_F(W).
\]
For any \( g \in G \) we now define a new \( F[N] \)-module \( g^*(W) \) by the composite homomorphism
\[
F[N] \longrightarrow F[N] \xrightarrow{\pi} \text{End}_F(W) \xrightarrow{\pi} \text{End}_F(W)
\]
or equivalently by
\[
F[N] \times g^*(W) \longrightarrow g^*(W)
\]
\[(h, w) \mapsto ghg^{-1}w.
\]

Remark 2.5.1

i. \( \dim_F g^*(W) = \dim_F W \).

ii. The map \( U \mapsto g^*(U) \) is a bijection between the set of \( F[N] \)-submodules of \( W \) and the set of \( F[N] \)-submodules of \( g^*(W) \).

iii. \( W \) is simple if and only if \( g^*(W) \) is simple.

iv. \( g_1^*(g_2^*(W)) = (g_2g_1)^*(W) \) for any \( g_1, g_2 \in G \).

v. Any \( F[N] \)-module homomorphism \( \alpha : W_1 \longrightarrow W_2 \) also is a homomorphism of \( F[N] \)-modules \( \alpha : g^*(W_1) \longrightarrow g^*(W_2) \).

Proof Trivial or straightforward.

Suppose that \( g \in N \). One checks that then
\[
W \xrightarrow{\cong} g^*(W)
\]
\[w \mapsto gw
\]
is an isomorphism of \( F[N] \)-modules. Together with Remark 2.5.1.iv/v this implies that
\[
G/N \times (\mathcal{M}_{F[N]}/\cong) \longrightarrow \mathcal{M}_{F[N]}/\cong
\]

\[
(gN, \{W\}) \longmapsto \{g^*(W)\}
\]
is a well-defined action of the group \(G/N\) on the set \(\mathcal{M}_{F[N]}/\cong\). By Remark 2.5.1.iii this action respects the subset \(\mathcal{F}[N]/\cong\). For any \(\{W\}\) in \(\mathcal{F}[N]\) we put

\[
I_G(W) := \{g \in G : \{g^*(W)\} = \{W\}\}.
\]

As a consequence of Remark 2.5.1.iv this is a subgroup of \(G\), which contains \(N\) of course.

**Remark 2.5.2** Let \(V\) be an \(F[G]\)-module, and let \(g \in G\) be any element; then the map

\[
\text{set of all } F[N]\text{-submodules of } V \cong \text{set of all } F[N]\text{-submodules of } V
\]

\[
W \longmapsto gW
\]
is an inclusion preserving bijection; moreover, for any \(F[N]\)-submodule \(W \subseteq V\) we have:

i. The map

\[
g^*(W) \cong g^{-1}W
\]

\[
w \longmapsto g^{-1}w
\]
is an isomorphism of \(F[N]\)-modules;

ii. \(gW\) is a simple \(F[N]\)-module if and only if \(W\) is a simple \(F[N]\)-module;

iii. if \(W_1 \cong W_2\) are isomorphic \(F[N]\)-submodules of \(V\) then also \(gW_1 \cong gW_2\) are isomorphic as \(F[N]\)-modules.

**Proof** For \(h \in N\) we have

\[
h(gW) = g(g^{-1}hg)W = gW
\]
since \(g^{-1}hg \in N\). Hence \(gW\) indeed is an \(F[N]\)-submodule, and the map in the assertion is well defined. It obviously is inclusion preserving. Its bijectivity is immediate from \(g^{-1}(gW) = W = g(g^{-1}W)\). The assertion ii. is a direct consequence. The map in i. clearly is an \(F\)-linear isomorphism. Because of

\[
g^{-1}(ghg^{-1}w) = h(g^{-1}w) \quad \text{for any } h \in N \text{ and } w \in W
\]
it is an \(F[N]\)-module isomorphism. The last assertion iii. follows from i. and Remark 2.5.1.v. \qed
2.5 Clifford Theory

**Theorem 2.5.3** (Clifford) Let $V$ be a simple $F[G]$-module; we then have:

i. $V$ is semisimple as an $F[G]$-module;

ii. let $W \subseteq V$ be a simple $F[N]$-submodule, and let $\tilde{W} \subseteq V$ be the \{W\}-isotypic component; then
   a. $\tilde{W}$ is a simple $F[I_G(W)]$-module, and
   b. $V \cong \text{Ind}_{I_G(W)}^G (\tilde{W})$ as $F[G]$-modules.

**Proof** Since $V$ is of finite length as an $F[N]$-module we find a simple $F[N]$-submodule $W \subseteq V$. Then $gW$, for any $g \in G$, is another simple $F[N]$-submodule by Remark 2.5.2.ii. Therefore $V_0 := \sum_{g \in G} gW$ is, on the one hand, a semisimple $F[N]$-module by Proposition 1.1.4. On the other hand it is, by definition, a nonzero $F[G]$-submodule of $V$. Since $V$ is simple we must have $V_0 = V$, which proves the assertion i. As an $F[N]$-submodule the \{W\}-isotypic component $\tilde{W}$ of $V$ is of the form $\tilde{W} = W_1 \oplus \cdots \oplus W_m$ with simple $F[N]$-submodules $W_i \cong W$. Let first $g$ be an element in $I_G(W)$. Using Remark 2.5.2.iii we obtain $gW_i \cong gW \cong W$ for any $1 \leq i \leq m$. It follows that $gW_i \subseteq \tilde{W}$ for any $1 \leq i \leq m$ and hence $g\tilde{W} \subseteq \tilde{W}$. We see that $\tilde{W}$ is an $F[I_G(W)]$-submodule of $V$. For a general $g \in G$ we conclude from Remark 2.5.2 that $g\tilde{W}$ is the \{gW\}-isotypic component of $V$. We certainly have

$$V = \sum_{g \in G/I_G(W)} g\tilde{W}.$$ 

Two such submodules $g_1 \tilde{W}$ and $g_2 \tilde{W}$, being isotypic components, either are equal or have zero intersection. If $g_1 \tilde{W} = g_2 \tilde{W}$ then $g_2^{-1}g_1 \tilde{W} = \tilde{W}$, hence $g_2^{-1}g_1 W \cong W$, and therefore $g_2^{-1}g_1 \in I_G(W)$. We see that in fact

$$V = \bigoplus_{g \in G/I_G(W)} g\tilde{W}.$$ (2.5.1)

The inclusion $\tilde{W} \subseteq V$ induces, by the first Frobenius reciprocity, the $F[G]$-module homomorphism

$$\text{Ind}_{I_G(W)}^G (\tilde{W}) \cong F[G] \otimes_{F[I_G(W)]} \tilde{W} \rightarrow V$$

$$\left(\sum_{g \in G} a_g g\right) \otimes \tilde{w} \mapsto \sum_{g \in G} a_g g\tilde{w}.$$ 

Since $V$ is simple it must be surjective. But both sides have the same dimension as $F$-vector spaces $[G : I_G(W)] \cdot \dim_F \tilde{W}$, the left-hand side by the argument in the proof of Remark 2.3.2 and the right-hand side by (2.5.1). Hence this map is an isomorphism which proves ii.b. Finally, since $F[G] \otimes_{F[I_G(W)]} \tilde{W} \cong V$ is a simple $F[G]$-module it follows from Remark 2.3.3 that $\tilde{W}$ is a simple $F[I_G(W)]$-module. \[\square\]

In the next section we will need the following particular consequence of this result. But first we point out that for an $F[N]$-module $W$ of dimension $\dim_F W = 1$
the describing algebra homomorphism \( \pi \) is of the form

\[
\pi : \ F[N] \longrightarrow F.
\]

The corresponding homomorphism for \( g^*(W) \) then is \( x \mapsto \pi(gxg^{-1}) \). Since an endomorphism of a one-dimensional \( F \)-vector space is given by multiplication by a scalar we have \( g \in I_G(W) \) if and only if there is a scalar \( a \in F^\times \) such that

\[
a \pi(h)w = \pi(ghg^{-1})aw \quad \text{for any } h \in N \text{ and } w \in W.
\]

It follows that

\[
I_G(W) = \{ g \in G : \pi(ghg^{-1}) = \pi(h) \text{ for any } h \in N \} \quad (2.5.2)
\]

if \( \dim_F W = 1 \).

**Remark 2.5.4** Suppose that \( N \) is abelian and that \( F \) is a splitting field for \( N \); then any simple \( F[N] \)-module \( W \) has dimension \( \dim_F W = 1 \).

**Proof** Let \( \pi : F[N] \longrightarrow \text{End}_F(W) \) be the algebra homomorphism describing \( W \). Since \( N \) is abelian we have \( \text{im}(\pi) \subseteq \text{End}_{F[N]}(W) \). By our assumption on the field \( F \) the latter is equal to \( F \). This means that any element in \( F[N] \) acts on \( W \) by multiplication by a scalar in \( F \). Since \( W \) is simple this forces it to be one-dimensional. \( \square \)

**Proposition 2.5.5** Let \( H \) be an \( \ell \)-hyper-elementary group with cyclic normal subgroup \( C \) such that \( \ell \nmid |C| \) and \( H/C \) is an \( \ell \)-group, and let \( V \) be a simple \( F[H] \)-module; we suppose that

a. \( F \) is a splitting field for \( C \),

b. \( V \) does not contain the trivial \( F[C] \)-module, and
c. the subgroup \( C_0 := \{ c \in C : cg = gc \text{ for any } g \in H \} \) acts trivially on \( V \);

then there exists a proper subgroup \( H' \subsetneq H \) and an \( F[H'] \)-module \( V' \) such that \( V \cong \text{Ind}_H^{H'}(V') \) as \( F[H] \)-modules.

**Proof** We pick any simple \( F[C] \)-submodule \( W \subseteq V \). By applying Clifford’s Theorem 2.5.3 to the normal subgroup \( C \) and the module \( W \) it suffices to show that

\[
I_H(W) \neq H.
\]

According to Remark 2.5.4 the module \( W \) is one-dimensional and given by an algebra homomorphism \( \pi : F[C] \longrightarrow F \), and by (2.5.2) we have

\[
I_H(W) = \{ g \in H : \pi( g cg^{-1} ) = \pi(c) \text{ for any } c \in C \}.
\]

The assumption c. means that

\[
C_0 \subseteq C_1 := \ker(\pi|C).
\]
We immediately note that any subgroup of the cyclic normal subgroup $C$ also is normal in $H$. By assumption b. we find an element $c_2 \in C \setminus C_1$ so that $\pi(c_2) \neq 1$. Let $L \subseteq H$ be an $\ell$-Sylow subgroup. We claim that we find an element $g_0 \in L$ such that $\pi(g_0c_2g_0^{-1}) \neq \pi(c_2)$. Then $g_0 \not\in I_H(W)$ which establishes what we wanted. We point out that, since $H = C \times L$ as sets, we have

$$C_0 = \{c \in C : cg = gc$ for any $g \in L\}.$$

Arguing by contradiction we assume that $\pi(gc_2g^{-1}) = \pi(c_2)$ for any $g \in L$. Then

$$gc_2C_1g^{-1} = gc_2g^{-1}C_1 = c_2C_1 \quad \text{for any } g \in L.$$

This means that we may consider $c_2C_1$ as an $L$-set with respect to the conjugation action. Since $L$ is an $\ell$-group the cardinality of any $L$-orbit in $c_2C_1$ is a power of $\ell$. On the other hand we have $\ell \nmid |C_1| = |c_2C_1|$. There must therefore exist an element $c_0 \in c_2C_1$ such that

$$g c_0 g^{-1} = c_0 \quad \text{for any } g \in L$$

(i.e. an $L$-orbit of cardinality one). We conclude that $c_0 \in C_0 \subseteq C_1$ and hence $c_2C_1 = c_0C_1 = C_1$. This is in contradiction to $c_2 \not\in C_1$. \(\square\)

### 2.6 Brauer’s Induction Theorem

In this section $F$ is a field of characteristic zero, and $G$ continues to be any finite group.

**Definition**

i. Let $\ell$ be a prime number. A finite group $H$ is called $\ell$-elementary if it is a direct product $H = C \times L$ of a cyclic group $C$ and an $\ell$-group $L$.

ii. A finite group is called elementary if it is $\ell$-elementary for some prime number $\ell$.

**Exercise**

i. If $H$ is $\ell$-elementary then $H = C \times L$ is the direct product of a cyclic group $C$ of order prime to $\ell$ and an $\ell$-group $L$.

ii. Any $\ell$-elementary group is $\ell$-hyper-elementary.

iii. Any subgroup of an $\ell$-elementary group is $\ell$-elementary.

Let $\mathcal{H}_e$ denote the family of elementary subgroups of $G$.

**Theorem 2.6.1** (Brauer) Suppose that $F$ is a splitting field for every subgroup of $G$; then

$$\sum_{H \in \mathcal{H}_e} \text{ind}_H^G(R_F(H)) = R_F(G).$$
Proof By Corollary 2.3.5 each \( \text{ind}^G_H(R_F(H)) \) is an ideal in the ring \( R_F(G) \). Hence the left-hand side of the asserted identity is an ideal in the right-hand side. To obtain equality we therefore need only to show that the unit element \( 1_G \in R_F(G) \) lies in the left-hand side. According to Solomon’s Theorem 2.4.8 together with Lemma 2.4.6.i we have

\[
1_G \in \sum_{H \in H_{he}} \mathbb{Z}[F[G/H]] = \sum_{H \in H_{he}} \mathbb{Z}b([G/H]) = \sum_{H \in H_{he}} \mathbb{Z} \text{ind}^G_H(1_H)
\]

\[
\subseteq \sum_{H \in H_{he}} \text{ind}^G_H(R_F(H)).
\]

By the transitivity of induction this reduces us to the case that \( G \) is \( \ell \)-hyper-elementary for some prime number \( \ell \). We now proceed by induction with respect to the order of \( G \) and assume that our assertion holds for all proper subgroups \( H' \subsetneq G \). We also may assume, of course, that \( G \) is not elementary. Using the transitivity of induction again it then suffices to show that

\[
1_G \in \sum_{H' \subsetneq G} \text{ind}^G_{H'}(R_F(H')).
\]

Let \( C \subseteq G \) be the cyclic normal subgroup of order prime to \( \ell \) such that \( G/C \) is an \( \ell \)-group. We fix an \( \ell \)-Sylow subgroup \( L \subseteq G \). Then \( G = C \times L \) as sets. In \( C \) we have the (cyclic) subgroup

\[
C_0 := \{ c \in C : cg = gc \text{ for any } g \in G \}.
\]

Then

\[
H_0 := C_0 \times L
\]

is an \( \ell \)-elementary subgroup of \( G \). Since \( G \) is not elementary we must have \( H_0 \subsetneq G \). We consider the induction \( \text{Ind}^G_{H_0}(F) \) of the trivial \( F[H_0] \)-module \( F \). By semisimplicity it decomposes into a direct sum

\[
\text{Ind}^G_{H_0}(F) = V_0 \oplus V_1 \oplus \cdots \oplus V_r
\]

of simple \( F[G] \)-modules \( V_i \). We recall that \( \text{Ind}^G_{H_0}(F) \) is the \( F \)-vector space of all functions \( \phi : G/H_0 \rightarrow F \) with the \( G \)-action given by

\[
g \cdot \phi(g'H_0) = \phi(g^{-1}g'H_0).
\]

This \( G \)-action fixes a function \( \phi \) if and only if \( \phi(gg'H_0) = \phi(g'H_0) \) for any \( g, g' \in G \), i.e. if and only if \( \phi \) is constant. It follows that the one-dimensional subspace of constant functions is the only simple \( F[G] \)-submodule in \( \text{Ind}^G_{H_0}(F) \) which is isomorphic to the trivial module. We may assume that \( V_0 \) is this trivial submodule. In \( R_F(G) \) we then have the equation

\[
1_G = \text{ind}^G_{H_0}(1_{H_0}) - [V_1] - \cdots - [V_r].
\]
This reduces us further to showing that, for any $1 \leq i \leq r$, we have

$$[V_i] \in \text{ind}^G_{H_i}(R_F(H_i))$$

for some proper subgroup $H_i \subsetneq G$. This will be achieved by applying the criterion in Proposition 2.5.5. By our assumption on $F$ it remains to verify the conditions b. and c. in that proposition for each $V_1, \ldots, V_r$. Since $C_0$ is central in $G$ and is contained in $H_0$ it acts trivially on $\text{Ind}^G_{H_0}(F)$ and a fortiori on any $V_i$. This is condition c. For b. we note that $CH_0 = G$ and $C \cap H_0 = C_0$. Hence the inclusion $C \subseteq G$ induces a bijection $C/C_0 \cong G/H_0$. It follows that the map

$$\text{Ind}^G_{H_0}(F) \cong \text{Ind}^C_{C_0}(F)$$

$$\phi \mapsto \phi|(C/C_0)$$

is an isomorphism of $F[C]$-modules. It maps constant functions to constant functions and hence the unique trivial $F[G]$-submodule $V_0$ to the unique trivial $F[C]$-submodule. Therefore $V_i$, for $1 \leq i \leq r$, cannot contain any trivial $F[C]$-submodule. □

**Lemma 2.6.2** Let $H$ be an elementary group, and let $N_0 \subseteq H$ be a normal subgroup such that $H/N_0$ is not abelian; then there exists a normal subgroup $N_0 \subseteq N \subseteq H$ such that $N/N_0$ is abelian but is not contained in the center $Z(H/N_0)$ of $H/N_0$.

**Proof** With $H$ also $H/N_0$ is elementary (if $H = C \times L$ with $C$ and $L$ having co-prime orders then $H/N_0 \cong C/C \cap N_0 \times L/L \cap N_0$). We therefore may assume without loss of generality that $N_0 = \{1\}$. **Step 1:** We assume that $H$ is an $\ell$-group for some prime number $\ell$. By assumption we have $Z(H) \neq H$ so that $H/Z(H)$ is an $\ell$-group $\neq \{1\}$. We pick a cyclic normal subgroup $\{1\} \neq N = \langle g \rangle \subseteq H/Z(H)$. Let $Z(H) \subseteq N \subseteq H$ be the normal subgroup such that $N/Z(H) = N$ and let $g \in N$ be a preimage of $\bar{g}$. Clearly $N = \langle Z(H), g \rangle$ is abelian. But $Z(H) \subsetneq N$ since $\bar{g} \neq 1$.

**Step 2:** In general let $H = C \times L$ where $C$ is cyclic and $L$ is an $\ell$-group. With $H$ also $L$ is not abelian. Applying Step 1 to $L$ we find a normal abelian subgroup $N_L \subseteq L$ such that $N_L \subsetneq Z(L)$. Then $N := C \times N_L$ is a normal abelian subgroup of $H$ such that $N \subsetneq Z(H) = C \times Z(L)$.

□

**Lemma 2.6.3** Let $H$ be an elementary group, and let $W$ be a simple $F[H]$-module; we suppose that $F$ is a splitting field for all subgroups of $H$; then there exists a subgroup $H' \subseteq H$ and a one-dimensional $F[H']$-module $W'$ such that

$$W \cong \text{Ind}^H_{H'}(W')$$

as $F[H]$-modules.
Proof We choose $H' \subseteq H$ to be a minimal subgroup (possibly equal to $H$) such that there exists an $F[H']$-module $W'$ with $W \cong \text{Ind}_{H'}^H(W')$, and we observe that $W'$ necessarily is a simple $F[H']$-module by Remark 2.3.3. Let

$$\pi': H' \rightarrow \text{End}_F(W')$$

be the corresponding algebra homomorphism, and put $N_0 := \ker(\pi')$. We claim that $H'/N_0$ is abelian. Suppose otherwise. Then, by Lemma 2.6.2, there exists a normal subgroup $N_0 \subseteq N \subseteq H'$ such that $N/N_0$ is abelian but is not contained in $Z(H'/N_0)$. Let $\tilde{W} \subseteq W'$ be a simple $F[N]$-submodule. By Clifford’s Theorem 2.5.3 we have

$$W' \cong \text{Ind}_{H'}^H(\text{Ind}_{H'}^H(\tilde{W}))(\tilde{\tilde{W}})$$

where $\tilde{\tilde{W}}$ denotes the $\{\tilde{W}\}$-isotypic component of $W'$. Transitivity of induction implies

$$W \cong \text{Ind}_H^H(\text{Ind}_{H'}^H(\tilde{W}))(\tilde{\tilde{W}}) \cong \text{Ind}_{H'}^H(\tilde{W}).$$

By the minimality of $H'$ we therefore must have $I_{H'}(\tilde{W}) = H'$ which means that $W' = \tilde{W}$ is $\{\tilde{W}\}$-isotypic.

On the other hand, $W'$ is an $F[H'/N_0]$-module. Hence $\tilde{W}$ is a simple $F[N/N_0]$-module for the abelian group $N/N_0$. Remark 2.5.4 then implies (note that $\text{End}_{F[N/N_0]}(\tilde{W}) = \text{End}_{F[N]}(\tilde{W}) = F$) that $\tilde{W}$ is one-dimensional given by an algebra homomorphism

$$\chi: F[N/N_0] \rightarrow F.$$

It follows that any $h \in N$ acts on the $\{\tilde{W}\}$-isotypic module $W'$ by multiplication by the scalar $\chi(hN_0)$. In other words the injective homomorphism

$$H'/N_0 \rightarrow \text{End}_F(W')$$

$$hN_0 \mapsto \pi'(h)$$

satisfies

$$\pi'(h) = \chi(hN_0) \cdot \text{id}_{W'}$$

for any $h \in N$.

But $\chi(hN_0) \cdot \text{id}_{W'}$ lies in the center of $\text{End}_F(W')$. The injectivity of the homomorphism therefore implies that $N/N_0$ lies in the center of $H'/N_0$. This is a contradiction.

We thus have established that $H'/N_0$ is abelian. Applying Remark 2.5.4 to $W'$ viewed as a simple $F[H'/N_0]$-module we conclude that $W'$ is one-dimensional. □

Theorem 2.6.4 (Brauer) Suppose that $F$ is a splitting field for any subgroup of $G$, and let $x \in R_F(G)$ be any element; then there exist integers $m_1, \ldots, m_r$, elementary
subgroups $H_1, \ldots, H_r$, and one-dimensional $F[H_i]$-modules $W_i$ such that

$$x = \sum_{i=1}^{r} m_i \text{ind}_{H_i}^G ([W_i]).$$

**Proof** Combine Theorem 2.6.1, Proposition 1.7.1, Lemma 2.6.3, and the transitivity of induction. □

### 2.7 Splitting Fields

Again $F$ is a field of characteristic zero.

**Lemma 2.7.1** Let $E/F$ be any extension field, and let $V$ and $W$ be two finitely generated $F[G]$-modules; we then have

$$\text{Hom}_{E[G]}(E \otimes_F V, E \otimes_F W) = E \otimes_F \text{Hom}_F(V, W).$$

**Proof** First of all we observe, by comparing dimensions, that

$$\text{Hom}_E(E \otimes_F V, E \otimes_F W) = E \otimes_F \text{Hom}_F(V, W)$$

holds true. We now consider $U := \text{Hom}_F(V, W)$ as an $F[G]$-module via

$$G \times U \longrightarrow U$$

$$(g, f) \longmapsto gf := g f (g^{-1}).$$

Then $\text{Hom}_{F[G]}(V, W) = U^G := \{ f \in U : g f = f \text{ for any } g \in G \}$ is the $\{F\}$-isotypic component of $U$ for the trivial $F[G]$-module $F$. Correspondingly we obtain

$$\text{Hom}_{E[G]}(E \otimes_F V, E \otimes_F W) = \text{Hom}_E(E \otimes_F V, E \otimes_F W)^G = (E \otimes_F U)^G.$$

This reduces us to proving that

$$(E \otimes_F U)^G = E \otimes_F U^G$$


$$\varepsilon_G := \frac{1}{|G|} \sum_{g \in G} g \in F[G] \subseteq E[G]$$

is an idempotent with the property that

$$U^G = \varepsilon_G \cdot U,$$
and hence

\[(E \otimes_F U)^G = \varepsilon_G \cdot (E \otimes_F U) = E \otimes_F \varepsilon_G \cdot U = E \otimes_F U^G.\]

\[\square\]

**Theorem 2.7.2** (Brauer) Let $e$ be the exponent of $G$, and suppose that $F$ contains a primitive $e$th root of unity; then $F$ is a splitting field for any subgroup of $G$.

**Proof** We fix an algebraic closure $\bar{F}$ of $F$. Step 1: We show that, for any finitely generated $\bar{F}[G]$-module $\bar{V}$, there is an $F[G]$-module $V$ such that

\[\bar{V} \cong \bar{F} \otimes_F V\] as $\bar{F}[G]$-modules.

According to Brauer’s Theorem 2.6.4 we find integers $m_1, \ldots, m_r$, subgroups $H_1, \ldots, H_r$ of $G$, and one-dimensional $\bar{F}[H_i]$-modules $\bar{W}_i$ such that

\[\bar{V} = \sum_{i=1}^r m_i \bar{F}[G] \otimes_{\bar{F}[H_i]} \bar{W}_i\]

\[= \sum_{m_i > 0} \left( \bar{F}[G] \otimes_{\bar{F}[H_i]} \bar{W}_i^m \right) - \sum_{m_i < 0} \left( \bar{F}[G] \otimes_{\bar{F}[H_i]} \bar{W}_i^{-m} \right)
\]

\[= \bigoplus_{m_i > 0} \left( \bar{F}[G] \otimes_{\bar{F}[H_i]} \bar{W}_i^m \right) - \bigoplus_{m_i < 0} \left( \bar{F}[G] \otimes_{\bar{F}[H_i]} \bar{W}_i^{-m} \right).
\]

Let $\pi_i : \bar{F}[H_i] \longrightarrow \bar{F}$ denote the $\bar{F}$-algebra homomorphism describing $\bar{W}_i$. We have $\pi_i(h^e) = \pi_i(h)^e = \pi_i(1) = 1$ for any $h \in H_i$. Our assumption on $F$ therefore implies that $\pi_i(F[H_i]) \subseteq F$. Hence the restriction $\pi_i|F[H_i]$ describes a one-dimensional $F[H_i]$-module $W_i$ such that

\[\bar{W}_i \cong \bar{F} \otimes_F W_i\] as $\bar{F}[H_i]$-modules.

We define the $F[G]$-modules

\[V_+ := \bigoplus_{m_i > 0} \left( F[G] \otimes_{F[H_i]} W_i^m \right)\] and \[V_- := \bigoplus_{m_i < 0} \left( F[G] \otimes_{F[H_i]} W_i^{-m} \right).
\]

Then

\[\bar{F} \otimes_F \bar{V}_+ \bar{F} = \bigoplus_{m_i > 0} \left( \bar{F} \otimes_F F[G] \otimes_{F[H_i]} W_i^m \right) = \bigoplus_{m_i > 0} \left( \bar{F}[G] \otimes_{\bar{F}[H_i]} \bar{F}[H_i] \otimes_{F[H_i]} W_i^m \right)
\]

\[= \bigoplus_{m_i > 0} \left( \bar{F}[G] \otimes_{\bar{F}[H_i]} \bar{F}[H_i] \otimes_{F[H_i]} W_i^m \right).
\]
\[ \bigoplus_{m_i > 0} (\bar{F}[G] \otimes \bar{F}[H_i] (\bar{F} \otimes_F W_i^{m_i})) \]

\[ \cong \bigoplus_{m_i > 0} (\bar{F}[G] \otimes \bar{F}[H_i] \tilde{W}_i^{m_i}) \]

and similarly

\[ \bar{F} \otimes_F V_{-} \cong \bigoplus_{m_i < 0} (\bar{F}[G] \otimes \bar{F}[H_i] \tilde{W}_i^{-m_i}). \]

It follows that

\[ [\tilde{V}] = [\bar{F} \otimes_F V_{-}] - [\bar{F} \otimes_F V_{-}] \]

or, equivalently, that

\[ [\tilde{V} \oplus (\bar{F} \otimes_F V_{-})] = [\bar{F} \otimes_F V_{+}]. \]

Using Corollary 2.3.7.i/iii we deduce that

\[ \tilde{V} \oplus (\bar{F} \otimes_F V_{-}) \cong \bar{F} \otimes_F V_{+} \]

as \( \bar{F}[G] \)-modules.

If \( V_{-} \) is nonzero then \( V_{-} = U \oplus V'_{-} \) is a direct sum of \( F[G] \)-modules where \( U \) is simple. On the other hand let \( V_{+} = U_1 \oplus \cdots \oplus U_m \) be a decomposition into simple \( F[G] \)-modules. Then \( \bar{F} \otimes_F U \) is a direct summand of \( \bar{F} \otimes_F V_{-} \) and hence is isomorphic to a direct summand of \( \bar{F} \otimes_F V_{+} \). We therefore must have \( \text{Hom}_{\bar{F}[G]}(\bar{F} \otimes_F U, \bar{F} \otimes_F U_j) \neq \{0\} \) for some \( 1 \leq j \leq m \). Lemma 2.7.1 implies that \( \text{Hom}_{F[G]}(U, U_j) \neq \{0\} \). Hence \( U \cong U_j \) as \( F[G] \)-modules. We conclude that \( V_{+} \cong U \oplus V'_{+} \) with \( V'_{+} := \bigoplus_{i \neq j} U_i \), and we obtain

\[ \tilde{V} \oplus (\bar{F} \otimes_F V'_{-}) \oplus (\bar{F} \otimes_F U) \cong (\bar{F} \otimes_F V'_{+}) \oplus (\bar{F} \otimes_F U). \]

The Jordan–Hölder Proposition 1.1.2 then implies that

\[ \tilde{V} \oplus (\bar{F} \otimes_F V'_{-}) \cong \bar{F} \otimes_F V'_{+}. \]

By repeating this argument we arrive after finitely many steps at an \( F[G] \)-module \( V \) such that

\[ \tilde{V} \cong \bar{F} \otimes_F V. \]

\textbf{Step 2:} Let now \( V \) be a simple \( F[G] \)-module, and let \( \bar{F} \otimes_F V = \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_m \) be a decomposition into simple \( \bar{F}[G] \)-modules. By Step 1 we find \( F[G] \)-modules \( V_i \) such that

\[ \tilde{V}_i \cong \bar{F} \otimes_F V_i. \]

For any \( 1 \leq i \leq m \), the \( F[G] \)-module \( V_i \) necessarily is simple, and we have \( \text{Hom}_{\bar{F}[G]}(\bar{F} \otimes_F V_i, \bar{F} \otimes_F V) \neq \{0\} \). Hence \( \text{Hom}_{F[G]}(V_i, V) \neq \{0\} \) by Lemma 2.7.1. It follows that \( V_i \cong V \) for any \( 1 \leq i \leq m \). By comparing dimensions we conclude
that \( m = 1 \). This means that \( \bar{F} \otimes_F V \) is a simple \( \bar{F}[G] \)-module. Using Lemma 2.7.1 again we obtain

\[
\bar{F} \otimes_F \text{End}_{\bar{F}[G]}(V) = \text{End}_{\bar{F}[G]}(\bar{F} \otimes_F V) = \bar{F}.
\]

Hence \( \text{End}_{F[G]}(V) = F \) must be one-dimensional. This shows that \( F \) is a splitting field for \( G \).

Step 3: Let \( H \subseteq G \) be any subgroup with exponent \( e_H \). Since \( e_H \) divides \( e \) the field \( F \) \textit{a fortiori} contains a primitive \( e_H \)th root of unity. Hence \( F \) also is a splitting field for \( H \).

\[ \square \]

2.8 Properties of the Cartan–Brauer Triangle

We go back to the setting from the beginning of this chapter: \( k \) is an algebraically closed field of characteristic \( p > 0 \), \( R \) is a \((0, p)\)-ring for \( k \) with maximal ideal \( m_R = R \pi_R \), and \( K \) denotes the field of fractions of \( R \).

The ring \( R \) will be called \textit{splitting} for our finite group \( G \) if \( K \) contains a primitive \( e \)th root of unity \( \zeta \) where \( e \) is the exponent of \( G \). By Theorem 2.7.2 the field \( K \), in this case, is a splitting field for any subgroup of \( G \). This additional condition can easily be achieved by defining \( K' := K(\zeta) \) and \( R' := \{ a \in K': \text{Norm}_{K'/K}(a) \in R \} \); then \( R' \) is a \((0, p)\)-ring for \( k \) which is splitting for \( G \).

It is our goal in this section to establish the deeper properties of the Cartan–Brauer triangle

\[
\begin{array}{ccc}
R_K(G) & \xrightarrow{d_G} & R_k(G) \\
\downarrow{e_G} & & \downarrow{c_G} \\
K_0(k[G]). & & \\
\end{array}
\]

\textbf{Lemma 2.8.1} For any subgroup \( H \subseteq G \) the diagram

\[
\begin{array}{ccc}
R_K(H) & \xrightarrow{d_H} & R_k(H) \\
\downarrow{\text{ind}_H^G} & & \downarrow{\text{ind}_H^G} \\
R_K(G) & \xrightarrow{d_G} & R_k(G) \\
\end{array}
\]

is commutative.

\textbf{Proof} Let \( W \) be a finitely generated \( k[H] \)-module. We choose a lattice \( L \subseteq W \) which is \( H \)-invariant. Then

\[
\text{ind}_H^G(d_H([W])) = \text{ind}_H^G([L/\pi_R L]) = [k[G] \otimes_{k[H]} (L/\pi_R L)].
\]
Moreover, \( R[G] \otimes_{R[H]} L \cong L^{[G:H]} \) is a \( G \)-invariant lattice in \( K[G] \otimes_{K[H]} W \cong W^{[G:H]} \) (compare the proof of Remark 2.3.2). Hence
\[
d_G(\text{ind}_H^G([W])) = d_G([K[G] \otimes_{K[H]} W])
\]
\[
= \left[ (R[G] \otimes_{R[H]} L)/\pi_R(R[G] \otimes_{R[H]} L) \right]
\]
\[
= [k[G] \otimes_{k[H]} (L/\pi_R L)].
\]

**Lemma 2.8.2** We have
\[
R_k(G) = \sum_{H \in \mathcal{H}_e} \text{ind}_H^G(R_k(H)).
\]

**Proof** Since the \( \text{ind}_H^G(R_k(H)) \) are ideals in \( R_k(G) \) it suffices to show that the unit element \( 1_{k[G]} \in R_k(G) \) lies in the right-hand side. We choose \( R \) to be splitting for \( G \).

By Brauer’s induction Theorem 2.6.1 we have
\[
1_{K[G]} \in \sum_{H \in \mathcal{H}_e} \text{ind}_H^G(R_K(H))
\]
where \( 1_{K[G]} \) is the unit element in \( R_K(G) \). Using Lemma 2.8.1 we obtain
\[
d_G(1_{K[G]}) \subseteq \sum_{H \in \mathcal{H}_e} d_G(\text{ind}_H^G(R_K(H))) = \sum_{H \in \mathcal{H}_e} \text{ind}_H^G(d_H(R_K(H)))
\]

It is trivial to see that \( d_G(1_{K[G]}) = 1_{k[G]} \). □

**Theorem 2.8.3** The decomposition homomorphism
\[
d_G: R_K(G) \to R_k(G)
\]
is surjective.

**Proof** By Lemma 2.8.1 we have
\[
d_G(R_K(G)) \supseteq \sum_{H \in \mathcal{H}_e} d_G(\text{ind}_H^G(R_K(H))) = \sum_{H \in \mathcal{H}_e} \text{ind}_H^G(d_H(R_K(H))).
\]

Because of Lemma 2.8.2 it therefore suffices to show that \( d_H(R_K(H)) = R_k(H) \) for any \( H \in \mathcal{H}_e \). This means we are reduced to proving our assertion in the case where the group \( G \) is elementary. Then \( G = H \times P \) is the direct product of a group \( H \) of order prime to \( p \) and a \( p \)-group \( P \). By Proposition 1.7.1 it suffices to show that the class \([W] \in R_k(G)\), for any simple \( k[G] \)-module \( W \), lies in the image of \( d_G \). Viewed
as a $k[P]$-module $W$ must contain the trivial $k[P]$-module by Proposition 2.2.7. We deduce that

$$W^P := \{ w \in W : gw = w \text{ for any } g \in P \} \neq \{0\}.$$  

Since $P$ is a normal subgroup of $G$ the $k[P]$-submodule $W^P$ in fact is a $k[G]$-submodule of $W$. But $W$ is simple. Hence $W^P = W$ which means that $k[G]$ acts on $W$ through the projection map $k[G] \longrightarrow k[H]$. According to Corollary 2.2.6 we find a simple $k[H]$-module $V$ together with a $G$-invariant lattice $L \subseteq V$ such that $L/\pi R L \cong W$ as $k[H]$-modules. Viewing $V$ as a $k[G]$-module through the projection map $K[G] \longrightarrow K[H]$ we obtain $[V] \in R_K(G)$ and $d_G([V]) = [W]$. □

**Theorem 2.8.4** Let $p^m$ be the largest power of $p$ which divides the order of $G$; the Cartan homomorphism $c_G : K_0(k[G]) \longrightarrow R_k(G)$ is injective, its cokernel is finite, and $p^m R_k(G) \subseteq \text{im}(c_G)$.

**Proof** Step 1: We show that $p^m R_k(G) \subseteq \text{im}(c_G)$ holds true. It is trivial from the definition of the Cartan homomorphism that, for any subgroup $H \subseteq G$, the diagram

$$
\begin{array}{ccc}
K_0(k[H]) & \xrightarrow{c_H} & R_k(H) \\
\downarrow & & \downarrow \\
[K] \longrightarrow [k[G] \otimes_k k[H], P] & \xrightarrow{\text{ind}_{H}^G} & \text{im}(c_H) \\
K_0(k[G]) & \xrightarrow{c_G} & R_k(G)
\end{array}
$$

is commutative. It follows that

$$\text{ind}_{H}^G(\text{im}(c_H)) \subseteq \text{im}(c_G).$$

Lemma 2.8.2 therefore reduces us to the case that $G$ is an elementary group. Let $W$ be any simple $k[G]$-module. With the notations of the proof of Theorem 2.8.3 we have seen there that $k[G]$ acts on $W$ through the projection map $k[G] \longrightarrow k[H]$. Viewed as a $k[H]$-module $W$ is projective by Remark 1.7.3. Hence $k[G] \otimes_k k[H] W$ is a finitely generated projective $k[G]$-module. We claim that

$$c_G([k[G] \otimes_k k[H] W]) = |G/H| \cdot [W]$$

holds true. Using the above commutative diagram as well as Proposition 2.3.4 we obtain

$$c_G([k[G] \otimes_k k[H] W]) = \text{ind}_{H}^G([W]) = [k[G] \otimes_k k[H] k] \cdot [W].$$

In order to analyze the $k[G]$-module $k[G] \otimes_k k[H] k$ let $h, h' \in H$, $g \in P$, and $a \in k$. Then

$$h(gh' \otimes a) = ghh' \otimes a = g \otimes a = gh' \otimes a.$$  

This shows that $H$ acts trivially on $k[G] \otimes_k k[H] k$. In other words, $k[G]$ acts on $k[G] \otimes_k k[H] k$ through the projection map $k[G] \longrightarrow k[P]$. It then follows from Proposition 2.2.7 that all simple subquotients in a composition series of the $k[G]$-
module $k[G] \otimes_{k[H]} k$ are trivial $k[G]$-modules. We conclude that

$$[k[G] \otimes_{k[H]} k] = \dim_k (k[G] \otimes_{k[H]} k) \cdot 1 = |G/H| \cdot 1$$

(where $1 \in R_k(G)$ is the unit element).

**Step 2:** We know from Proposition 1.7.1 that, as an abelian group, $R_k(G) \cong \mathbb{Z}^r$ for some $r \geq 1$. It therefore follows from Step 1 that $R_k(G)/\text{im}(c_G)$ is isomorphic to a factor group of the finite group $\mathbb{Z}^r/p^m \mathbb{Z}^r$.

**Step 3:** It is a consequence of Proposition 1.7.4 that $K_0(k[G])$ and $R_k(G)$ are isomorphic to $\mathbb{Z}^r$ for the same integer $r \geq 1$. Hence

$$\text{id} \otimes c_G : \mathbb{Q} \otimes \mathbb{Z} K_0(k[G]) \longrightarrow \mathbb{Q} \otimes \mathbb{Z} R_k(G)$$

is a linear map between two $\mathbb{Q}$-vector spaces of the same finite dimension $r$. Its injectivity is equivalent to its surjectivity. Let $a \in \mathbb{Q}$ and $x \in R_k(G)$. By Step 1 we find an element $y \in K_0(k[G])$ such that $c_G(y) = p^m x$. Then

$$(\text{id} \otimes c_G)(\frac{a}{p^m} \otimes y) = \frac{a}{p^m} \otimes p^m x = a \otimes x.$$ 

This shows that $\text{id} \otimes c_G$ and consequently $c_G$ are injective. \qed

In order to discuss the third homomorphism $e_G$ we first introduce two bilinear forms. We start from the maps

$$(\mathcal{M}_K[G]/\cong) \times (\mathcal{M}_K[G]/\cong) \longrightarrow \mathbb{Z}$$

$$([V], [W]) \longmapsto \dim_K \text{Hom}_{K[G]}(V, W)$$

and

$$(\mathcal{M}_k[G]/\cong) \times (\mathcal{M}_k[G]/\cong) \longrightarrow \mathbb{Z}$$

$$([P], [V]) \longmapsto \dim_k \text{Hom}_{k[G]}(P, V).$$

They extend to $\mathbb{Z}$-bilinear maps

$$\mathbb{Z}[\mathcal{M}_K[G]] \times \mathbb{Z}[\mathcal{M}_K[G]] \longrightarrow \mathbb{Z}$$

and

$$\mathbb{Z}[\mathcal{M}_k[G]] \times \mathbb{Z}[\mathcal{M}_k[G]] \longrightarrow \mathbb{Z}.$$ 

Since $K[G]$ is semisimple we have, for any exact sequence $0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$ in $\mathcal{M}_K[G]$, that $V \cong V_1 \oplus V_2$ and hence that

$$\dim_K \text{Hom}_{K[G]}(V, W) - \dim_K \text{Hom}_{K[G]}(V_1, W) - \dim_K \text{Hom}_{K[G]}(V_2, W) = 0.$$
The corresponding fact in the “variable” $W$ holds as well, of course. The first map therefore induces a well-defined $\mathbb{Z}$-bilinear form

$$\langle \cdot, \cdot \rangle_{K[G]} : R_K(G) \times R_K(G) \longrightarrow \mathbb{Z}$$

$$([V], [W]) \mapsto \dim_K \text{Hom}_{K[G]}(V, W).$$

Even though $k[G]$ might not be semisimple any exact sequence $0 \rightarrow P_1 \rightarrow P \rightarrow P_2 \rightarrow 0$ in $\mathcal{M}_{k[G]}$ still satisfies $P \cong P_1 \oplus P_2$ as a consequence of Lemma 1.6.2.ii. Hence we again have

$$\dim_k \text{Hom}_{k[G]}(P, V) - \dim_k \text{Hom}_{k[G]}(P, V_1) - \dim_k \text{Hom}_{k[G]}(P, V_2) = 0$$

for any $V$ in $\mathcal{M}_{k[G]}$. Furthermore, for any $P$ in $\mathcal{M}_{k[G]}$ and any exact sequence $0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$ in $\mathcal{M}_{k[G]}$ we have, by the definition of projective modules, the exact sequence

$$0 \longrightarrow \text{Hom}_{k[G]}(P, V_1) \longrightarrow \text{Hom}_{k[G]}(P, V) \longrightarrow \text{Hom}_{k[G]}(P, V_2) \longrightarrow 0.$$

Hence once more

$$\dim_k \text{Hom}_{k[G]}(P, V) - \dim_k \text{Hom}_{k[G]}(P, V_1) - \dim_k \text{Hom}_{k[G]}(P, V_2) = 0.$$

This shows that the second map also induces a well-defined $\mathbb{Z}$-bilinear form

$$\langle \cdot, \cdot \rangle_{k[G]} : K_0(k[G]) \times R_k(G) \longrightarrow \mathbb{Z}$$

$$([P], [V]) \mapsto \dim_k \text{Hom}_{k[G]}(P, V).$$

If $\{V_1\}, \ldots, \{V_r\}$ are the isomorphism classes of the simple $K[G]$-modules then $\{V_1\}, \ldots, \{V_r\}$ is a $\mathbb{Z}$-basis of $R_K(G)$ by Proposition 1.7.1. We have

$$\langle [V_i], [V_j] \rangle_{K[G]} = \begin{cases} \dim_K \text{End}_{K[G]}(V_i) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

In particular, if $K$ is a splitting field for $G$ then

$$\langle [V_i], [V_j] \rangle_{K[G]} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\{P_1\}, \ldots, \{P_t\}$ be the isomorphism classes of finitely generated indecomposable projective $k[G]$-modules. By Proposition 1.7.4.iii the $\{P_1\}, \ldots, \{P_t\}$ form a $\mathbb{Z}$-basis of $K_0(k[G])$, and the $\{P_1/\text{Jac}(k[G])P_1\}, \ldots, \{P_t/\text{Jac}(k[G])P_t\}$ form a $\mathbb{Z}$-basis of $R_k(G)$ by Proposition 1.7.4.i. We have
\[
\text{Hom}_{k[G]}(P_i, P_j / \text{Jac}(k[G]) P_j) = \text{End}_{k[G]}(P_i / \text{Jac}(k[G]) P_i) \quad \text{if } i = j,
\]
\[
= \begin{cases} 
0 & \text{if } i \neq j 
\end{cases}
\]
\[
= \begin{cases} 
k & \text{if } i = j, 
0 & \text{if } i \neq j,
\end{cases}
\]

where the latter identity comes from the fact that the algebraically closed field \( k \) is a splitting field for \( G \). Hence

\[
\langle [P_i], [P_j / \text{Jac}(k[G]) P_j] \rangle_{k[G]} = \begin{cases} 
1 & \text{if } i = j, 
0 & \text{if } i \neq j.
\end{cases}
\]

**Exercise 2.8.5**

i. If \( K \) is a splitting field for \( G \) then the map

\[
R_K(G) \cong \text{Hom}_\mathbb{Z}(R_K(G), \mathbb{Z})
\]

\[
x \mapsto \langle x, \cdot \rangle_{K[G]}
\]

is an isomorphism of abelian groups.

ii. The maps

\[
K_0(k[G]) \cong \text{Hom}_\mathbb{Z}(R_k(G), \mathbb{Z}) \quad \text{and} \quad R_k(G) \cong \text{Hom}_\mathbb{Z}(K_0(k[G]), \mathbb{Z})
\]

\[
y \mapsto \langle y, \cdot \rangle_{k[G]} \quad \text{and} \quad z \mapsto \langle \cdot, z \rangle_{k[G]}
\]

are isomorphisms of abelian groups.

**Lemma 2.8.6** We have

\[
\langle y, d_G(x) \rangle_{k[G]} = \langle e_G(y), x \rangle_{k[G]}
\]

for any \( y \in K_0(k[G]) \) and \( x \in R_K(G) \).

**Proof** It suffices to consider elements of the form \( y = [P / \pi R P] \) for some finitely generated projective \( R[G] \)-module \( P \) (see Proposition 2.1.1) and \( x = [V] \) for some finitely generated \( K[G] \)-module \( V \). We pick a \( G \)-invariant lattice \( L \subseteq V \). The asserted identity then reads

\[
\dim_k \text{Hom}_{k[G]}(P / \pi R P, L / \pi R L) = \dim_K \text{Hom}_{K[G]}(K \otimes_R P, V).
\]
We have

\[ \text{Hom}_{k[G]}(P \times R P, L \times R L) = \text{Hom}_{R[G]}(P, L \times R L) \]

\[ = \text{Hom}_{R[G]}(P, L) / \text{Hom}_{R[G]}(P, \pi R L) \]

\[ = \text{Hom}_{R[G]}(P, L) / \pi R \text{Hom}_{R[G]}(P, L) \]

\[ = k \otimes_R \text{Hom}_{R[G]}(P, L) \quad (2.8.1) \]

where the second identity comes from the projectivity of \( P \) as an \( R[G] \)-module. On the other hand

\[ \text{Hom}_{K[G]}(K \otimes_R P, V) = \text{Hom}_{R[G]}(P, V) \]

\[ = \text{Hom}_{R[G]}(P, \bigcup_{i \geq 0} \pi^{-i} R L) \]

\[ = \bigcup_{i \geq 0} \text{Hom}_{R[G]}(P, \pi^{-i} R L) \]

\[ = \bigcup_{i \geq 0} \pi^{-i} \text{Hom}_{R[G]}(P, L) \]

\[ = K \otimes_R \text{Hom}_{R[G]}(P, L). \quad (2.8.2) \]

For the third identity one has to observe that, since \( P \) is finitely generated as an \( R \)-module, any \( R \)-module homomorphism \( P \to V = \bigcup_{i \geq 0} \pi^{-i} R L \) has to have its image inside \( \pi^{-i} R L \) for some sufficiently large \( i \).

Both, \( P \) being a direct summand of some \( R[G] \) (Remark 2.1.2) and \( L \) by definition are free \( R \)-modules. Hence \( \text{Hom}_R(P, L) \) is a finitely generated free \( R \)-module. The ring \( R \) being noetherian the \( R \)-submodule \( \text{Hom}_{R[G]}(P, L) \) is finitely generated as well. Lemma 2.2.1.i then implies that \( \text{Hom}_{R[G]}(P, L) \cong R^s \) is a free \( R \)-module. We now deduce from (2.8.1) and (2.8.2) that

\[ \text{Hom}_{k[G]}(P / \pi R P, L / \pi R L) \cong k^s \quad \text{and} \quad \text{Hom}_{K[G]}(K \otimes_R P, V) \cong K^s, \]

respectively. \( \square \)

**Theorem 2.8.7** The homomorphism \( e_G : K_0(k[G]) \to R_k(G) \) is injective and its image is a direct summand of \( R_k(G) \).

**Proof** Step 1: We assume that \( R \) is splitting for \( G \). By Theorem 2.8.3 the map \( d_G : R_k(G) \to R_k(G) \) is surjective. Since, by Proposition 1.7.1, \( R_k(G) \) is a free abelian group we find a homomorphism \( s : R_k(G) \to R_k(G) \) such that \( d_G \circ s = \text{id} \). It follows that

\[ \text{Hom}(s, \mathbb{Z}) \circ \text{Hom}(d_G, \mathbb{Z}) = \text{Hom}(d_G \circ s, \mathbb{Z}) = \text{Hom}(\text{id}, \mathbb{Z}) = \text{id}. \]
Hence the map

\[ \text{Hom}(d_G, \mathbb{Z}) : \text{Hom}_\mathbb{Z}(R_k(G), \mathbb{Z}) \rightarrow \text{Hom}_\mathbb{Z}(R_K(G), \mathbb{Z}) \]

is injective and

\[ \text{Hom}_\mathbb{Z}(R_K(G), \mathbb{Z}) = \text{im}(\text{Hom}(d_G, \mathbb{Z})) \oplus \ker(\text{Hom}(\sigma, \mathbb{Z})). \]

But because of Lemma 2.8.6 the map \( \text{Hom}(d_G, \mathbb{Z}) \) corresponds under the isomorphisms in Exercise 2.8.5 to the homomorphism \( e_G \).

**Step 2:** For general \( R \) we use, as described at the beginning of this section a larger \((0, \, p)\)-ring \( R' \) for \( k \) which contains \( R \) and is splitting for \( G \). Let \( K' \) denote the field of fractions of \( R' \). If \( P \) is a finitely generated projective \( R[G] \)-module then \( R' \otimes_R P = R'[G] \otimes_{R[G]} P \) is a finitely generated projective \( R'[G] \)-module such that

\[
\left( R' \otimes_R P \right) / \pi_R R' \otimes_R P = \left( R'/\pi_R R' \right) \otimes_R P = (R/\pi_R R) \otimes_R P
\]

(recall that \( k = R/\pi_R R = R'/\pi_R R' \)). This shows that the diagram

\[
\begin{array}{ccc}
R_K(G) & \xleftarrow{\kappa} & K_0(R[G]) \\
[\{V\}] \rightarrow [K' \otimes_K V] & & [P] \rightarrow [R' \otimes_R P] \\
R_K'(G) & \xleftarrow{\kappa} & K_0(R'[G])
\end{array}
\]

is commutative. Hence

\[
\begin{array}{ccc}
K_0(k[G]) & \xrightarrow{e_G} & R_K(G) & \xrightarrow{\text{Tr}} & \text{Cl}(G, K) \\
& & [\{V\}] \rightarrow [K' \otimes_K V] & & \subseteq \\
& & R_K'(G) & \xrightarrow{\text{Tr}} & \text{Cl}(G, K')
\end{array}
\]

is commutative. The oblique arrow is injective by Step 1 and so then is the upper left horizontal arrow. The two right horizontal arrows are injective by Corollary 2.3.7.i. This implies that the middle vertical arrow is injective and therefore induces an injective homomorphism

\[ R_K(G)/\text{im}(e_G) \rightarrow R_K'(G)/\text{im}(e_G). \]
By Step 1 the target $R_K'(G)/\text{im}(e_G)$ is isomorphic to a direct summand of the free abelian group $R_K'(G)$. It follows that $R_K(G)/\text{im}(e_G)$ is isomorphic to a subgroup in a finitely generated free abelian group and hence is a free abelian group by the elementary divisor theorem. We conclude that

$$R_K(G) \cong \text{im}(e_G) \oplus R_K(G)/\text{im}(e_G).$$

We choose $R$ to be splitting for $G$, and we fix the $\mathbb{Z}$-bases of the three involved Grothendieck groups as described before Exercise 2.8.5. Let $E$, $D$, and $C$ denote the matrices which describe the homomorphisms $e_G$, $d_G$, and $c_G$, respectively, with respect to these bases. We, of course, have

$$DE = C.$$

Lemma 2.8.6 says that $D$ is the transpose of $E$. It follows that the quadratic Cartan matrix $C$ of $k[G]$ is symmetric.
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