2. The Theory of Noetherian Rings

As we have seen in 1.5/8, a ring is called *Noetherian* if all its ideals are finitely generated or, equivalently by 1.5/9, if its ideals satisfy the ascending chain condition. The aim of the present chapter is to show that the Noetherian hypothesis, as simple as it might look, nevertheless has deep impacts on the structure of ideals and their inclusions, culminating in the theory of *Krull dimension*, to be dealt with in Section 2.4.

To discuss some standard examples of Noetherian and non-Noetherian rings, recall from Hilbert's Basis Theorem 1.5/14 that all polynomial rings of type $R[X_1, \ldots, X_n]$ in *finitely* many variables $X_1, \ldots, X_n$ over a Noetherian ring R are Noetherian. The result extends to algebras of *finite type* over a Noetherian ring R, i.e. R-algebras of type $R[X_1, \ldots, X_n]/\mathfrak{a}$ where $\mathfrak{a}$ is an ideal in $R[X_1, \ldots, X_n]$. In particular, algebras of finite type over a field K or over the ring of integers $\mathbb{Z}$ are Noetherian. One also knows that all rings of integral algebraic numbers in *finite* extensions of $\mathbb{Q}$ are Noetherian (use Atiyah–Macdonald [2], 5.17), whereas the integral closure of $\mathbb{Z}$ in any infinite algebraic extension of $\mathbb{Q}$ is not; see Section 3.1 for the notion of integral dependence and in particular 3.1/8 for the one of integral closure. Also note that any polynomial ring $R[\mathfrak{X}]$ in an infinite family of variables $\mathfrak{X}$ over a non-zero ring R will not be Noetherian. Other interesting examples of non-Noetherian rings belong to the class of (general) valuation rings, as introduced in 9.5/13.

To approach the subject of Krull dimension for Noetherian rings, the technique of *primary decomposition*, developed in Section 2.1, is used as a key tool. We will show in 2.1/6 that such a primary decomposition exists for all ideals $\mathfrak{a}$ of a Noetherian ring R. It is of type

$$(*) \qquad \mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are so-called *primary ideals* in R. Primary ideals generalize the notion of prime powers in principal ideal domains, whereas the concept of primary decomposition generalizes the one of prime factorization.

Looking at a primary decomposition $(*)$, the nilradicals $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$ are of particular significance; they are prime in R and we say that $\mathfrak{q}_i$ is $\mathfrak{p}_i$-*primary*. As any finite intersection of $\mathfrak{p}$-primary ideals, for any prime ideal $\mathfrak{p} \subset R$, is $\mathfrak{p}$-primary again (see 2.1/4), we may assume that all $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ belonging to the primary decomposition $(*)$ are different. In addition, we can require that the decomposition $(*)$ is *minimal* in the sense that it cannot be shortened any

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further. In such a situation we will show in 2.1/8 that the set of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ is uniquely determined by $\mathfrak{a}$; it is denoted by $\mathrm{Ass}(\mathfrak{a})$, referring to the members of this set as the prime ideals *associated* to $\mathfrak{a}$. There is a uniqueness assertion for some of the primary ideals $\mathfrak{q}_i$ as well (see 2.1/15), although not all of them will be unique in general.

Now look at the primary decomposition (∗) and pass to nilradicals, thereby obtaining the decomposition

$$\mathrm{rad}(\mathfrak{a}) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(\mathfrak{a})} \mathfrak{p}$$

for the nilradical of $\mathfrak{a}$. This is again a primary decomposition, but maybe not a minimal one. Anyway, we know already from 1.3/6 that $\mathrm{rad}(\mathfrak{a})$ equals the intersection of all prime ideals in R containing $\mathfrak{a}$ or even better, of all minimal prime divisors of $\mathfrak{a}$; the latter are the prime ideals $\mathfrak{p} \subset R$ that are minimal with respect to the inclusion $\mathfrak{a} \subset \mathfrak{p}$. Using 1.3/8, it follows that the minimal prime divisors of $\mathfrak{a}$ all belong to $\mathrm{Ass}(\mathfrak{a})$. Thus, their number is *finite* since $\mathrm{Ass}(\mathfrak{a})$ is finite; see 2.1/12. It is this finiteness assertion, which is of utmost importance in the discussion of Krull dimensions for Noetherian rings. Translated to the world of schemes it corresponds to the fact that every Noetherian scheme consists of only finitely many irreducible components; see 7.5/5.

To give an application of the just explained finiteness of sets of associated prime ideals, consider a Noetherian ring R where every prime ideal is maximal; we will say that R is of Krull dimension 0. Then all prime ideals of R are associated to the zero ideal in R and, hence, there can exist only finitely many of them. Using this fact in conjunction with some standard arguments, we can show in 2.2/8 that R satisfies the descending chain condition for ideals and, thus, is *Artinian*. Conversely, it is shown that every Artinian ring is Noetherian of Krull dimension 0.

The Krull dimension of a general ring R is defined as the supremum of all lengths n of chains of prime ideals $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n$ in R and is denoted by $\dim R$. Restricting to chains ending at a given prime ideal $\mathfrak{p} \subset R$, the corresponding supremum is called the *height* of $\mathfrak{p}$, denoted by $\mathrm{ht}\,\mathfrak{p}$. As a first major result in dimension theory we prove Krull's Dimension Theorem 2.4/6, implying that $\mathrm{ht}\,\mathfrak{p}$ is finite if R is Noetherian. From this we conclude that the Krull dimension of any Noetherian *local* ring is finite; see 2.4/8. On the other hand, it is not too hard to construct (non-local) Noetherian rings R where $\dim R = \infty$. Namely, following Nagata [22], Appendix A1, Example 1, we consider a polynomial ring $R = K[\mathfrak{X}_1, \mathfrak{X}_2, \ldots]$ over a field K where each $\mathfrak{X}_i$ is a finite system of variables, say of length n_i, such that $\lim_i n_i = \infty$. Let $\mathfrak{p}_i$ be the prime ideal that is generated by $\mathfrak{X}_i$ in R and let $S \subset R$ be the multiplicative system given by the complement of the union $\bigcup_{i=1}^{\infty} \mathfrak{p}_i$. Then we claim that the localization R_S is a Noetherian ring of infinite dimension. Indeed, looking at residue rings of type $R/(\mathfrak{X}_r, \mathfrak{X}_{r+1}, \ldots)$ for sufficiently large indices r, we can use 1.3/7 in order to show that the $\mathfrak{p}_i$ are just those ideals in R that are maximal with respect to the property of being disjoint from S. Therefore the ideals $\mathfrak{m}_i = \mathfrak{p}_i R_S$, $i = 1, 2, \ldots$, represent all maximal ideals of R_S. Identifying each localization $(R_S)_{\mathfrak{m}_i}$ with the localization $K(\mathfrak{X}_j; j \neq i)[\mathfrak{X}_i]_{(\mathfrak{X}_i)}$, we see with the help of Hilbert's Basis Theorem 1.5/14 that all these rings are Noetherian. It follows, as remarked after defining Krull dimensions in 2.4/2, that $\dim(R_S)_{\mathfrak{m}_i} = \operatorname{ht}(\mathfrak{m}_i) \geq n_i$. Hence, we get $\dim R_S = \infty$. To show that R_S is Noetherian, indeed, one can use the facts that all localizations $(R_S)_{\mathfrak{m}_i}$ are Noetherian and that any non-zero element $a \in R_S$ is contained in at most finitely many of the maximal ideals $\mathfrak{m}_i \subset R_S$. See the reference of Nagata [22] given above in conjunction with Exercise 4.3/5.

Krull's Dimension Theorem 2.4/6 reveals a very basic fact: if R is a Noetherian ring and $\mathfrak{a} \subset R$ an ideal generated by r elements, then $\operatorname{ht} \mathfrak{p} \leq r$ for every minimal prime divisor $\mathfrak{p}$ of $\mathfrak{a}$. For the proof we need the finiteness of $\operatorname{Ass}(\mathfrak{a})$ as discussed in 2.1/12 and, within the context of localizations, the characterization of Noetherian rings of dimension 0 in terms of Artinian rings. Another technical ingredient is Krull's Intersection Theorem 2.3/2, which in turn is based on Nakayama's Lemma 1.4/10 and the Artin–Rees Lemma 2.3/1. Both, Krull's Intersection Theorem and the Artin–Rees Lemma allow nice topological interpretations in terms of ideal-adic topologies; see the corresponding discussion in Section 2.3.

For Noetherian local rings there is a certain converse of Krull's Intersection Theorem. Consider such a ring R of dimension d, and let $\mathfrak{m}$ be its maximal ideal so that $\operatorname{ht} \mathfrak{m} = d$. Then every $\mathfrak{m}$-primary ideal $\mathfrak{q} \subset R$ satisfies $\operatorname{rad}(\mathfrak{q}) = \mathfrak{m}$ and, thus, by Krull's Dimension Theorem, cannot be generated by less than d elements. On the other hand, a simple argument shows in 2.4/11 that there always exist $\mathfrak{m}$-primary ideals in R that are generated by a system of d elements. Alluding to the situation of polynomial rings over fields, such systems are called systems of *parameters* of the local ring R. Using parameters, the dimension theory of Noetherian local rings can be handled quite nicely; see for example the results 2.4/13 and 2.4/14. Furthermore, we can show that a polynomial ring $R[X_1, \ldots, X_n]$ in finitely many variables $X_1, \ldots, X_n$ over a Noetherian ring R has dimension $\dim R + n$, which by examples of Seidenberg may fail to be true if R is not Noetherian any more.

A very particular class of Noetherian local rings is given by the subclass of *regular* local rings, where a Noetherian local ring is called regular if its maximal ideal can be generated by a system of parameters. Such rings are integral domains, as we show in 2.4/19. They are quite close to (localizations of) polynomial rings over fields and are useful to characterize the geometric notion of smoothness in ALGEBRAIC GEOMETRY; see for example 8.5/15.

2.1 Primary Decomposition of Ideals

Let R be a principal ideal domain. Then R is factorial and any non-zero element $a \in R$ admits a factorization $a = \varepsilon p_1^{n_1} \ldots p_r^{n_r}$ with a unit $\varepsilon \in R^*$, pairwise non-equivalent prime elements $p_i \in R$, and exponents $n_i > 0$ where these quantities are essentially unique. Passing to ideals, it follows that every ideal $\mathfrak{a} \subset R$ admits a decomposition

$$\mathfrak{a} = \mathfrak{p}_1^{n_1} \cap \ldots \cap \mathfrak{p}_r^{n_r}$$

with pairwise different prime ideals $\mathfrak{p}_i$ that are unique up to order, and exponents $n_i > 0$ that are unique as well. The purpose of the present section is to study similar decompositions for more general rings R where the role of the above prime powers $\mathfrak{p}_i^{n_i}$ is taken over by the so-called *primary ideals*. In the following we start with a general ring R (commutative and with a unit element, as always). Only later, when we want to show the existence of primary decompositions, R will be assumed to be Noetherian. For a generalization of primary decompositions to the context of modules see Serre [24], I.B.

Definition 1. *A proper ideal $\mathfrak{q} \subset R$ is called a* primary ideal *if $ab \in \mathfrak{q}$ for any elements $a, b \in R$ implies $a \in \mathfrak{q}$ or, if the latter is not the case, that there is an exponent $n \in \mathbb{N}$ such that $b^n \in \mathfrak{q}$.*

Clearly, any prime ideal is primary. Likewise, for a prime element p of a factorial ring, all powers $(p)^n$, $n > 0$, are primary. But for general rings we will see that the higher powers of prime ideals may fail to be primary. Also note that an ideal $\mathfrak{q} \subset R$ is primary if and only if the zero ideal in $R/\mathfrak{q}$ is primary. The latter amounts to the fact that all zero-divisors in $R/\mathfrak{q}$ are nilpotent. More generally, if $\pi\colon R \longrightarrow R/\mathfrak{a}$ is the canonical projection from R onto its quotient by any ideal $\mathfrak{a} \subset R$, then an ideal $\mathfrak{q} \subset R/\mathfrak{a}$ is primary if and only if its preimage $\pi^{-1}(\mathfrak{q})$ is primary in R.

Remark 2. *Let $\mathfrak{q} \subset R$ be a primary ideal. Then $\mathfrak{p} = \operatorname{rad}(\mathfrak{q})$ is a prime ideal in R. It is the unique smallest prime ideal $\mathfrak{p}$ containing $\mathfrak{q}$.*

Proof. First, $\operatorname{rad}(\mathfrak{q})$ is a proper ideal in R, since the same holds for $\mathfrak{q}$. Now let $ab \in \operatorname{rad}(\mathfrak{q})$ for some elements $a, b \in R$ where $a \notin \operatorname{rad}(\mathfrak{q})$. Then there is an exponent $n \in \mathbb{N}$ such that $a^n b^n \in \mathfrak{q}$. Since $a \notin \operatorname{rad}(\mathfrak{q})$ implies $a^n \notin \mathfrak{q}$, there exists an exponent $n' \in \mathbb{N}$ such that $b^{nn'} \in \mathfrak{q}$. However, the latter shows $b \in \operatorname{rad}(\mathfrak{q})$ and we see that $\operatorname{rad}(\mathfrak{q})$ is prime. Furthermore, if $\mathfrak{p}'$ is a prime ideal in R containing $\mathfrak{q}$, it must contain $\mathfrak{p} = \operatorname{rad}(\mathfrak{q})$ as well. Hence, the latter is the unique smallest prime ideal containing $\mathfrak{q}$. □

Given a primary ideal $\mathfrak{q} \subset R$, we will say more specifically that $\mathfrak{q}$ is $\mathfrak{p}$-*primary* for the prime ideal $\mathfrak{p} = \operatorname{rad}(\mathfrak{q})$.

Remark 3. *Let $\mathfrak{q} \subset R$ be an ideal such that its radical $\operatorname{rad}(\mathfrak{q})$ coincides with a maximal ideal $\mathfrak{m} \subset R$. Then $\mathfrak{q}$ is $\mathfrak{m}$-primary.*

Proof. Replacing R by $R/\mathfrak{q}$ and $\mathfrak{m}$ by $\mathfrak{m}/\mathfrak{q}$, we may assume $\mathfrak{q} = 0$. Then $\mathfrak{m}$ is the only prime ideal in R by 1.3/4 and we see that R is a local ring with maximal ideal $\mathfrak{m}$. Since $R - \mathfrak{m}$ consists of units, all zero divisors of R are contained in $\mathfrak{m}$ and therefore nilpotent. This shows that $\mathfrak{q} = 0$ is a primary ideal in R. □

In particular, the powers $\mathfrak{m}^n$, $n > 0$, of any maximal ideal $\mathfrak{m} \subset R$ are $\mathfrak{m}$-primary. However, there might exist $\mathfrak{m}$-primary ideals of more general type, as can be read from the example of the polynomial ring $R = K[X, Y]$ over a field K. Namely, $\mathfrak{q} = (X, Y^2)$ is $\mathfrak{m}$-primary for $\mathfrak{m} = (X, Y)$, although it is not a power of $\mathfrak{m}$. On the other hand, there are rings R admitting a (non-maximal) prime ideal $\mathfrak{p}$ such that the higher powers of $\mathfrak{p}$ are *not* primary. To construct such a ring, look at the polynomial ring $R = K[X, Y, Z]$ in three variables over a field K. The ideal $\mathfrak{q} = (X^2, XZ, Z^2, XY - Z^2) \subset R$ has radical $\mathfrak{p} = \mathrm{rad}(\mathfrak{q}) = (X, Z)$, which is prime. However, $\mathfrak{q}$ is not primary, since $XY \in \mathfrak{q}$, although $X \notin \mathfrak{q}$ and $Y \notin \mathrm{rad}(\mathfrak{q}) = \mathfrak{p}$. Now observe that that $\mathfrak{q} = (X, Z)^2 + (XY - Z^2)$. Since $(XY - Z^2) \subset \mathfrak{q} \subset \mathfrak{p}$, we may pass to the residue ring $\overline{R} = R/(XY - Z^2)$, obtaining $\overline{\mathfrak{q}} = \overline{\mathfrak{p}}^2$ for the ideals induced from $\mathfrak{p}$ and $\mathfrak{q}$. Then $\overline{\mathfrak{p}}$ is prime in $\overline{R}$, but $\overline{\mathfrak{p}}^2$ cannot be primary since its preimage $\mathfrak{q} \subset R$ is not primary.

Lemma 4. *Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be $\mathfrak{p}$-primary ideals in R for some prime ideal $\mathfrak{p} \subset R$. Then the intersection $\mathfrak{q} = \bigcap_{i=1}^r \mathfrak{q}_i$ is $\mathfrak{p}$-primary.*

Proof. First observe that $\mathrm{rad}(\mathfrak{q}) = \bigcap_{i=1}^r \mathrm{rad}(\mathfrak{q}_i) = \mathfrak{p}$, since we have $\mathrm{rad}(\mathfrak{q}_i) = \mathfrak{p}$ for all i. Next consider elements $a, b \in R$ such that $ab \in \mathfrak{q}$ or, in other words, $ab \in \mathfrak{q}_i$ for all i. Now if $b \notin \mathrm{rad}(\mathfrak{q}) = \mathfrak{p}$, we get $b \notin \mathrm{rad}(\mathfrak{q}_i)$ for all i. But then, as all $\mathfrak{q}_i$ are primary, we must have $a \in \mathfrak{q}_i$ for all i and, hence, $a \in \mathfrak{q}$. This shows that $\mathfrak{q}$ is $\mathfrak{p}$-primary. □

Definition 5. *A* primary decomposition *of an ideal $\mathfrak{a} \subset R$ is a decomposition*

$$\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$$

into primary ideals $\mathfrak{q}_i \subset R$. The decomposition is called minimal *if the corresponding prime ideals $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$ are pairwise different and the decomposition is unshortenable; the latter means that $\bigcap_{i \neq j} \mathfrak{q}_i \not\subset \mathfrak{q}_j$ for all $j = 1, \ldots, r$.*

An ideal $\mathfrak{a} \subset R$ is called decomposable *if it admits a primary decomposition.*

In particular, it follows from Lemma 4 that any primary decomposition of an ideal $\mathfrak{a} \subset R$ can be reduced to a minimal one. On the other hand, to show the existence of primary decompositions, special assumptions are necessary such as R being Noetherian.

Theorem 6. *Let R be a Noetherian ring. Then any ideal $\mathfrak{a} \subset R$ is decomposable, i.e. it admits a primary decomposition and, in particular, a minimal one.*

Proof. Let us call a proper ideal $\mathfrak{q} \subset R$ *irreducible* if from any relation $\mathfrak{q} = \mathfrak{a}_1 \cap \mathfrak{a}_2$ with ideals $\mathfrak{a}_1, \mathfrak{a}_2 \subset R$ we get $\mathfrak{q} = \mathfrak{a}_1$ or $\mathfrak{q} = \mathfrak{a}_2$. In a first step we show that every irreducible ideal $\mathfrak{q} \subset R$ is primary. To achieve this we may pass to the quotient $R/\mathfrak{q}$, thereby assuming that $\mathfrak{q}$ is the zero ideal in R.

Now consider elements $a, b \in R$, $a \neq 0$, such that $ab = 0$ and look at the chain of ideals

$$\mathrm{Ann}(b) \subset \mathrm{Ann}(b^2) \subset \ldots \subset R$$

where the annihilator $\mathrm{Ann}(b^i)$ consists of all elements $x \in R$ such that $xb^i = 0$. The chain becomes stationary, since R is Noetherian. Hence, there is an index $n \in \mathbb{N}$ such that $\mathrm{Ann}(b^i) = \mathrm{Ann}(b^n)$ for all $i \geq n$. We claim that $(a) \cap (b^n) = 0$. Indeed, for any $y \in (a) \cap (b^n)$ there are elements $c, d \in R$ such that $y = ca = db^n$. Then $yb = 0$ and, hence, $db^{n+1} = 0$ so that $d \in \mathrm{Ann}(b^{n+1}) = \mathrm{Ann}(b^n)$. It follows $y = db^n = 0$ and therefore $(a) \cap (b^n) = 0$, as claimed. Assuming that the zero ideal of R is irreducible, the latter relation shows $(b^n) = 0$, since $a \neq 0$. Consequently, b is nilpotent and we see that $0 \subset R$ is primary.

It remains to show that every ideal $\mathfrak{a} \subset R$ is a finite intersection of irreducible ideals. To do this, let M be the set of all ideals in R that are *not* representable as a finite intersection of irreducible ideals. If M is non-empty, we can use the fact that R is Noetherian and thereby show that M contains a maximal element $\mathfrak{a}$. Then $\mathfrak{a}$ cannot be irreducible, due to the definition of M. Therefore we can write $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$ with ideals $\mathfrak{a}_1, \mathfrak{a}_2 \subset R$ that do not belong to M. But then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are finite intersections of irreducible ideals in R and the same is true for $\mathfrak{a}$. However, this contradicts the definition of $\mathfrak{a}$ and therefore yields $M = \emptyset$. Thus, every ideal in R is a finite intersection of irreducible and, hence, primary ideals. □

To give an example of a non-trivial primary decomposition, consider again the polynomial ring $R = K[X, Y, Z]$ in three variables over a field K and its ideals

$$\mathfrak{p} = (X, Y), \qquad \mathfrak{p}' = (X, Z), \qquad \mathfrak{m} = (X, Y, Z).$$

Then $\mathfrak{a} = \mathfrak{p} \cdot \mathfrak{p}'$ satisfies

$$\mathfrak{a} = \mathfrak{p} \cap \mathfrak{p}' \cap \mathfrak{m}^2$$

and, as is easily checked, this is a minimal primary decomposition.

In the following we will discuss uniqueness assertions for primary decompositions of decomposable ideals. In particular, this applies to the Noetherian case, although the uniqueness results themselves do not require the Noetherian assumption. We will need the notion of ideal quotients, as introduced in Section 1.1. Recall that for an ideal $\mathfrak{q} \subset R$ and an element $x \in R$ the ideal quotient $(\mathfrak{q} : x)$ consists of all elements $a \in R$ such that $ax \in \mathfrak{q}$.

Lemma 7. *Consider a primary ideal $\mathfrak{q} \subset R$ as well as the corresponding prime ideal $\mathfrak{p} = \mathrm{rad}(\mathfrak{q})$. Then, for any $x \in R$,*

(i) $(\mathfrak{q} : x) = R$ *if* $x \in \mathfrak{q}$,

(ii) $(\mathfrak{q} : x)$ *is* $\mathfrak{p}$*-primary if* $x \notin \mathfrak{q}$,

(iii) $(\mathfrak{q} : x) = \mathfrak{q}$ *if* $x \notin \mathfrak{p}$.

Furthermore, if R is Noetherian, there exists an element $x \in R$ such that $(\mathfrak{q} : x) = \mathfrak{p}$.

Proof. The case (i) is trivial, whereas (iii) follows directly from the definition of primary ideals, since $\mathfrak{p} = \mathrm{rad}(\mathfrak{q})$. To show (ii), assume $x \notin \mathfrak{q}$ and observe that

$$\mathfrak{q} \subset (\mathfrak{q} : x) \subset \mathfrak{p} = \mathrm{rad}(\mathfrak{q}),$$

where the first inclusion is trivial and the second follows from the fact that $\mathfrak{q}$ is $\mathfrak{p}$-primary. In particular, we can conclude that $\mathrm{rad}\big((\mathfrak{q} : x)\big) = \mathfrak{p}$. Now let $a, b \in R$ such that $ab \in (\mathfrak{q} : x)$, and assume $b \notin \mathfrak{p}$. Then $abx \in \mathfrak{q}$ implies $ax \in \mathfrak{q}$ as $b \notin \mathfrak{p}$ and, hence, $a \in (\mathfrak{q} : x)$. In particular, $(\mathfrak{q} : x)$ is $\mathfrak{p}$-primary.

Next assume that R is Noetherian. Then $\mathfrak{p} = \mathrm{rad}(\mathfrak{q})$ is finitely generated and there exists an integer $n \in \mathbb{N}$ such that $\mathfrak{p}^n \subset \mathfrak{q}$. Assume that n is minimal so that $\mathfrak{p}^{n-1} \not\subset \mathfrak{q}$, and choose $x \in \mathfrak{p}^{n-1} - \mathfrak{q}$. Then $\mathfrak{p} \subset (\mathfrak{q} : x)$, since $\mathfrak{p} \cdot x \in \mathfrak{p}^n \subset \mathfrak{q}$, and we see from (ii) that $(\mathfrak{q} : x) \subset \mathfrak{p}$. Hence, $(\mathfrak{q} : x) = \mathfrak{p}$. □

Theorem 8. *For a decomposable ideal* $\mathfrak{a} \subset R$, *let*

$$\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$$

be a minimal primary decomposition. Then the set of corresponding prime ideals $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$ *is independent of the chosen decomposition and, thus, depends only on the ideal* $\mathfrak{a}$. *More precisely, it coincides with the set of all prime ideals in* R *that are of type* $\mathrm{rad}\big((\mathfrak{a} : x)\big)$ *for* x *varying over* R.

Proof. Let $x \in R$. The primary decomposition of $\mathfrak{a}$ yields $(\mathfrak{a} : x) = \bigcap_{i=1}^{r} (\mathfrak{q}_i : x)$ and therefore by Lemma 7 a primary decomposition

$$\mathrm{rad}\big((\mathfrak{a} : x)\big) = \bigcap_{i=1}^{r} \mathrm{rad}\big((\mathfrak{q}_i : x)\big) = \bigcap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i.$$

Now assume that the ideal $\mathrm{rad}\big((\mathfrak{a} : x)\big)$ is prime. Then we see from 1.3/8 that it must coincide with one of the prime ideals $\mathfrak{p}_i$. Conversely, fix any of the ideals $\mathfrak{p}_i$, say $\mathfrak{p}_j$ where $j \in \{1, \ldots, r\}$, and choose $x \in (\bigcap_{i \neq j} \mathfrak{q}_i) - \mathfrak{q}_j$. Then, as we have just seen, we get the primary decomposition

$$\mathrm{rad}\big((\mathfrak{a} : x)\big) = \bigcap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i = \mathfrak{p}_j$$

and we are done. □

The prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ in the situation of Theorem 8 are uniquely associated to the ideal $\mathfrak{a}$. We use this fact to give them a special name.

Definition and Proposition 9. *Let* $\mathfrak{a} \subset R$ *be an ideal.*

(i) *The set of all prime ideals in* R *that are of type* $\mathrm{rad}\big((\mathfrak{a} : x)\big)$ *for some* $x \in R$ *is denoted by* $\mathrm{Ass}(\mathfrak{a})$; *its members are called the* prime ideals associated *to* $\mathfrak{a}$.[1]

[1] Let us point out that usually the set $\mathrm{Ass}(\mathfrak{a})$ is defined by the condition in (ii), which is equivalent to (i) only in the Noetherian case. However, we prefer to base the definition of

(ii) *If R is Noetherian, a prime ideal $\mathfrak{p} \subset R$ belongs to* $\operatorname{Ass}(\mathfrak{a})$ *if and only if it is of type* $(\mathfrak{a} : x)$ *for some* $x \in R$.

(iii) $\operatorname{Ass}(\mathfrak{a})$ *is finite if R is Noetherian or, more generally, if $\mathfrak{a}$ is decomposable.*

Proof. Assume that R is Noetherian and let $\mathfrak{a} = \bigcap_{i=1}^r \mathfrak{q}_i$ be a minimal primary decomposition of $\mathfrak{a}$; see Theorem 6. Choosing $x \in (\bigcap_{i \neq j} \mathfrak{q}_i) - \mathfrak{q}_j$ for any $j \in \{1, \ldots, r\}$, we obtain

$$(\mathfrak{a} : x) = \bigcap_{x \notin \mathfrak{q}_i} (\mathfrak{q}_i : x) = (\mathfrak{q}_j : x)$$

where $(\mathfrak{q}_j : x)$ is $\mathfrak{p}_j$-primary by Lemma 7. Furthermore, by the same result, we can find an element $x' \in R$ such that $(\mathfrak{q}_j : xx') = \big((\mathfrak{q}_j : x) : x'\big) = \mathfrak{p}_j$. But then, since $xx' \in \mathfrak{q}_i$ for all $i \neq j$, we conclude that

$$(\mathfrak{a} : xx') = \bigcap_{xx' \notin \mathfrak{q}_i} (\mathfrak{q}_i : xx') = (\mathfrak{q}_j : xx') = \mathfrak{p}_j.$$

This shows that any prime ideal in $\operatorname{Ass}(\mathfrak{a})$ is as stated in (ii). Since prime ideals of type (ii) are also of type (i), assertion (ii) is clear. Furthermore, (iii) follows from Theorem 8. □

There is a close relationship between the prime ideals associated to an ideal $\mathfrak{a} \subset R$ and the set of zero divisors in $R/\mathfrak{a}$. For an arbitrary ideal $\mathfrak{a} \subset R$ let us call

$$Z(\mathfrak{a}) = \bigcup_{x \in R - \mathfrak{a}} (\mathfrak{a} : x) = \{z \in R\,;\ zx \in \mathfrak{a} \text{ for some } x \in R - \mathfrak{a}\},$$

the set of *zero divisors modulo* $\mathfrak{a}$ in R. Indeed, an element $z \in R$ belongs to $Z(\mathfrak{a})$ if and only if its residue class $\overline{z} \in R/\mathfrak{a}$ is a zero divisor. Furthermore, if a power z^n of some element $z \in R$ belongs to $Z(\mathfrak{a})$, then z itself must belong to $Z(\mathfrak{a})$ and therefore

$$Z(\mathfrak{a}) = \bigcup_{x \in R - \mathfrak{a}} \operatorname{rad}\big((\mathfrak{a} : x)\big).$$

Corollary 10. *If $\mathfrak{a}$ is a decomposable ideal in R, then*

$$Z(\mathfrak{a}) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(\mathfrak{a})} \mathfrak{p}.$$

Proof. Let $\mathfrak{p} \in \operatorname{Ass}(\mathfrak{a})$. Then, by the definition of $\operatorname{Ass}(\mathfrak{a})$, there exists some $x \in R$ such that $\mathfrak{p} = \operatorname{rad}\big((\mathfrak{a} : x)\big)$. Since $\mathfrak{p}$ is a proper ideal in R, the element x cannot belong to $\mathfrak{a}$ and, hence, $\mathfrak{p} \subset Z(\mathfrak{a})$.

$\operatorname{Ass}(\mathfrak{a})$ on the condition in (i), since this is more appropriate for handling primary decompositions in non-Noetherian situations.

Conversely, consider an element $z \in Z(\mathfrak{a})$ and let $x \in R - \mathfrak{a}$ such that $z \in (\mathfrak{a} : x)$. Furthermore, let $\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$ be a minimal primary decomposition of $\mathfrak{a}$. Then, writing $\mathfrak{p}_i = \operatorname{rad}(\mathfrak{q}_i)$, we get

$$z \in (\mathfrak{a} : x) = \bigcap_{x \notin \mathfrak{q}_i} (\mathfrak{q}_i : x) \subset \bigcap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i,$$

since $(\mathfrak{q}_i : x)$ is $\mathfrak{p}_i$-primary for $x \notin \mathfrak{q}_i$ and coincides with R for $x \in \mathfrak{q}_i$; see Lemma 7. Using $x \notin \mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$, the set of all i such that $x \notin \mathfrak{q}_i$ is non-empty and we are done. □

Although for a minimal primary decomposition $\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$ of some ideal $\mathfrak{a} \subset R$, proper inclusions of type $\mathfrak{q}_i \subsetneq \mathfrak{q}_j$ are not allowed, these can nevertheless occur on the level of prime ideals associated to $\mathfrak{a}$. For example, look at the polynomial ring $R = K[X, Y]$ in two variables over a field K. Then the ideal $\mathfrak{a} = (X^2, XY)$ admits the primary decomposition $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2^2$ where $\mathfrak{p}_1 = (X)$ and $\mathfrak{p}_2 = (X, Y)$. Clearly, this primary decomposition is minimal and, hence, we get $\operatorname{Ass}(\mathfrak{a}) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ with $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$.

Definition 11. *Given any ideal $\mathfrak{a} \subset R$, the subset of all prime ideals that are minimal in* $\operatorname{Ass}(\mathfrak{a})$ *is denoted by* $\operatorname{Ass}'(\mathfrak{a})$ *and its members are called the* isolated *prime ideals associated to $\mathfrak{a}$. All other elements of* $\operatorname{Ass}(\mathfrak{a})$ *are said to be* embedded *prime ideals.*

The notion of isolated and embedded prime ideals is inspired from geometry. Indeed, passing to the spectrum $\operatorname{Spec} R$ of all prime ideals in R and looking at zero sets of type $V(E)$ for subsets $E \subset R$ as done in Section 6.1, a strict inclusion of prime ideals $\mathfrak{p} \subsetneq \mathfrak{p}'$ is reflected on the level of zero sets as a strict inclusion $V(\mathfrak{p}) \supsetneq V(\mathfrak{p}')$. In particular, the zero set $V(\mathfrak{p}')$ is "embedded" in the bigger one $V(\mathfrak{p})$ and, likewise, $\mathfrak{p}'$ is said to be an embedded prime ideal associated to $\mathfrak{a}$ if we have $\mathfrak{p}, \mathfrak{p}' \in \operatorname{Ass}(\mathfrak{a})$ with $\mathfrak{p} \subsetneq \mathfrak{p}'$. Also recall from Theorem 8 that if $\mathfrak{a}$ is decomposable with primary decomposition $\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$, then $\operatorname{Ass}(\mathfrak{a})$ consists of the finitely many radicals $\mathfrak{p}_i = \operatorname{rad}(\mathfrak{q}_i)$. Since $V(\mathfrak{q}_i) = V(\mathfrak{p}_i)$, one obtains

$$V(\mathfrak{a}) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(\mathfrak{a})} V(\mathfrak{p}) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}'(\mathfrak{a})} V(\mathfrak{p}),$$

as explained at the end of Section 1.3.

Proposition 12. *Let $\mathfrak{a} \subset R$ be an ideal and assume that it is decomposable; for example, the latter is the case by Theorem 6 if R is Noetherian. Then:*

(i) *The set* $\operatorname{Ass}(\mathfrak{a})$ *and its subset* $\operatorname{Ass}'(\mathfrak{a})$ *of isolated prime ideals $\mathfrak{a}$ are finite.*

(ii) *Every prime ideal $\mathfrak{p} \subset R$ satisfying $\mathfrak{a} \subset \mathfrak{p}$ contains a member of* $\operatorname{Ass}'(\mathfrak{a})$. *Consequently,* $\operatorname{Ass}'(\mathfrak{a})$ *consists of all prime ideals $\mathfrak{p} \subset R$ that are minimal among the ones satisfying $\mathfrak{a} \subset \mathfrak{p} \subset R$.*

(iii) $\operatorname{rad}(\mathfrak{a}) = \bigcap_{\mathfrak{p} \in \operatorname{Ass}'(\mathfrak{a})} \mathfrak{p}$ *is the primary decomposition of the nilradical of $\mathfrak{a}$.*

Proof. Let $\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$ be a minimal primary decomposition and set $\mathfrak{p}_i = \operatorname{rad}(\mathfrak{q}_i)$. Then $\operatorname{Ass}(\mathfrak{a})$ is finite, since it consists of the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ by Theorem 8. Furthermore, we obtain $\mathfrak{a} \subset \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Ass}(\mathfrak{a})$. Conversely, consider a prime ideal $\mathfrak{p} \subset R$ such that $\mathfrak{a} \subset \mathfrak{p}$. Then we see from $\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i \subset \mathfrak{p}$ by 1.3/8 that there is an index i satisfying $\mathfrak{q}_i \subset \mathfrak{p}$ and, hence, $\mathfrak{p}_i = \operatorname{rad}(\mathfrak{q}_i) \subset \mathfrak{p}$. In particular, $\mathfrak{p}$ contains an element of $\operatorname{Ass}(\mathfrak{a})$ and, thus, of $\operatorname{Ass}'(\mathfrak{a})$. This settles (ii), whereas (iii) is derived from the primary decomposition of $\mathfrak{a}$ by passing to radicals. □

Looking at a primary decomposition $\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$ of some ideal $\mathfrak{a} \subset R$, we know from Theorem 8 that the associated prime ideals $\mathfrak{p}_i = \operatorname{rad}(\mathfrak{q}_i)$ are unique; this led us to the definition of the set $\operatorname{Ass}(\mathfrak{a})$. In the remainder of this section we want to examine the uniqueness of the primary ideals $\mathfrak{q}_i$ themselves. Not all of them will be unique, as is demonstrated by the two different primary decompositions

$$(X^2, XY) = (X) \cap (X, Y)^2 = (X) \cap (X^2, Y)$$

in the polynomial ring $R = K[X, Y]$ over a field K. We start with the following generalization of 1.2/5:

Lemma 13. *Let $S \subset R$ be a multiplicative system and R_S the localization of R by S. Furthermore, consider a prime ideal $\mathfrak{p} \subset R$ and a $\mathfrak{p}$-primary ideal $\mathfrak{q} \subset R$.*

(i) *If $S \cap \mathfrak{p} \neq \emptyset$, then $\mathfrak{q}R_S = R_S$.*

(ii) *If $S \cap \mathfrak{p} = \emptyset$, then $\mathfrak{p}R_S$ is a prime ideal in R_S satisfying $\mathfrak{p}R_S \cap R = \mathfrak{p}$ and $\mathfrak{q}R_S$ is a $\mathfrak{p}R_S$-primary ideal in R_S satisfying $\mathfrak{q}R_S \cap R = \mathfrak{q}$.*

Proof. Since $\mathfrak{p} = \operatorname{rad}(\mathfrak{q})$, we see that $S \cap \mathfrak{p} \neq \emptyset$ implies $S \cap \mathfrak{q} \neq \emptyset$ and, hence, $\mathfrak{q}R_S = R_S$ by 1.2/5. This settles (i). To show (ii), assume that $S \cap \mathfrak{p} = \emptyset$. Then we conclude from 1.2/5 that $\mathfrak{p}R_S$ is a prime ideal in R_S containing $\mathfrak{q}R_S$ and satisfying $\mathfrak{p}R_S \cap R = \mathfrak{p}$. In particular, $\mathfrak{q}R_S$ is a proper ideal in R_S and we can even see that $\mathfrak{p}R_S = \operatorname{rad}(\mathfrak{q}R_S)$.

To show $\mathfrak{q}R_S \cap R = \mathfrak{q}$, observe first that $\mathfrak{q} \subset \mathfrak{q}R_S \cap R$ for trivial reasons. Conversely, consider an element $a \in \mathfrak{q}R_S \cap R$. Then there are elements $a' \in \mathfrak{q}$ and $s \in S$ such that $\frac{a}{1} = \frac{a'}{s}$ in R_S. So there exists $t \in S$ such that $(as - a')t = 0$ in R and, hence, $ast \in \mathfrak{q}$. Since $st \in S$, no power of it will be contained in $\mathfrak{q}$ and we get $a \in \mathfrak{q}$ from the primary ideal condition of $\mathfrak{q}$. Therefore $\mathfrak{q}R_S \cap R = \mathfrak{q}$, as claimed. It remains to show that $\mathfrak{q}R_S$ is a $\mathfrak{p}R_S$-primary ideal. Indeed, look at elements $\frac{a}{s}, \frac{a'}{s'} \in R_S$ such that $\frac{aa'}{ss'} \in \mathfrak{q}R_S$, and assume $\frac{a}{s} \notin \mathfrak{q}R_S$. Then $\frac{aa'}{1} \in \mathfrak{q}R_S$ and, hence, $aa' \in \mathfrak{q}$, as we just have shown. Furthermore, $\frac{a}{s} \notin \mathfrak{q}R_S$ implies $a \notin \mathfrak{q}$ so that $a' \in \mathfrak{p}$. But then $\frac{a'}{s'} \in \mathfrak{p}R_S$ and we are done. □

Proposition 14. *Let $S \subset R$ be a multiplicative system and R_S the localization of R by S. Furthermore, consider a minimal primary decomposition $\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$ of some ideal $\mathfrak{a} \subset R$. Let $\mathfrak{p}_i = \operatorname{rad}(\mathfrak{q}_i)$ for $i = 1, \ldots, r$ and assume that r' is an integer, $0 \leq r' \leq r$, such that $\mathfrak{p}_i \cap S = \emptyset$ for $i = 1, \ldots, r'$ and $\mathfrak{p}_i \cap S \neq \emptyset$ for*

$i = r'+1,\dots,r$. Then $\mathfrak{q}_i R_S$ is $\mathfrak{p}_i R_S$-primary for $i = 1,\dots,r'$ and

$$\text{(i) } \mathfrak{a}R_S = \bigcap_{i=1}^{r'} \mathfrak{q}_i R_S \qquad \textit{as well as} \qquad \text{(ii) } \mathfrak{a}R_S \cap R = \bigcap_{i=1}^{r'} \mathfrak{q}_i$$

are minimal primary decompositions in R_S and R.

Proof. The equations (i) and (ii) can be deduced from Lemma 13 if we know that the formation of extended ideals in R_S commutes with finite intersections. However, to verify this is an easy exercise. If $\mathfrak{b}, \mathfrak{c}$ are ideals in R, then clearly $(\mathfrak{b} \cap \mathfrak{c})R_S \subset \mathfrak{b}R_S \cap \mathfrak{c}R_S$. To show the reverse inclusion, consider an element $\frac{b}{s} = \frac{c}{s'} \in \mathfrak{b}R_S \cap \mathfrak{c}R_S$ where $b \in \mathfrak{b}$, $c \in \mathfrak{c}$, and $s, s' \in S$. Then there is some $t \in S$ such that $(bs' - cs)t = 0$ and, hence, $bs't = cst \in \mathfrak{b} \cap \mathfrak{c}$. But then $\frac{b}{s} = \frac{bs't}{ss't} \in (\mathfrak{b} \cap \mathfrak{c})R_S$, which justifies our claim.

Now we see from Lemma 13 that (i) and (ii) are primary decompositions. The primary decomposition (ii) is minimal, since the decomposition of $\mathfrak{a}$ we started with was assumed to be minimal. But then it is clear that the decomposition (i) is minimal as well. Indeed, the $\mathfrak{p}_i R_S$ for $i = 1,\dots,r'$ are prime ideals satisfying $\mathfrak{p}_i R_S \cap R = \mathfrak{p}_i$. So they must be pairwise different, since $\mathfrak{p}_1,\dots,\mathfrak{p}_{r'}$ are pairwise different. Furthermore, the decomposition (i) is unshortenable, since the same is true for (ii) and since $\mathfrak{q}_i R_S \cap R = \mathfrak{q}_i$ for $i = 1,\dots,r'$ by Lemma 13. □

Now we are able to derive the following strengthening of the uniqueness assertion in Theorem 8:

Theorem 15. *Let $\mathfrak{a} \subset R$ be a decomposable ideal and $\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$ a minimal primary decomposition where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. Then $\mathfrak{q}_i = \mathfrak{a}R_{\mathfrak{p}_i} \cap R$ for all indices i such that $\mathfrak{p}_i \in \mathrm{Ass}'(\mathfrak{a})$.*

In particular, a member $\mathfrak{q}_i$ of the primary decomposition of $\mathfrak{a}$ is unique, depending only on $\mathfrak{a}$ and the prime ideal $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i) \in \mathrm{Ass}(\mathfrak{a})$, if $\mathfrak{p}_i$ belongs to $\mathrm{Ass}'(\mathfrak{a})$. In the latter case, we call $\mathfrak{q}_i$ the $\mathfrak{p}_i$-primary part *of $\mathfrak{a}$.*

Proof. Consider an index $j \in \{1,\dots,r\}$ such that $\mathfrak{p}_j = \mathrm{rad}(\mathfrak{q}_j) \in \mathrm{Ass}'(\mathfrak{a})$ and set $S = R - \mathfrak{p}_j$. Then $S \cap \mathfrak{p}_j = \emptyset$, while $S \cap \mathfrak{p}_i \neq \emptyset$ for all $i \neq j$. Therefore we can conclude from Proposition 14 that $\mathfrak{a}R_S \cap R = \mathfrak{q}_j$. □

Let R be a Noetherian ring. As an application to the above uniqueness result we can define for $n \geq 1$ the nth symbolic power of any prime ideal $\mathfrak{p} \subset R$. Namely, consider a primary decomposition of $\mathfrak{p}^n$, which exists by Theorem 6. Then $\mathrm{Ass}'(\mathfrak{a}) = \{\mathfrak{p}\}$ by Proposition 12 and the $\mathfrak{p}$-primary part of $\mathfrak{p}^n$ is unique, given by $\mathfrak{p}^n R_{\mathfrak{p}} \cap R$. The latter is denoted by $\mathfrak{p}^{(n)}$ and called the nth *symbolic power* of $\mathfrak{p}$. Of course, $\mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ and we have $\mathfrak{p}^n = \mathfrak{p}^{(n)}$ if and only if $\mathfrak{p}^n$ is primary itself.

Exercises

1. Show for a decomposable ideal $\mathfrak{a} \subset R$ that $\operatorname{rad}(\mathfrak{a}) = \mathfrak{a}$ implies $\operatorname{Ass}'(\mathfrak{a}) = \operatorname{Ass}(\mathfrak{a})$. Is the converse of this true as well?

2. Let $\mathfrak{p} \subset R$ be a prime ideal and assume that R is Noetherian. Show that the nth symbolic power $\mathfrak{p}^{(n)}$ is the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$. Can we expect that there is a smallest primary ideal containing $\mathfrak{p}^n$?

3. For a non-zero Noetherian ring R let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be its (pairwise different) prime ideals that are associated to the zero ideal $0 \subset R$. Show that there is an integer $n_0 \in \mathbb{N}$ such that $\bigcap_{i=1}^r \mathfrak{p}_i^{(n)} = 0$ for all $n \geq n_0$ and, hence, that the latter are minimal primary decompositions of the zero ideal. Conclude that $\mathfrak{p}_i^{(n)} = \mathfrak{p}_i^{(n_0)}$ for all $n \geq n_0$ if $\mathfrak{p}_i$ is an isolated prime ideal and show that the symbolic powers $\mathfrak{p}_i^{(n)}$ are pairwise different if $\mathfrak{p}_i$ is embedded.

4. Consider a decomposable ideal $\mathfrak{a} \subset R$ and a multiplicative subset $S \subset R$ such that $S \cap \mathfrak{a} = \emptyset$. Then the map $\mathfrak{p} \longmapsto \mathfrak{p} \cap R$ from prime ideals $\mathfrak{p} \subset R_S$ to prime ideals in R restricts to an injective map $\operatorname{Ass}(\mathfrak{a}R_S) \hookrightarrow \operatorname{Ass}(\mathfrak{a})$.

5. Let $R = K[X_1, \ldots X_r]$ be the polynomial ring in finitely many variables over a field K and let $\mathfrak{p} \subset R$ be an ideal that is generated by some of the variables X_i. Show that $\mathfrak{p}$ is prime and that all its powers $\mathfrak{p}^n$, $n \geq 1$, are $\mathfrak{p}$-primary. In particular, $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ for $n \geq 1$.

6. Let M be an R-module. A submodule $N \subset M$ is called *primary* if $N \neq M$ and if the multiplication by any element $a \in R$ is either injective or nilpotent on M/N; in other words, if there exists an element $x \in M - N$ such that $ax \in N$, then there is an exponent $n \geq 1$ satisfying $a^n M \subset N$.

   (a) Let $(N : M) = \{a \in R\,;\, aM \subset N\}$ and show that $\mathfrak{p} = \operatorname{rad}(N : M)$ is a prime ideal in R if N is primary in M; we say that N is $\mathfrak{p}$-*primary* in M.

   (b) Show that an ideal $\mathfrak{q} \subset R$, viewed as a submodule of R, is $\mathfrak{p}$-primary in the sense of (a) for some prime ideal $\mathfrak{p} \subset R$ if and only if it is $\mathfrak{p}$-primary in the sense of Definition 1.

   (c) Generalize the theory of primary decompositions from ideals to submodules of M; see for example Serre [24], I.B.

2.2 Artinian Rings and Modules

As we have seen in 1.5/9, a module M over a ring R is Noetherian if and only if it satisfies the ascending chain condition, i.e. if and only if every ascending chain of submodules $M_1 \subset M_2 \subset \ldots \subset M$ becomes stationary. Switching to descending chains we arrive at the notion of *Artinian* modules.

Definition 1. *An R-module M is called* Artinian *if every descending chain of submodules $M \supset M_1 \supset M_2 \supset \ldots$ becomes stationary, i.e. if there is an index i_0 such that $M_{i_0} = M_i$ for all $i \geq i_0$. A ring R is called* Artinian *if, viewed as an R-module, it is Artinian.*

Clearly, an R-module M is Artinian if and only if every non-empty set of submodules admits a minimal element. In a similar way, Noetherian modules are characterized by the existence of maximal elements. This suggests a close analogy between Noetherian and Artinian modules. However, let us point out right away that such an analogy does not exist, as is clearly visible on the level of rings. Namely, the class of Artinian rings is, in fact, a subclass of the one of Noetherian rings, as we will prove below in Theorem 8. Thus, for rings the ascending chain condition is, in fact, a consequence of the descending chain condition.

Looking at some simple examples, observe that any field is Artinian. Furthermore, a vector space over a field is Artinian if and only if it is of finite dimension. Since, for any prime p, the ring of integers $\mathbb{Z}$ admits the strictly descending sequence of ideals $\mathbb{Z} \supsetneq (p) \supsetneq (p^2) \supsetneq \ldots$, we see that $\mathbb{Z}$ cannot be Artinian. However, any quotient $\mathbb{Z}/n\mathbb{Z}$ for some $n \neq 0$ is Artinian, since it consists of only finitely many elements.

For the discussion of general properties of Artinian modules, the Artinian analogue of 1.5/10 is basic:

Lemma 2. *Let* $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *be an exact sequence of R-modules. Then the following conditions are equivalent*:

(i) *M is Artinian.*

(ii) *M' and M'' are Artinian.*

Proof. If M is Artinian, M' is Artinian as well, since any chain of submodules in M' can be viewed as a chain of submodules in M. Furthermore, given a chain of submodules in M'', its preimage in M becomes stationary if M is Artinian. But then the chain we started with in M'' must become stationary as well. Thus, it is clear that condition (i) implies (ii).

Conversely, assume that M' and M'' are Artinian. Then, for any descending chain of submodules $M \supset M_1 \supset M_2 \supset \ldots$, we know that the chains

$$M' \supset f^{-1}(M_1) \supset f^{-1}(M_2) \supset \ldots, \qquad M'' \supset g(M_1) \supset g(M_2) \supset \ldots$$

become stationary, say at some index i_0, and we can look at the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^{-1}(M_i) & \longrightarrow & M_i & \longrightarrow & g(M_i) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \sigma_i} & & \| & & \\ 0 & \longrightarrow & f^{-1}(M_{i_0}) & \longrightarrow & M_{i_0} & \longrightarrow & g(M_{i_0}) & \longrightarrow & 0 \end{array}$$

for $i \geq i_0$. Since the rows are exact, all inclusions $\sigma_i \colon M_i \longrightarrow M_{i_0}$ must be bijective. Hence, the chain $M \supset M_1 \supset M_2 \supset \ldots$ becomes stationary at i_0 as well and we see that M is Artinian. □

Using the fact that Noetherian modules can be characterized by the ascending chain condition, as shown in 1.5/9, the above line of arguments yields an alternative method for proving the Noetherian analogue 1.5/10 of Lemma 2. Just as in the Noetherian case, let us derive some standard consequences from the above lemma.

Corollary 3. *Let M_1, M_2 be Artinian R-modules. Then*:

(i) *$M_1 \oplus M_2$ is Artinian.*

(ii) *If M_1, M_2 are given as submodules of some R-module M, then $M_1 + M_2$ and $M_1 \cap M_2$ are Artinian.*

Proof. Due to Lemma 2, the proof of 1.5/11 carries over literally, just replacing Noetherian by Artinian. □

Corollary 4. *For a ring R, the following conditions are equivalent*:

(i) *R is Artinian.*

(ii) *Every R-module M of finite type is Artinian.*

Proof. Use the argument given in the proof of 1.5/12. □

There is a straightforward generalization of Lemma 2 to chains of modules.

Proposition 5. *Let $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ be a chain of submodules of an R-module M. Then M is Artinian if and only if all quotients M_i/M_{i-1}, $i = 1, \ldots, n$, are Artinian.*

The same is true for Artinian replaced by Noetherian.

Proof. The only-if part is trivial. To prove the if part we use induction by n, observing that the case $n = 1$ is trivial. Therefore let $n > 1$ and assume by induction hypothesis that M_{n-1} is Artinian (resp. Noetherian). Then the exact sequence

$$0 \longrightarrow M_{n-1} \longrightarrow M \longrightarrow M/M_{n-1} \longrightarrow 0$$

shows that M is Artinian by Lemma 2 (resp. Noetherian by 1.5/10). □

Corollary 6. *Let R be a ring with (not necessarily distinct) maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \subset R$ such that $\mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_n = 0$. Then R is Artinian if and only if it is Noetherian.*

Proof. Look at the chain of ideals

$$0 = \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_n \subset \ldots \subset \mathfrak{m}_1 \cdot \mathfrak{m}_2 \subset \mathfrak{m}_1 \subset R$$

and, for $i = 1, \ldots, n$, consider the quotient $V_i = (\mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_i)$ as an $R/\mathfrak{m}_i$-vector space. Being a vector space, V_i is Artinian if and only if it is of finite dimension and the latter is equivalent to V_i being Noetherian. As there is no essential difference between V_i as an $R/\mathfrak{m}_i$-module or as an R-module, Proposition 5 shows that R is Artinian if and only if it is Noetherian. □

Proposition 7. *Let R be an Artinian ring. Then R contains only finitely many prime ideals. All of them are maximal and, hence, minimal as well. In particular, the Jacobson radical $j(R)$ coincides with the nilradical* $\mathrm{rad}(R)$ *of R.*

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. In order to show that $\mathfrak{p}$ is a maximal ideal, we have to show that the quotient $R/\mathfrak{p}$ is a field. So consider a non-zero element $x \in R/\mathfrak{p}$. Since $R/\mathfrak{p}$ is Artinian, the chain $(x^1) \supset (x^2) \supset \ldots$ will become stationary, say at some exponent i_0, and there is an element $y \in R/\mathfrak{p}$ such that $x^{i_0} = x^{i_0+1} \cdot y$. Since $R/\mathfrak{p}$ is an integral domain, this implies $1 = x \cdot y$ and, hence, that x is a unit. Therefore $R/\mathfrak{p}$ is a field and $\mathfrak{p}$ a maximal ideal in R. Thus, all prime ideals of R are maximal and it follows with the help of 1.3/4 that the Jacobson radical of R coincides with the nilradical.

Finally, assume that there is an infinite sequence $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \ldots$ of pairwise distinct prime ideals in R. Then

$$\mathfrak{p}_1 \supset \mathfrak{p}_1 \cap \mathfrak{p}_2 \supset \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3 \supset \ldots$$

is a descending sequence of ideals in R and, thus, must become stationary, say at some index i_0. The latter means $\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_{i_0} \subset \mathfrak{p}_{i_0+1}$ and we see from 1.3/8 that one of the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_{i_0}$ will be contained in $\mathfrak{p}_{i_0+1}$. However, this is impossible, since the $\mathfrak{p}_i$ are pairwise distinct maximal ideals in R. Consequently, there exist only finitely many prime ideals in R. □

In Section 2.4 we will introduce the *Krull dimension* $\dim R$ of a ring R. It is defined as the supremum of all integers n such that there is a chain of prime ideals $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n$ of length n in R. At the moment we are only interested in rings of dimension 0, i.e. rings where every prime ideal is maximal. For example, any non-zero Artinian ring R is of dimension 0 by Proposition 7.

Theorem 8. *For a non-zero ring R the following conditions are equivalent*:
(i) *R is Artinian.*
(ii) *R is Noetherian and* $\dim R = 0$.

Proof. Since Artinian rings are of dimension 0, we must show that, for rings of Krull dimension 0, Artinian is equivalent to Noetherian. To do this we use Corollary 6 and show that in a Noetherian or Artinian ring R of dimension 0 the zero ideal is a product of maximal ideals.

Assume first that R is a Noetherian ring of dimension 0. Then we see from 2.1/12 applied to the ideal $\mathfrak{a} = 0$ that there are only finitely many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ in R and that the intersection of these is the nilradical $\mathrm{rad}(R)$. The latter is finitely generated and, hence, nilpotent, say $(\mathrm{rad}(R))^r = 0$ for some integer $r > 0$. Then

$$(\mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_n)^r \subset (\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n)^r = \mathrm{rad}(R)^r = 0$$

shows that the zero ideal in R is a product of maximal ideals.

Now consider the case, where R is Artinian. Then R contains only finitely many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ by Proposition 7 and all of these are maximal in R. In particular, we get $\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n = \mathrm{rad}(R)$ from 1.3/4. As exercised in the Noetherian case, it is enough to show that the nilradical $\mathrm{rad}(R)$ is nilpotent.

To do this, we proceed indirectly. Writing $\mathfrak{a} = \mathrm{rad}(R)$, we assume that $\mathfrak{a}^i \neq 0$ for all $i \in \mathbb{N}$. Since R is Artinian, the descending chain $\mathfrak{a}^1 \supset \mathfrak{a}^2 \supset \ldots$ becomes stationary, say at some index i_0. Furthermore, let $\mathfrak{b} \subset \mathfrak{a}^{i_0}$ be a (non-zero) minimal ideal such that $\mathfrak{a}^{i_0}\mathfrak{b} \neq 0$. We claim that $\mathfrak{p} = (0 : \mathfrak{a}^{i_0}\mathfrak{b})$ is a prime ideal in R. Indeed, $\mathfrak{p} \neq R$, since $\mathfrak{a}^{i_0}\mathfrak{b} \neq 0$. Furthermore, let $x, y \in R$ such that $xy \in \mathfrak{p}$. Assuming $x \notin \mathfrak{p}$, we get $\mathfrak{a}^{i_0}\mathfrak{b}xy = 0$, but $\mathfrak{a}^{i_0}\mathfrak{b}x \neq 0$. Then $\mathfrak{b}x = \mathfrak{b}$ by the minimality of $\mathfrak{b}$ and therefore $\mathfrak{a}^{i_0}\mathfrak{b}y = 0$ so that $y \in \mathfrak{p}$. Thus, $\mathfrak{p}$ is prime and necessarily belongs to the set of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. In particular, we have $\mathfrak{a}^{i_0} \subset \mathfrak{p}$. But then

$$\mathfrak{a}^{i_0}\mathfrak{b} = \mathfrak{a}^{i_0}\mathfrak{a}^{i_0}\mathfrak{b} \subset \mathfrak{p}\mathfrak{a}^{i_0}\mathfrak{b} = 0,$$

which contradicts the choice of $\mathfrak{b}$. Therefore it follows that $\mathfrak{a} = \mathrm{rad}(R)$ is nilpotent and we are done. □

Exercises

1. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Show that $R/\mathfrak{q}$ is Artinian for every $\mathfrak{m}$-primary ideal $\mathfrak{q} \subset R$.

2. Let $R_1, \ldots, R_n$ be Artinian rings. Show that the cartesian product $\prod_{i=1}^n R_i$ is Artinian again.

3. Let R be an Artinian ring and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be its (pairwise different) prime ideals. Show:

   (a) The canonical homomorphism $R \longrightarrow \prod_{i=1}^r R/\mathfrak{p}_i^n$ is an isomorphism if n is large enough.

   (b) The canonical homomorphisms $R_{\mathfrak{p}_i} \longrightarrow R/\mathfrak{p}_i^n$, $i = 1, \ldots, r$, are isomorphisms if n is large enough. Consequently, the isomorphism of (a) can be viewed as a canonical isomorphism $R \xrightarrow{\sim} \prod_{i=1}^r R_{\mathfrak{p}_i}$.

4. Let A be an algebra of finite type over a field K, i.e. a quotient of type $K[X_1, \ldots, X_n]/\mathfrak{a}$ by some ideal $\mathfrak{a}$ of the polynomial ring $K[X_1, \ldots, X_n]$. Show that A is Artinian if and only if the vector space dimension $\dim_K A$ is finite. *Hint*: Use Exercise 3 in conjunction with the fact to be proved in 3.2/4 that $\dim_K A/\mathfrak{m} < \infty$ for all maximal ideals $\mathfrak{m} \subset A$.

5. Call an R-module M *simple* if it admits only 0 and M as submodules. A *Jordan–Hölder sequence* for an R-module M consists of a decreasing sequence of R-modules $M = M_0 \supset M_1 \supset \ldots \supset M_n = 0$ such that all quotients M_{i-1}/M_i, $i = 1, \ldots, n$, are simple. Show that an R-module M admits a Jordan–Hölder sequence if and only if M is Artinian and Noetherian.

2.3 The Artin–Rees Lemma

We will prove the Artin–Rees Lemma in order to derive Krull's Intersection Theorem from it. The latter in turn is a basic ingredient needed for characterizing the Krull dimension of Noetherian rings in Section 2.4.

Lemma 1 (Artin–Rees Lemma). *Let R be a Noetherian ring, $\mathfrak{a} \subset R$ an ideal, M a finitely generated R-module, and $M' \subset M$ a submodule. Then there exists an integer $k \in \mathbb{N}$ such that*

$$(\mathfrak{a}^i M) \cap M' = \mathfrak{a}^{i-k}(\mathfrak{a}^k M \cap M')$$

for all exponents $i \geq k$.

Postponing the proof for a while, let us give some explanations concerning this lemma. The descending sequence of ideals $\mathfrak{a}^1 \supset \mathfrak{a}^2 \supset \ldots$ defines a topology on R, the so-called $\mathfrak{a}$-*adic topology*; see 6.1/3 for the definition of a topology. Indeed, a subset $E \subset R$ is called *open* if for every element $x \in E$ there exists an exponent $i \in \mathbb{N}$ such that $x + \mathfrak{a}^i \subset E$. Thus, the powers $\mathfrak{a}^i$ for $i \in \mathbb{N}$ are the basic open neighborhoods of the zero element in R. In a similar way, one defines the $\mathfrak{a}$-adic topology on any R-module M by taking the submodules $\mathfrak{a}^i M$ for $i \in \mathbb{N}$ as basic open neighborhoods of $0 \in M$. Now if M' is a submodule of M, we may restrict the $\mathfrak{a}$-adic topology on M to a topology on M' by taking the intersections $\mathfrak{a}^i M \cap M'$ as basic open neighborhoods of $0 \in M'$. Thus, a subset $E \subset M'$ is open if and only if for every $x \in E$ there exists an exponent $i \in \mathbb{N}$ such that $x + (\mathfrak{a}^i M \cap M') \subset E$.

However, on M' the $\mathfrak{a}$-adic topology exists as well and we may try to compare both topologies. Clearly, since $\mathfrak{a}^i M' \subset \mathfrak{a}^i M \cap M'$, any subset $E \subset M'$ that is open with respect to the restriction of the $\mathfrak{a}$-adic topology on M to M' will be open with respect to the $\mathfrak{a}$-adic topology on M'. Moreover, in the situation of the Artin–Rees Lemma, both topologies coincide, as follows from the inclusions

$$(\mathfrak{a}^i M) \cap M' = \mathfrak{a}^{i-k}(\mathfrak{a}^k M \cap M') \subset \mathfrak{a}^{i-k} M'$$

for $i \geq k$.

A topology on a set X is said to satisfy the *Hausdorff separation axiom* if any different points $x, y \in X$ admit disjoint open neighborhoods. Since two cosets with respect to $\mathfrak{a}^i$ in R, or with respect to $\mathfrak{a}^i M$ in M are disjoint as soon as they do not coincide, it is easily seen that the $\mathfrak{a}$-adic topology on R (resp. M) is Hausdorff if and only if we have $\bigcap_{i \in \mathbb{N}} \mathfrak{a}^i = 0$ (resp. $\bigcap_{i \in \mathbb{N}} \mathfrak{a}^i M = 0$). In certain cases, the latter relations can be derived from the Artin–Rees Lemma:

Theorem 2 (Krull's Intersection Theorem). *Let R be a Noetherian ring, $\mathfrak{a} \subset R$ an ideal such that $\mathfrak{a} \subset j(R)$, and M a finitely generated R-module. Then*

$$\bigcap_{i \in \mathbb{N}} \mathfrak{a}^i M = 0.$$

Proof. Applying the Artin–Rees Lemma to the submodule $M' = \bigcap_{i\in\mathbb{N}} \mathfrak{a}^i M$ of M, we obtain an index $k \in \mathbb{N}$ such that $M' = \mathfrak{a}^{i-k}M'$ for $i \geq k$. Since M as a finitely generated module over a Noetherian ring is Noetherian by 1.5/12, we see that the submodule $M' \subset M$ is finitely generated. Therefore Nakayama's Lemma 1.4/10 yields $M' = 0$. □

Now in order to prepare the proof of the Artin–Rees Lemma, consider a *Noetherian* ring R, an ideal $\mathfrak{a} \subset R$, and an R-module M *of finite type* together with an $\mathfrak{a}$-filtration $(M_i)_{i\in\mathbb{N}}$. By the latter we mean a descending sequence of submodules $M = M_0 \supset M_1 \supset \ldots$ such that $\mathfrak{a}^i M_j \subset M_{i+j}$ for all $i, j \in \mathbb{N}$. The filtration $(M_i)_{i\in\mathbb{N}}$ is called $\mathfrak{a}$-*stable* if there is an index k such that $M_{i+1} = \mathfrak{a} \cdot M_i$ for all $i \geq k$ or, equivalently, if $M_{k+i} = \mathfrak{a}^i \cdot M_k$ for all $i \in \mathbb{N}$. Furthermore, let us consider the direct sum $R_\bullet = \bigoplus_{i\in\mathbb{N}} \mathfrak{a}^i$ as a ring by using componentwise addition and by distributively extending the canonical multiplication maps $\mathfrak{a}^i \times \mathfrak{a}^j \longrightarrow \mathfrak{a}^{i+j}$ for $i, j \in \mathbb{N}$ that are given on components. In a similar way, we can view the direct sum $M_\bullet = \bigoplus_{i\in\mathbb{N}} M_i$ as an $R_\bullet$-module. Note that $R_\bullet$ is a graded ring in the sense of 9.1/1 and $M_\bullet$ a graded $R_\bullet$-module in the sense of Section 9.2.

Observe that the ring $R_\bullet$ is Noetherian. Indeed, R is Noetherian and, hence, the ideal $\mathfrak{a}$ admits a finite system of generators $a_1, \ldots, a_n$. Thus, the canonical injection $R \longrightarrow R_\bullet$ extends to a surjection $R[X_1, \ldots, X_n] \longrightarrow R_\bullet$ by sending the polynomial variable X_ν to a_ν for $\nu = 1, \ldots, n$. At this point a_ν has to be viewed as an element in the first power $\mathfrak{a}^1$, which is the component of index 1 in $R_\bullet$. Since the polynomial ring $R[X_1, \ldots, X_n]$ is Noetherian by Hilbert's Basis Theorem 1.5/14, we see that $R_\bullet$ is Noetherian as well.

Lemma 3. *For a Noetherian ring R, an ideal $\mathfrak{a} \subset R$, and an R-module M of finite type, let $R_\bullet$ and $M_\bullet$ be as before. Then the following conditions are equivalent*:

(i) *The filtration $(M_i)_{i\in\mathbb{N}}$ is $\mathfrak{a}$-stable.*

(ii) *$M_\bullet$ is an $R_\bullet$-module of finite type.*

Proof. Starting with the implication (i) $\Longrightarrow$ (ii), let k be an integer such that $M_{k+i} = \mathfrak{a}^i \cdot M_k$ for all $i \in \mathbb{N}$. Then $M_\bullet$ is generated as an $R_\bullet$-module by the subgroup $\bigoplus_{i\leq k} M_i \subset M_\bullet$. Since M is a finitely generated R-module over a Noetherian ring, it is Noetherian by 1.5/12. Hence, all its submodules M_i are finitely generated and the same is true for the finite direct sum $\bigoplus_{i\leq k} M_i$. Choosing a finite system of R-generators for the latter, it generates $M_\bullet$ as an $R_\bullet$-module.

Conversely, assume that $M_\bullet$ is a finitely generated $R_\bullet$-module. Then we can choose a finite system of "homogeneous generators", namely of generators $x_1, \ldots, x_n$, where $x_\nu \in M_{\sigma(\nu)}$ for certain integers $\sigma(1), \ldots, \sigma(n)$. It follows that $\mathfrak{a}M_i = M_{i+1}$ for $i \geq \max\{\sigma(1), \ldots, \sigma(n)\}$ and, hence, that the filtration $(M_i)_{i\in\mathbb{N}}$ of M is $\mathfrak{a}$-stable. □

Now the *proof of the Artin–Rees Lemma* is easy to achieve. Consider the filtration $(M_i)_{i\in\mathbb{N}}$ of M given by $M_i = \mathfrak{a}^i M$; it is $\mathfrak{a}$-stable by its definition. Furthermore, consider the induced filtration $(M'_i)_{i\in\mathbb{N}}$ on M'; the latter is given by $M'_i = M_i \cap M'$. Then $M'_\bullet$ is canonically an $R_\bullet$-submodule of $M_\bullet$. Since $M_\bullet$ is of finite type by the Lemma 3 and $R_\bullet$ is Noetherian, we see from 1.5/12 that $M'_\bullet$ is of finite type as well. Thus, by Lemma 3 again, the filtration $(M'_i)_{i\in\mathbb{N}}$ is $\mathfrak{a}$-stable, and there exists an integer k such that $\mathfrak{a}(\mathfrak{a}^i M \cap M') = \mathfrak{a}^{i+1}M \cap M'$ for $i \geq k$. Iteration yields $\mathfrak{a}^i(\mathfrak{a}^k M \cap M') = \mathfrak{a}^{k+i}M \cap M'$ for $i \in \mathbb{N}$ and, thus, the assertion of the Artin–Rees Lemma. □

Exercises

1. Let M be an R-module. Verify for an ideal $\mathfrak{a} \subset R$ that the $\mathfrak{a}$-adic topology on M, as defined above, is a topology in the sense of 6.1/3. Show that the $\mathfrak{a}$-adic topology on M satisfies the Hausdorff separation axiom if and only if $\bigcap_{i\in\mathbb{N}} \mathfrak{a}^i M = 0$.

2. Consider the $\mathfrak{a}$-adic topology on an R-module M for some ideal $\mathfrak{a} \subset R$. Show that a submodule $N \subset M$ is closed in M if and only if $\bigcap_{i\in\mathbb{N}}(\mathfrak{a}^i M + N) = N$. For R Noetherian, M finitely generated and $\mathfrak{a} \subset j(R)$, deduce that every submodule of M is closed.

3. *Generalization of Krull's Intersection Theorem*: Consider an ideal $\mathfrak{a}$ of a Noetherian ring R and a finitely generated R-module M. Then $\bigcap_{i\in\mathbb{N}} \mathfrak{a}^i M$ consists of all elements $x \in M$ that are annihilated by $1 + a$ for some element $a \in \mathfrak{a}$. In particular, show $\bigcap_{i\in\mathbb{N}} \mathfrak{a}^i = 0$ for any proper ideal $\mathfrak{a}$ of a Noetherian integral domain. *Hint*: Use the generalized version of Nakayama's Lemma in Exercise 1.4/5.

4. For a field K and an integer $n > 1$ consider the homomorphism of formal power series rings $\sigma\colon K[\![X]\!] \longrightarrow K[\![X]\!]$ given by $f(X) \longmapsto f(X^n)$. Show that σ is an injective homomorphism between local rings and satisfies $\sigma(X) \subset (X)$ for the maximal ideal $(X) \subset K[\![X]\!]$. Construct a ring R as the union of the infinite chain of rings

$$K[\![X]\!] \overset{\sigma}{\hookrightarrow} K[\![X]\!] \overset{\sigma}{\hookrightarrow} K[\![X]\!] \overset{\sigma}{\hookrightarrow} \ldots$$

and show that R is a local integral domain where Krull's Intersection Theorem is not valid any more. In particular, R cannot be Noetherian. *Hint*: For the definition of formal power series rings see Exercise 1.5/8.

5. Given an ideal $\mathfrak{a} \subset R$, show that $\mathrm{gr}_\mathfrak{a}(R) = \bigoplus_{i\in\mathbb{N}} \mathfrak{a}^i/\mathfrak{a}^{i+1}$ is canonically a ring; it is called the *graded ring associated to* $\mathfrak{a}$. Show that $\mathrm{gr}_\mathfrak{a}(R)$ is Noetherian if R is Noetherian. Furthermore, let $M = M_0 \supset M_1 \supset \ldots$ be an $\mathfrak{a}$-filtration on an R-module M. Show that $\mathrm{gr}(M) = \bigoplus_{i\in\mathbb{N}} M_i/M_{i+1}$ is canonically a $\mathrm{gr}_\mathfrak{a}(R)$-module and that the latter is finitely generated if M is a finite R-module and the filtration on M is $\mathfrak{a}$-stable.

2.4 Krull Dimension

In order to define the *dimension* of a ring R, we use strictly ascending chains $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n$ of prime ideals in R where the integer n is referred to as the *length* of the chain.

Remark 1. *Let R be a ring and $\mathfrak{p} \subset R$ a prime ideal. Then:*

(i) *The chains of prime ideals in R starting with $\mathfrak{p}$ correspond bijectively to the chains of prime ideals in $R/\mathfrak{p}$ starting with the zero ideal.*

(ii) *The chains of prime ideals in R ending with $\mathfrak{p}$ correspond bijectively to the chains of prime ideals in the localization $R_\mathfrak{p}$ ending with $\mathfrak{p}R_\mathfrak{p}$.*

Proof. Assertion (i) is trivial, whereas (ii) follows from 1.2/5. □

Definition 2. *For a ring R, the supremum of lengths n of chains*

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n,$$

where the $\mathfrak{p}_i$ are prime ideals in R, is denoted by $\dim R$ *and called the* Krull dimension *or simply the* dimension *of R.*

For example, fields are of dimension 0, whereas a principal ideal domain is of dimension 1, provided it is not a field. In particular, we have $\dim \mathbb{Z} = 1$, as well as $\dim K[X] = 1$ for the polynomial ring over a field K. Also we know that $\dim K[X_1, \ldots, X_n] \geq n$, since the polynomial ring $K[X_1, \ldots, X_n]$ contains the chain of prime ideals $0 \subsetneq (X_1) \subsetneq (X_1, X_2) \subsetneq \ldots \subsetneq (X_1, \ldots, X_n)$. In fact, we will show $\dim K[X_1, \ldots, X_n] = n$ in Corollary 16 below. Likewise, the polynomial ring $K[X_1, X_2, \ldots]$ in an infinite sequence of variables is of infinite dimension, whereas the zero ring 0 is a ring having dimension $-\infty$, since by convention the supremum over an empty subset of $\mathbb{N}$ is $-\infty$. Any non-zero ring contains at least one prime ideal and therefore is of dimension ≥ 0.

Definition 3. *Let R be a ring.*

(i) *For a prime ideal $\mathfrak{p} \subset R$ its* height $\operatorname{ht} \mathfrak{p}$ *is given by* $\operatorname{ht} \mathfrak{p} = \dim R_\mathfrak{p}$.

(ii) *For an ideal $\mathfrak{a} \subset R$, its* height $\operatorname{ht} \mathfrak{a}$ *is given by the infimum of all heights* $\operatorname{ht} \mathfrak{p}$, *where $\mathfrak{p}$ varies over all prime ideals in R containing $\mathfrak{a}$.*

(iii) *For an ideal $\mathfrak{a} \subset R$, its* coheight $\operatorname{coht} \mathfrak{a}$ *is given by* $\operatorname{coht} \mathfrak{a} = \dim R/\mathfrak{a}$.

Thus, for a prime ideal $\mathfrak{p} \subset R$, its height $\operatorname{ht} \mathfrak{p}$ (resp. its coheight $\operatorname{coht} \mathfrak{p}$) equals the supremum of all lengths of chains of prime ideals in R ending at $\mathfrak{p}$ (resp. starting at $\mathfrak{p}$). In particular, we see that $\operatorname{ht} \mathfrak{p} + \operatorname{coht} \mathfrak{p} \leq \dim R$.

In order to really work with the notions of dimension and height, it is necessary to characterize these in terms of so-called *parameters*. For example, we will show that an ideal $\mathfrak{a}$ of a Noetherian ring satisfies $\operatorname{ht} \mathfrak{a} \leq r$ if it can be generated by r elements; see Krull's Dimension Theorem 6 below. Furthermore, we will prove that any Noetherian local ring R of dimension r admits a system of

r elements, generating an ideal whose radical coincides with the maximal ideal $\mathfrak{m} \subset R$; see Proposition 11 below. If $\mathfrak{m}$ itself can be generated by r elements, we face a special case, namely, where R is a so-called *regular local ring.*

To prepare the discussion of such results, we need a special case of Krull's Intersection Theorem 2.3/2.

Lemma 4. *Let R be a Noetherian ring and $\mathfrak{p} \subset R$ a prime ideal. Then, looking at the canonical map $R \longrightarrow R_{\mathfrak{p}}$ from R into its localization by $\mathfrak{p}$, we have*

$$\ker(R \longrightarrow R_{\mathfrak{p}}) = \bigcap_{i \in \mathbb{N}} \mathfrak{p}^{(i)},$$

where $\mathfrak{p}^{(i)} = \mathfrak{p}^i R_{\mathfrak{p}} \cap R$ is the ith *symbolic power of $\mathfrak{p}$, as considered at the end of Section* 2.1.

Proof. Since $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ by 1.2/7 and since we can conclude from 1.2/5 (ii) that $R_{\mathfrak{p}}$ is Noetherian, Krull's Intersection Theorem 2.3/2 is applicable and yields $\bigcap_{i \in \mathbb{N}} \mathfrak{p}^i R_{\mathfrak{p}} = 0$. Using the fact that taking preimages with respect to $R \longrightarrow R_{\mathfrak{p}}$ is compatible with intersections, we are done. □

Now we can prove a first basic result on the height of ideals in Noetherian rings, which is a special version of *Krull's Principal Ideal Theorem* to be derived in Corollary 9 below. For any ideal $\mathfrak{a} \subset R$ we consider the set of *minimal prime divisors* of $\mathfrak{a}$; thereby we mean the set of prime ideals $\mathfrak{p} \subset R$ that are minimal among those satisfying $\mathfrak{a} \subset \mathfrak{p}$. In the setting of 2.1/12, this is the set $\operatorname{Ass}'(\mathfrak{a})$ of *isolated* prime ideals associated to $\mathfrak{a}$. Also note that, for a prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ there is always a minimal prime divisor of $\mathfrak{a}$ contained in $\mathfrak{p}$. This follows from 2.1/12 (ii) or without using the theory of primary decompositions by applying Zorn's Lemma.

Lemma 5. *Let R be a Noetherian integral domain and consider a non-zero element $a \in R$ that is not a unit. Then* $\operatorname{ht} \mathfrak{p} = 1$ *for every minimal prime divisor $\mathfrak{p}$ of (a).*

Proof. For a minimal prime divisor $\mathfrak{p}$ of (a), we can pass from R to its localization $R_{\mathfrak{p}}$ and thereby assume that R is a Noetherian local integral domain with maximal ideal $\mathfrak{p}$. Then we have to show that any prime ideal $\mathfrak{p}_0 \subset R$ that is strictly contained in $\mathfrak{p}$ satisfies $\mathfrak{p}_0 = 0$. To do this, look at the descending sequence of ideals

$$\mathfrak{a}_i = \mathfrak{p}_0^{(i)} + (a), \qquad i \in \mathbb{N},$$

where $\mathfrak{p}_0^{(i)}$ is the ith symbolic power of $\mathfrak{p}_0$, and observe that $R/(a)$ is Artinian by 2.2/8. Indeed, $R/(a)$ is Noetherian and satisfies $\dim R/(a) = 0$, as $\mathfrak{p}$ is the only prime ideal containing a. Therefore the sequence of the $\mathfrak{a}_i$ becomes stationary, say at some index i_0. Hence,

$$\mathfrak{p}_0^{(i_0)} \subset \mathfrak{p}_0^{(i)} + (a), \qquad i \geq i_0,$$

and, in fact

$$\mathfrak{p}_0^{(i_0)} = \mathfrak{p}_0^{(i)} + a\mathfrak{p}_0^{(i_0)}, \qquad i \geq i_0,$$

using the equality $(\mathfrak{p}_0^{(i_0)} : a) = \mathfrak{p}_0^{(i_0)}$; the latter relation follows from 2.1/7, since $\mathfrak{p}_0^{(i_0)}$ is $\mathfrak{p}_0$-primary and since $a \notin \mathfrak{p}_0$. Now Nakayama's Lemma in the version of 1.4/11 implies $\mathfrak{p}_0^{(i_0)} = \mathfrak{p}_0^{(i)}$ for $i \geq i_0$. Furthermore, since R is an integral domain, Lemma 4 shows

$$\mathfrak{p}_0^{(i_0)} = \bigcap_{i \in \mathbb{N}} \mathfrak{p}_0^{(i)} = \ker(R \longrightarrow R_{\mathfrak{p}_0}) = 0.$$

But then we conclude from $\mathfrak{p}_0^{i_0} \subset \mathfrak{p}_0^{(i_0)}$ that $\mathfrak{p}_0 = 0$ and, thus, that $\operatorname{ht}\mathfrak{p} = 1$. □

Theorem 6 (Krull's Dimension Theorem). *Let R be a Noetherian ring and $\mathfrak{a} \subset R$ an ideal generated by elements $a_1, \ldots, a_r$. Then* $\operatorname{ht}\mathfrak{p} \leq r$ *for every minimal prime divisor $\mathfrak{p}$ of $\mathfrak{a}$.*

Proof. We conclude by induction on the number r of generators of $\mathfrak{a}$. The case $r = 0$ is trivial. Therefore let $r > 0$ and consider a minimal prime divisor $\mathfrak{p}$ of $\mathfrak{a}$ with a strictly ascending chain of prime ideals $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_t = \mathfrak{p}$ in R. It has to be shown that $t \leq r$. To do this, we can assume that R is local with maximal ideal $\mathfrak{p}$ and, hence, that $\mathfrak{p}$ is the only prime ideal in R containing $\mathfrak{a}$. Furthermore, we may suppose $t > 0$ and, using the Noetherian hypothesis, that there is no prime ideal strictly between $\mathfrak{p}_{t-1}$ and $\mathfrak{p}_t$. Since $\mathfrak{a} \not\subset \mathfrak{p}_{t-1}$, there exists a generator of $\mathfrak{a}$, say a_r, that is not contained in $\mathfrak{p}_{t-1}$. Then $\mathfrak{p} = \mathfrak{p}_t$ is the only prime ideal containing $\mathfrak{p}_{t-1} + (a_r)$, and it follows from 1.3/6 (iii) that $\operatorname{rad}(\mathfrak{p}_{t-1} + (a_r)) = \mathfrak{p}$. Thus, there are equations of type

$$a_i^n = a_i' + a_r y_i, \qquad i = 1, \ldots, r-1,$$

for some exponent $n > 0$ where $a_i' \in \mathfrak{p}_{t-1}$ and $y_i \in R$. Now consider the ideal $\mathfrak{a}' = (a_1', \ldots, a_{r-1}') \subset R$. Since $\mathfrak{a}' \subset \mathfrak{p}_{t-1}$, there is a minimal prime divisor $\mathfrak{p}'$ of $\mathfrak{a}'$ such that $\mathfrak{p}' \subset \mathfrak{p}_{t-1}$. By the above equations we have

$$\mathfrak{p} = \operatorname{rad}(\mathfrak{a}) \subset \operatorname{rad}\big(\mathfrak{a}' + (a_r)\big) \subset \operatorname{rad}\big(\mathfrak{p}' + (a_r)\big) \subset \mathfrak{p}.$$

In particular, $\mathfrak{p}$ is a minimal prime divisor of $\mathfrak{p}' + (a_r)$ or, in other words, $\mathfrak{p}/\mathfrak{p}'$ is a minimal prime divisor of $a_r \cdot R/\mathfrak{p}'$ in $R/\mathfrak{p}'$. But then we get $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}') = 1$ from Theorem 5 and we can conclude $\mathfrak{p}' = \mathfrak{p}_{t-1}$ from $\mathfrak{p}' \subset \mathfrak{p}_{t-1} \subsetneq \mathfrak{p}_t = \mathfrak{p}$. Thus, $\mathfrak{p}_{t-1}$ turns out to be a minimal prime divisor of $\mathfrak{a}'$. Since the latter ideal is generated by $r-1$ elements, we get $t - 1 \leq r - 1$ by induction hypothesis and, thus, are done. □

Let us list some immediate consequences from Krull's Dimension Theorem:

Corollary 7. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Then* $\operatorname{ht}\mathfrak{a} < \infty$.

Corollary 8. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then* $\dim R \leq \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 < \infty$.

Proof. First note that $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 < \infty$, since R is Noetherian and, hence, $\mathfrak{m}$ is finitely generated. Choosing elements $a_1, \ldots, a_r \in \mathfrak{m}$ giving rise to an $R/\mathfrak{m}$-basis of $\mathfrak{m}/\mathfrak{m}^2$, we conclude from Nakayama's Lemma in the version of 1.4/12 that the a_i generate $\mathfrak{m}$. But then $\dim R = \operatorname{ht} \mathfrak{m} \leq r = \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$ by Krull's Dimension Theorem. □

Corollary 9 (Krull's Principal Ideal Theorem). *Let R be a Noetherian ring and a an element of R that is neither a zero divisor nor a unit. Then every minimal prime divisor $\mathfrak{p}$ of (a) satisfies* $\operatorname{ht} \mathfrak{p} = 1$.

Proof. We get $\operatorname{ht} \mathfrak{p} \leq 1$ from Krull's Dimension Theorem. Since the minimal prime ideals in R are just the isolated prime ideals associated to the zero ideal $0 \subset R$ by 2.1/12, the assertion follows from the characterization of zero divisors given in 2.1/10. □

Next, to approach the subject of parameters, we prove a certain converse of Krull's Dimension Theorem.

Lemma 10. *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal in R of height* $\operatorname{ht} \mathfrak{a} = r$. *Assume there are elements $a_1, \ldots, a_{s-1} \in \mathfrak{a}$ such that* $\operatorname{ht}(a_1, \ldots, a_{s-1}) = s - 1$ *for some $s \in \mathbb{N}$ where $1 \leq s \leq r$. Then there exists an element $a_s \in \mathfrak{a}$ such that* $\operatorname{ht}(a_1, \ldots, a_s) = s$.

In particular, there are elements $a_1, \ldots, a_r \in \mathfrak{a}$ such that $\operatorname{ht}(a_1, \ldots, a_r) = r$.

Proof. We have only to justify the first statement, as the second one follows from the first by induction. Therefore consider elements $a_1, \ldots, a_{s-1} \in \mathfrak{a}$, $1 \leq s \leq r$, such that $\operatorname{ht}(a_1, \ldots, a_{s-1}) = s - 1$, and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset R$ be the minimal prime divisors of $(a_1, \ldots, a_{s-1})$. Then, using Krull's Dimension Theorem, we have $\operatorname{ht} \mathfrak{p}_i \leq s - 1 < r$ for all i. Since $\operatorname{ht} \mathfrak{a} = r$, this implies $\mathfrak{a} \not\subset \mathfrak{p}_i$ for all i and, hence, $\mathfrak{a} \not\subset \bigcup_{i=1}^n \mathfrak{p}_i$ by 1.3/7. Thus, choosing $a_s \in \mathfrak{a} - \bigcup_{i=1}^n \mathfrak{p}_i$, we get $\operatorname{ht}(a_1, \ldots, a_s) \geq s$ and, in fact $\operatorname{ht}(a_1, \ldots, a_s) = s$ by Krull's Dimension Theorem. □

For the discussion of parameters, which follows below, recall from 2.1/3 that an ideal $\mathfrak{a}$ of a local ring R with maximal ideal $\mathfrak{m}$ is $\mathfrak{m}$-primary if and only if $\operatorname{rad}(\mathfrak{a}) = \mathfrak{m}$.

Proposition 11. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then there exists an $\mathfrak{m}$-primary ideal in R generated by $d = \dim R$ elements, but no such ideal that is generated by less than d elements.*

Proof. Combine Lemma 10 with Krull's Dimension Theorem 6. □

Definition 12. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. A set of elements $x_1, \ldots, x_d \in \mathfrak{m}$ is called a system of* parameters *of R if $d = \dim R$ and the ideal $(x_1, \ldots, x_d) \subset R$ is $\mathfrak{m}$-primary.*

In particular, we see from the Proposition 11 that every Noetherian local ring admits a system of parameters.

Proposition 13. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then, for given elements $x_1, \ldots, x_r \in \mathfrak{m}$, the following conditions are equivalent*:

(i) *The system $x_1, \ldots, x_r$ can be enlarged to a system of parameters of R.*

(ii) $\dim R/(x_1, \ldots, x_r) = \dim R - r$.

Furthermore, conditions (i) *and* (ii) *are satisfied if* $\operatorname{ht}(x_1, \ldots, x_r) = r$.

Proof. First observe that condition (i) follows from $\operatorname{ht}(x_1, \ldots, x_r) = r$, using Lemma 10. Next consider elements $y_1, \ldots, y_s \in \mathfrak{m}$. Writing $\overline{R} = R/(x_1, \ldots, x_r)$ and $\overline{\mathfrak{m}} = \mathfrak{m}/(x_1, \ldots, x_r)$, the residue classes $\overline{y}_1, \ldots, \overline{y}_s$ generate an $\overline{\mathfrak{m}}$-primary ideal in $\overline{R}$ if and only if $x_1, \ldots, x_r, y_1, \ldots, y_s$ generate an $\mathfrak{m}$-primary ideal in R. In particular, if $x_1, \ldots, x_r, y_1, \ldots, y_s$ is a system of parameters of R, as we may assume in the situation of (i), we have $r + s = \dim R$ and can conclude from Proposition 11 that $\dim \overline{R} \leq \dim R - r$. On the other hand, if the residue classes of $y_1, \ldots, y_s$ form a system of parameters in $\overline{R}$, we have $s = \dim \overline{R}$ and see that $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ is an $\mathfrak{m}$-primary ideal in R. Therefore $\dim R \leq r + \dim \overline{R}$ by Proposition 11. Combining both estimates yields $\dim \overline{R} \leq \dim R - r \leq \dim \overline{R}$ and, thus, $\dim \overline{R} = \dim R - r$, as needed in (ii).

Conversely, assume (ii) and consider elements $y_1, \ldots, y_s \in \mathfrak{m}$, whose residue classes form a system of parameters in $\overline{R}$ so that $s = \dim \overline{R}$. Then the system $x_1, \ldots, x_r, y_1, \ldots, y_s$ generates an $\mathfrak{m}$-primary ideal in R. Since $r + s = \dim R$ by (ii), the system is, in fact, a system of parameters and (i) follows. □

Corollary 14. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$, and let $a \in \mathfrak{m}$ be an element that is not a zero divisor. Then* $\dim R/(a) = \dim R - 1$.

Proof. Use the fact that $\operatorname{ht}(a) = 1$ by Corollary 9. □

As an application of the theory of parameters, let us discuss the dimension of polynomial rings.

Proposition 15. *Consider the polynomial ring $R[X_1, \ldots, X_n]$ in n variables over a Noetherian ring R. Then* $\dim R[X_1, \ldots, X_n] = \dim R + n$.

Proof. We will show $\dim R[X] = \dim R + 1$ for one variable X, from which the general case follows by induction. Let $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_r$ be a strictly ascending chain of prime ideals in R of length r. Then

$$\mathfrak{p}_0 R[X] \subsetneq \ldots \subsetneq \mathfrak{p}_r R[X] \subsetneq \mathfrak{p}_r R[X] + X R[X]$$

is a strictly ascending chain of prime ideals of length $r+1$. From this we see that $\dim R[X] \geq \dim R+1$. To show that this is actually an equality, consider a maximal ideal $\mathfrak{m} \subset R[X]$ and set $\mathfrak{p} = \mathfrak{m} \cap R$. Then $\mathfrak{p}$ is a prime ideal in R and it is enough to show that $\operatorname{ht} \mathfrak{m} \leq \operatorname{ht} \mathfrak{p} + 1$. To do this, we replace R by its localization $R_\mathfrak{p}$ and interpret the polynomial ring $R_\mathfrak{p}[X]$ as the localization of $R[X]$ by the multiplicative system $R - \mathfrak{p}$. Thereby we have reduced our claim to the case where R is a Noetherian local ring with maximal ideal $\mathfrak{p}$; use 1.2/7. Then $R/\mathfrak{p}$ is a field and $R[X]/\mathfrak{p}R[X] \simeq R/\mathfrak{p}[X]$ a principal ideal domain. Hence, there is a polynomial $f \in \mathfrak{m}$ such that

$$\mathfrak{m} = \mathfrak{p}R[X] + fR[X].$$

Now let $x_1, \ldots, x_r$ be a system of parameters of R where $r = \dim R$. Then $x_1, \ldots, x_r, f$ generate an ideal in $R[X]$ whose radical is $\mathfrak{m}$. Therefore Theorem 6 shows $\operatorname{ht} \mathfrak{m} \leq r+1 = \operatorname{ht} \mathfrak{p} + 1$, as claimed. □

Corollary 16. $\dim K[X_1, \ldots, X_n] = n$ *for the polynomial ring in* n *variables over a field* K.

In particular, we can conclude for finitely generated K-algebras, i.e. quotients of polynomial rings of type $K[X_1, \ldots, X_n]$, that their dimension is finite. Making use of the fact to be proved later in 3.2/4 that any maximal ideal $\mathfrak{m} \subset K[X_1, \ldots, X_n]$ leads to a residue field $K[X_1, \ldots, X_n]/\mathfrak{m}$ that is *finite* over K, we can even derive the following more specific version of Corollary 16:

Proposition 17. *Consider the polynomial ring* $K[X_1, \ldots, X_n]$ *in* n *variables over a field* K *and a maximal ideal* $\mathfrak{m} \subset K[X_1, \ldots, X_n]$. *Then*:

(i) $\mathfrak{m}$ *is generated by* n *elements.*

(ii) $\operatorname{ht} \mathfrak{m} = n$.

(iii) *The localization* $K[X_1, \ldots, X_n]_\mathfrak{m}$ *is a local ring of dimension* n, *whose maximal ideal is generated by a system of parameters.*

Proof. To establish (i) and (ii), we use induction on n, the case $n=0$ being trivial. So assume $n \geq 1$. Let $\mathfrak{n} = \mathfrak{m} \cap K[X_1, \ldots, X_{n-1}]$ and consider the inclusions

$$K \hookrightarrow K[X_1, \ldots, X_{n-1}]/\mathfrak{n} \hookrightarrow K[X_1, \ldots, X_n]/\mathfrak{m}.$$

Then $K[X_1, \ldots, X_n]/\mathfrak{m}$ is a field that is finite over K by 3.2/4 and, hence, finite over $K[X_1, \ldots, X_{n-1}]/\mathfrak{n}$. In particular, using 3.1/2, the latter is a field as well. Therefore $\mathfrak{n}$ is a maximal ideal in $K[X_1, \ldots, X_{n-1}]$, and we may assume by induction hypothesis that $\mathfrak{n}$, as an ideal in $K[X_1, \ldots, X_{n-1}]$, is generated by $n-1$ elements.

Now look at the canonical surjection

$$\big(K[X_1, \ldots, X_{n-1}]/\mathfrak{n}\big)[X_n] \longrightarrow K[X_1, \ldots, X_n]/\mathfrak{m}$$

sending X_n onto its residue class in $K[X_1, \dots, X_n]/\mathfrak{m}$. Since on the left-hand side we are dealing with a principal ideal domain, there is a polynomial $f \in K[X_1, \dots, X_n]$ such that $\mathfrak{m}$ is generated by $\mathfrak{n}$ and f. In particular, $\mathfrak{m}$ is generated by n elements and assertion (i) is clear. Furthermore, we see that $\operatorname{ht} \mathfrak{m} \leq n$ by Krull's Dimension Theorem 6. On the other hand, since

$$K[X_1, \dots, X_n]/(\mathfrak{n}) \simeq \big(K[X_1, \dots, X_{n-1}]/\mathfrak{n}\big)[X_n],$$

it is clear that $\mathfrak{n}$ generates a prime ideal in $K[X_1, \dots, X_n]$ different from $\mathfrak{m}$ so that $\mathfrak{n} \subsetneq \mathfrak{m}$. Using $\operatorname{ht} \mathfrak{n} = n - 1$ from the induction hypothesis, we get $\operatorname{ht} \mathfrak{m} \geq n$ and, hence, $\operatorname{ht} \mathfrak{m} = n$, showing (ii).

Finally, (iii) is a consequence of (i) and (ii). □

To end this section, we briefly want to touch the subject of regular local rings.

Proposition and Definition 18. *For a Noetherian local ring R with maximal ideal $\mathfrak{m}$ and dimension d the following conditions are equivalent*:

(i) *There exists a system of parameters of length d in R generating the maximal ideal $\mathfrak{m}$; in other words, $\mathfrak{m}$ is generated by d elements.*

(ii) $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = d$.

R is called regular *if it satisfies the equivalent conditions* (i) *and* (ii).

Proof. Assume (i) and let $x_1, \dots, x_d$ be a system of parameters generating $\mathfrak{m}$. Then, as an $R/\mathfrak{m}$-vector space, $\mathfrak{m}/\mathfrak{m}^2$ is generated by d elements and, hence, $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 \leq d$. Since $\dim R \leq \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$ by Corollary 8, we get $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = d$ and therefore (ii).

Conversely, assume (ii). Then $\mathfrak{m}/\mathfrak{m}^2$ is generated by d elements and the same is true for $\mathfrak{m}$ by Nakayama's Lemma in the version of 1.4/12. In particular, such a system of generators is a system of parameters then. □

For example, we see from Hilbert's Basis Theorem 1.5/14 in conjunction with Proposition 17 that for a polynomial ring $K[X_1, \dots, X_n]$ over a field K its localization $K[X_1, \dots, X_n]_{\mathfrak{m}}$ at a maximal ideal $\mathfrak{m}$ is an example of a regular Noetherian local ring of dimension n. It is known that any localization $R_{\mathfrak{p}}$ of a regular Noetherian local ring R by a prime ideal $\mathfrak{p} \subset R$ is regular again; see Serre [24], Prop. IV.23. Also, let us mention the Theorem of Auslander–Buchsbaum stating that any regular Noetherian local ring is factorial; see Serre [24], Cor. 4 of Thm. IV.9 for a proof of this fact.

To deal with some more elementary properties of regular local rings, it is quite convenient to characterize Krull dimensions of rings not only via lengths of ascending chains of prime ideals, or systems of parameters, as we have done, but also in terms of the so-called *Hilbert polynomial.* For example, see Atiyah–Macdonald [2], Chapter 11, for such a treatment. From this it is easily seen that regular Noetherian local rings are integral domains. Since we need it later on, we will prove this fact by an *ad hoc* method.

Proposition 19. *Let R be a regular Noetherian local ring. Then R is an integral domain.*

Proof. We argue by induction on $d = \dim R$. Let $\mathfrak{m}$ be the maximal ideal of R. If $d = 0$, then $\mathfrak{m}$ is generated by 0 elements and, hence, $\mathfrak{m} = 0$. Therefore R is a field. Now let $\dim R > 0$ and let $x_1, \ldots, x_d$ be a system of parameters generating $\mathfrak{m}$. Furthermore, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal prime ideals in R; their number is finite by 2.1/12. We claim that we can find an element of type

$$a = x_1 + \sum_{i=2}^{d} c_i x_i \in \mathfrak{m} - \mathfrak{m}^2$$

for some coefficients $c_i \in R$ such that a is not contained in any of the minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. Clearly, any element a of this type cannot be contained in $\mathfrak{m}^2$ since $x_1, \ldots, x_d$ give rise to an $R/\mathfrak{m}$-vector space basis of $\mathfrak{m}/\mathfrak{m}^2$.

Using a recursive construction for a, assume that $a \notin \mathfrak{p}_1, \ldots, \mathfrak{p}_t$ for some $t < r$, but that $a \in \mathfrak{p}_{t+1}$. Applying 1.3/8, there exists an element

$$c \in \bigcap_{j=1}^{t} \mathfrak{p}_j - \mathfrak{p}_{t+1}$$

and we can find an index $i_0 \in \{2, \ldots, d\}$ such that $cx_{i_0} \notin \mathfrak{p}_{t+1}$. Otherwise, since $c \notin \mathfrak{p}_{t+1}$ and $cx_i \in \mathfrak{p}_{t+1}$ implies $x_i \in \mathfrak{p}_{t+1}$, all elements $a, x_2, \ldots, x_d$ would be contained in $\mathfrak{p}_{t+1}$ so that $\mathfrak{m} \subset \mathfrak{p}_{t+1}$. However, this contradicts the fact that $\dim R = d > 0$. Then we see that $a' = a + cx_{i_0}$ is of the desired type: a' is not contained in $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$, since this is true for a and since $c \in \bigcap_{j=1}^{t} \mathfrak{p}_j$. Furthermore, $a' \notin \mathfrak{p}_{t+1}$, since $a \in \mathfrak{p}_{t+1}$ and $cx_{i_0} \notin \mathfrak{p}_{t+1}$.

Thus, we have seen that there exists an element $a \in \mathfrak{m} - \mathfrak{m}^2$ as specified above that is not contained in any of the minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of R. Now look at the quotient $R/(a)$. Its dimension is $d - 1$ by Proposition 13, since $a, x_2, \ldots, x_d$ form a system of parameters of R. Furthermore, $x_2, \ldots, x_d$ give rise to a system of parameters of $R/(a)$ generating the maximal ideal of this ring. Therefore $R/(a)$ is regular and, thus, by induction hypothesis, an integral domain. In particular, we see that the ideal $(a) \subset R$ is prime. Now consider a minimal prime ideal of R that is contained in (a), say $\mathfrak{p}_1 \subset (a)$. Then any element $y \in \mathfrak{p}_1$ is of type ab for some $b \in R$ and, in fact $b \in \mathfrak{p}_1$, since $a \notin \mathfrak{p}_1$. Therefore $\mathfrak{p}_1 = (a) \cdot \mathfrak{p}_1$ and Nakyama's Lemma 1.4/10 shows $\mathfrak{p}_1 = 0$. Hence, R is an integral domain. □

Exercises

1. Consider the polynomial ring $R[X]$ in one variable over a not necessarily Noetherian ring. Show $\dim R + 1 \leq \dim R[X] \leq 2 \cdot \dim R + 1$. *Hint*: Let $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subset R[X]$ be two different prime ideals in $R[X]$ restricting to the same prime ideal $\mathfrak{p} \subset R$. Deduce that $\mathfrak{p}_1 = \mathfrak{p}R[X]$.

2. Let $K[X,Y]$ be the polynomial ring in two variables over a field K. Show $\dim K[X,Y]/(f) = 1$ for any non-zero polynomial $f \in K[X,Y]$ that is not constant.

3. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Show for $a_1, \ldots, a_r \in \mathfrak{m}$ that $\dim R/(a_1, \ldots, a_r) \geq \dim R - r$. *Hint*: Assume $r = 1$. Show for any chain of prime ideals $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n$ where $a_1 \in \mathfrak{p}_n$ that there is a chain of prime ideals $\mathfrak{p}'_1 \subsetneq \ldots \subsetneq \mathfrak{p}'_n$ satisfying $a_1 \in \mathfrak{p}'_1$ and $\mathfrak{p}_n = \mathfrak{p}'_n$; use induction on n.

4. Consider the formal power series ring $R = K[\![X_1, \ldots, X_n]\!]$ in finitely many variables over a field K. Show that R is a regular Noetherian local ring of dimension n. *Hint*: Use Exercise 1.5/8 for the fact that R is Noetherian.

5. Let R be a regular Noetherian local ring. Show that R is a field if $\dim R = 0$. Show that R is a discrete valuation ring, i.e. a local principal ideal domain, if $\dim R = 1$.

6. Let R be a regular Noetherian local ring of dimension d with maximal ideal $\mathfrak{m} \subset R$. Show for elements $a_1, \ldots, a_r \in \mathfrak{m}$ that the quotient $R/(a_1, \ldots, a_r)$ is regular of dimension $d - r$ if and only if the residue classes $\overline{a}_1, \ldots, \overline{a}_r \in \mathfrak{m}/\mathfrak{m}^2$ are linearly independent over the field $R/\mathfrak{m}$.

7. Let R be a regular Noetherian local ring of dimension d with maximal ideal $\mathfrak{m}$. Let $\mathfrak{m} = (a_1, \ldots, a_d)$ and set $k = R/\mathfrak{m}$. Show for polynomial variables $X_1, \ldots, X_d$ that the canonical k-algebra homomorphism

$$k[X_1, \ldots, X_d] \longrightarrow \operatorname{gr}_{\mathfrak{m}}(R) = \bigoplus_{i \in \mathbb{N}} \mathfrak{m}^i/\mathfrak{m}^{i+1}, \qquad X_j \longmapsto \overline{a}_j,$$

where $\overline{a}_j$ is the residue class of a_j in $\mathfrak{m}/\mathfrak{m}^2$, is an isomorphism of k-algebras. *Hints*: See Exercise 2.3/5 for the fact that the graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$ associated to $\mathfrak{m}$ is Noetherian if R is Noetherian. Proceed by induction, similarly as in the proof of Proposition 19, and look at the maximal ideal $\bigoplus_{i>0} \mathfrak{m}^i/\mathfrak{m}^{i+1} \subset \operatorname{gr}_{\mathfrak{m}}(R)$. Show that this maximal ideal cannot be a minimal prime ideal in $\operatorname{gr}_{\mathfrak{m}}(R)$ if $d = \dim R > 0$.

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