Chapter 2
Background on Nonlinear Systems and Control

In this chapter, we review some basic results on the analysis and control of nonlinear systems. This review is not intended to be exhaustive but to provide the reader with the necessary background for the results presented in the subsequent chapters. The results presented in this chapter are standard in the nonlinear systems and control literature. For detailed discussion and proofs of the results, the reader may refer to the classic books [72, 76].

2.1 Notation

Throughout this book, the operator $|\cdot|$ is used to denote the absolute value of a scalar and the operator $\|\cdot\|$ is used to denote Euclidean norm of a vector, while we use $\|\cdot\|_Q$ to denote the square of a weighted Euclidean norm, i.e., $\|x\|_Q = x^T Q x$ for all $x \in \mathbb{R}^n$. The symbol Ω_r is used to denote the set $\Omega_r := \{x \in \mathbb{R}^n : V(x) \leq r\}$ where V is a scalar positive definite, continuous differentiable function and $V(0) = 0$, and the operator '/' denotes set subtraction, that is, $A/B := \{x \in \mathbb{R}^n : x \in A, x \notin B\}$. The notation $R = [r_1\ r_2]$ is used to denote the augmented vector $R \in \mathbb{R}^{m+n}$ comprising the vectors $r_1 \in \mathbb{R}^m$ and $r_2 \in \mathbb{R}^n$. The notation $x(T^+)$ denotes the limit of the trajectory $x(t)$ as T is approached from the right, i.e., $x(T^+) = \lim_{t \to T^+} x(t)$. The notation $L_f h$ denotes the standard Lie derivative of a scalar function $h(\cdot)$ with respect to the vector function $f(\cdot)$, i.e., $L_f h(x) = \frac{\partial h}{\partial x} f(x)$.

2.2 Nonlinear Systems

In this book, we deal with a class of time invariant nonlinear systems that can be described by the following state-space model:

$$\dot{x} = f(x, u), \tag{2.1}$$

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where $x \in \mathbb{R}^n$ denotes the vector of state variables, $u \in \mathbb{R}^m$ denotes the vector of control (manipulated) input variables, and f is a locally Lipschitz vector function on $\mathbb{R}^n \times \mathbb{R}^m$ such that $f(0,0) = 0$. This implies that the origin is an equilibrium point for the unforced system. The input vector is restricted to be in a nonempty convex set $U \subseteq \mathbb{R}^m$ which is defined as follows:

$$U := \{u \in \mathbb{R}^m : \|u\| \leq u^{\max}\}, \tag{2.2}$$

where $u^{\max}$ is the magnitude of the input constraint. Another version of the set that we will use is

$$U_{con} := \{u \in \mathbb{R}^m : u_i^{\min} \leq u_i \leq u_i^{\max}, i = 1, \dots, m\}, \tag{2.3}$$

where $u_i^{\min}$ and $u_i^{\max}$ denote the constraints on the minimum and maximum value of the ith input.

In many chapters, we will restrict our analysis to a special case of the system of Eq. (2.1) where the input vector u enters the dynamics of the state x in an affine fashion as follows:

$$\dot{x} = f(x) + G(x)u, \tag{2.4}$$

where f is a locally Lipschitz vector function on $\mathbb{R}^n$ such that $f(0) = 0$ and G is an $n \times m$ matrix of locally Lipschitz vector functions on $\mathbb{R}^n$.

2.3 Stability of Nonlinear Systems

For all control systems, stability is the primary requirement. One of the most widely used stability concepts in control theory is that of *Lyapunov stability*, which we employ throughout the book. In this section, we briefly review basic facts from Lyapunov's stability theory. To begin with, we note that Lyapunov stability and asymptotic stability are properties not of a dynamical system as a whole, but rather of its individual solutions. We restrict our attention to the class of time-invariant nonlinear systems:

$$\dot{x} = f(x), \tag{2.5}$$

where the control input u does not appear explicitly. This does not necessarily mean that the input to the system is zero. It could be that the input u has been specified as a given function of the state x, $u = u(x)$, and could be considered as a special case of the system of Eq. (2.1).

The solution of Eq. (2.5), starting from x_0 at time $t_0 \in \mathbb{R}$, is denoted as $x(t; x_0, t_0)$, so that $x(t_0; x_0, t_0) = x_0$. Because the solutions of Eq. (2.5) are invariant under a translation of t_0, that is, $x(t + T; x_0, t_0 + T) = x(t; x_0, t_0)$, the stability properties of $x(t; x_0, t_0)$ are *uniform*, i.e., they do not depend on t_0. Therefore, without loss of generality, we assume $t_0 = 0$ and write $x(t; x_0)$ instead of $x(t; x_0, 0)$.

Lyapunov stability concepts describe continuity properties of $x(t; x_0, t_0)$ with respect to x_0. If the initial state x_0 is perturbed to $\tilde{x}_0$, then, for stability, the perturbed solution $\tilde{x}(t; x_0)$ is required to stay close to $x(t; x_0)$ for all $t \geq 0$. In addition, for asymptotic stability, the error $\tilde{x}(t; x_0) - x(t; x_0)$ is required to vanish as $t \to \infty$. Some solutions of Eq. (2.5) may be stable and some unstable. We are particularly interested in studying and characterizing the stability properties of *equilibria*, that is, constant solutions $x(t; x_e) \equiv x_e$ satisfying $f(x_e) = 0$.

For convenience, we state all definitions and theorems for the case when the equilibrium point is at the origin of $\mathbb{R}^n$; that is, $x_e = 0$. There is no loss of generality in doing so since any equilibrium point under investigation can be translated to the origin via a change of variables. Suppose $x_e \neq 0$, and consider the change of variables, $z = x - x_e$. The derivative of z is given by:

$$\dot{z} = \dot{x} = f(x) = f(z + x_e) := g(z),$$

where $g(0) = 0$. In the new variable z, the system has an equilibrium point at the origin. Therefore, for simplicity and without loss of generality, we will always assume that $f(x)$ satisfies $f(0) = 0$ and confine our attention to the stability properties of the origin $x_e = 0$.

2.3.1 Stability Definitions

The origin is said to be a *stable* equilibrium point of the system of Eq. (2.5), in the sense of Lyapunov, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that we have:

$$\|x(0)\| \leq \delta \quad \Longrightarrow \quad \|x(t)\| \leq \varepsilon, \quad \forall t \geq 0. \tag{2.6}$$

In this case, we will also simply say that the system of Eq. (2.5) is stable. A similar convention will apply to other stability concepts introduced below. The origin is said to be *unstable* if it is not stable. The ε–δ requirement for stability takes a challenge–answer form. To demonstrate that the origin is stable, for every value of ε that a challenger may care to design, we must produce a value of δ, possibly dependent on ε, such that a trajectory starting in a δ neighborhood of the origin will never leave the ε neighborhood.

The origin of the system of Eq. (2.5) is said to be *asymptotically stable* if it is stable and δ in Eq. (2.6) can be chosen so that (attractivity property of the origin):

$$\|x(0)\| \leq \delta \quad \Longrightarrow \quad x(t) \to 0 \quad \text{as } t \to \infty. \tag{2.7}$$

When the origin is asymptotically stable, we are often interested in determining how far from the origin the trajectory can be and still converge to the origin as t approaches ∞. This gives rise to the definition of the *region of attraction* (also called *region of asymptotic stability*, *domain of attraction*, and *basin*). Let $\phi(t; x)$ be the solution of Eq. (2.5) that starts at initial state x at time $t = 0$. Then the region of attraction is defined as the set of all points x such that $\lim_{t\to\infty}\phi(t;x)=0$. If the origin is a stable equilibrium and its domain of attraction is the entire state-space, then the origin is called *globally asymptotically stable*.

If the system is not necessarily stable but has the property that all solutions with initial conditions in some neighborhood of the origin converge to the origin, then it is called (locally) attractive. We say that the system is *globally attractive* if its solutions converge to the origin from all initial conditions.

The system of Eq. (2.5) is called *exponentially stable* if there exist positive real constants δ, c, and λ such that all solutions of Eq. (2.5) with $\|x(0)\| \le \delta$ satisfy the inequality:

$$\left\|x(t)\right\| \le c\left\|x(0)\right\|e^{-\lambda t}, \quad \forall t \ge 0. \tag{2.8}$$

If this exponential decay estimate holds for any $x(0) \in \mathbb{R}^n$, the system is said to be *globally exponentially stable*.

2.3.2 *Stability Characterizations Using Function Classes $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$*

Scalar comparison functions, known as class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$, are important stability analysis tools that are frequently used to characterize the stability properties of a nonlinear system.

Definition 2.1 A function $\alpha : [0,a) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0)=0$. It is said to belong to class $\mathcal{K}_\infty$ if $a=\infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Definition 2.2 A function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if, for each fixed $t \ge 0$, the mapping $\beta(r,t)$ is of class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r,t)$ is decreasing with respect to t and $\beta(r,t) \to 0$ as $t \to \infty$.

We will write $\alpha \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ to indicate that α is a class $\mathcal{K}$ function and β is a class $\mathcal{KL}$ function, respectively. As an immediate application of these function classes, we can rewrite the stability definitions of the previous section in a more compact way. For example, stability of the system of Eq. (2.5) is equivalent to the property that there exist a $\delta > 0$ and a class $\mathcal{K}$ function, α, such that all solutions with $\|x(0)\| \le \delta$ satisfy:

$$\left\|x(t)\right\| \le \alpha\left(\left\|x(0)\right\|\right), \quad \forall t \ge 0. \tag{2.9}$$

Asymptotic stability is equivalent to the existence of a $\delta > 0$ and a class $\mathcal{KL}$ function, β, such that all solutions with $\|x(0)\| \le \delta$ satisfy:

$$\left\|x(t)\right\| \le \beta\left(\left\|x(0)\right\|, t\right), \quad \forall t \ge 0. \tag{2.10}$$

Global asymptotic stability amounts to the existence of a class $\mathcal{KL}$ function, β, such that the inequality of Eq. (2.10) holds for all initial conditions. Exponential stability means that the function β takes the form $\beta(r, s) = cre^{-\lambda s}$ for some $c, \lambda > 0$.

2.3.3 Lyapunov's Direct (Second) Method

Having defined stability and asymptotic stability of equilibrium points, the next task is to find ways to determine stability. To be of practical interest, stability conditions must not require that we explicitly solve Eq. (2.5). The direct method of Lyapunov aims at determining the stability properties of an equilibrium point from the properties of $f(x)$ and its relationship with a positive-definite function $V(x)$.

Definition 2.3 Consider a $\mathcal{C}^1$ (i.e., continuously differentiable) function $V : \mathbb{R}^n \to \mathbb{R}$. It is called *positive-definite* if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. If $V(x) \to \infty$ as $\|x\| \to \infty$, then V is said to *be radially unbounded*.

If V is both positive-definite and radially unbounded, then there exist two class $\mathcal{K}_\infty$ functions α_1, α_2 such that V satisfies:

$$\alpha_1\big(\|x\|\big) \leq V(x) \leq \alpha_2\big(\|x\|\big) \tag{2.11}$$

for all x. We write $\dot{V}$ for the derivative of V along the solutions of the system of Eq. (2.5), i.e.:

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x). \tag{2.12}$$

The main result of Lyapunov's stability theory is expressed by the following statement.

Theorem 2.1 (Lyapunov) *Let $x = 0$ be an equilibrium point for the system of Eq. (2.5) and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$ in its interior. Suppose that there exists a positive-definite $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ whose derivative along the solutions of the system of Eq. (2.5) satisfies*:

$$\dot{V}(x) \leq 0, \quad \forall x \in D \tag{2.13}$$

then $x = 0$ of the system of Eq. (2.5) is stable. If the derivative of V satisfies:

$$\dot{V}(x) < 0, \quad \forall x \in D/\{0\} \tag{2.14}$$

then $x = 0$ of the system of Eq. (2.5) is asymptotically stable. If in the latter case, V is also radially unbounded, then $x = 0$ of the system of Eq. (2.5) is globally asymptotically stable.

A continuously differentiable positive-definite function $V(x)$ satisfying Eq. (2.13) is called a *Lyapunov function*. The surface $V(x) = c$, for some $c > 0$, is called a *Lyapunov surface* or a level surface. The condition $\dot{V} \leq 0$ implies that when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ and can never come out again. When $\dot{V} < 0$, the trajectory moves from one Lyapunov surface to an inner Lyapunov surface with smaller c. As c decreases, the Lyapunov surface $V(x) = c$ shrinks to the origin, showing that the trajectory approaches the origin as time progresses. If we only know that $\dot{V}(x) \leq 0$, we cannot be sure that the trajectory will approach the origin, but we can conclude that the origin is stable since the trajectory can be contained inside any ball, B_ε, by requiring that the initial state x_0 lie inside a Lyapunov surface contained in that ball.

The utility of a Lyapunov function arises from the need (or difficulty) of specifying a unique (necessary and sufficient) direction of movement of states for stability. To understand this, consider any scalar system (whether linear or nonlinear). The necessary and sufficient condition for stability is that, for any value of the state x, the value of $\dot{x}$ should be opposite in sign to x, and greater than zero in magnitude (unless $x = 0$). A Lyapunov function that allows readily capturing this requirement is $V(x) = \frac{x^2}{2}$, resulting in $\dot{V}(x) = x\dot{x}$. If and only if the origin of the systems is stable (i.e., x is opposite in sign to $\dot{x}$), it will result in $\dot{V}(x) < 0$.

For non-scalar systems, this 'unique' direction of movement of states, while possible for linear systems (see Remark 2.1), is in general difficult to identify for nonlinear systems. For instance, if one considers a simple two state system, and restricts the choice of the Lyapunov function to quadratic forms, it is clear that the square of the distance to the origin (resulting in 'circles' as level curves) is not necessarily the only choice of the Lyapunov-function, and there is no unique way to find a necessary and sufficient direction of the movement of states to achieve stability. This is the problem that lies at the core of the Lyapunov-stability theory—the inability to define (and/or construct) a unique Lyapunov function for a given system that is necessary and sufficient to establish stability. Having recognized this limitation, it is important to note that the Lyapunov-based analysis at least provides sufficient conditions to ascertain stability.

In this direction, various converse Lyapunov theorems show that the conditions of Theorem 2.1 are also necessary. For example, if the system is asymptotically stable, then there exists a positive-definite $\mathcal{C}^1$ function V that satisfies the inequality of Eq. (2.14). The theorems, however, do not provide a way of constructing this Lyapunov function.

Remark 2.1 It is well-known that for the linear time-invariant system

$$\dot{x} = Ax \tag{2.15}$$

asymptotic stability, exponential stability, and their global versions are all equivalent and amount to the property that A is a Hurwitz matrix, i.e., all eigenvalues of A have negative real parts. Fixing an arbitrary positive-definite symmetric matrix Q and finding the unique positive-definite symmetric matrix P that satisfies the Lyapunov equation

$$A^T P + PA = -Q,$$

one obtains a quadratic Lyapunov function $V(x) = x^T Px$ whose time derivative along the solutions of the system of Eq. (2.15) is $\dot{V} = -x^T Qx$. The explicit formula for P is

$$P = \int_0^\infty e^{A^T t} Q e^{At}\, dt.$$

Indeed, we have

$$A^T P + PA = \int_0^\infty \frac{d}{dt}\left(e^{A^T t} Q e^{At}\right) dt = -Q,$$

because A is Hurwitz.

2.3.4 LaSalle's Invariance Principle

With some additional knowledge about the behavior of solutions, it is possible to prove asymptotic stability using a Lyapunov function which satisfies the nonstrict inequality of Eq. (2.13). This is facilitated by *LaSalle's invariance principle*. To state this principle, we first recall the definition of an invariant set.

Definition 2.4 A set M is called (positively) invariant with respect to the given system if all solutions starting in M remain in M for all future times.

We now state a version of LaSalle's theorem.

Theorem 2.2 (LaSalle) *Suppose that there exists a positive-definite $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ whose derivative along the solutions of the system of Eq.* (2.5) *satisfies the inequality of Eq.* (2.13)*. Let M be the largest invariant set contained in the set $\{x : \dot{V}(x) = 0\}$. Then the system of Eq.* (2.5) *is stable and every solution that remains bounded for $t \geq 0$ approaches M as $t \to \infty$. In particular, if all solutions remain bounded and $M = \{0\}$, then the system of Eq.* (2.5) *is globally asymptotically stable.*

To deduce global asymptotic stability with the help of this result, one needs to check two conditions. First, all solutions of the system must be bounded. This property follows automatically from the inequality of Eq. (2.13) if V is chosen to be radially unbounded; however, radial boundedness of V is not necessary when boundedness of solutions can be established by other means. The second condition is that V be not identically zero along any nonzero solution. We also remark that if one only wants to prove asymptotic convergence of bounded solutions to zero and is not concerned with Lyapunov stability of the origin, then positive-definiteness of V is not needed (this is in contrast to Theorem 2.1).

While Lyapunov's stability theorem readily generalizes to time-varying systems, for LaSalle's invariance principle this is not the case. Instead, one usually works with the weaker property that all solutions approach the set $\{x : \dot{V}(x) = 0\}$.

2.3.5 Lyapunov's Indirect (First) Method

Lyapunov's indirect method allows one to deduce stability properties of the nonlinear system of Eq. (2.5), where f is $\mathcal{C}^1$, from stability properties of its *linearization*, which is the linear system of Eq. (2.15) with

$$A := \frac{\partial f}{\partial x}(0). \tag{2.16}$$

By the mean value theorem, we can write

$$f(x) = Ax + g(x)x,$$

where g is given componentwise by $g_i(x) := \frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)$ for some point, z_i, on the line segment connecting x to the origin, $i = 1, \ldots, n$. Since $\frac{\partial f}{\partial x}$ is continuous, we have $\|g(x)\| \to 0$ as $x \to 0$. From this it follows that if the matrix A is Hurwitz (i.e., all its eigenvalues lie in the open left half of the complex plane), then a quadratic Lyapunov function for the linearization serves—locally—as a Lyapunov function for the original nonlinear system. Moreover, its rate of decay in a neighborhood of the origin can be bounded below by a quadratic function, which implies that stability is, in fact, exponential. This is summarized by the following result.

Theorem 2.3 *If f is $\mathcal{C}^1$ and the matrix of Eq.* (2.16) *is Hurwitz, then the system of Eq.* (2.5) *is locally exponentially stable.*

It is also known that if the matrix A has at least one eigenvalue with a positive real part, the origin of the nonlinear system of Eq. (2.5) is not stable. If A has eigenvalues on the imaginary axis but no eigenvalues in the open right half-plane, the linearization test is inconclusive. However, in this critical case, the system of Eq. (2.5) cannot be exponentially stable since exponential stability of the linearization is not only a sufficient but also a necessary condition for (local) exponential stability of the nonlinear system.

2.3.6 Input-to-State Stability

It is of interest to extend stability concepts to systems with disturbance inputs. In the linear case represented by the system

$$\dot{x} = Ax + B\theta,$$

it is well known that if the matrix A is Hurwitz, i.e., if the unforced system, $\dot{x} = Ax$, is asymptotically stable, then bounded inputs θ lead to bounded states while inputs converging to zero produce states converging to zero. Now, consider a nonlinear system of the form

$$\dot{x} = f(x, \theta), \tag{2.17}$$

where θ is a measurable bounded disturbance input. In general, global asymptotic stability of the unforced system $\dot{x} = f(x, 0)$ does not guarantee input-to-state stability with respect to θ of the kind mentioned above. For example, the scalar system

$$\dot{x} = -x + x\theta \tag{2.18}$$

has unbounded trajectories under the bounded input $\theta \equiv 2$. This motivates the following important concept, introduced by Sontag [151].

Definition 2.5 The system of Eq. (2.17) is called *input-to-state stable* (ISS) with respect to θ if for some functions $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for every initial state $x(0)$, and every input θ, the corresponding solution of the system of Eq. (2.17) satisfies the inequality

$$\|x(t)\| \le \beta\big(\|x(0)\|, t\big) + \gamma\big(\|\theta\|^s_{[0,t]}\big), \tag{2.19}$$

where $\|\theta\|^s_{[0,t]} := \text{ess.sup}\{\|\theta(s)\| : s \in [0, t]\}$ (supremum norm on $[0, t]$ except for a set of measure zero).

Since the system of Eq. (2.17) is time-invariant, the same property results if we write

$$\|x(t)\| \le \beta\big(\|x(t_0)\|, t - t_0\big) + \gamma\big(\|\theta\|^s_{[t_0,t]}\big), \quad \forall t \ge t_0 \ge 0. \tag{2.20}$$

The ISS property admits the following Lyapunov-like equivalent characterization: The system of Eq. (2.17) is ISS if and only if there exists a positive-definite radially unbounded $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some class $\mathcal{K}_\infty$ functions α and χ we have

$$\frac{\partial V}{\partial x} f(x, \theta) \le -\alpha\big(\|x\|\big) + \chi\big(\|\theta\|\big), \quad \forall x, \theta. \tag{2.21}$$

This is, in turn, equivalent to the following "gain margin" condition:

$$\|x\| \ge \rho\big(\|\theta\|\big) \quad \Longrightarrow \quad \frac{\partial V}{\partial x} f(x, \theta) \le -\alpha\big(\|x\|\big), \tag{2.22}$$

where $\alpha, \rho \in \mathcal{K}_\infty$. Such functions V are called *ISS-Lyapunov functions*. If the system of Eq. (2.17) is ISS, then $\theta(t) \to 0$ implies $x(t) \to 0$.

The system of Eq. (2.17) is said to be *locally input-to-state stable* (locally ISS) if the bound of Eq. (2.19) is valid for solutions with sufficiently small initial conditions and inputs, i.e., if there exists a $\delta > 0$ such that Eq. (2.19) is satisfied whenever $\|x(0)\| \le \delta$ and $\|\theta\|_{[0,t]}^{s} \le \delta$. It turns out that (local) asymptotic stability of the unforced system $\dot{x} = f(x, 0)$ implies local ISS.

2.4 Stabilization of Nonlinear Systems

This book is primarily about control *design*. Our objective is to create closed-loop systems with desirable stability and performance properties, rather than analyze the properties of a given system. For this reason, we are interested in an extension of the Lyapunov function concept, called a *control Lyapunov function* (CLF).

Suppose that our problem for the time-invariant system

$$\dot{x} = f(x, u), \tag{2.23}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ (i.e., we consider the unconstrained problem), $f(0, 0) = 0$, is to design a feedback control law $\alpha(x)$ for the control variable u such that the equilibrium $x = 0$ of the closed-loop system

$$\dot{x} = f\bigl(x, \alpha(x)\bigr) \tag{2.24}$$

is globally asymptotically stable. We can pick a function $V(x)$ as a Lyapunov function candidate, and require that its derivative along the solutions of the system of Eq. (2.24) satisfies $\dot{V} \le -W(x)$, where $W(x)$ is a positive-definite function. We therefore need to find $\alpha(x)$ to guarantee that for all $x \in \mathbb{R}^n$

$$\frac{\partial V}{\partial x}(x) f\bigl(x, \alpha(x)\bigr) \le -W(x). \tag{2.25}$$

This is a difficult task. A stabilizing control law for the system of Eq. (2.23) may exist, but it may fail to satisfy Eq. (2.25) because of a poor choice of $V(x)$ and $W(x)$. A system for which a good choice of $V(x)$ and $W(x)$ exists is said to possess a CLF. This notion is made more precise below.

Definition 2.6 A smooth positive-definite radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ is called a control Lyapunov function (CLF) for the system of Eq. (2.23) if

$$\inf_{u \in \mathbb{R}} \left\{ \frac{\partial V}{\partial x}(x) f(x, u) \right\} < 0, \quad \forall x \ne 0. \tag{2.26}$$

The CLF concept of Artstein [9] is a generalization of Lyapunov design results by Jacobson and Judjevic and Quinn. Artstein showed that Eq. (2.26) is not only necessary, but also sufficient for the existence of a control law satisfying Eq. (2.25), that is, the existence of a CLF is equivalent to global asymptotic stabilizability.

For systems affine in the control, namely,

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0, \tag{2.27}$$

the CLF inequality of Eq. (2.25) becomes

$$L_f V(x) + L_g V(x)u \leq -W(x). \tag{2.28}$$

If V is a CLF for the system of Eq. (2.27), then a particular stabilizing control law $\alpha(x)$, smooth for all $x \neq 0$, is given by Sontag's formula [150]:

$$u = \alpha_s(x) = \begin{cases} -\frac{L_f V(x) + \sqrt{(L_f V)^2(x) + (L_g V)^4(x)}}{(L_g V)^2(x)} L_g V(x), & L_g V(x) \neq 0, \\ 0, & L_g V(x) = 0. \end{cases} \tag{2.29}$$

It should be noted that Eq. (2.28) can be satisfied only if

$$L_g V(x) = 0 \quad \Longrightarrow \quad L_f V(x) < 0, \quad \forall x \neq 0. \tag{2.30}$$

The intuitive interpretation of the existence of a CLF is as follows: For any x such that $L_g V(x) \neq 0$, since there are no constraints on the input, $\dot{V}$ can be made negative by picking a 'large enough' control action, with an appropriate sign, to counter the effect of possibly positive $L_f V(x)$ term. For all x such that $L_g V(x) = 0$, the control action has no effect on the Lyapunov-function derivative. For it to be possible to show stability using the CLF V, it should therefore be true that whenever $L_g V(x) = 0$, we also have that $L_f V(x) < 0$. This is the requirement that is formalized in Eq. (2.30). With such a CLF, Eq. (2.29) results in

$$W(x) = \sqrt{(L_f V)^2(x) + (L_g V)^4(x)} > 0, \quad \forall x \neq 0. \tag{2.31}$$

A further characterization of a stabilizing control law $\alpha(x)$ for the system of Eq. (2.27) with a given CLF V is that $\alpha(x)$ is continuous at $x = 0$ if and only if the CLF satisfies the *small control property*: For each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that, if $x \neq 0$ satisfies $|x| \leq \delta$, then there is some u with $|u| < \varepsilon$ such that

$$L_f V(x) + L_g V(x)u < 0. \tag{2.32}$$

The main deficiency of the CLF concept as a design tool is that for most nonlinear systems a CLF is not known. The task of finding an appropriate CLF maybe as complex as that of designing a stabilizing feedback law. In the next section, we review one commonly used tool for designing a Lyapunov-based control law that utilizes coordinate transformations. We also note that in the presence of input constraints, the concept of a CLF needs to be revisited, and this issue is discussed in Sect. 2.6.

2.5 Feedback Linearization and Zero Dynamics

One of the popular methods for nonlinear control design (or alternatively, one way to construct a Lyapunov-function for the purpose of control design) is *feedback linearization*, which employs a change of coordinates and feedback control to transform a nonlinear system into a system whose dynamics are linear (at least partially). This transformation allows the construction and use of a Lyapunov function for the control design utilizing results from linear systems analysis. A great deal of research has been devoted to this subject over the last four decades, as evidenced by the comprehensive books [72, 126] and the references therein. In this section, we briefly review some of the basic geometric concepts that will be used in subsequent chapters. While this book does not require the formalism of differential geometry, we will employ Lie derivatives only for notational convenience. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field and $h : \mathbb{R}^n \to \mathbb{R}$ is a scalar function, the notation $L_f h$ is used for $\frac{\partial h}{\partial x} f(x)$. It is recursively extended to

$$L_f^k h(x) = L_f\big(L_f^{k-1} h(x)\big) = \frac{\partial}{\partial x}\big(L_f^{k-1} h(x)\big) f(x).$$

Let us consider the following nonlinear system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \tag{2.33}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, f, g, h are analytic (i.e., infinitely differentiable) vector functions. The derivative of the output $y = h(x)$ is given by

$$\begin{aligned} \dot{y} &= \frac{\partial h}{\partial x}(x) f(x) + \frac{\partial h}{\partial x}(x) g(x) u \\ &= L_f h(x) + L_g h(x) u. \end{aligned} \tag{2.34}$$

If $L_g h(x_0) \neq 0$, then the system of Eq. (2.33) is said to have *relative degree one at* x_0 (note that since the functions are smooth $L_g h(x_0) \neq 0$ implies that there exists a neighborhood of x_0 on which $L_g h(x) \neq 0$). In our terminology, this implies that the output y is separated form the input u by one integration only. If $L_g h(x_0) = 0$, there are two cases:

(i) If there exist points arbitrarily close to x_0 such that $L_g h(x) \neq 0$, then the system of Eq. (2.33) does not have a well-defined relative degree at x_0.
(ii) If there exists a neighborhood B_0 of x_0 such that $L_g h(x) = 0$ for all $x \in B_0$, then the relative degree of the system of Eq. (2.33) may be well-defined.

In case (ii), we define

$$\psi_1(x) = h(x), \qquad \psi_2(x) = L_f h(x) \tag{2.35}$$

and compute the second derivative of y

$$\begin{aligned}\ddot{y} &= \frac{\partial \psi_2}{\partial x}(x)f(x) + \frac{\partial \psi_2}{\partial x}(x)g(x)u \\ &= L_f^2 h(x) + L_g L_f h(x)u. \end{aligned} \tag{2.36}$$

If $L_g L_f h(x_0) \neq 0$, then the system of Eq. (2.33) is said to have *relative degree two at* x_0. If $L_g L_f h(x) = 0$ in a neighborhood of x_0, then we continue the differentiation procedure.

Definition 2.7 The system of Eq. (2.33) is said to have relative degree r at the point x_0 if there exists a neighborhood B_0 of x_0 on which

$$L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{r-2} h(x) = 0, \tag{2.37}$$

$$L_g L_f^{r-1} h(x) \neq 0. \tag{2.38}$$

If Eq. (2.37)–(2.38) are valid for all $x \in \mathbb{R}^n$, then the relative degree of the system of Eq. (2.33) is said to be globally defined.

Suppose now that the system of Eq. (2.33) has relative degree r at x_0. Then we can use a change of coordinates and feedback control to locally transform this system into the *cascade interconnection* of an r-dimensional linear system and an $(n-r)$-dimensional nonlinear system. In particular, after differentiating r times the output $y = h(x)$, the control appears:

$$y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x)u. \tag{2.39}$$

Since $L_g L_f^{r-1} h(x) \neq 0$ in a neighborhood of x_0, we can linearize the input–output dynamics of the system of Eq. (2.33) using feedback to cancel the nonlinearities in Eq. (2.39):

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \left[-L_f^r h(x) + v \right]. \tag{2.40}$$

Then the dynamics of y and its derivatives are governed by a chain of r integrators: $y^{(r)} = v$. Since our original system of Eq. (2.33) has dimension n, we need to account for the remaining $n-r$ states. Using differential geometry tools, it can be shown that it is always possible to find $n-r$ functions $\psi_{r+1}, \ldots, \psi_n(x)$ with $\frac{\partial \psi_i}{\partial x}(x)g(x) = 0$, for $i = r+1, \ldots, n$ such that the change of coordinates

$$\begin{aligned} &\zeta_1 = y = h(x), \qquad \zeta_2 = \dot{y} = L_f h(x), \ldots, \zeta_r = y^{(r-1)} = L_f^{r-1} h(x), \\ &\eta_1 = \psi_{r+1}, \ldots, \eta_{n-r} = \psi_n(x) \end{aligned} \tag{2.41}$$

is locally invertible and transforms, along with the feedback law of Eq. (2.40), the system of Eq. (2.33) into

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_2,\\ &\vdots\\ \dot{\zeta}_r &= v,\\ \dot{\eta}_1 &= \Psi_1(\zeta, \eta),\\ &\vdots\\ \dot{\eta}_{n-r} &= \Psi_{n-r}(\zeta, \eta),\\ y &= \zeta_1,\end{aligned} \tag{2.42}$$

where $\Psi_1(\zeta, \eta) = L_f^{r+1} h(x)$, $\Psi_{n-r}(\zeta, \eta) = L_f^n h(x)$.

The states $\eta_1, \ldots, \eta_{n-r}$ have been rendered *unobservable* from the output y by the control of Eq. (2.40). Hence, feedback linearization in this case is the nonlinear equivalent of placing $n-r$ poles of a linear system at the origin and canceling the r zeros with the remaining poles. Of course, to guarantee stability, the canceled zeros must be stable. In the nonlinear case, using the new control input v to stabilize the linear subsystem of Eq. (2.42) does not guarantee stability of the whole system, unless the stability of the nonlinear part of the system of Eq. (2.42) has been established separately.

When v is used to keep the output y equal to zero for all $t > 0$, that is, when $\zeta_1 \equiv \cdots \equiv \zeta_r \equiv 0$, the dynamics of $\eta_1, \ldots, \eta_{n-r}$ are described by

$$\begin{aligned}\dot{\eta}_1 &= \Psi_1(0, \eta),\\ &\vdots\\ \dot{\eta}_{n-r} &= \Psi_{n-r}(0, \eta).\end{aligned} \tag{2.43}$$

They are called the zero dynamics of the system of Eq. (2.33) because they evolve on the subset of the state-space on which the output of the system is identically zero. If the equilibrium at $\eta_1 = \cdots = \eta_{n-r} = 0$ of the zero dynamics of Eq. (2.43) is asymptotically stable, the system of Eq. (2.33) is said to be *minimum phase*.

Remark 2.2 Most nonlinear analytical controllers emanating from the area of geometric control are input–output linearizing and induce a linear input–output response in the absence of constraints [72, 81]. For the class of processes modeled by equations of the form of Eq. (2.33) with relative order r and under the minimum phase assumption, the appropriate linearizing state feedback controller is given by

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \big(v - L_f^r h(x) - \beta_1 L_f^{r-1} h(x) - \cdots - \beta_{r-1} L_f h(x) - \beta_r h(x) \big) \tag{2.44}$$

and induces the linear rth order response

$$\frac{d^r y}{dt^r} + \beta_1 \frac{d^{r-1} y}{dt^{r-1}} + \cdots + \beta_{r-1} \frac{dy}{dt} + \beta_r y = v, \tag{2.45}$$

where the tunable parameters, $\beta_1, \ldots, \beta_r$, are essentially closed-loop time constants that influence and shape the output response. The nominal stability of the process is guaranteed by placing the roots of the polynomial $s^r + \beta_1 s^{r-1} + \cdots + \beta_{r-1} s + \beta_r$ in the open left-half of the complex plane.

2.6 Input Constraints

The presence of input constraints requires revisiting the concept of the CLF for both linear and nonlinear systems. To understand this, consider a scalar linear system of the form $\dot{x} = \alpha x + \beta u$, with $u^{\min} \le u \le u^{\max}$. For the sake of simplicity and without loss of generality, let us assume $u^{\min} < 0 < u^{\max}$ and $\beta > 0$. For the case of scalar systems, it is possible to determine the entire set of initial conditions from where the system can be driven to the origin subject to input constraints (regardless of the choice of the control law). This set is generally referred to as the null controllable region (NCR). An explicit computation of the NCR is possible in this case because for scalar systems (as discussed earlier) there exists a unique direction in which the system states needs to move to achieve stability.

To determine this set, one can simply analyze the system trajectory to the left and right of zero. Consider first $x > 0$, and the requirement that for $x > 0$, $\dot{x} < 0$. If $\alpha < 0$, $\dot{x} < 0\ \forall x > 0$ (and also $\dot{x} > 0\ \forall x < 0$). On the other hand, if $\alpha > 0$, $\dot{x} < 0$ can only be achieved for $x < \frac{-u^{\min}\beta}{\alpha}$. Similarly, $\dot{x} > 0$ can only be achieved for $x > \frac{-u^{\max}\beta}{\alpha}$. The analysis reveals what was perhaps intuitive to begin with: For linear systems, if the steady state is open-loop stable, the NCR is the entire state space, while if the steady state is open-loop unstable, it has a finite NCR, which in this case is $\{x : \frac{-u^{\max}\beta}{\alpha} < x < \frac{-u^{\min}\beta}{\alpha}\}$. The same result for the NCR can also be obtained using a CLF $V(x) = \frac{x^2}{2}$ and determining the states for which $\dot{V} < 0$ is achievable using the available control action. Furthermore, it points to the requirement of additional considerations when defining CLFs for systems with constrained inputs. In particular, requiring that $\dot{V}(x) < 0\ \forall x$ is simply not achievable for certain cases, at best what is achievable is that $\dot{V}(x) < 0\ \forall x \in \mathrm{NCR} - \{0\}$. The definition of a CLF (or more appropriately, a constrained CLF) then becomes intricately linked with the characterization of the NCR. The characterization of the NCR, however, is an increasingly difficult (although possible, see [71]) problem when considering non-scalar linear systems, and currently an open problem for nonlinear systems.

To understand the impact of the lack of availability of constrained CLFs (CCLFs), let us first consider again the linear scalar system under a feedback law of the form $u_c(x) = -kx$, with $k > 0$ such that $(\alpha - k\beta) < 0$ under two possible scenarios: (i) $\alpha < 0$ (i.e., for the unforced system, there is an isolated equilibrium point at the origin and the system is stable at that operating point) and (ii) $\alpha > 0$ (i.e., for the unforced system, there is an isolated equilibrium point at the origin and the system is unstable at that operating point). Due to the presence of input constraints, the closed-loop system is no longer a linear system, but operates in three 'modes', depending on the state, described by the following set of equations:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x + \beta u_c, \quad u^{\min} \le u_c \le u^{\max}, \\ \frac{dx}{dt} &= \alpha x + \beta u^{\max}, \quad u_c > u^{\max}, \\ \frac{dx}{dt} &= \alpha x + \beta u^{\min}, \quad u^{\min} > u_c. \end{aligned} \tag{2.46}$$

Let us analyze the three possible modes of operation of the closed-loop system for scenario (i). For $-\frac{|u^{\max}|}{k} \le x \le \frac{|u^{\min}|}{k}$, we have that $\frac{dx}{dt} = \alpha x + \beta u_c = (\alpha - k\beta)x$, which establishes that for all initial conditions x_0 such that $-\frac{|u^{\max}|}{k} \le x_0 \le \frac{|u^{\min}|}{k}$, the prescribed control action u_c is within the constraints and the system state will be driven to the origin. For $\frac{|u^{\min}|}{k} < x \le \frac{-u^{\min}\beta}{\alpha}$, $u_c > u^{\max}$ resulting in $u = u^{\max}$, in turn resulting in $\dot{x} < 0$. A similar result is obtained for $\frac{-u^{\max}\beta}{\alpha} < x < -\frac{|u^{\max}|}{k}$. The analysis shows that for scalar systems, while the region of unconstrained operation for a particular control law might depend on the specific control law chosen, the stability region under the control law might still possibly be the entire NCR.

The issue of directionality again crops up when considering non-scalar systems. While it is relatively easy to determine the region of unconstrained operation for a particular control law, and, in certain cases, the region of attraction for the closed-loop system, it is not necessary that the region of attraction for the closed-loop system match the NCR. This happens due to the fact that it is in general difficult to determine, for a particular value of the state, the unique direction in which the inputs should saturate to achieve closed-loop stability. To achieve this objective, recent control designs have utilized the explicit characterization of the NCR [71] in designing CCLF based control laws that ensure stabilization from all initial conditions in the NCR [93, 94]. For nonlinear systems, where the characterization of the NCR is still an open problem, a meaningful control objective is to be able to explicitly account for the constraints in the control design and provide an explicit characterization of the closed-loop stability region.

2.7 Model Predictive Control

One of the control methods useful for accounting for constraints and optimality simultaneously is that of model predictive control (MPC). MPC is an approach which accounts for optimality considerations explicitly and is widely adopted in industry as an effective approach to deal with large multivariable constrained optimal control problems. The main idea of MPC is to choose control actions by repeatedly solving an online a constrained optimization problem, which aims at minimizing a performance index over a finite prediction horizon based on predictions obtained by a system model. In general, an MPC design is composed of three components:

1. A model of the system. This model is used to predict the future evolution of the system in open-loop and the efficiency of the calculated control actions of an MPC depends highly on the accuracy of the model.
2. A performance index over a finite horizon. This index is minimized subject to constraints imposed by the system model, restrictions on control inputs and system state, and other considerations at each sampling time to obtain a trajectory of future control inputs.
3. A receding horizon scheme. This scheme introduces the notion of feedback into the control law to compensate for disturbances and modeling errors, whereby only the first piece of the future input trajectory is implemented and the constrained optimization problem is resolved at the next sampling instance.

Consider the control of the system of Eq. (2.1) and assume that the state measurements of the system of Eq. (2.1) are available at synchronous sampling time instants $\{t_{k\geq 0}\}$, a standard MPC is formulated as follows [60]:

$$\min_{u\in S(\Delta)} \int_{t_k}^{t_{k+N}} \left[\left\|\tilde{x}(\tau)\right\|_{Q_c} + \left\|u(\tau)\right\|_{R_c}\right] d\tau + F\big(x(t_{k+N})\big) \tag{2.47}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f\big(\tilde{x}(t), u(t)\big), \tag{2.48}$$

$$u(t) \in U, \tag{2.49}$$

$$\tilde{x}(t_k) = x(t_k), \tag{2.50}$$

where $S(\Delta)$ is the family of piece-wise constant functions with sampling period Δ, N is the prediction horizon, Q_c and R_c are strictly positive definite symmetric weighting matrices, $\tilde{x}$ is the predicted trajectory of the system due to control input u with initial state $x(t_k)$ at time t_k, and $F(\cdot)$ denotes the terminal penalty.

The optimal solution to the MPC optimization problem defined by Eq. (2.47)–(2.50) is denoted as $u^*(t|t_k)$ which is defined for $t \in [t_k, t_{k+N})$. The first step value of $u^*(t|t_k)$ is applied to the closed-loop system for $t \in [t_k, t_{k+1})$. At the next sampling time t_{k+1}, when a new measurement of the system state $x(t_{k+1})$ is available, and the control evaluation and implementation procedure is repeated. The manipulated input of the system of Eq. (2.1) under the control of the MPC of Eq. (2.47)–(2.50) is defined as follows:

$$u(t) = u^*(t|t_k), \quad \forall t \in [t_k, t_{k+1}), \tag{2.51}$$

which is the standard receding horizon scheme.

In the MPC formulation of Eq. (2.47)–(2.50), Eq. (2.47) defines a performance index or cost index that should be minimized. In addition to penalties on the state and control actions, the index may also include penalties on other considerations; for example, the rate of change of the inputs. Equation (2.48) is the model of the system of Eq. (2.1) which is used in the MPC to predict the future evolution of the system. Equation (2.49) takes into account the constraint on the control input, and Eq. (2.50) provides the initial state for the MPC which is a measurement of the actual system state. Note that in the above MPC formulation, state constraints are not considered but can be readily taken into account.

It is well known that the MPC of Eq. (2.47)–(2.50) is not necessarily stabilizing. To understand this, let us consider a discrete time version of the MPC implementation, for a scalar system described by $x(k+1) = \alpha x(k) + u(k)$, in the absence of input constraints. Also, let $N = 1$, q and r denote the horizon, penalty on the state deviation and input deviation, respectively. The objective function then simplifies to $q(\alpha^2 x(k)^2 + u(k)^2 + 2\alpha x(k)u(k)) + ru(k)^2$, and the minimizing control action is $u(k) = \frac{-q\alpha x(k)}{q+r}$, resulting in the closed-loop system $x(k+1) = \frac{r\alpha x(k)}{q+r}$. The minimizing solution will result in stabilizing control action only if $q > r(\alpha - 1)$. Note that for $\alpha < 1$, this trivially holds (i.e., the result trivially holds for stabilization around an open-loop stable steady state). For $\alpha > 1$, the result establishes how large the penalty on the set point deviation should be compared to the penalty on the control action for the controller to be stabilizing. The analysis is meant to bring out the fact that generally speaking, the stability of the closed-loop system in the MPC depends on the MPC parameters (penalties and the control horizon) as well as the system dynamics. Note also that even though we have analyzed an unconstrained system, the prediction horizon we used was finite (in comparison to linear quadratic regulator designs, where the infinite horizon cost is essentially captured in computing the control action, and therefore results in stabilizing controller in the absence of constraints). Finally, also note that for the case of infinite horizon, the optimum solution is also the stabilizing one, and it can be shown that such an MPC will stabilize the system with the NCR as the stability region (albeit at an impractical computational burden).

To achieve closed-loop stability without relying on the objective function parameters, different approaches have been proposed in the literature. One class of approaches is to use well-designed terminal penalty terms that capture infinite horizon costs; please, see [16, 100] for surveys of these approaches. Another class of approaches is to impose stability constraints in the MPC optimization problem [3, 14, 100]. There are also efforts focusing on getting explicit stabilizing MPC laws using offline computations [92]. However, the implicit nature of MPC control law makes it very difficult to explicitly characterize, a priori, the admissible initial conditions starting from where the MPC is guaranteed to be feasible and stabilizing. In practice, the initial conditions are usually chosen in an ad hoc fashion and tested through extensive closed-loop simulations.

2.8 Lyapunov-Based MPC

In this section, we introduce Lyapunov-based MPC (LMPC) designs proposed in [93, 108, 110] which allow for an explicit characterization of the stability region and guarantee controller feasibility and closed-loop stability.

For the predictive control of the system of Eq. (2.1), the key idea in LMPC-based designs is to utilize a Lyapunov-function based constraint and achieve immediate decay of the Lyapunov function. The set of initial conditions for which it is possible to achieve an instantaneous decay in the Lyapunov function value can be computed explicitly, and picking the (preferably largest) level curve contained in this set can provide the explicitly characterized feasibility and stability region for the LMPC.

The following example of the LMPC design is based on an existing explicit control law $h(x)$ which is able to stabilize the closed-loop system [108, 110]. The formulation of the LMPC is as follows:

$$\min_{u \in S(\Delta)} \int_{t_k}^{t_{k+N}} \left[\left\| \tilde{x}(\tau) \right\|_{Q_c} + \left\| u(\tau) \right\|_{R_c} \right] d\tau \tag{2.52}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f\left(\tilde{x}(t), u(t)\right), \tag{2.53}$$

$$u(t) \in U, \tag{2.54}$$

$$\tilde{x}(t_k) = x(t_k), \tag{2.55}$$

$$\frac{\partial V(x(t_k))}{\partial x} f\left(x(t_k), u(t_k)\right) \le \frac{\partial V(x(t_k))}{\partial x} f\left(x(t_k), h\left(x(t_k)\right)\right), \tag{2.56}$$

where $V(x)$ is a Lyapunov function associated with the nonlinear control law $h(x)$. The optimal solution to this LMPC optimization problem is denoted as $u_l^*(t|t_k)$ which is defined for $t \in [t_k, t_{k+N})$. The manipulated input of the system of Eq. (2.1) under the control of the LMPC of Eq. (2.52)–(2.56) is defined as follows:

$$u(t) = u_l^*(t|t_k), \quad \forall t \in [t_k, t_{k+1}), \tag{2.57}$$

which implies that this LMPC also adopts a standard receding horizon strategy.

In the LMPC defined by Eq. (2.52)–(2.56), the constraint of Eq. (2.56) guarantees that the value of the time derivative of the Lyapunov function, $V(x)$, at time t_k is smaller than or equal to the value obtained if the nonlinear control law $u = h(x)$ is implemented in the closed-loop system in a sample-and-hold fashion. This is a constraint that allows one to prove (when state measurements are available every synchronous sampling time) that the LMPC inherits the stability and robustness properties of the nonlinear control law $h(x)$ when it is applied in a sample-and-hold fashion; please, see [30, 125] for results on sampled-data systems.

Let us denote the stability region of $h(x)$ as Ω_ρ. The stability properties of the LMPC implies that the origin of the closed-loop system is guaranteed to be stable and the LMPC is guaranteed to be feasible for any initial state inside Ω_ρ when the sampling time Δ is sufficiently small. Note that the region Ω_ρ can be explicitly characterized; please, refer to [110] for more discussion on this issue. The main advantage of the LMPC approach with respect to the nonlinear control law $h(x)$ is that optimality considerations can be taken explicitly into account (as well as constraints on the inputs and the states [110]) in the computation of the control actions within an online optimization framework while improving the closed-loop performance of the system. Since the closed-loop stability and feasibility of the

LMPC of Eq. (2.52)–(2.56) are guaranteed by the nonlinear control law $h(x)$, it is unnecessary to use a terminal penalty term in the cost index (see Eq. (2.52) and compare it with Eq. (2.47)) and the length of the horizon N does not affect the stability of the closed-loop system but it affects the closed-loop performance.

2.9 Hybrid Systems

Hybrid systems are characterized by the co-existence of continuous modes of operation along with discrete switches between the distinct modes of operation and arise frequently in the design and analysis of fault-tolerant control systems. The class of hybrid systems of interest to the focus of this book–switched systems–can be described by

$$\dot{x} = f_{i(x,t)}(x) + g_{i(x,t)}(x)u_{i(x,t)}, \tag{2.58}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$ are the continuous variables and $i \in N$ are the discrete variables indexing the mode of operation. The nature of the function $i(x,t)$ and, in particular, its two specific forms $i(x)$ and $i(t)$ result in the so-called state-dependent and time-dependent switching. What is of more interest from a stability analysis and design point of view (both when considering the design of control laws and, in the case of time-dependent switching, the switching signal) is the possibility of infinitely many switches where it becomes crucial to explicitly consider the switched nature of the system in the stability analysis. In particular, when the possibility of infinitely many switches exists, establishing stability in the individual modes of operation is not sufficient [19], and additional conditions on the behavior of the Lyapunov-functions (used to establish stability in the individual modes of operation) during the switching (as well as of sufficient dwell-time [68]) need to be satisfied for the stability of the switched system. For the case of finite switches, the considerations include ensuring stability requirements at the onset of a particular mode are satisfied and, in particular, satisfied for the terminal (last) mode of operation.

2.10 Conclusions

In this chapter, some fundamental results on nonlinear systems analysis and control were briefly reviewed. First, the class of nonlinear systems that will be considered in this book was presented; then the definitions of stability of nonlinear systems were introduced; and following that, techniques for stabilizing nonlinear systems, for example, Lyapunov-based control, feedback linearization, handling constraints, model predictive control and Lyapunov-based model predictive control and stability of hybrid (switched) systems were discussed.

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