# Chapter 2
# The Analytic Approach

**Abstract** Homogeneous coordinates for a projective plane are introduced and extended to more than two dimensions. This leads to the concept of projective collineations. The theorems of Pappus and Desargues are proved using the method of homogeneous coordinates. The relation between analytic Projective geometry of $N$-dimensions and vector space theory is established. The concept of projective collineations is introduced and the classification of collineations in the projective plane described. Plücker's line coordinates are then briefly discussed and their $N$-dimensional generalization to Grassmann algebra is explained.

## 2.1 Homogeneous Coordinates

The analytic approach to projective geometry is based on a system of *homogeneous coordinates*.

To begin thinking about how coordinates can be assigned in projective geometry, it is illuminating to recall the historical origins of projective geometry in the techniques used by artists and architects for representing three-dimensional objects in a picture.

In Fig. 2.1, we can think of the grey plane p as a picture plane and E as the viewing point. Then any point P′ in space is represented on the plane by the point P where the line EP′ intersects the plane. Take E as the origin and set up three coordinate axes through E. These may be axes for an *affine* coordinate system in three-dimensional space, so they need not be orthogonal and the length units along the three axes—given by the line segments connecting the origin to the points (100), (010) and (001)—need not be related. Hence the positions of these three points along the three axes can be chosen arbitrarily.

It will be convenient to avoid the constant repetition of the phrase 'the point with coordinates $(X\ Y\ Z)$'. We shall often simply say 'the point $(X\ Y\ Z)$'. Indeed, no harm is done by regarding a 'point' as a set of numbers.

For the present, we shall take the coordinates to be *real numbers*. We are dealing with the 'two-dimensional projective space over the reals', P(2, R).

Now if P′ is $(X\ Y\ Z)$, its image $P$ in the plane is $(\lambda X\ \lambda Y\ \lambda Z)$ for some $\lambda$. The important thing to realize is that it *does not matter* what $\lambda$ is. So long as it is not zero it can be chosen *arbitrarily*, and $(\lambda X\ \lambda Y\ \lambda Z)$ still serves to identify the point P

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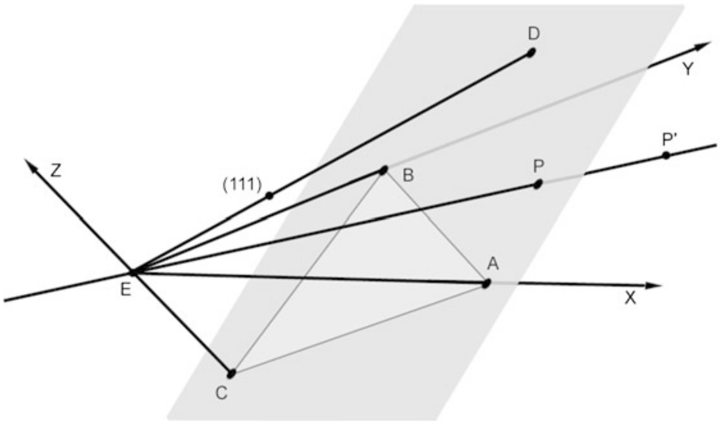


**Fig. 2.1** Projection of three-dimensional affine space onto a plane

in the plane. Only the ratios $X : Y : Z$ are significant, which is why the points of the plane p—*a two*-dimensional space—now have *three* 'homogeneous' coordinates.

Notice there are no coordinate *axes* in the projective plane $p$. Instead, there is a *reference triangle* ABC, given by the intersections of the axes EX, EY and EZ of the affine space with the plane p. The vertices A, B and C of the reference triangle are (100), (010) and (001). The point (111) in the three-dimensional affine space can be chosen arbitrarily by adjusting the scales along the three axes. This does not change the position of the reference triangle ABC, and completes the reference system in the projective plane p by introducing the *unit point* D, with homogeneous coordinates (111). It is obvious from Fig. 2.1 that *any* four points in p, no three of which are collinear, can be chosen to be the reference triangle ABC and the unit point D. That's just a matter of choosing the set of axes through the origin E and the position of the point (111) in the three-dimensional affine space.

The *lines* in the projective plane also have homogeneous coordinates. A plane through $E$ has an equation $lX + mY + nZ = 0$. It intersect the projective plane p in a line L, which acquires the homogeneous coordinates $[l\ m\ n]$. The condition for a point $(X\ Y\ Z)$ and a line $[l\ m\ n]$ in the projective plane to be incident is then simply

$$lX + mY + nZ = 0.$$

Now, if we choose another plane p′ (not through E), with a different position and orientation, any diagram (drawing) in p is projected to a diagram in p′—a distorted version of the drawing in p. But the two diagrams look the same when viewed from E. We regard the two diagrams as *projectively equivalent*. This kind of equivalence has a role similar to *congruence* in Euclidean geometry.

If two points $(X_1\ Y_1\ Z_1)$ and $(X_2\ Y_2\ Z_2)$ are given, the line $[l\ m\ n]$ through them is easily found. The two conditions

$$lX_1 + mY_1 + nZ_1 = 0$$

$$lX_2 + mY_2 + nZ_2 = 0,$$

are satisfied by

$$[l\ m\ n] = [Y_1Z_2 - Z_1Y_2, Z_1X_2 - X_1Z_2, X_1Y_2 - Y_1X_2].$$

It is convenient to write this as

$$[l\ m\ n] = \begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}.$$

The coordinates $[l\ m\ n]$ are then got by 'cross multiplication', as in the cross product of two vectors, or, equivalently, they are the coefficients of $X$, $Y$ and $Z$ when the determinant

$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_3 \end{vmatrix}$$

is expanded:

$$[l\ m\ n] = \left[\begin{vmatrix} Y_1 & Z_1 \\ Y_2 & Z_2 \end{vmatrix}, \begin{vmatrix} Z_1 & X_1 \\ Z_2 & X_2 \end{vmatrix}, \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix}\right].$$

Similarly (and *dually*), the point of intersection of two lines $[l_1\ m_1\ n_1]$ and $[l_2\ m_2\ n_2]$ is given by

$$(X\ Y\ Z) = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = (m_1n_2 - n_1m_2, n_1l_2 - l_1n_2, l_1m_2 - m_1l_2).$$

The sides of the reference triangle, of course, are the lines [100], [010] and [001].

There is one plane through E that does not intersect the plane p. It is the plane through E *parallel* to p. In a manner of speaking we may say that this plane corresponds to the 'line at infinity' in p. This fictitious line has coordinates $[l\ m\ n]$ just like any other line, and under the projection from p to p′ it can correspond to an actual line. So that, to maintain the concept of *projective equivalence*, we need to regard the 'line at infinity' as no different from any other line. The geometry in any plane p not through E is then a true *projective geometry*—satisfying Axioms **1**, **2** and **3**.

Three points $(X_1\ Y_1\ Z_1)$, $(X_2\ Y_2\ Z_2)$ and $(X_3\ Y_3\ Z_3)$ are collinear if and only if

$$lX_1 + mY_1 + nZ_1 = 0$$
$$lX_2 + mY_2 + nZ_2 = 0$$
$$lX_3 + mY_3 + nZ_3 = 0$$

for some triple $[l\ m\ n]$. This condition is the same as the vanishing of the determinant

$$\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix}.$$

That is to say, the rows of this matrix are linearly dependent; the coordinates of the three points satisfy a relation of the form

$$\lambda(X_1\ Y_1\ Z_1) + \mu(X_2\ Y_2\ Z_2) + \nu(X_3\ Y_3\ Z_3) = 0.$$

The *dual* of all this is:

Three lines $[l_1\ m_1\ n_1]$, $[l_2\ m_2\ n_2]$ and $[l_3\ m_3\ n_3]$ are concurrent if and only if

$$Xl_1 + Ym_1 + Zn_1 = 0$$
$$Xl_2 + Ym_2 + Zn_2 = 0$$
$$Xl_3 + Ym_3 + Zn_3 = 0$$

for some triple $(X\ Y\ Z)$. This condition is the same as the vanishing of the determinant

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}.$$

That is to say, the rows of this matrix are linearly dependent; the coordinates of three concurrent lines satisfy a relation of the form

$$\lambda[l_1 \quad m_1 \quad n_1] + \mu[l_2 \quad m_2 \quad n_2] + \nu[l_3 \quad m_3 \quad n_3] = 0.$$

## 2.2 More than Two Dimensions

All this is readily generalizable, in an obvious way, to $N$-dimensional projective spaces, in which the points and *hyperplanes* ($(N-1)$-dimensional subspaces) have $N+1$ homogeneous coordinates. In projective 3-space, for example, a point has four homogeneous coordinates $(X\ Y\ Z\ W)$ and a *plane* has four homogeneous coordinates $[l\ m\ n\ k]$. Any four non-coplanar points can be chosen as the vertices (1000), (0100), (0010) and (0001) of the *reference tetrahedron* and the freedom remains to designate any other point not on a face of the tetrahedron as the unit point (1111). A point $(X\ Y\ Z\ W)$ lies in a plane $[l\ m\ n\ k]$ if and only if

$$Xl + Ym + Zn + Wk = 0.$$

The plane $[l\ m\ n\ k]$ through three non-collinear points is given by the coefficients of $X$, $Y$, $Z$ and $W$ in the expansion of the determinant

$$\begin{vmatrix} X & Y & Z & W \\ X_1 & Y_1 & Z_1 & W_1 \\ X_2 & Y_2 & Z_2 & W_2 \\ X_3 & Y_3 & Z_3 & W_3 \end{vmatrix}.$$

In particular, the plane faces of the reference tetrahedron are [1000], [0100], [0010] and [0001].

Similarly, the common point $(X\ Y\ Z\ W)$ of three planes that do not contain a common line is given by the coefficients $l$, $m$, $n$, and $k$ in the expansion of

$$\begin{vmatrix} l & m & n & k \\ l_1 & m_1 & n_1 & k_1 \\ l_2 & m_2 & n_2 & k_2 \\ l_3 & m_3 & n_3 & k_3 \end{vmatrix}.$$

Four points are coplanar if

$$\begin{vmatrix} X_1 & Y_1 & Z_1 & W_1 \\ X_2 & Y_2 & Z_2 & W_2 \\ X_3 & Y_3 & Z_3 & W_3 \\ X_4 & Y_4 & Z_4 & W_4 \end{vmatrix} = 0$$

and four planes are concurrent if

$$\begin{vmatrix} l_1 & m_1 & n_1 & k_1 \\ l_2 & m_2 & n_2 & k_2 \\ l_3 & m_3 & n_3 & k_3 \\ l_4 & m_4 & n_4 & k_4 \end{vmatrix} = 0.$$

## 2.3 Collineations

Let us write the coordinates $(X\ Y\ Z)$ of a point P in a projective plane as a *column* (i.e., a $3 \times 1$ matrix) P and the coordinates $[l\ m\ n]$ of a line $L$ as a row (i.e., a $1 \times 3$ matrix) L. The condition for the point P to lie on the line L is then simply $\mathrm{LP} = 0$. If T is any non-singular $3 \times 3$ matrix, we can apply the transformation $\mathrm{P} \to \mathrm{P}' = \mathrm{TP}$, $\mathrm{L} \to \mathrm{L}' = \mathrm{LT}^{-1}$. Since $\mathrm{L}'\mathrm{P}' = 0$ the transformation, applied to all points and lines of the projective plane, is a mapping of the plane onto itself, or a mapping of one plane onto another, that *preserves incidences*. It is a *projective transformation*, or *collineation*. Any figure in the plane and its image under a collineation are *projectively equivalent*. Since an overall factor $\lambda$ is irrelevant for homogeneous coordinates, we may take the matrices T to be unimodular, $|\mathrm{T}| = 1$. Then we can say that all the collineations of a real projective plane constitute the special linear group $\mathrm{SL}(3,\ \mathrm{R})$ of all real unimodular matrices.

In general, a collineation T will leave some points unchanged. These are given by the *eigenvectors* of the matrix T, which are the points whose coordinates satisfy

$$\mathrm{TP} = \lambda \mathrm{P}$$

for some $\lambda$. These are the *fixed points* of the collineation. In the general case they will be three non-collinear points. A line through two fixed points is an *invariant* *line*. If two of the *eigenvalues* λ happen to be equal, then any linear combination of the two corresponding eigenvectors is also an eigenvector, and we get a line of fixed points—a *fixed line*. There are several cases to consider. Collineations may be classified according to their pattern of fixed points and invariant lines. By appropriate choice of the vertices and edges of the reference triangle, relating them to the fixed points and invariant lines, the matrix T takes on simple *canonical forms*:

(a) The general case: the points (100), (010) and (001) are fixed points and the lines [100]. [010] and [001] are invariant lines;

$$\mathrm{T} = \begin{pmatrix} \mu & & \\ & \nu & \\ & & \rho \end{pmatrix}, \qquad \mathrm{T}^{-1} = \begin{pmatrix} \mu^{-1} & & \\ & \nu^{-1} & \\ & & \rho^{-1} \end{pmatrix}.$$

(b) A fixed point (001) and a fixed line [001];

$$\mathrm{T} = \begin{pmatrix} \mu & & \\ & \mu & \\ & & \rho \end{pmatrix}, \qquad \mathrm{T}^{-1} = \begin{pmatrix} \mu^{-1} & & \\ & \mu^{-1} & \\ & & \rho^{-1} \end{pmatrix}.$$

(c) A fixed line [010] and an invariant line [001];

$$\mathrm{T} = \begin{pmatrix} \mu & & \\ 1 & \mu & \\ & & \rho \end{pmatrix}, \qquad \mathrm{T}^{-1} = \begin{pmatrix} \mu^{-1} & -1 & \\ & \mu^{-1} & \\ & & \rho^{-1} \end{pmatrix}.$$

(d) A fixed line [100] and all lines through the point (001) on it invariant;

$$\mathrm{T} = \begin{pmatrix} \mu & & \\ 1 & \mu & \\ & & \mu \end{pmatrix}, \qquad \mathrm{T}^{-1} = \begin{pmatrix} \mu^{-1} & -1 & \\ & \mu^{-1} & \\ & & \mu^{-1} \end{pmatrix}.$$

(e) A fixed point (001) and an invariant line [100] through it;

$$\mathrm{T} = \begin{pmatrix} \mu & & \\ 1 & \mu & \\ & 1 & \mu \end{pmatrix}, \qquad \mathrm{T}^{-1} = \begin{pmatrix} \mu^{-1} & -1 & \\ & \mu^{-1} & -1 \\ & & \rho^{-1} \end{pmatrix}.$$

(f) Finally, we have the trivial case of the collineation that changes nothing: all points and lines are fixed and T is just (a multiple of) the unit matrix.

In Fig. 2.2 fixed points and lines are indicated in black, invariant lines in grey.

Obviously, all this is readily generalizable to collineations in $N$-dimensional projective space. But it gets complicated!

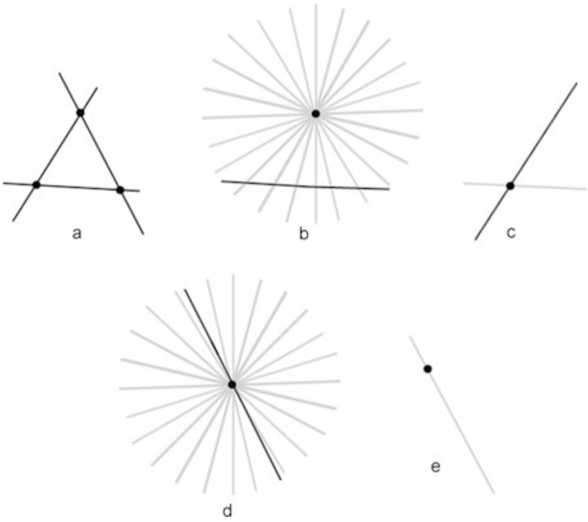


**Fig. 2.2** Classification of plane collineations

## 2.4 A Proof of Pappus's Theorem

We are now ready to look at a proof of Pappus's theorem, using the methods of homogeneous coordinates. There are several ways of going about it. In Fig. 2.3 there are two sets of three collinear points ABC and DEF. Let these two lines be respectively the sides [001] and [100] of the reference triangle (i.e. the lines $Z = 0$ and $X = 0$). The coordinates of the six points then have the form

$$\begin{aligned} \mathrm{A} &= (1 \quad \alpha_1 \quad 0), & \mathrm{D} &= (0 \quad \beta_1 \quad 1) \\ \mathrm{B} &= (1 \quad \alpha_2 \quad 0), & \mathrm{E} &= (0 \quad \beta_2 \quad 1) \\ \mathrm{C} &= (1 \quad \alpha_3 \quad 0), & \mathrm{F} &= (0 \quad \beta_3 \quad 1). \end{aligned}$$

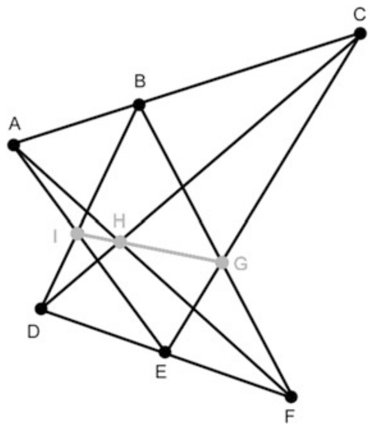


**Fig. 2.3** Pappus's theorem

The coordinates of the lines BF and CE are then

$$\begin{vmatrix} 1 & \alpha_2 & 0 \\ 0 & \beta_3 & 1 \end{vmatrix} = [\alpha_2 \quad -1 \quad \beta_3] \quad \text{and} \quad \begin{vmatrix} 1 & \alpha_3 & 0 \\ 0 & \beta_2 & 1 \end{vmatrix} = [\alpha_3 \quad -1 \quad \beta_2],$$

so that

$$\mathrm{G} = \begin{vmatrix} \alpha_2 & -1 & \beta_3 \\ \alpha_3 & -1 & \beta_2 \end{vmatrix} = (\beta_3 - \beta_2,\ \beta_3\alpha_3 - \alpha_2\beta_2,\ \alpha_3 - \beta_2).$$

Similarly,

$$\mathrm{H} = (\beta_1 - \beta_3,\ \beta_1\alpha_1 - \alpha_3\beta_3,\ \alpha_1 - \alpha_3) \quad \text{and} \quad I = (\beta_2 - \beta_1,\ \beta_2\alpha_2 - \alpha_1\beta_1,\ \alpha_2 - \alpha_1).$$

Therefore, G, H and I are collinear, because

$$\begin{vmatrix} \beta_3 - \beta_2 & \beta_3\alpha_3 - \alpha_2\beta_2 & \alpha_3 - \alpha_2 \\ \beta_1 - \beta_3 & \beta_1\alpha_1 - \alpha_3\beta_3 & \alpha_1 - \alpha_3 \\ \beta_2 - \beta_1 & \beta_2\alpha_2 - \alpha_1\beta_1 & \alpha_2 - \alpha_1 \end{vmatrix} = 0$$

(the sum of the three rows vanishes).

Notice something peculiar and subtle here. We have tacitly assumed that the numbers used as coordinates *commute*, $\alpha\beta = \beta\alpha$, otherwise the three rows of this determinant would not sum to zero and Pappus's theorem would fail. That is not a problem because we've been taking it for granted that the coordinates are real numbers. The geometry is the 'projective geometry of two dimensions over the reals', P(2, R). But geometries can be constructed *that satisfy the projective geometry axioms*, using coordinates from other number fields, such as complex numbers or finite 'Galois' fields (e.g. numbers modulo $p$). We then have the projective geometries $\mathrm{P}(N, \mathrm{C})$ and $\mathrm{P}(N, \mathrm{GF}(p))$. Commutation is still valid for these fields, so Pappus's theorem remains valid. However, for the *quaternions*, for example, commutation fails, so that Pappus's theorem would not be true in $\mathrm{P}(N, \mathrm{Q})$.

## 2.5 Proofs of Desargues' Theorem

Proving Desargues' theorem by means of homogeneous coordinates is very simple. In Fig. 2.4, ABC and DEF are two triangles perspective from a point V. Choose ABC as the reference triangle and V as the unit point. Then D is on VA, so its coordinates are a linear combination of (111) and (100); since homogeneous coordinates can have an arbitrary overall factor we can take them to be of the form $(\alpha\ 1\ 1)$. Similarly, E and F can be taken to be $(1\ \beta\ 1)$ and $(1\ 1\ \gamma)$. Now BC is [100] and EF is

$$\begin{vmatrix} 1 & 1 & \gamma \\ 1 & \beta & 1 \end{vmatrix} = [1 - \beta\gamma \quad \gamma - 1 \quad \beta - 1].$$

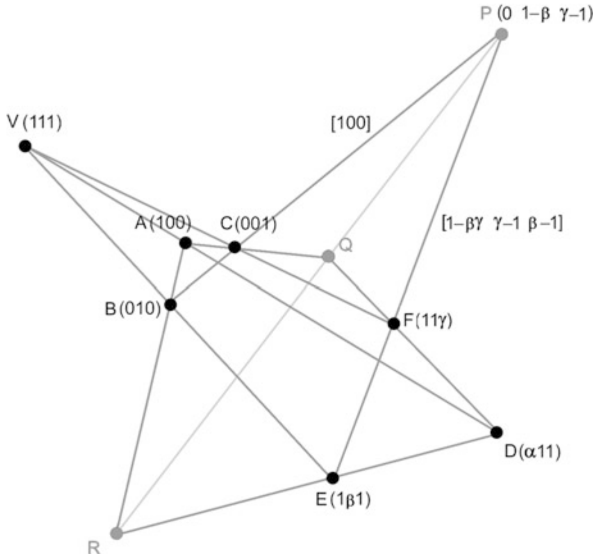


**Fig. 2.4** Proof of Desargues' theorem

Then P is

$$\begin{vmatrix} 1 & 0 & 0 \\ 1-\beta\gamma & \gamma-1 & \beta-1 \end{vmatrix} = [0 \quad 1-\beta \quad \gamma-1].$$

Similarly, Q is $[\alpha\ -1\ 0\ 1-\gamma]$ and R is $[1-\alpha\ \beta-1\ 0]$. Therefore P, Q and R are collinear because

$$\begin{vmatrix} 0 & 1-\beta & \gamma-1 \\ \alpha-1 & 0 & 1-\gamma \\ 1-\alpha & \beta-1 & 0 \end{vmatrix} = 0.$$

A very elegant proof of the theorem, that does not depend on a particular choice of reference system, is the following. If ABC and DEF are two triangles perspective from V, the homogeneous coordinates of these seven points can be adjusted by multiplying them by arbitrary overall factors so that

$$\mathrm{A+D+V=0}$$
$$\mathrm{B+E+V=0}$$
$$\mathrm{C+F+V=0.}$$

Subtracting these equations in pairs gives

$$\mathrm{A+D=B+E=C+F.}$$

Therefore

$$\mathrm{B-C=F-E}, \qquad \mathrm{C-A=D-F}, \qquad \mathrm{A-B=E-D}.$$

These are the three points P, Q and R in the figure. They are collinear because

$$(\mathrm{B}-\mathrm{C})+(\mathrm{C}-\mathrm{A})+(\mathrm{A}-\mathrm{B})=0.$$

## 2.6 Affine Coordinates

We have already observed that an affine plane can be regarded as a specialization of a projective plane in which a line has been singled out, regarded as special, and called the 'line at infinity'. This allows the reintroduction of the affine concept of 'parallel lines'—they are lines that intersect 'at infinity'. Three-dimensional affine space can similarly be treated as a specialization of a projective three-space with a chosen 'plane at infinity', and so on, for higher dimensions.

An affine coordinate system for an affine space derived in this way can be readily obtained from the homogeneous coordinates of the projective space. Take for example the projective plane P(2, R) (the generalization to P($N$, F) is obvious). Let (X Y Z) be homogeneous coordinates for the points of the projective plane. Choose the coordinate system so that the 'line at infinity' is $Z=0$, that is, [0 0 1]. Then all the finite points in the affine plane have $Z\neq 0$, and the homogeneous coordinates for the affine plane can be taken to be

$$(\mathrm{X},\ \mathrm{Y})=(X/Z,\ Y/Z)$$

the line [$l$ $m$ $n$] becomes the line with equation

$$l\mathrm{X}+m\mathrm{Y}+n=0.$$

It is often convenient and fruitful to employ the converse of this procedure, making use of homogeneous coordinates to deal with problems in affine geometry. (An example will be demonstrated in Sect. 3.4.)

## 2.7 Subspaces of a Vector Space

In the above construction of homogeneous coordinates, described in Sect. 2.1, *any* plane p that does not pass through E can be used—all the planes that do not contain E are projectively equivalent. They are related by collineations. It follows that we can remove the plane p from Fig. 2.1 and we still have a plane projective geometry if we simply *call* the lines through E 'points' and the planes through E 'lines'. The axioms of plane projective geometry are valid for these 'points' and 'lines'. In fact, we are dealing with the three-dimensional *vector space* of position vectors, with $E$ as origin. The 'points' are its one-dimensional subspaces and the 'lines' are its two-dimensional subspaces. We can therefore say that a one-dimensional *vector* space is a zero-dimensional *projective* space (a 'point') and that a two-dimensional vector space is a one-dimensional projective space (a 'line'). This can be immediately generalized to a definition of higher-dimensional projective geometries:

*The set of all vector subspaces of an* $(N+1)$*-dimensional vector space is an* $N$*-dimensional projective space.*

The *union* of two vector subspaces is the smallest subspace that contains them both and their *intersection* is the largest subspace that is contained in both. The union and intersection of two vector subspaces then correspond, respectively, to the join and intersection of the corresponding subspaces of a projective space. It is then a simple matter to deduce that *the axioms for* $N$*-dimensional projective space that we presented in Sect. 1.10 are all simple consequences of the properties of a vector space.*

We need now to review briefly some of the basic ideas of vector space theory.

A *vector space* $V$ over the field $R$ of real numbers is the set of all vectors of the form

$$x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 + \cdots + x^{N+1}\mathbf{e}_{N+1} = x^i\mathbf{e}_i$$

where the $x^i$ are real numbers and the $\mathbf{e}_i$ are $(N+1)$ vectors, which are *linearly independent* but otherwise arbitrary. (By linear independence we mean that the above expression can be zero only if all the $x^i$ are zero.) The set of vectors $\mathbf{e}_i$ is a *basis* for $V$. The indices $i$ $(i = 1, 2, \ldots, N+1)$ in the shorthand expression $x^i\mathbf{e}_i$ have been written as superscripts on the $x^i$ so that we can use the *summation convention*: if an index appears twice in an expression, once as a subscript and once as a superscript, a summation is implied. (This convention, introduced by Einstein, allows us to omit summation signs $\sum$ and simplifies the look of many expressions in mathematics and theoretical physics.)

A one-dimensional subspace of $V$ is the set of all vectors of the form $\lambda\mathbf{v}$, for *any* real number $\lambda$ and some fixed vector $\mathbf{v}$. This is a *point* in the $N$-dimensional projective space $\mathrm{P}(N, \mathrm{R})$. The components $x^i$ of $\mathbf{v}$ are its homogeneous coordinates, as also are $\lambda x^i$ for any non-zero real number $\lambda$. Similarly, a line in $\mathrm{P}(N, \mathrm{R})$ is given by all linear combinations $\lambda\mathbf{v} + \mu\mathbf{w}$ of two linearly independent fixed vectors ('there is a unique line through any two distinct points'...), a plane is given by all linear combinations of three linearly independent fixed vectors, and so on.

In the projective plane $\mathrm{P}(2, \mathrm{R})$, we denoted the homogeneous coordinates of points by $(X\ Y\ Z)$ and introduced homogeneous coordinates $[l\ m\ n]$ of a line in such a way that the condition for the point to lie on the line is $lX + mY + nZ = 0$. This readily generalizes: in $\mathrm{P}(N, \mathrm{R})$, we introduce homogeneous coordinates $[\omega_1,\ \omega_2, \ldots, \omega_{N+1}]$ for a hyperplane ($(N-1)$-dimensional subspace), so that the condition for the point $(x^1,\ x^2, \ldots, x^{N+1})$ to lie in the hyperplane is $\omega_i x^i = 0$. In terms of the vector space V, this suggests that the hyperplanes can be thought of as *dual* vectors. We can associate, with any vector space V, a *dual* vector space V*. A vector $\mathbf{v}$ in V and a vector $\boldsymbol{\omega}$ in V* can be combined to give a scalar (in this case, a real number) $\boldsymbol{\omega}\mathbf{v}$, and this product preserves linearity, in the sense that $(\lambda\boldsymbol{\omega})\mathbf{v} = \boldsymbol{\omega}(\lambda\mathbf{v}) = \lambda(\boldsymbol{\omega}\mathbf{v})$, $(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)\mathbf{v} = \boldsymbol{\omega}_1\mathbf{v} + \boldsymbol{\omega}_2\mathbf{v}$ and $\boldsymbol{\omega}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v}_1 + \mathbf{v}_2$, where $\lambda$ is any real number, $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are any two vectors in V* and $\mathbf{v}_1 + \mathbf{v}_2$ are any two vectors in V. A basis $\mathbf{e}^i$ for V* can be chosen so that

$$\mathbf{e}^i\mathbf{e}_j = \delta^i_j$$

($\delta^i_j$ is defined to be 1 if $i = j$, otherwise 0). Then, if $\boldsymbol{\omega} = \omega_i \mathbf{e}^i$ and $\mathbf{v} = x^i \mathbf{e}_i$

$$\boldsymbol{\omega}\mathbf{v} = \omega_i x^i .$$

It follows that, just as the points in P($N$, R) can be identified with one-dimensional subspaces of V, the hyperplanes of P($N$, R) can be identified with one-dimensional subspaces of V*, and that the condition for a point $\mathbf{v}$ to lie in a hyperplane $\boldsymbol{\omega}$ is $\boldsymbol{\omega}\mathbf{v} = 0$.

## 2.8 Plücker Coordinates

In a three-dimensional projective space (P(3, R), say), points are assigned four homogeneous coordinates $(x^1\ x^2\ x^3\ x^4)$, planes are assigned four homogeneous coordinates $[\pi_1\ \pi_2\ \pi_3\ \pi_4]$, and the condition for a point $x^i$ to lie in a plane $\pi_i$ is $x^i \pi_i = 0$ (where a summation is implied over the repeated index $i$). In this notation, coordinates of points are indexed with a superscript and coordinates of planes are indexed with a subscript. The significance of this convention is that it indicates the nature of the change of coordinates under the action of a collineation (given by a $4 \times 4$ matrix T):

$$x^i \to \mathrm{T}^i_j x^j, \qquad \pi_i \to \left(\mathrm{T}^{-1}\right)^j_i \pi_j .$$

The *lines* of a three-dimensional space can be assigned a set of *six* homogeneous coordinates. If $x^i$ and $y^i$ are any two distinct points on a line P, the *Plücker coordinates* of P are

$$\mathrm{P}^{ij} = -\mathrm{P}^{ji} = x^i y^j - x^j y^i .$$

Observe that it does not matter which two points on $P$ are chosen. With a different choice we get, apart from an irrelevant overall factor, the same set of Plücker coordinates. Defining $x'^i = \alpha x^i + \beta y^i$, $y'^i = \gamma x^i + \delta y^i$, then $\mathrm{P}'^{ij} = (\alpha\delta - \beta\gamma)\mathrm{P}^{ij}$.

It is not difficult to see that the condition for a line P to lie in a plane $\pi$ is

$$\mathrm{P}^{ij} \pi_j = 0$$

and that if this condition is not satisfied, $\mathrm{P}^{ij}\pi_j$ gives the coordinates of the point of intersection of the line P with the plane $\pi$.

The $\mathrm{P}^{ij}$ defined in this way satisfy

$$\mathrm{P}^{23}\mathrm{P}^{14} + \mathrm{P}^{31}\mathrm{P}^{24} + \mathrm{P}^{12}\mathrm{P}^{34} = 0.$$

Moreover, this is a necessary and sufficient condition for a skewsymmetric $\mathrm{P}^{ij}$ to be a set of Plücker coordinates of a line—that is, for $\mathrm{P}^{ij}$ to have the form $x^i y^j - x^j y^i$. To see this, take two planes $\pi$ and $\mu$ such that $x^i = \mathrm{P}^{ij}\pi_j$ and $y^i = \mathrm{P}^{ij}\mu_j$ are nonzero. We assume, of course, that the $\mathrm{P}^{ij}$ are not all zero, so we lose no generality by taking $P^{34} = 1$ and choosing the reference system so that $\pi$ and $\mu$ are the reference planes [0010] and [0001]. Then

$$\begin{pmatrix} x^1 & x^2 & x^3 & x^4 \end{pmatrix} = \begin{pmatrix} P^{13} & P^{23} & 0 & -1 \end{pmatrix} \quad \text{and}$$
$$\begin{pmatrix} y^1 & y^2 & y^3 & y^4 \end{pmatrix} = \begin{pmatrix} P^{14} & P^{24} & 1 & 0 \end{pmatrix}.$$

Hence $Q^{ij} = x^i y^j - x^j y^i$ is given by the skewsymmetric matrix

$$\begin{pmatrix} 0 & P^{13}P^{24} - P^{23}P^{14} & P^{13} & P^{14} \\ P^{23}P^{14} - P^{13}P^{24} & 0 & P^{23} & P^{24} \\ P^{31} & P^{32} & 0 & P^{34} \\ P^{41} & P^{42} & P^{43} & 0 \end{pmatrix}$$

applying $P^{ij} = -P^{ji}$ and the condition $P^{23}P^{14} + P^{31}P^{24} + P^{12}P^{34} = 0$ then gives $Q^{ij} = P^{ij}$.

The same line P can be given *dual* Plücker coordinates, by taking any two distinct *planes* $\pi_i$ and $\mu_i$ whose intersection is the line P:

$$P_{ij} = \pi_i \mu_j - \pi_j \mu_i.$$

These two kinds of coordinate for a line are related through

$$P_{23} = P^{14}, \qquad P_{31} = P^{24}, \qquad P_{12} = P^{34},$$
$$P_{14} = P^{23}, \qquad P_{24} = P^{31}, \qquad P_{34} = P^{12}.$$

This is more briefly expressed if we introduce the *alternating symbol*

$\varepsilon^{ijkl} = \varepsilon_{ijkl} = \pm 1$ according as $ijkl$ is an even or an odd permutation of 1234, otherwise zero.

Then

$$P_{ij} = \left(\frac{1}{2}\right)\varepsilon_{ijkl}P^{kl}, \qquad P^{ij} = \left(\frac{1}{2}\right)\varepsilon^{ijkl}P_{kl}.$$

The condition given above for a skewsymmetric $P^{ij}$ to be Plücker coordinates of a line is then $P^{ij}P_{ij} = 0$. The condition for a point $x$ to lie on a line P is $P_{ij}x^j = 0$, the condition for a line P to be contained in a plane $\pi$ is $P^{ij}\pi_j = 0$. If a plane $\lambda_i$ does not contain the line P ($P^{ij}\lambda_j \neq 0$), then $x^i = P^{ij}\lambda_j$ is the point of intersection of the line and the plane. The condition $P^{23}P^{14} + P^{31}P^{24} + P^{12}P^{34} = 0$ for a given skewsymmetric $P^{ij}$ to be a set of Plucker coordinates of a line is $P^{ij}P_{ij} = 0$. The condition for two lines P and Q to intersect is $P^{ij}Q_{ij} = 0$.

## 2.9 Grassmann Coordinates

Two questions arise. When deriving the projective geometry P(N, R) from an $(N+1)$-dimensional vector space V, we have introduced two quite different definitions of a hyperplane: as an $N$-dimensional subspace of V, or as a one-dimensional subspace of V*. How are these apparently different definitions to be reconciled? And how can one assign homogeneous coordinates for lines, planes, etc., in a projective space of more than two dimensions? The generalization of Plücker's coordinates for lines in 3-space provides the answer to these questions.

The $n$-dimensional subspace determined by $n+1$ general points of an $N$-dimensional projective space has a set of $\binom{N+1}{n}$ homogeneous coordinates.

To investigate this general case we introduce the *alternating symbols*

$\varepsilon_{i_1 i_2 i_3 \ldots i_{N+1}} = \varepsilon^{i_1 i_2 i_3 \ldots i_{N+1}} = \pm 1$ according as $i_1 i_2 i_3 \ldots i_{N+1}$ is an even or an odd permutation of $123, \ldots, N+1$, otherwise zero.

Now, if $x^i$ and $y^i$ $(i = 1, 2, \ldots, N+1)$ are the coordinates of two distinct points, the $\binom{N+1}{2}$ numbers

$$\omega^{ij} = x^{[i} y^{j]} = \frac{1}{2}\left(x^i y^j - x^j y^i\right)$$

are not all zero if the points are distinct, and serve to specify the unique line through the two points. We can adopt the $\omega^{ij}$ as the homogeneous coordinates of the line through the two points. Similarly, given three non-collinear points with coordinate $x^i$, $y^i$ and $z^i$, we can take as homogeneous coordinates of the plane through these three points the $\binom{N+1}{3}$ numbers

$$\omega^{ijk} = x^{[i} y^j z^{k]} = (1/6)\left(x^i y^j z^k + x^j y^k z^i + x^k y^i z^j - x^i y^k z^j - x^k y^j z^i - x^j y^i z^k\right).$$

And so on. The alternating symbols allow us to construct alternative sets of homogeneous coordinates, the *dual* coordinates. Dual coordinates of points, lines, planes, etc., are (using the summation convention) given by

$$\begin{aligned} x_{i_1 i_2 i_3 \ldots i_N} &= \varepsilon_{i_1 i_2 i_3 \ldots i_N j} x^j \\ \omega_{i_1 i_2 i_3 \ldots i_{N-1}} &= (1/2)\varepsilon_{i_1 i_2 i_3 \ldots i_{N-1} jk} \omega^{jk} \\ \omega_{i_1 i_2 i_3 \ldots i_{N-2}} &= (1/6)\varepsilon_{i_1 i_2 i_3 \ldots i_{N-2} jkl} \omega^{jkl} \end{aligned}$$

and so on. The properties of the alternating symbol imply

$$\begin{aligned} x^j &= (1/N!) x_{i_1 i_2 i_3 \ldots i_N} \varepsilon^{i_1 i_2 i_3 \ldots i_N j} \\ \omega^{jk} &= \left(1/(N-1)!\right) \omega_{i_1 i_2 i_3 \ldots i_{N-1}} \varepsilon^{i_1 i_2 i_3 \ldots i_{N-1} jk} \\ \omega^{jk} &= \left(1/(N-2)!\right) \omega_{i_1 i_2 i_3 \ldots i_{N-2}} \varepsilon^{i_1 i_2 i_3 \ldots i_{N-2} jkl} \end{aligned}$$

*et cetera*. These expressions are simpler than they look. Taking $N = 4$ as an example, they are just $x_{1234} = x^5$, $\omega_{123} = \omega^{45}$ and $\omega_{12} = \omega^{345}$ and the expressions obtained from these by cyclically permuting 12345. For $N = 3$ we have the Plücker coordinates of a line.

(Readers who are familiar with the algebra of ‘exterior forms’ will recognize all this. A ‘one-form’ is a dual vector $\omega_i \mathbf{e}^i$. A basis for 2-forms $\omega_{ij}\mathbf{e}^{ij}$ is a set of symbolic quantities $\mathbf{e}^{ij}$ $(= -\mathbf{e}^{ji})$, usually written as $\mathbf{e}^i \wedge \mathbf{e}^j$. For 3-forms $\omega_{ijk}\mathbf{e}^{ijk}$ we have $\mathbf{e}^{ijk}$ $(= \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k)$, completely skewsymmetric in the index set $ijk$. *Et cetera*. Dualizing with the alternating symbol corresponds to the Hodge $*$ operation.)

We shall have no further use for Grassman’s algebraic ideas in what follows—they have been introduced here simply to hint at how the principle of duality in $N$-dimensional projective geometry can be expressed algebraically in terms of homogeneous coordinates. In particular, we see now how a hyperplane in $\mathrm{P}(N, \mathrm{R})$ can be regarded without contradiction either as an $N$-dimensional subspace of a vector space V or as a one-dimensional subspace of V*.

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