Chapter 2
Kinematic Synthesis

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2.1 Kinematics Equations of a Serial Chain

To study the relative movement at each joint of a spatial linkage,[1] we introduce three 4×4 matrices that we call *coordinate screw displacements*. Each of these matrices defines a translation along one coordinate axis combined with a rotation about that axis. This is the movement allowed by an RP open chain that has the axis of the revolute joint parallel to the guide of the slider. This assembly is called a cylindric joint, or C-joint, because trajectories traced by points in the moving body lie on cylinders about the joint axis.

Let S_1 be the axis of a cylindric joint that connects a link $\mathsf{S}_1\mathsf{S}_2$ to ground. Locate the fixed frame F so that its z-axis is along S_1 and its origin is the point $\mathbf{p}$. Attach the link frame B so that its z-axis is along S_1 and its x-axis is along the common normal N from S_1 and S_2. The displacement of B relative to F consists of a slide d and rotation θ along and around the z-axis of F. Combine the rotation matrix and translation vector for this displacement to form the 4×4 *homogeneous transform*, given by

$$\begin{Bmatrix} X \\ Y \\ Z \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}, \tag{2.1}$$

or

$$\mathbf{X} = \big[Z(\theta, d)\big]\mathbf{x}. \tag{2.2}$$

[1]This chapter combines excerpts from J.M. McCarthy and G.S. Soh, *Geometric Design of Linkages*, Springer, 2010, and is based on the Ph.D. research by Alba Perez and Haijun Su. Reproduced by kind permission of Springer © 2012.

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This defines the transformation of coordinates $\mathbf{x}$ in B to $\mathbf{X}$ in F that represents the movement allowed by a cylindric joint. Notice that we do not distinguish between point coordinate vectors with and without the fourth component of 1. In what follows the difference should be clear from the context of our calculations.

The transform $[Z(\theta, d)]$ is the *coordinate screw displacement* about the z-axis. We can define similar screw displacements $[X(\cdot,\cdot)]$ and $[Y(\cdot,\cdot)]$ about the x- and y-axes,

$$\begin{aligned}\big[X(\theta,d)\big] &= \begin{bmatrix} 1 & 0 & 0 & d \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ \big[Y(\theta,d)\big] &= \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & d \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.\end{aligned} \tag{2.3}$$

We use these coordinate screw displacements to formulate the kinematics equations for spatial linkages.

It is useful to note that the inverse of a coordinate screw displacement can be obtained by negating its parameters. For example,

$$\big[Z(\theta,d)^{-1}\big] = \big[Z(-\theta,-d)\big] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.4}$$

Notice that $[Z(\theta,d)^{-1}]$ is not the transpose of $[Z(\theta,d)]$.

2.1.1 The Denavit-Hartenberg Convention

A spatial open chain can be viewed as a sequence of joint axes S_i connected by common normal lines, Fig. 2.1. Let A_{ij} be the common normal from joint axis S_i to S_j. The Denavit-Hartenberg convention attaches the link frame B_i such that its z-axis is directed along the axis S_i and its x-axis is directed along the common normal A_{ij}. This convention leaves undefined the initial and final coordinate frames F and M. These frames usually have their z-axes aligned with the first and last axes of the chain. However, their x-axes can be assigned any convenient direction.

This assignment of standard frames B_i allows us to define the 4×4 transformation $[D]$ that locates the end-link of a spatial open chain as the sequence of transformations

$$[D] = \big[Z(\theta_1,d_1)\big]\big[X(\alpha_{12},a_{12})\big]\big[Z(\theta_2,d_2)\big]\cdots\big[X(\alpha_{n-1,n},a_{n-1,n})\big]\big[Z(\theta_n,d_n)\big], \tag{2.5}$$

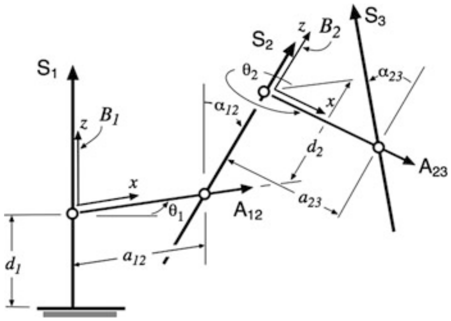


Fig. 2.1 Joint axes S_1, S_2, and S_3 and the link frames B_1 and B_2

where α_{ij} and a_{ij} are the *twist angle* and *offset* between the axes S_i and S_j. This matrix equation defines the *kinematics equations* of the open chain.

The 4×4 transform $[T_j] = [X(\alpha_{ij}, a_{ij})][Z(\theta_j, d_j)]$ is the transformation from frame B_i to B_j. Equation (2.5) is often written as

$$[D] = [T_1][T_2]\cdots[T_n]. \tag{2.6}$$

Notice that $[T_1] = [Z(\theta_1, d_1)]$.

2.2 The Product of Exponentials Form of the Kinematics Equations

The synthesis equations for a spatial serial chain are obtained from the matrix exponential form of its kinematics equations. This form of the kinematics equations replaces the Denavit-Hartenberg parameters with the coordinates of the n joint axes, S_i, $i = 1, \ldots n$. It is the coordinates of these axes that are the unknowns of the design problem.

Consider a displacement defined such that the moving body rotates the angle ϕ and slides the distance k around and along the screw axis $\mathsf{S} = (\mathbf{S}, \mathbf{C} \times \mathbf{S})$. Let $\mu = k/\phi$, then we can introduce the screw $\mathsf{J} = (\mathbf{S}, \mathbf{V}) = (\mathbf{S}, \mathbf{C} \times \mathbf{S} + \mu\mathbf{S})$, where μ is called the *pitch* of the screw. The components of J define the 4×4 twist matrix,

$$J = \begin{bmatrix} 0 & -s_z & s_y & v_x \\ s_z & 0 & -s_x & v_y \\ -s_y & s_x & 0 & v_z \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.7}$$

and we find that the 4×4 homogeneous transform representing a rotation ϕ and a translation k about and along an axis S, $[T(\phi, k, \mathsf{S})]$, is defined as the matrix exponential

$$\big[T(\phi, k, \mathsf{S})\big] = e^{\phi J}. \tag{2.8}$$

The matrix exponential takes a simple form for the matrices $[Z(\theta_i, d_i)]$ and $[X(\alpha_{i,i+1}, a_{i,i+1})]$. The screws defined for these two transformations are $\mathsf{K} = (\mathbf{k}, \nu\mathbf{k})$ and $\mathsf{I} = (\boldsymbol{\imath}, \lambda\boldsymbol{\imath})$, where $\nu = d_i/\theta_i$ and $\lambda = a_{i,i+1}/\alpha_{i,i+1}$ are their respective pitches. Thus, we have

$$\big[Z(\theta_i, d_i)\big] = e^{\theta_i K} \quad \text{and} \quad \big[X(\alpha_{i,i+1}, a_{i,i+1})\big] = e^{\alpha_{i,i+1} I}, \tag{2.9}$$

and the kinematics equations become

$$[D] = [G]e^{\theta_1 K} e^{\alpha_{12} I} e^{\theta_2 K} \dots e^{\alpha_{n-1,n} I} e^{\theta_n K}[H]. \tag{2.10}$$

This is one way to write the product of exponentials form of the kinematics equations. In the next section, we modify this slightly for use as our design equations.

2.2.1 Relative Displacements

If we choose a reference position for the end-effector, denoted by $[D_0]$, then the associated joint angle vector $\boldsymbol{\theta}_0$ can be determined, as well as the world frame coordinates of each of the joint axes. The transformation $[D_0]$ is often selected to be the configuration in which the joint parameters are zero and is called the *zero reference position* by Gupta (1986) [1].

The displacement of the serial chain relative to this reference configuration is defined by $[D(\Delta\boldsymbol{\theta})] = [D(\boldsymbol{\theta})][D(\boldsymbol{\theta}_0)]^{-1}$ and yields a convenient formulation for the kinematics equations. Assume that $[D_0]$ is a general position of the end-effector defined by joint parameters $\boldsymbol{\theta}_0$, so $\Delta\boldsymbol{\theta} = \boldsymbol{\theta} - \boldsymbol{\theta}_0$. Then, using the usual kinematics equations, we have

$$\begin{aligned}\big[D(\Delta\boldsymbol{\theta})\big] = \big([G]\big[Z(\theta_1, d_1)\big]\dots\big[Z(\theta_n, d_n)\big][H]\big)\\ \big([G]\big[Z(\theta_{10}, d_{10})\big]\dots\big[Z(\theta_{n0}, d_{n0})\big][H]\big)^{-1}.\end{aligned} \tag{2.11}$$

In order to expand this equation, we introduce the partial displacements

$$[A_{i0}] = [G]\big[Z(\theta_{10}, d_{10})\big]\big[X(\alpha_{12}, a_{12})\big]\dots\big[X(\alpha_{i-1,i}, a_{i-1,i})\big], \tag{2.12}$$

where, for example,

$$[A_{10}] = [G], \quad \text{and} \quad [A_{20}] = [G]\big[Z(\theta_{10}, d_{10})\big]\big[X(\alpha_{12}, a_{12})\big].$$

Now, insert the identity $[Z(\theta_{i,0})]^{-1}[A_{i0}]^{-1}[A_{i0}][Z(\theta_{i0})] = [I]$ after the first $n-1$ joint transforms $[Z(\theta_i, d_i)]$ in (2.11), in order to obtain the sequence of terms

$$\big[T(\Delta\theta_i, \mathsf{S}_i)\big] = [A_{i0}]\big[Z(\theta_i, d_i)\big]\big[Z(\theta_{i,0})\big]^{-1}[A_{i0}]^{-1} = [A_{i0}]\big[Z(\Delta\theta_i, \Delta d_i)\big][A_{i0}]^{-1}. \tag{2.13}$$

The result is the relative transformation that takes the form

$$\big[D(\Delta\boldsymbol{\theta})\big] = \big[T(\Delta\theta_1, \mathsf{S}_1)\big]\big[T(\Delta\theta_2, \mathsf{S}_2)\big]\dots\big[T(\Delta\theta_n, \mathsf{S}_n)\big], \tag{2.14}$$

where S_i are the Plucker coordinates of each joint axis obtained by transforming the joint screw K to the world frame by the coordinate transformations defined in (2.13).

Using the exponential form the transformations $[T(\Delta\theta_i, \mathsf{S}_i)]$, we write the relative kinematics equations (2.14) as

$$\big[D(\Delta\boldsymbol{\theta})\big] = e^{\Delta\theta_1 S_1} e^{\Delta\theta_2 S_2} \dots e^{\Delta\theta_n S_n}, \tag{2.15}$$

where the matrices S_i are defined as

$$S_i = A_{i0} K A_{i0}^{-1}. \tag{2.16}$$

The product of exponentials form of the kinematics equations (2.10) is now obtained as

$$[D] = \big[D(\Delta\boldsymbol{\theta})\big][D_0] = e^{\Delta\theta_1 S_1} e^{\Delta\theta_2 S_2} \dots e^{\Delta\theta_n S_n}[D_0]. \tag{2.17}$$

The difference between this equation and (2.10) is that here the coordinates of the joint axes of the serial chain are defined in the world frame.

2.3 The Even Clifford Algebra $C^+(P^3)$

The Clifford algebra of the projective three space P^3 is a sixteen-dimensional vector space with a product operation that is defined in terms of a scalar product, see McCarthy (1990) [2]. The elements of even rank form an eight-dimensional subalgebra $C^+(P^3)$ that can be identified with the set of 4×4 homogeneous transforms.

The typical element of $C^+(P^3)$ can be written as the eight dimensional vector given by

$$\hat{A} = a_0 + a_1 i + a_2 j + a_3 k + a_4 \varepsilon + a_5 i\varepsilon + a_6 j\varepsilon + a_7 k\varepsilon, \tag{2.18}$$

where the basis elements i, j, and k are the well-known quaternion units, and ε is called the dual unit. The quaternion units satisfy the multiplication relations

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j, \quad \text{and} \quad ijk = -1. \tag{2.19}$$

The dual number ε commutes with i, j, and k, and multiplies by the rule $\varepsilon^2 = 0$.

In our calculations, it is convenient to consider the linear combination of quaternion units to be a vector in three dimensions, so we use the notation $\mathbf{A} = a_1 i + a_2 j + a_3 k$ and $\mathbf{A}^\circ = a_5 i + a_6 j + a_7 k$—the small circle superscript is often used to distinguish coefficients of the dual unit. This allows us to write the Clifford algebra element (2.18) as

$$\hat{A} = a_0 + \mathbf{A} + a_4\varepsilon + \mathbf{A}^\circ\varepsilon. \tag{2.20}$$

Now, collect the scalar and vector terms so this element takes the form

$$\hat{A} = (a_0 + a_4\varepsilon) + \big(\mathbf{A} + \mathbf{A}^\circ\varepsilon\big) = \hat{a} + \mathsf{A}. \tag{2.21}$$

The dual vector $\mathsf{A} = \mathbf{A} + \mathbf{A}^\circ\varepsilon$ can be identified with the pairs of vectors that define lines and screws.

Using this notation the Clifford algebra product of elements $\hat{A} = \hat{a} + \mathsf{A}$ and $\hat{B} = \hat{b} + \mathsf{B}$ takes the form

$$\hat{C} = (\hat{b} + \mathsf{B})(\hat{a} + \mathsf{A}) = (\hat{b}\hat{a} - \mathsf{B}\cdot\mathsf{A}) + (\hat{a}\mathsf{B} + \hat{b}\mathsf{A} + \mathsf{B}\times\mathsf{A}), \tag{2.22}$$

where the usual vector dot and cross products are extended linearly to dual vectors.

2.3.1 Exponential of a Vector

The product operation in the Clifford algebra allows us to compute the exponential of a vector $\theta\mathbf{S}$, where $|\mathbf{S}| = 1$, as

$$e^{\theta\mathbf{S}} = 1 + \theta\mathbf{S} + \frac{\theta^2}{2}\mathbf{S}^2 + \frac{\theta^3}{3!}\mathbf{S}^3 + \cdots. \tag{2.23}$$

Using (2.22) we can write $\mathbf{S} = 0 + \mathbf{S}$ and compute

$$\mathbf{S}^2 = (0+\mathbf{S})(0+\mathbf{S}) = -1, \qquad \mathbf{S}^3 = -\mathbf{S}, \qquad \mathbf{S}^4 = 1, \quad \text{and} \quad \mathbf{S}^5 = \mathbf{S}, \tag{2.24}$$

which means we have

$$\begin{aligned} e^{\theta\mathbf{S}} &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots\right) + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)\mathbf{S} \\ &= \cos\theta + \sin\theta\,\mathbf{S}. \end{aligned} \tag{2.25}$$

This is the well-known *unit quaternion* that represents a rotation around the axis $\mathbf{S}$ by the angle $\phi = 2\theta$. The rotation angle ϕ is double that given in the quaternion, because the Clifford algebra form of a rotation requires multiplication by both $Q = \cos\theta + \sin\theta\mathbf{S}$ and its conjugate $Q^* = \cos\theta - \sin\theta\mathbf{S}$. In particular, if $\mathbf{x}$ and $\mathbf{X}$ are the coordinates of a point before and after the rotation, then we have the quaternion coordinate transformation equation

$$\mathbf{X} = Q\mathbf{x}Q^*. \tag{2.26}$$

For this reason the quaternion is often written in terms of one-half the rotation angle, that is $Q = \cos\frac{\phi}{2} + \sin\frac{\phi}{2}\mathbf{S}$.

2.3.2 Exponential of a Screw

The Plücker coordinates $\mathsf{S} = (\mathbf{S}, \mathbf{C} \times \mathbf{S})$ of a line can be identified with the Clifford algebra element $\mathsf{S} = \mathbf{S} + \varepsilon \mathbf{C} \times \mathbf{S}$. Similarly, the screw $\mathsf{J} = (\mathbf{S}, \mathbf{V}) = (\mathbf{S}, \mathbf{C} \times \mathbf{S} + \mu \mathbf{S})$ becomes the element $\mathsf{J} = \mathbf{S} + \varepsilon \mathbf{V} = (1 + \mu\varepsilon)\mathsf{S}$. Using the Clifford product we can compute the exponential of the screw $\theta\mathsf{J}$,

$$e^{\theta \mathsf{J}} = 1 + \mathsf{J} + \frac{\theta^2}{2}\mathsf{J}^2 + \frac{\theta^3}{3!}\mathsf{J}^3 + \cdots. \tag{2.27}$$

Notice that $\mathsf{S}^2 = -1$, therefore

$$\begin{aligned} &\mathsf{J}^2 = -(1+\mu\varepsilon)^2 = -(1+2\mu\varepsilon), \qquad \mathsf{J}^3 = -(1+3\mu\varepsilon)\mathsf{S}, \\ &\mathsf{J}^4 = 1 + 4\mu\varepsilon, \quad \text{and} \quad \mathsf{J}^5 = (1+5\mu\varepsilon)\mathsf{S}, \end{aligned} \tag{2.28}$$

and, we obtain

$$\begin{aligned} e^{\theta \mathsf{J}} &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots\right) + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)\mathsf{S} \\ &\quad - \theta\mu\varepsilon\left(\theta - \frac{\theta^3}{3!} + \cdots\right) + \theta\mu\varepsilon\left(1 - \frac{\theta^2}{2} + \cdots\right)\mathsf{S} \\ &= (\cos\theta - d\sin\theta\varepsilon) + (\sin\theta + d\cos\theta\varepsilon)\mathsf{S}. \end{aligned} \tag{2.29}$$

Let $d = \theta\mu$ be the slide along the screw axis associated with the angle θ. At this point it is convenient to introduce the dual angle $\hat{\theta} = \theta + d\varepsilon$, so we have the identities

$$\sin\hat{\theta} = \sin\theta + d\cos\theta\varepsilon, \quad \text{and} \quad \cos\hat{\theta} = \cos\theta - d\sin\theta\varepsilon, \tag{2.30}$$

which are derived using the series expansions of sine and cosine.

Equation (2.29) introduces the *unit dual quaternion* which is identified with spatial displacements. To see the relationship we factor out the rotation term to obtain

$$\hat{Q} = \cos\hat{\theta} + \sin\hat{\theta}\mathsf{S} = (1 + \mathbf{t}\varepsilon)(\cos\theta + \sin\theta\mathbf{S}), \tag{2.31}$$

where

$$\mathbf{t} = d\mathbf{S} + \sin\theta\cos\theta\mathbf{C} \times \mathbf{S} - \sin^2\theta(\mathbf{C} \times \mathbf{S}) \times \mathbf{S}. \tag{2.32}$$

This vector is one-half the translation $\mathbf{d} = 2\mathbf{t}$ of the spatial displacement associated with this dual quaternion in the same way that we saw that the rotation angle is $\phi = 2\theta$. This is because the Clifford algebra form of the transformation of line coordinates x to X by the rotation ϕ around an axis S with the translation $\mathbf{d}$ involves multiplication by both the Clifford algebra element $\hat{Q} = \cos\hat{\theta} + \sin\hat{\theta}\mathsf{S}$ and its conjugate $\hat{Q}^* = \cos\hat{\theta} - \sin\hat{\theta}\mathsf{S}$, given by

$$\mathsf{X} = \hat{Q}\mathsf{x}\hat{Q}^*. \tag{2.33}$$

For this reason the unit dual quaternion is usually written in terms of the half rotation angle and half displacement vector,

$$\hat{Q} = \cos\frac{\hat{\phi}}{2} + \sin\frac{\hat{\phi}}{2}\mathsf{S} = \left(1 + \frac{1}{2}\mathbf{d}\varepsilon\right)\left(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\mathbf{S}\right), \tag{2.34}$$

where

$$\mathbf{d} = 2\left(\frac{k}{2}\mathbf{S} + \sin\frac{\phi}{2}\cos\frac{\phi}{2}\mathbf{C}\times\mathbf{S} - \sin^2\frac{\phi}{2}(\mathbf{C}\times\mathbf{S})\times\mathbf{S}\right). \tag{2.35}$$

Notice that we introduced the slide along S given by $k = \phi\mu$, so we have the dual angle $\hat{\phi} = \phi + k\varepsilon$.

2.3.3 Clifford Algebra Kinematics Equations

The exponential of a screw defines a relative displacement from an initial position to a final position in terms of a rotation around and slide along an axis. This means the composition of Clifford algebra elements defines the relative kinematics equations for a serial chain that are equivalent to (2.15) [21].

Consider the nC serial chain in which each joint can rotate an angle θ_i around, and slide the distance d_i along, the axis S_i, for $i = 1, \ldots, n$. Let $\boldsymbol{\theta}_0$ and $\mathbf{d}_0$ be the joint parameters of this chain when in the reference configuration, so we have

$$\Delta\hat{\boldsymbol{\theta}} = (\boldsymbol{\theta} + \mathbf{d}\varepsilon) - (\boldsymbol{\theta}_0 + \mathbf{d}_0\varepsilon) = (\Delta\hat{\theta}_1, \Delta\hat{\theta}_2, \ldots, \Delta\hat{\theta}_n). \tag{2.36}$$

Then, the movement from this reference configuration is defined by the kinematics equation,

$$\begin{aligned}\hat{D}(\Delta\hat{\boldsymbol{\theta}}) &= e^{\frac{\Delta\hat{\theta}_1}{2}\mathsf{S}_1} e^{\frac{\Delta\hat{\theta}_2}{2}\mathsf{S}_2} \cdots e^{\frac{\Delta\hat{\theta}_n}{2}\mathsf{S}_n}\\ &= \left(\mathrm{c}\frac{\Delta\hat{\theta}_1}{2} + \mathrm{s}\frac{\Delta\hat{\theta}_1}{2}\mathsf{S}_1\right)\left(\mathrm{c}\frac{\Delta\hat{\theta}_2}{2} + \mathrm{s}\frac{\Delta\hat{\theta}_2}{2}\mathsf{S}_2\right)\cdots\left(\mathrm{c}\frac{\Delta\hat{\theta}_n}{2} + \mathrm{s}\frac{\Delta\hat{\theta}_n}{2}\mathsf{S}_n\right).\end{aligned} \tag{2.37}$$

Note that s and c denote the sine and cosine functions, respectively.

2.4 Design Equations for a Serial Chain

The goal of our design problem is to determine the dimensions of a spatial serial chain that can position a tool held by its end-effector in a given set of task positions. The location of the base of the robot, the position of the tool frame, as well as the link dimensions and joint angles are considered to be design variables.

2.4.1 Specified Task Positions

Identify a set of task positions $[P_j]$, $j=1,\ldots,m$. Then, the physical dimensions of the chain are defined by the requirement that for each position $[P_j]$ there is a joint parameter vector $\boldsymbol{\theta}_j$ such that the kinematics equations of the chain satisfy the relations

$$[P_j]=\left[D(\boldsymbol{\theta}_j)\right],\quad i=1,\ldots,m. \tag{2.38}$$

Now, choose $[P_1]$ as the reference position and compute the relative displacements $[P_j][P_1^{-1}]=[P_{1j}]$, $j=2,\ldots,m$.

For each of these relative displacements $[P_{1j}]$ we can determine the dual unit quaternion $\hat{P}_{1j}=\cos\frac{\Delta\hat{\phi}_{1j}}{2}+\sin\frac{\Delta\hat{\phi}_{1j}}{2}\mathsf{P}_{1j}$, $j=2,\ldots,m$. The dual angle $\Delta\hat{\phi}_{1j}$ defines the rotation about and slide along the axis P_{1j} that defines the displacement from the first to the jth position. Now writing (2.37) for the $m-1$ relative displacements, we obtain

$$\hat{P}_{1j}=e^{\frac{\Delta\hat{\theta}_{1j}}{2}\mathsf{S}_1}e^{\frac{\Delta\hat{\theta}_{2j}}{2}\mathsf{S}_2}\cdots e^{\frac{\Delta\hat{\theta}_{nj}}{2}\mathsf{S}_n},\quad j=2,\ldots,m. \tag{2.39}$$

The result is $8(m-1)$ design equations. The unknowns are the n joint axes S_i, $i=1,\ldots,n$, and the $n(m-1)$ pairs of joint parameters $\Delta\hat{\theta}_{ij}=\Delta\theta_{ij}+\Delta d_{ij}\varepsilon$.

2.4.2 The Independent Synthesis Equations

The eight components of the unit Clifford algebra kinematics equations (2.39) are not independent. It is easy to see that a dual unit quaternion satisfies the identity,

$$\hat{Q}\hat{Q}^*=e^{\frac{\Delta\hat{\phi}}{2}\mathsf{S}}e^{-\frac{\Delta\hat{\phi}}{2}\mathsf{S}}=1, \tag{2.40}$$

which imposes a two constraints. Thus, only six of the eight synthesis equations obtained for each relative task position are independent, which means there are only $6(n-1)$ independent synthesis equations for an n position task. Furthermore, the axis S has unit magnitude with means that only four of its six components are independent.

In order to count the number of independent equations and unknowns in the Clifford algebra synthesis equations, it is useful to identify the relationship between the constraints on a dual unit quaternion and the constraints on the dual unit vector that generates it. Therefore, we take a moment as verify

Remark 2.1 (Normality Condition) The dual quaternion arising from the product of dual quaternions has unit magnitude if and only if each factor is the exponential of dual unit vector.

Proof For the screw displacement $\hat{Q} = e^{\frac{\Delta\phi}{2}\mathsf{S}}$ the unit condition yields,

$$\hat{Q}\hat{Q}^* = \left(\mathrm{c}\frac{\Delta\hat{\phi}}{2} + \mathrm{s}\frac{\Delta\hat{\phi}}{2}\mathsf{S}\right)\left(\mathrm{c}\frac{\Delta\hat{\phi}}{2} - \mathrm{s}\frac{\Delta\hat{\phi}}{2}\mathsf{S}\right) = \mathrm{c}\frac{\Delta\hat{\phi}}{2}\mathrm{c}\frac{\Delta\hat{\phi}}{2} + \mathrm{s}\frac{\Delta\hat{\phi}}{2}\mathrm{s}\frac{\Delta\hat{\phi}}{2}\mathsf{S}\cdot\mathsf{S}. \quad (2.41)$$

Notice that, if $\mathsf{S}\cdot\mathsf{S} = 1$, then

$$\hat{Q}\hat{Q}^* = \mathrm{c}\frac{\Delta\hat{\phi}}{2}\mathrm{c}\frac{\Delta\hat{\phi}}{2} + \mathrm{s}\frac{\Delta\hat{\phi}}{2}\mathrm{s}\frac{\Delta\hat{\phi}}{2} = \mathrm{c}\frac{\Delta\phi}{2}^2 + \mathrm{s}\frac{\Delta\phi}{2}^2 = 1. \quad (2.42)$$

Now, for a dual quaternion obtained as the composition of transformations about n joint axes, we have

$$\hat{Q}\hat{Q}^* = \left(e^{\frac{\Delta\phi_1}{2}\mathsf{S}_1}\dots e^{\frac{\Delta\phi_n}{2}\mathsf{S}_n}\right)\left(e^{\frac{\Delta\phi_1}{2}\mathsf{S}_1}\dots e^{\frac{\Delta\phi_n}{2}\mathsf{S}_n}\right)^*. \quad (2.43)$$

Expand this product and use the associative property of the Clifford algebra to obtain

$$\hat{Q}\hat{Q}^* = e^{\frac{\Delta\phi_1}{2}\mathsf{S}_1}\dots\left(e^{\frac{\Delta\phi_n}{2}\mathsf{S}_n}e^{\frac{-\Delta\phi_n}{2}\mathsf{S}_n}\right)\dots e^{\frac{-\Delta\phi_1}{2}\mathsf{S}_1}, \quad (2.44)$$

such that the terms $e^{\frac{\Delta\phi_n}{2}\mathsf{S}_n}e^{\frac{-\Delta\phi_n}{2}\mathsf{S}_n} = 1$ when $\mathsf{S}_n\cdot\mathsf{S}_n = 1$. The result is

$$\hat{Q}\hat{Q}^* = 1 \quad \Longleftrightarrow \quad \mathsf{S}_i\cdot\mathsf{S}_i = 1, \quad i = 1\dots,n. \quad (2.45)$$

□

This condition shows that six of the eight components of the dual quaternion kinematics equations combine with the normal conditions on the Plücker coordinates of the joint axes to define the minimum set of independent synthesis equations for the serial chain problem.

2.4.3 Counting the Equations and Unknowns

Consider a spatial serial chain that consists of r revolute joints and p prismatic joints. A purely prismatic joint is defined by the unit vector $\mathbf{S}$ that defines the slide direction, so it has two independent parameters. The revolute joint axis is defined by Plücker coordinate vectors, $\mathsf{S}_i = \mathbf{S} + \mathbf{C}\times\mathbf{S}\varepsilon$, that have four independent components due to the normal conditions

$$|\mathbf{S}| = 1 \quad \text{and} \quad \mathbf{S}\cdot(\mathbf{C}\times\mathbf{S}) = 0. \quad (2.46)$$

Thus, the joint axes that define this chain have $K = 6r + 3p$ components, minus $2r + p$ Plücker constraints, which yields $4r + 2p$ independent unknowns.

Revolute and prismatic joints each have a single joint parameter, either a rotation angle or slide distance, which means that our chain has $(r + p)(m - 1)$ unknown joint parameters that define the m relative positions.

Table 2.1 The number of task positions that determine the structural parameters for five degree-of-freedom serial chains

Chain	K	Task positions	Total equations
PRPRP	21	15	91
RPRPR	24	17	104
RRRRP	27	19	117
RRRRR	30	21	130

Subtracting the number of equations from the number of unknowns, we obtain

$$\begin{aligned} E &= 4r + 2p + (r + p)(m - 1) - 6(m - 1) \\ &= (3r + p + 6) + (r + p - 6)m, \end{aligned} \tag{2.47}$$

where E excess of unknowns over equations. This excess can be made to equal zero for chains with degree of freedom dof $= r + p \leq 5$, in which case we specify

$$m = \frac{3r + p + 6 - c}{6 - (r + p)}, \tag{2.48}$$

task positions. If fewer than this number of task positions are defined, or if the chain has six or more degrees of freedom, then we are free to select values for the excess design parameters. In (2.48) we have added c to denote any extra constraint that may be imposed on the axes. Table 2.1 presents the maximum number of positions that can be defined for some chains with 5 degrees of freedom.

It is interesting to notice that, because the composition of displacements has structure of semi-direct product, the rotations are obtained by operating rotations only. A specific counting scheme can be generated for the rotations by considering the first quaternion of the dual quaternion only. We obtain that the maximum number of task rotations is

$$m_R = \frac{3 + r}{3 - r}. \tag{2.49}$$

In some cases with $r = 1$ or 2, the rotation part of the design equations can be used to determine the directions of these axes independently. Perez and McCarthy [3] call these chains "orientation limited."

2.5 Assembling the Design Equations

The structure of the Clifford algebra design equations provides a systematic approach to assembling the design equations for a broad range of serial chains. The basic approach is to formulate the design equations for the nC serial chain, and then (i) restrict the joint variables to form prismatic or sliding joints, and (ii) impose geometric conditions on the axes to form universal or spherical joints or to account for specific geometry. The result is a systematic way of defining the design equations for a broad range of chains. Here we present the procedure for the 3C serial chain, but it has been implemented in our numerical solver for the 2C, 4C and 5C cases, as well.

2.5.1 The 3C Chain

The Clifford algebra form of the relative kinematics equations for the 3C chain can be written as

$$\hat{D}(\Delta\hat{\boldsymbol{\theta}}) = \left(\mathrm{c}\frac{\Delta\hat{\theta}_1}{2} + \mathrm{s}\frac{\Delta\hat{\theta}_1}{2}\mathsf{S}_1\right)\left(\mathrm{c}\frac{\Delta\hat{\theta}_2}{2} + \mathrm{s}\frac{\Delta\hat{\theta}_2}{2}\mathsf{S}_2\right)\left(\mathrm{c}\frac{\Delta\hat{\theta}_3}{2} + \mathrm{s}\frac{\Delta\hat{\theta}_3}{2}\mathsf{S}_3\right), \quad (2.50)$$

where $\mathsf{S}_i = \mathbf{S}_i + \mathbf{S}_i^\circ \varepsilon$ define the joint axes in the reference position, and $\Delta\hat{\theta}_i = \Delta\theta_i + \Delta d_i$ define the rotation and slide of the cylindric joint around the ith axis.

Expand the right side of (2.50) using the Clifford product to obtain

$$\begin{aligned}
\hat{D}(\Delta\hat{\boldsymbol{\theta}}) &= (\hat{c}_1\hat{c}_2 - \hat{s}_1\hat{s}_2\mathsf{S}_1\cdot\mathsf{S}_2 + \hat{s}_1\hat{c}_2\mathsf{S}_1 + \hat{c}_1\hat{s}_2\mathsf{S}_2 + \hat{s}_1\hat{s}_2\mathsf{S}_1\times\mathsf{S}_2)(\hat{c}_3 + \hat{s}_3\mathsf{S}_3)\\
&= \hat{c}_1\hat{c}_2\hat{c}_3 - \hat{s}_1\hat{s}_2\hat{c}_3\mathsf{S}_1\cdot\mathsf{S}_2 - \hat{s}_1\hat{c}_2\hat{s}_3\mathsf{S}_1\cdot\mathsf{S}_3 - \hat{c}_1\hat{s}_2\hat{s}_3\mathsf{S}_2\cdot\mathsf{S}_3\\
&\quad - \hat{s}_1\hat{s}_2\hat{s}_3\mathsf{S}_1\times\mathsf{S}_2\cdot\mathsf{S}_3 + \hat{s}_1\hat{c}_2\hat{c}_3\mathsf{S}_1 + \hat{c}_1\hat{s}_2\hat{c}_3\mathsf{S}_2 + \hat{c}_1\hat{c}_2\hat{s}_3\mathsf{S}_3\\
&\quad + \hat{s}_1\hat{s}_2\hat{c}_3\mathsf{S}_1\times\mathsf{S}_2 + \hat{s}_1\hat{c}_2\hat{s}_3\mathsf{S}_1\times\mathsf{S}_3 + \hat{c}_1\hat{s}_2\hat{s}_3\mathsf{S}_2\times\mathsf{S}_3\\
&\quad + \hat{s}_1\hat{s}_2\hat{s}_3\big((\mathsf{S}_1\times\mathsf{S}_2)\times\mathsf{S}_3 - (\mathsf{S}_1\cdot\mathsf{S}_2)\,\mathsf{S}_3\big). \qquad (2.51)
\end{aligned}$$

For convenience, we have introduced the notation $\hat{c}_i = \cos\frac{\Delta\hat{\theta}_i}{2}$ and $\hat{s}_i = \sin\frac{\Delta\hat{\theta}_i}{2}$.

Equation (2.51) can be written in matrix form to emphasize that it is the linear combination of the eight monomials formed as products of the joint angles, which we assemble into an array in reversed lexicographic order obtained by reading right to left,

$$\hat{\mathbf{V}} = (\hat{c}_1\hat{c}_2\hat{c}_3, \hat{s}_1\hat{c}_2\hat{c}_3, \hat{c}_1\hat{s}_2\hat{c}_3, \hat{c}_1\hat{c}_2\hat{s}_3, \hat{s}_1\hat{s}_2\hat{c}_3, \hat{s}_1\hat{c}_2\hat{s}_3, \hat{c}_1\hat{s}_2\hat{s}_3, \hat{s}_1\hat{s}_2\hat{s}_3)^T. \quad (2.52)$$

To do this, we must introduce the vector form of the dual unit quaternion $\hat{Q} = \cos\frac{\Delta\hat{\theta}}{2} + \sin\frac{\Delta\hat{\theta}}{2}\mathsf{S}$ given by

$$\hat{\mathbf{Q}} = \begin{Bmatrix} \sin\frac{\Delta\hat{\theta}}{2}(S_x + S_x^\circ\varepsilon)\\ \sin\frac{\Delta\hat{\theta}}{2}(S_y + S_y^\circ\varepsilon)\\ \sin\frac{\Delta\hat{\theta}}{2}(S_z + S_z^\circ\varepsilon)\\ \cos\frac{\Delta\hat{\theta}}{2} \end{Bmatrix} = \begin{Bmatrix} \sin\frac{\Delta\hat{\theta}}{2}\mathsf{S}\\ \cos\frac{\Delta\hat{\theta}}{2} \end{Bmatrix}. \quad (2.53)$$

Collecting terms in (2.51), we obtain the matrix equation

$$\hat{\mathbf{D}}(\Delta\hat{\boldsymbol{\theta}}) = \begin{bmatrix} 0 & \mathsf{S}_1 & \mathsf{S}_2 & \mathsf{S}_3 & \mathsf{S}_1\times\mathsf{S}_1 & \mathsf{S}_1\times\mathsf{S}_3 & \mathsf{S}_2\times\mathsf{S}_3 & -(\mathsf{S}_1\cdot\mathsf{S}_2)\,\mathsf{S}_3+(\mathsf{S}_1\times\mathsf{S}_2)\times\mathsf{S}_3 \\ 1 & 0 & 0 & 0 & -\mathsf{S}_1\cdot\mathsf{S}_2 & -\mathsf{S}_1\cdot\mathsf{S}_3 & -\mathsf{S}_2\cdot\mathsf{S}_3 & -\mathsf{S}_1\times\mathsf{S}_2\cdot\mathsf{S}_3 \end{bmatrix}\hat{\mathbf{V}}. \tag{2.54}$$

The Clifford algebra notation is compact in that each column of this matrix actually forms a column of four dual coefficients, or eight real coefficients if we write the dual components of the dual quaternion after the real components, forming an eight-dimensional vector. Similarly, each of the monomials in $\hat{\mathbf{V}}$ expands into four real terms, which we can list as

$$\mathbf{N} = \left(\mathbf{V},\, \frac{\Delta d_1}{2}\mathbf{V},\, \frac{\Delta d_2}{2}\mathbf{V},\, \frac{\Delta d_3}{2}\mathbf{V}\right), \tag{2.55}$$

where $\mathbf{V}$ is the array of real parts of $\hat{\mathbf{V}}$. Thus, (2.54) expands to an 8×32 matrix equation. The number k of joint variable monomials in an nC serial chain is given by

$$k = (n+1)2^n. \tag{2.56}$$

Thus, these equations become 8×12 for 2C, 8×80 for 4C and 8×192 for 5C chains.

The kinematics equations (2.54) can be used directly for the design of a 3C chain. In what follows, we specialize these equations to obtain design equations for a variety of special serial chains.

2.5.2 RCC, RRC and RRR Chains

The ith cylindric joint in the 3C chain is converted to a revolute joint simply by setting $\Delta d_i = 0$. This can be done in seven different ways to define three permutations of the RRC chain, three permutations of the RRC chain and the RRR chain [23–25].

For example, the monomials in (2.54) that define the RCC, CRC or CCR chains are given by

$$\begin{aligned} \text{RCC:}\quad \mathbf{N} &= \left(\mathbf{V},\, \frac{\Delta d_2}{2}\mathbf{V},\, \frac{\Delta d_3}{2}\mathbf{V}\right),\\ \text{CRC:}\quad \mathbf{N} &= \left(\mathbf{V},\, \frac{\Delta d_1}{2}\mathbf{V},\, \frac{\Delta d_3}{2}\mathbf{V}\right),\\ \text{CCR:}\quad \mathbf{N} &= \left(\mathbf{V},\, \frac{\Delta d_1}{2}\mathbf{V},\, \frac{\Delta d_2}{2}\mathbf{V}\right). \end{aligned} \tag{2.57}$$

Similarly, the RRC, RCR and CRR chains have the monomials

$$\begin{aligned}
\text{RRC:}\quad \mathbf{N} &= \left(\mathbf{V}, \frac{\Delta d_3}{2}\mathbf{V}\right),\\
\text{RCR:}\quad \mathbf{N} &= \left(\mathbf{V}, \frac{\Delta d_2}{2}\mathbf{V}\right),\\
\text{CRR:}\quad \mathbf{N} &= \left(\mathbf{V}, \frac{\Delta d_1}{2}\mathbf{V}\right).
\end{aligned} \tag{2.58}$$

Finally, the RRR chain is defined by the monomial list

$$\text{RRR:}\quad \mathbf{N} = \mathbf{V}. \tag{2.59}$$

Notice that if an nC chain is specialized to have r revolute joints, then the number of monomials is given by

$$k = (n - r + 1)2^n. \tag{2.60}$$

2.5.3 PCC, PPC and PPP Chains

A two-step process is required to convert the ith cylindric joint to a prismatic joint. The first step is to set $\Delta\theta_i = 0$. The second step consists of specializing the joint axis $\mathsf{S}_i = \mathbf{S}_i$, so that its dual part is zero. This latter constraint arises because the pure translation defined by a prismatic joint depends only on the direction, not the location in space, of its axis.

In order to define the monomials for the three permutations of the PCC chain, we introduce $\mathbf{W}_1 = (c_1c_2c_3, c_1s_2c_3, c_1c_2s_3, c_1s_2s_3)$, and similarly define $\mathbf{W}_2$ and $\mathbf{W}_3$, where the subscript i indicates that we make $s_i = 0$. This allows us to define the arrays of monomials,

$$\begin{aligned}
\text{PCC:}\quad \mathbf{N} &= \left(\mathbf{W}_1, \frac{\Delta d_1}{2}\mathbf{W}_1, \frac{\Delta d_2}{2}\mathbf{W}_1, \frac{\Delta d_3}{2}\mathbf{W}_1\right),\\
\text{CPC:}\quad \mathbf{N} &= \left(\mathbf{W}_2, \frac{\Delta d_1}{2}\mathbf{W}_2, \frac{\Delta d_2}{2}\mathbf{W}_2, \frac{\Delta d_3}{2}\mathbf{W}_2\right),\\
\text{CCP:}\quad \mathbf{N} &= \left(\mathbf{W}_3, \frac{\Delta d_1}{2}\mathbf{W}_3, \frac{\Delta d_2}{2}\mathbf{W}_3, \frac{\Delta d_3}{2}\mathbf{W}_3\right).
\end{aligned} \tag{2.61}$$

The monomials for the three permutations of the PPC chain are easily determined by introducing the set of monomials $\mathbf{W}_{12} = (c_1c_2c_3, c_1c_2s_3)$, and similarly $\mathbf{W}_{13}$

Table 2.2 Constraints that specialize C-joints to R, P, T and S joints

Joint	Axes	Constraints
R	S_i	$\Delta d_i = 0$
P	S_i	$\Delta\theta_i = 0$
C	S_i	None
T	$\mathsf{S}_i, \mathsf{S}_{i+1}$	$\Delta d_i = 0$, $\Delta d_{i+1} = 0$, $\mathsf{S}_i \cdot \mathsf{S}_{i+1} = 0$
S	$\mathsf{S}_i, \mathsf{S}_{i+1}, \mathsf{S}_{i+2}$	$\Delta d_i = 0$, $\Delta d_{i+1} = 0$, $\Delta d_{i+2} = 0$ $\mathsf{S}_i \cdot \mathsf{S}_{i+1} = 0$, $\mathsf{S}_{i+1} \cdot \mathsf{S}_{i+2} = 0$, $\mathsf{S}_i \cdot \mathsf{S}_{i+2} = 0$

and $\mathbf{W}_{23}$,

$$\begin{aligned}
\text{PPC:}\quad \mathbf{N} &= \left(\mathbf{W}_{12}, \frac{\Delta d_1}{2}\mathbf{W}_{12}, \frac{\Delta d_2}{2}\mathbf{W}_{12}, \frac{\Delta d_3}{2}\mathbf{W}_{12}\right),\\
\text{PCP:}\quad \mathbf{N} &= \left(\mathbf{W}_{13}, \frac{\Delta d_1}{2}\mathbf{W}_{13}, \frac{\Delta d_2}{2}\mathbf{W}_{13}, \frac{\Delta d_3}{2}\mathbf{W}_{13}\right),\\
\text{CPP:}\quad \mathbf{N} &= \left(\mathbf{W}_{23}, \frac{\Delta d_1}{2}\mathbf{W}_{23}, \frac{\Delta d_2}{2}\mathbf{W}_{23}, \frac{\Delta d_3}{2}\mathbf{W}_{23}\right).
\end{aligned} \tag{2.62}$$

Finally, the PPP chain is defined by the monomial list

$$\text{PPP:}\quad \mathbf{N} = \left((\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3), \frac{\Delta d_1}{2}(\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3), \frac{\Delta d_2}{2}(\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3), \frac{\Delta d_3}{2}(\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3)\right). \tag{2.63}$$

The number of monomials in an nC chain with p of the joints restricted to be prismatic is seen to be

$$k = (n+1)2^{n-p}. \tag{2.64}$$

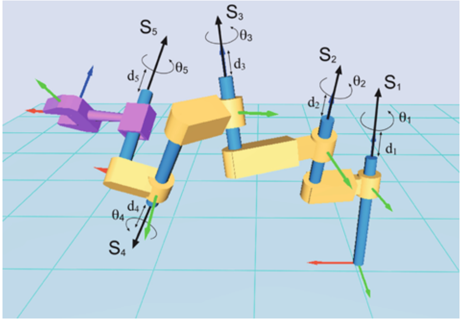
Table 2.2 summarizes the constraints needed to transform the C joint into the most common types of joints. Notice that, for the spherical joint and other special cases, we use the approach of adding constraints between consecutive joint axes. This will not yield the minimum set of joint parameters, but it gives satisfactory results with the numerical solver.

This approach to the formulation of the design equations for special cases of the CCC chain can be extended to any nC chain [22].

2.6 The Synthesis of 5C and Related Chains

In this section, we present a numerical synthesis algorithm which uses the Clifford algebra exponential design equations for the 5C serial chain, see Fig. 2.2. The special cases of this chain include robots with up to five joints and up to ten degrees of freedom.

Fig. 2.2 The 5C serial robot



The design equations for a specific serial robot are be obtained from the 5C robot equations by imposing conditions on some of the axes or joint variables. The kinematics equations for the 5C robot are given by

$$\hat{Q}_{5C} = e^{\frac{\Delta\hat{\theta}_1}{2}\mathsf{S}_1} e^{\frac{\Delta\hat{\theta}_2}{2}\mathsf{S}_2} e^{\frac{\Delta\hat{\theta}_3}{2}\mathsf{S}_3} e^{\frac{\Delta\hat{\theta}_4}{2}\mathsf{S}_4} e^{\frac{\Delta\hat{\theta}_5}{2}\mathsf{S}_5}, \tag{2.65}$$

or

$$\hat{Q}_{5C} = \left(\cos\frac{\Delta\hat{\theta}_1}{2} + \sin\frac{\Delta\hat{\theta}_1}{2}\mathsf{S}_1\right)\left(\cos\frac{\Delta\hat{\theta}_2}{2} + \sin\frac{\Delta\hat{\theta}_2}{2}\mathsf{S}_2\right)\cdots \left(\cos\frac{\Delta\hat{\theta}_5}{2} + \sin\frac{\Delta\hat{\theta}_5}{2}\mathsf{S}_5\right). \tag{2.66}$$

The kinematics equations for a serial chain consisting of revolute R, prismatic P, universal T, cylindrical C or spherical S joints can be obtained from the 5C robot using the approach presented in the previous section. For example, the kinematics equation of the TPR serial chain are obtained by requiring the axes S_1 and S_2 to be perpendicular and coincident, which is obtained by setting the joint variables d_1, d_2, θ_3 and d_4 to zero. The extra joint is eliminated by setting θ_5 and d_5 to zero. Other joints, like the helical H or planar E joints can also be modeled by imposing constraints on the axes and joint parameters.

In order to facilitate the specialization of the general 5C robot to a specific serial chain topology,the kinematics equations are organized as a linear combination of the products of joint angles and slides, which form the *monomials* of these equations with coefficients that are given by the structural parameters of the chain. In this way, the kinematics equations of the 5C serial chain is a linear combination of 192 monomials, which can be organized into six sets of 32 products of sines and cosines of the $\Delta\theta_i$ joint angles, given by,

$$\mathbf{V} = \big(s_1s_2s_3s_4s_5, (s_1s_2s_3s_4c_5)_5, (s_1s_2s_3c_4c_5)_{10}, (s_1s_2c_3c_4c_5)_{10}, (s_1c_2c_3c_4c_5)_5, c_1c_2c_3c_4c_5\big), \tag{2.67}$$

where $c_i = \cos\frac{\Delta\theta_i}{2}$, $s_i = \sin\frac{\Delta\theta_i}{2}$. The notation $()_j$ denotes j permutations of each set of sines and cosines. The remaining five sets of monomials are obtained by multiplying $\mathbf{V}$ by the joint slides $\frac{\Delta d_i}{2}$, so we have a total set of monomials $\mathbf{N}$, where

$$\mathbf{N} = \left(\mathbf{V}, \frac{\Delta d_1}{2}\mathbf{V}, \frac{\Delta d_2}{2}\mathbf{V}, \frac{\Delta d_3}{2}\mathbf{V}, \frac{\Delta d_4}{2}\mathbf{V}, \frac{\Delta d_5}{2}\mathbf{V}\right). \tag{2.68}$$

The kinematics equations of the 5C robot can now be written as the linear combination,

$$\hat{Q}_{5C} = \sum_{i=1}^{192} \mathbf{K}_i m_i, \quad m_i \in \mathbf{N}. \tag{2.69}$$

The coefficients $\mathbf{K}_i$ are 8-dimensional vectors containing the structural variables defining the joint axes.

This equation is adjusted to accommodate a revolute or prismatic joints inserted as the jth joint axis by selecting the non-zero components of the vector $\mathbf{N}$. Notice if the jth C joint is restricted to be a revolute joint, then the slide Δd_j is zero, which eliminates 32 components in $\mathbf{N}$. Similarly, if this joint is replaced by a prismatic joint the angle becomes $\Delta\theta_j = 0$, which eliminates 16 terms from the vector $\mathbf{V}$.

In order to construct these equations start with the array $L_{5C} = \{1, 2, \ldots, 192\}$ of indices that denote the components of $\mathbf{N}$ for the general 5C chain, sorted as shown above. Next define the arrays L_{R_j}, L_{P_j} and L_{C_j} that denote the non-zero components of $\mathbf{N}$ for the cases when joint j is either a revolute, prismatic or cylindrical joint, given by

$$\begin{aligned} L_{R_j} &= \left\{ i : \left(\cos\frac{\Delta\theta_j}{2} \wedge \sin\frac{\Delta\theta_j}{2} \right) \in m_i \vee \frac{\Delta d_j}{2} \notin m_i \right\}, \\ L_{P_j} &= \left\{ i : \left(\frac{\Delta d_j}{2} \wedge \cos\frac{\Delta\theta_j}{2} \right) \in m_i \vee \sin\frac{\Delta\theta_j}{2} \notin m_i \right\}, \\ L_{C_j} &= \left\{ i : \left(\frac{\Delta d_j}{2} \wedge \cos\frac{\Delta\theta_j}{2} \wedge \sin\frac{\Delta\theta_j}{2} \right) \in m_i \right\}, \end{aligned} \tag{2.70}$$

where $\wedge$ and $\vee$ are the logical or, and operations, respectively. Finally, compute the array of indices L for a specific serial chain topology by intersecting the arrays obtained for all of the joints, that is,

$$L = \bigcap_{j=1}^{5} (L_{R_j} \cup L_{P_j} \cup L_{C_j}), \tag{2.71}$$

where $L_{P_j} = \emptyset$ and $L_{C_j} = \emptyset$ if j is a revolute joint, for example.

The kinematics equations for the specific serial chain is now given by

$$\hat{Q} = \sum_{i \in L} \mathbf{K}_i m_i. \tag{2.72}$$

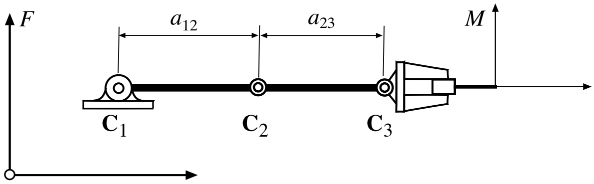


Fig. 2.3 A planar 3R chain in the reference configuration

The synthesis equations for the chain are obtained by equating the kinematics equations in (2.72) to the set of task positions $\hat{P}_{1i}$, that is

$$\hat{Q} = \hat{P}_{1i}, \quad i = 2, \ldots, m, \tag{2.73}$$

where the maximum number of task positions, m, is obtained for the chosen topology using (2.48) and (2.49). Additional constraint equations may be added to account for the specialized geometry of T and S joints or for any other geometric constraint present in the robot.

These synthesis equations are solved to determine the joint axes S_i in the reference configuration, as well as for values for the joint variables that ensure that the serial chain reaches each of the task positions.

2.6.1 The Synthesis Process

It is possible to automate the generation of the synthesis equations as cases of the four classes of 2C, 3C, 4C and 5C related serial chains. The synthesis equations can then be solved numerically given a random start value. The input data consists of a set of task positions and topology of the serial chain. The topology of the chains is used to construct its kinematics equations $\hat{Q}$. These equations are set equal to the task positions $\hat{P}_{1i}$ to yield the synthesis equations as the difference $\hat{Q} - \hat{P}_{1i}$, $i = 2, \ldots, m$. The numerical solver finds values for the components of the joint axes and joint variables that minimize this difference.

It is not necessary that the numerical solver use the minimum set of design equations as defined by (2.48). In fact, it is convenient to use all $8(m-1)+c$ synthesis equations. For the cases of 3R, 4R and 5R serial chains this approach introduces two, eight and 30 redundant equations, respectively. Experience shows that the additional equations enhance the convergence of the numerical algorithm.

2.7 Planar Serial Chains

We now specialize the kinematics equations defined above to the case of planar serial chains. It is convenient for our purposes to focus on chains consisting only of revolute joints, the nR chain [22].

The Plucker coordinates of the axis of a typical revolute joint in a planar chain are given by $\mathsf{J} = (\mathbf{k}, \mathbf{C} \times \mathbf{k})$, where $\mathbf{k} = (0, 0, 1)$ is directed along the z-axis of the base frame, and $\mathbf{C} = (c_x, c_y, 0)$ is the point of intersection of this axis with the x-y plane. The associated twist matrix $\hat{J}$ is

$$\hat{J} = \begin{bmatrix} 0 & -1 & 0 & -c_y \\ 1 & 0 & 0 & c_x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.74}$$

Let the transformation to the base of the chain be a translation by the vector $\mathbf{G} = (g_x, g_y, 0)$, then he zero configuration of the nR planar chain has the points $\mathbf{C}_i$, $i = 1, \dots, n$ on the joint axes J_i distributed along a line parallel to x-axis (see Fig. 2.3), such that

$$\begin{aligned} &\mathbf{C}_1 = \begin{Bmatrix} g_x \\ g_y \\ 0 \end{Bmatrix}, \qquad \mathbf{C}_2 = \begin{Bmatrix} g_z + a_{12} \\ g_y \\ 0 \end{Bmatrix}, \quad \dots, \\ &\mathbf{C}_n = \begin{Bmatrix} g_x + a_{12} + a_{23} + \cdots + a_{n-1,n} \\ g_y \\ 0 \end{Bmatrix}. \end{aligned} \tag{2.75}$$

Substituting these points into (2.74) we obtain a twist matrix $\hat{J}_i$ for each revolute joint, and the product of exponentials kinematics equations

$$\big[D(\boldsymbol{\theta})\big] = e^{\Delta\theta_1 \hat{J}_1} e^{\Delta\theta_2 \hat{J}_2} \dots e^{\Delta\theta_n \hat{J}_n} [D_0]. \tag{2.76}$$

The zero frame transformation $[D_0]$ can be define by introducing $[C]$ which is the translation by the vector $\mathbf{c} = (a_{12} + a_{23} + \cdots + a_{n-1,n})\boldsymbol{\imath}$ along the chain in the zero configuration, so we have

$$[D_0] = [G][C][H]. \tag{2.77}$$

The matrix exponential defining the rotation about J by the angle $\Delta\theta$ can be computed using formulas in Murray et al. (1994) [4] to yield,

$$e^{\Delta\theta \hat{J}} = \begin{bmatrix} \cos\Delta\theta & -\sin\Delta\theta & 0 & (1-\cos\Delta\theta)c_x + \sin\Delta\theta c_y \\ \sin\Delta\theta & \cos\Delta\theta & 0 & -\sin\Delta\theta c_x + (1-\cos\Delta\theta)c_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.78}$$

This matrix defines a displacement consisting of a planar rotation about the point $\mathbf{C}$, called the *pole of the displacement*.

2.7.1 Complex Number Kinematics Equations

It is convenient at this point to introduce the complex numbers $e^{i\Delta\theta} = \cos\Delta\theta + i\sin\Delta\theta$ and $\mathbf{C} = c_x + ic_y$ to simplify the representation of the displacement (2.78).

Let $\mathbf{X}_1 = x + iy$ be the coordinates of a point in the world frame in the first position and $\mathbf{X}_2 = X + iY$ be its coordinates in the second position, then this transformation becomes

$$\mathbf{X}_2 = e^{i\Delta\theta}\mathbf{X}_1 + \left(1 - e^{i\Delta\theta}\right)\mathbf{C}. \tag{2.79}$$

The complex numbers $[e^{i\Delta\theta}, (1 - e^{i\Delta\theta})\mathbf{C}]$ define the rotation and translation, that form the planar displacement $e^{\Delta\theta\hat{J}}$. The point $\mathbf{C}$ is the pole of the displacement, and the translation vector $\mathbf{D}$ associated with this displacement is given by

$$\mathbf{D} = \left(1 - e^{i\Delta\theta}\right)\mathbf{C}. \tag{2.80}$$

The composition of the exponentials $e^{\theta_1\hat{C}_1}$ and $e^{\theta_2\hat{C}_2}$ that define rotations about the points $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively, yields

$$\begin{aligned} e^{\phi\hat{P}} &= e^{\theta_1\hat{C}_1}e^{\theta_2\hat{C}_2}, \quad \text{or} \\ \left[e^{i\phi}, (1 - e^{i\phi})\mathbf{P}\right] &= \left[e^{i\theta_1}, \left(1 - e^{i\theta_1}\right)\mathbf{C}_1\right]\left[e^{i\theta_2}, \left(1 - e^{i\theta_2}\right)\mathbf{C}_2\right] \\ &= \left[e^{i(\theta_1+\theta_2)}, \left(1 - e^{i\theta_1}\right)\mathbf{C}_1 + e^{i\theta_1}\left(1 - e^{i\theta_2}\right)\mathbf{C}_2\right]. \end{aligned} \tag{2.81}$$

Here $\mathbf{P}$ denotes the pole of the composite displacement.

The complex form of the relative kinematics equations (2.15) is seen to be

$$\left[D(\Delta\boldsymbol{\theta})\right] = \left[e^{i\Delta\theta_1}, \left(1-e^{i\Delta\theta_1}\right)\mathbf{C}_1\right]\left[e^{i\Delta\theta_2}, \left(1-e^{i\Delta\theta_2}\right)\mathbf{C}_2\right]\ldots\left[e^{i\Delta\theta_n}, \left(1-e^{i\Delta\theta_n}\right)\mathbf{C}_n\right]. \tag{2.82}$$

If we define the relative displacement of the end-effector to be $[D] = [e^{i\Delta\phi}, (1 - e^{i\Delta\phi})\mathbf{P}]$, then we can expand this equation and equate the rotation and translation components to obtain,

$$\begin{aligned} e^{i\Delta\phi} &= e^{i\Delta\theta_1}e^{i\Delta\theta_2}\ldots e^{i\Delta\theta_n} = e^{i(\Delta\theta_1+\Delta\theta_2+\cdots+\Delta\theta_n)}, \\ \left(1 - e^{i\Delta\phi}\right)\mathbf{P} &= \left(1 - e^{i\Delta\theta_1}\right)\mathbf{C}_1 + e^{i\Delta\theta_1}\left(1 - e^{i\Delta\theta_2}\right)\mathbf{C}_2 + \cdots \\ &\quad + e^{i(\Delta\theta_1+\Delta\theta_2+\cdots+\Delta\theta_{n-1})}\left(1 - e^{i\Delta\theta_n}\right)\mathbf{C}_n. \end{aligned} \tag{2.83}$$

These complex vector equations can be used to design planar nR serial chains. We will see shortly that they are exactly Sandor and Erdman's standard form equations. However, in the next section we introduce an equivalent set of design equations using the Clifford algebra form of the kinematics equations.

2.8 The Even Clifford Algebra $C^+(P^2)$

The even Clifford algebra of the projective plane P^2 is a generalization of complex numbers. It is a vector space with a product operation that is linked to a scalar product. The elements of this Clifford algebra can be identified with the complex vectors that define points in the plane, and with rotations and translations of these coordinates. Our goal is a structure for the design equations that facilitates treating the relative joint angles as design parameters and can be generalized to the design of spatial serial chains [3].

Using homogeneous coordinates of points in the projective plane as the vectors and a degenerate scalar product, we obtain an eight dimensional Clifford algebra, $C(P^2)$. See McCarthy (1990) [2]. This Clifford algebra has an even sub-algebra, $C^+(P^2)$, which is a set of four dimensional elements of the form

$$A = a_1 i\varepsilon + a_2 j\varepsilon + a_3 k + a_4. \tag{2.84}$$

The basis elements $i\varepsilon$, $j\varepsilon$, k and 1 satisfy the following multiplication table,

$$\begin{array}{c|cccc} & i\varepsilon & j\varepsilon & k & 1 \\ \hline i\varepsilon & 0 & 0 & -j\varepsilon & i\varepsilon \\ j\varepsilon & 0 & 0 & i\varepsilon & j\varepsilon \\ k & j\varepsilon & -i\varepsilon & -1 & k \\ 1 & i\varepsilon & j\varepsilon & k & 1 \end{array} \tag{2.85}$$

Notice that the set of Clifford algebra elements $\mathbf{z} = x + ky$ formed using the basis element k ($k^2 = -1$) is isomorphic to the usual set of complex numbers. This means that we have $e^{k\theta} = \cos\theta + k\sin\theta$.

Translation by the vector $\mathbf{d} = d_x + kd_y$ and rotation by the angle ϕ are represented by the Clifford algebra elements

$$T(\mathbf{d}) = 1 + \frac{1}{2}\mathbf{d}i\varepsilon \quad \text{and} \quad R(\phi) = e^{k\phi/2}, \tag{2.86}$$

and a general planar displacement $D = T(\mathbf{d})R(\phi)$ is given by

$$D = \left(1 + \frac{1}{2}\mathbf{d}i\varepsilon\right)e^{k\phi/2}. \tag{2.87}$$

McCarthy (1993) [12] shows that a displacement defined to be a rotation by $\Delta\theta$ about a point $\mathbf{C}$ has the associated Clifford algebra element

$$D = \left(1 + \frac{1}{2}\left(1 - e^{k\Delta\theta}\right)\mathbf{C}i\varepsilon\right)e^{k\Delta\theta/2}, \tag{2.88}$$

which is the Clifford algebra version of the matrix exponential (2.78). Expand this equation to obtain the four dimensional vector

$$D = \frac{1}{2}\left(e^{-k\Delta\theta/2} - e^{k\Delta\theta/2}\right)\mathbf{C}i\varepsilon + e^{k\Delta\theta/2}$$

$$
\begin{aligned}
&= -\sin\frac{\Delta\theta}{2}\mathbf{C}j\varepsilon + e^{k\Delta\theta/2}\\
&= c_y \sin\frac{\Delta\theta}{2} i\varepsilon - c_x \sin\frac{\Delta\theta}{2} j\varepsilon + \sin\frac{\Delta\theta}{2}k + \cos\frac{\Delta\theta}{2}.
\end{aligned} \tag{2.89}
$$

The components of this vector form the kinematic mapping used by Bottema and Roth (1979) [26] to study planar displacements. Also see DeSa and Roth (1981) [5] and Ravani and Roth (1983) [6].

2.8.1 Clifford Algebra Kinematics Equations

The relative kinematics equations of an nR planar chain (2.82) can be written in terms of the Clifford algebra elements (2.89) to define,

$$
\begin{aligned}
-\sin\frac{\Delta\phi}{2}\mathbf{P}j\varepsilon + e^{k\Delta\phi/2} = &\left(-\sin\frac{\Delta\theta_{1j}}{2}\mathbf{C}_1 j\varepsilon + e^{k\Delta\theta_1/2}\right)\\
&\times\left(-\sin\frac{\Delta\theta_2}{2}\mathbf{C}_2 j\varepsilon + e^{k\Delta\theta_2/2}\right)\\
&\cdots\left(-\sin\frac{\Delta\theta_n}{2}\mathbf{C}_n j\varepsilon + e^{k\Delta\theta_n/2}\right).
\end{aligned} \tag{2.90}
$$

Expand this equations and equate coefficients of the basis elements to obtain

$$
\begin{aligned}
&e^{k\Delta\phi/2} = e^{k(\Delta\theta_1+\Delta\theta_2+\cdots+\Delta\theta_n)/2},\\
&\sin\frac{\Delta\phi}{2}\mathbf{P} = \sin\frac{\Delta\theta_1}{2}\mathbf{C}_1 e^{-k(\Delta\theta_2+\cdots+\Delta\theta_n)/2} + e^{k\Delta\theta_1/2}\sin\frac{\Delta\theta_2}{2}\mathbf{C}_2 e^{-k(\Delta\theta_3+\cdots+\Delta\theta_n)/2}\\
&\qquad + \cdots + e^{k(\Delta\theta_1+\Delta\theta_2+\cdots+\Delta\theta_{n-1})/2}\sin\frac{\Delta\theta_n}{2}\mathbf{C}_n.
\end{aligned} \tag{2.91}
$$

These equations are equivalent to the complex vector equations presented above. In fact, multiplication of (2.91) by $e^{k\Delta\phi/2}$ directly yields (2.83), note we must replace k by in the usual complex number i.

2.9 Design Equations for the Planar nR Chain

The goal of our design problem is to determine the dimensions of the planar nR chain that can position a tool held by its end-effector in a given set of task positions. The location of the base of the robot, the position of the tool frame, as well as the link dimensions and joint angles are considered to be design variables [22].

2.9.1 Relative Kinematics Equations for Specified Task Positions

Identify a set of planar task positions $[P_j]$, $j=1,\dots,m$. Then, the physical dimensions of the chain are defined by the requirement that for each position $[P_j]$ there is a joint parameter vector $\boldsymbol{\theta}_j$ such that the kinematics equations of the chain yield

$$[P_j]=\left[D(\boldsymbol{\theta}_j)\right],\quad i=1,\dots,m. \tag{2.92}$$

Now, choose $[P_1]$ as the reference position and compute the relative displacements $[P_j][P_1^{-1}]=[P_{1j}]$, $j=2,\dots,m$. This formulation of the linkage design equations can be found in Suh and Racliffe (1978) [7]. The result is the relative kinematics equations

$$[P_{1j}]=e^{\Delta\theta_{1j}\hat{J}_1}e^{\Delta\theta_{2j}\hat{J}_2}\dots e^{\Delta\theta_{nj}\hat{J}_n},\quad j=2,\dots,m, \tag{2.93}$$

where

$$\Delta\boldsymbol{\theta}_j=\boldsymbol{\theta}_j-\boldsymbol{\theta}_1=(\Delta\theta_{1j},\dots,\Delta\theta_{nj}).$$

The complex number form of (2.93) yields the equations

$$\begin{aligned} e^{i\Delta\phi_j}&=e^{i(\Delta\theta_{1j}+\Delta\theta_{2j}+\dots+\Delta\theta_{nj})},\\ \left(1-e^{i\Delta\phi_j}\right)\mathbf{P}_{1j}&=\left(1-e^{i\Delta\theta_{1j}}\right)\mathbf{C}_1+e^{i\Delta\theta_{1j}}\left(1-e^{i\Delta\theta_{2j}}\right)\mathbf{C}_2+\cdots\\ &\quad+e^{i(\Delta\theta_{1j}+\Delta\theta_{2j}+\dots+\Delta\theta_{n-1,j})}\left(1-e^{i\Delta\theta_{nj}}\right)\mathbf{C}_n,\quad j=2,\dots,m, \end{aligned} \tag{2.94}$$

where $\Delta\phi_j=\phi_j-\phi_1$ and $\mathbf{P}_{1j}$ is the pole of the relative displacement $[P_{1j}]$. These are the equations we use to design the planar nR chain.

In terms of elements of the Clifford algebra we obtain the equivalent set of design equations,

$$\begin{aligned} e^{k\Delta\phi_j/2}&=e^{k(\Delta\theta_{1j}+\Delta\theta_{2j}+\dots+\Delta\theta_{nj})/2},\\ \sin\frac{\Delta\phi_j}{2}\mathbf{P}_{1j}&=\sin\frac{\Delta\theta_{1j}}{2}\mathbf{C}_1e^{-k(\Delta\theta_{2j}+\dots+\Delta\theta_{nj})/2}\\ &\quad+e^{k\Delta\theta_{1j}/2}\sin\frac{\Delta\theta_{2j}}{2}\mathbf{C}_2e^{-k(\Delta\theta_{3j}+\dots+\Delta\theta_{nj})/2}\\ &\quad+\dots+e^{k(\Delta\theta_{1j}+\Delta\theta_{2j}+\dots+\Delta\theta_{n-1,j})/2}\sin\frac{\Delta\theta_{nj}}{2}\mathbf{C}_n,\quad j=2,\dots,m. \end{aligned} \tag{2.95}$$

Equations (2.95) allow the introduction of $\sin\frac{\Delta\theta_{ij}}{2}$ and $\cos\frac{\Delta\theta_{ij}}{2}$ as algebraic unknowns so these equations can be solved for the various joint angles as well as the coordinates of the joints. This is demonstrated below in our algebraic solution of the five position synthesis of a 2R chain.

2.9.2 The Number of Design Positions and Free Parameters

If we specify m task positions, then (2.94) provide $m-1$ rotation and $2(m-1)$ translation equations. The unknowns consist of the $n(m-1)$ relative joint angles, and the $2n$ coordinates $\mathbf{C}_i$, $i = 1, \ldots, n$.

It is useful to notice that the rotation equations are solved independently, which means that they determine $m-1$ of the relative joint angles. Thus, we have $2(m-1)$ translation equations to solve for $(n-1)(m-1)$ joint variables and $2n$ coordinates $\mathbf{C}_i$, that is

$$E = 2n + (n-1)(m-1) - 2(m-1) = m(n-3) + n + 3, \tag{2.96}$$

where E excess of unknowns over equations.

Notice that except for $n = 1$ and $n = 2$ the excess of variables over equations is greater than zero. For $n = 1$, we see that $m = 2$ yields an exact formula for what is equivalent to the pole of a relative displacement. For $n = 2$, we find that an exact solution is possible for $m = 5$, which is Burmester's result that a 2R chain can be designed to reach five specified positions (Burmester 1886 [8], Hartenberg and Denavit 1964 [9]).

Now consider the case $n = 3$, which has six unknown coordinates $\mathbf{C}_i$, $i = 1, 2, 3$, and $2(m-1)$ joint variables that are determined by $2(m-1)$ equations. The excess is $E = 6$ no matter how many positions are specified. In order to formulate this design problem, we specify the $m-1$ relative joint angles around $\mathbf{C}_1$. This is equivalent to adding $m-1$ design equations, which means that (2.47) takes the form $E = 6-(m-1)$. The result is that given seven positions, $m = 7$, we obtain a set of equations that determine the six coordinates $\mathbf{C}_i$, $i = 1, 2, 3$.

2.9.3 The Standard Form Equations

The synthesis of planar 2R chains is the primary step in the design of four-bar linkages, which are constructed by joining the end links of two 2R chains to form the floating link, or coupler. Specializing the relative kinematics equations (2.94) to this case, we obtain

$$\begin{gathered} e^{i\Delta\phi_j} = e^{i(\Delta\theta_{1j}+\Delta\theta_{2j})}, \\ \left(1-e^{i\Delta\phi_j}\right)\mathbf{P}_{1j} = \left(1-e^{i\Delta\theta_{1j}}\right)\mathbf{C}_1 + e^{i\Delta\theta_{1j}}\left(1-e^{i\Delta\theta_{2j}}\right)\mathbf{C}_2, \quad j = 2, \ldots, m. \end{gathered} \tag{2.97}$$

We now show that this is the standard form equation used by Sandor and Erdman for planar mechanism synthesis.

The standard form equation is obtained by equating the relative displacement vector between two positions to the difference of vectors along the chain in the two

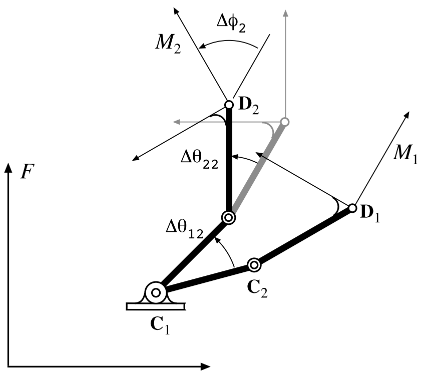


Fig. 2.4 Two positions of a planar 2R chain

positions. See Fig. 2.4. Let $\mathbf{C}_1$ be the fixed pivot and $\mathbf{C}_2$ the moving pivot when the tool frame of 2R chain is aligned with the first position.

Introduce the relative vectors $\mathbf{W} = \mathbf{C}_2 - \mathbf{C}_1$ and $\mathbf{Z} = \mathbf{D}_1 - \mathbf{C}_2$, where $\mathbf{D}_1$ is the translation vector to the first task position. We can now form the vector equations

$$\begin{aligned}
&\mathbf{D}_1 = \mathbf{C}_1 + \mathbf{W} + \mathbf{Z},\\
&\mathbf{D}_2 = \mathbf{C}_1 + \mathbf{W}e^{i\Delta\theta_{12}} + \mathbf{Z}e^{i(\Delta\theta_{12}+\Delta\theta_{22})},\\
&\dots\\
&\mathbf{D}_m = \mathbf{C}_1 + \mathbf{W}e^{i\Delta\theta_{1m}} + \mathbf{Z}e^{i(\Delta\theta_{1m}+\Delta\theta_{2m})}.
\end{aligned} \tag{2.98}$$

Recall that multiplication by the complex exponential rotates a vector by an angle measured relative to the x-axis of the fixed frame.

Subtract the first equation from the remaining m to obtain

$$\delta_{1j} = \mathbf{W}\big(e^{i\Delta\theta_{1j}} - 1\big) + \mathbf{Z}\big(e^{i(\Delta\theta_{1j}+\Delta\theta_{2j})} - 1\big), \quad j = 2, \dots, m, \tag{2.99}$$

where $\delta_{1j} = \mathbf{D}_j - \mathbf{D}_1$. Notice that the rotation of the jth task frame relative to the first position is

$$\Delta\phi_j = \Delta\theta_{1j} + \Delta\theta_{2j}. \tag{2.100}$$

Sandor and Erdman (1984) [11] call (2.99) the *standard form equation* and they use it to formulate a range of linkage synthesis problems based on the planar 2R chain.

Now substitute the definition of the relative vectors $\mathbf{W}$, $\mathbf{Z}$ and δ_{ij} back into the standard form equation to obtain

$$\mathbf{D}_j - \mathbf{D}_1 = (\mathbf{C}_2 - \mathbf{C}_1)\big(e^{i\Delta\theta_{1j}} - 1\big) + (\mathbf{D}_1 - \mathbf{C}_2)\big(e^{i(\Delta\theta_{1j}+\Delta\theta_{2j})} - 1\big),$$

and simplify to obtain

$$\mathbf{D}_j - \mathbf{D}_1 e^{i\Delta\phi_j} = \big(1 - e^{i\Delta\theta_{1j}}\big)\mathbf{C}_1 + e^{i\Delta\theta_{1j}}\big(1 - e^{i\Delta\theta_{2j}}\big)\mathbf{C}_2, \quad j = 1, \dots, m. \tag{2.101}$$

In order to show that this equation is identical to (2.97) we compute the pole $\mathbf{P}_{1j}$ in terms of the translation vectors $\mathbf{D}_j$ and $\mathbf{D}_1$.

Let $[D_j] = [e^{i\phi_j}, \mathbf{D}_j]$, $j = 1, \dots, m$, and compute

$$[D_{1j}] = [D_j][D_1]^{-1} = \left[e^{i(\phi_j-\phi_1)}, \mathbf{D}_j - \mathbf{D}_1 e^{i(\phi_j-\phi_1)}\right]. \tag{2.102}$$

Now the pole $\mathbf{P}_{1j}$ of this relative displacement is defined as the point that has the same coordinates before and after the displacement, which means it satisfies the condition

$$\mathbf{P}_{1j} = e^{i(\phi_j-\phi_1)}\mathbf{P}_{1j} + \mathbf{D}_j - \mathbf{D}_1 e^{i(\phi_j-\phi_1)}. \tag{2.103}$$

Thus, we obtain

$$\left(1 - e^{i\Delta\phi_j}\right)\mathbf{P}_{1j} = \mathbf{D}_j - \mathbf{D}_1 e^{\Delta\phi_j}, \tag{2.104}$$

and substituting this into (2.101), we find that the relative kinematics equations (2.97) are exactly Sandor and Erdman's standard form equations.

2.9.4 Synthesis of 3R Serial Chains

The planar 3R robot has three degrees of freedom and can reach any set of positions within its workspace boundary. The design equations for m task positions take the form

$$\begin{aligned} e^{i\Delta\phi_j} &= e^{i(\Delta\theta_{1j}+\Delta\theta_{2j}+\Delta\theta_{3j})}, \\ \left(1 - e^{i\Delta\phi_j}\right)\mathbf{P}_{1j} &= \left(1 - e^{i\Delta\theta_{1j}}\right)\mathbf{C}_1 + e^{i\Delta\theta_{1j}}\left(1 - e^{i\Delta\theta_{2j}}\right)\mathbf{C}_2 \\ &\quad + e^{i(\Delta\theta_{1j}+\Delta\theta_{2j})}\left(1 - e^{i\Delta\theta_{3j}}\right)\mathbf{C}_3, \quad j = 2, \dots, m. \end{aligned} \tag{2.105}$$

We consider the design of this chain for three, five and seven task positions with the condition that the relative joint angles around $\mathbf{C}_1$ are specified by the designer.

Three Task Positions If we specify three task positions, the result is four translation design equations, or two complex equations, which determine the six coordinates of $\mathbf{C}_i$ and the $2(3-1) = 4$ relative joint angles around $\mathbf{C}_1$ and $\mathbf{C}_2$. The joint angles around $\mathbf{C}_3$ are determined by the rotation design equations.

If we specify the four unknown relative joint angles and $\mathbf{C}_1$, then these four design equations are linear in the coordinates of $\mathbf{C}_2$ and $\mathbf{C}_3$. The result is two complex linear equations in two complex unknowns,

$$\begin{aligned} \kappa_{12} &= e^{i\Delta\theta_{12}}\left(1 - e^{i\Delta\theta_{22}}\right)\mathbf{C}_2 + e^{i(\Delta\theta_{12}+\Delta\theta_{22})}\left(1 - e^{i\Delta\theta_{32}}\right)\mathbf{C}_3, \\ \kappa_{13} &= e^{i\Delta\theta_{13}}\left(1 - e^{i\Delta\theta_{23}}\right)\mathbf{C}_2 + e^{i(\Delta\theta_{13}+\Delta\theta_{23})}\left(1 - e^{i\Delta\theta_{33}}\right)\mathbf{C}_3, \end{aligned} \tag{2.106}$$

where κ_{1j} are the known complex numbers,

$$\kappa_{1j} = \left(1 - e^{i\Delta\phi_j}\right)\mathbf{P}_{1j} - \left(1 - e^{i\Delta\theta_{1j}}\right)\mathbf{C}_1. \tag{2.107}$$

Five Task Positions If five task positions are specified, then we have eight translation design equations in fourteen unknowns, the six coordinate $\mathbf{C}_i$ and eight relative joint angles. Now specify the coordinates of $\mathbf{C}_1$ and the four relative angles around it to define six parameters. The result is the four complex equations

$$\begin{aligned}
\kappa_{12} &= e^{i\Delta\theta_{12}}\left(1 - e^{i\Delta\theta_{22}}\right)\mathbf{C}_2 + e^{i(\Delta\theta_{12}+\Delta\theta_{22})}\left(1 - e^{i\Delta\theta_{32}}\right)\mathbf{C}_3, \\
&\cdots \\
\kappa_{15} &= e^{i\Delta\theta_{15}}\left(1 - e^{i\Delta\theta_{25}}\right)\mathbf{C}_2 + e^{i(\Delta\theta_{15}+\Delta\theta_{25})}\left(1 - e^{i\Delta\theta_{35}}\right)\mathbf{C}_3,
\end{aligned} \tag{2.108}$$

where κ_{1j} are known complex number defined by (2.107). These equations have exactly the same structure as Sandor and Erdman's standard form (2.101) for five position synthesis and are solved in the same way.

Seven Task Positions If seven task positions are specified as well as the six relative joint angles around $\mathbf{C}_1$, then we obtain 12 translation design equations in the twelve unknowns consisting of the six joint coordinates $\mathbf{C}_i$ and six relative joint angles around $\mathbf{C}_2$. The result is six complex equations

$$\begin{aligned}
\left(1 - e^{i\Delta\phi_2}\right)\mathbf{P}_{12} &= \left(1 - e^{i\Delta\theta_{12}}\right)\mathbf{C}_1 + e^{i\Delta\theta_{12}}\left(1 - e^{i\Delta\theta_{22}}\right)\mathbf{C}_2 \\
&\quad + e^{i(\Delta\theta_{12}+\Delta\theta_{22})}\left(1 - e^{i\Delta\theta_{32}}\right)\mathbf{C}_3, \\
&\cdots \\
\left(1 - e^{i\Delta\phi_7}\right)\mathbf{P}_{17} &= \left(1 - e^{i\Delta\theta_{17}}\right)\mathbf{C}_1 + e^{i\Delta\theta_{17}}\left(1 - e^{i\Delta\theta_{27}}\right)\mathbf{C}_2 \\
&\quad + e^{i(\Delta\theta_{17}+\Delta\theta_{27})}\left(1 - e^{i\Delta\theta_{37}}\right)\mathbf{C}_3.
\end{aligned} \tag{2.109}$$

This problem has been solved using matrix resultants by Lin and Erdman (1987) [13] and using homotopy continuation by Subbian and Flugrad (1994) [14].

2.9.5 Single DOF Coupled Serial Chains

Krovi et al. (2002) [15] expand the standard form equations to nR chains in which the joints are coupled by cable transmissions so the system has one degree of freedom. They call the chain a *single degree-of-freedom coupled serial chain*, or SD-CSC. We formulate an equivalent form of their design equations using the relative kinematics equations (2.94).

Consider a planar nR serial chain in which each joint is connected to ground through a series of cables and pulleys located at each joint. Let each pulley have the same diameter and the cables routed through the links so they form parallelogram linkages. The result is n drive pulleys at the base of the chain that control the angle α_i of the ith link relative to the x-axis of the world frame, which means each joint angle is given by

$$\theta_i = \alpha_i - \alpha_{i-1}. \tag{2.110}$$

We now introduce a single drive angle β such that each joint angle is given by relation $\theta_i = R_i\beta$, where R_i denotes a constant speed ratio. The relations (2.110) yield the transmission matrix $[C]$ to the base drive angles are given by

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{Bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \beta, \tag{2.111}$$

or

$$\boldsymbol{\alpha} = [C][R]\beta, \tag{2.112}$$

where $[R]$ is the column matrix formed by the speed ratios. Our formulation differs slightly from Krovi et al. (2002) [15] in that we have added the drive variable β and therefore an additional speed ratio R_1.

Consider the design of an nR chain in which the speed ratios R_i, $i = 1, \dots, n$, are specified. Substitute these speed ratios into the rotation term of the design equations (2.94) to obtain

$$e^{i\Delta\phi_j} = e^{i(R_1+R_2+\cdots+R_m)\Delta\beta_j}, \quad j = 2, \dots, m, \tag{2.113}$$

where $\Delta\beta_j = \beta_j - \beta_1$ is the relative rotation of the drive angle. We find for each relative task position that

$$\Delta\beta_j = \frac{\Delta\phi_j}{R_1 + R_2 + \cdots + R_n}. \tag{2.114}$$

Substitute this into the translation terms of (2.94) to define a linear equation in the coordinates $\mathbf{C}_i$, $i = 1, \dots, n$ for each relative task position,

$$\begin{aligned}(1 - e^{i\Delta\phi_j})\mathbf{P}_{1j} = (1 - e^{iR_1\Delta\beta j})\mathbf{C}_1 + e^{iR_1\Delta\beta j}(1 - e^{iR_2\Delta\beta j})\mathbf{C}_2 + \cdots \\ + e^{i(R_1+R_2+\cdots+R_{n-1})\Delta\beta j}(1 - e^{iR_n\Delta\beta_j})\mathbf{C}_n, \quad j = 2, \dots, m.\end{aligned} \tag{2.115}$$

Given $m = n + 1$ task positions, we can solve these equations for the n complex unknowns $\mathbf{C}_i$. The result is a coupled serial nR chain designed to reach $n + 1$ arbitrarily specified task positions.

2.10 Reachable Surfaces

In this section, we consider the design of spatial serial chains that guide a body such that a point in the body moves on a specific algebraic surface. The problem originates with Schoenflies [16], who sought points that remained in a given configuration for a given set of spatial displacements. Burmester [8] applied this idea to planar mechanism design by seeking the points in a planar moving body that remain on a circle. Chen and Roth [10] generalized this problem to find points and lines in a moving body that take positions on surfaces associated with the articulated chains used to build robot manipulators.

2.11 Spatial Serial Chains

For our purposes, we consider five degree-of-freedom spatial serial chains that include a spherical wrist. Thus, the reachable surface is traced by the point, **P**, at the wrist center of this chain under the movement of two remaining joints. Considering only, revolute and prismatic joints, we can enumerate the seven possibilities:

1. The PPS chain, for which the wrist center, **P**, lies on a plane—notice that the angle between the slide can be any angle α except zero, similarly the distance ρ between the slides can be any value because a prismatic joint guides all points in the body in the same direction;
2. The TS chain that has **P** on a sphere—recall the T joint is constructed from two perpendicular intersecting revolute joints, that is with link angle $\alpha = \pi/2$ and length $\rho = 0$;
3. The CS chain for which **P** lies on a cylinder—the C joint is constructed from a PR chain for which direction of the prismatic slide is parallel to the axis of the revolute joint, that is $\alpha = 0$, note ρ can be any value;
4. The RPS chain that guides **P** on the surface of a right circular hyperboloid—the link angle α can be any value except zero;
5. The PRS chain in which the angle between of the prismatic slide and the axis of the revolute is not zero guides **P** on an elliptic cylinder—the link angle α can be any value except zero;
6. The "right" RRS chain in which the revolute joints are perpendicular but do not intersect traces has **P** trace a right circular torus—the linkage angle $\alpha = \phi/2$; and
7. The general RRS chain in which the revolute joint axes are not perpendicular nor intersecting guides the wrist center on a general circular torus—the linkage angle cannot be $\alpha = 0, \pi/2$.

The result is seven articulated chains and the associated algebraic surfaces that are reachable by their wrist centers, Table 2.3. These algebraic equations of these surfaces can be used to formulate the synthesis equations for these seven spatial serial chains. In what follows, we determine the number of free parameters for each chain, the associated number of task positions that define these parameters, and assemble the synthesis equations. These equations can be solved using numerical homotopy.

Table 2.3 The basic serial chains and their associated reachable surfaces

Case	Chain	Angle	Length	Surface
1	PPS	$\alpha \neq 0$	ρ	Plane
2	TS	$\alpha = \pi/2$	$\rho = 0$	Sphere
3	CS	$\alpha = 0$	ρ	Circular cylinder
4	RPS	$\alpha \neq 0$	ρ	Circular hyperboloid
5	PRS	$\alpha \neq 0$	ρ	Elliptic cylinder
6	Right RRS	$\alpha = \pi/2$	ρ	Circular torus
7	RRS	α	ρ	General torus

2.11.1 Linear Product Decomposition

The synthesis equations for the seven spatial serial chains describe above result in polynomial systems of very high degree. Bezout's theorem states that the number of solutions to a polynomial system is less than or equal to the degree of the polynomial system, which is obtained by multiplying the degrees of each of the polynomials in the system. In what follows, we find that the synthesis equations of these serial chains have so much internal structure that the total degree over-estimates the number of solutions by two orders of magnitude.

In order to efficiently use numerical homotopy techniques to find all of the solutions to our synthesis equations, it is useful to have a better estimate for the number of solutions than the total degree. Here we present the linear product decomposition of a polynomial system and then use it to determine a bound on the number of solutions for each of our systems of synthesis equations. The linear product decomposition also serves as a convenient start system for numerical homotopy algorithms.

Morgan et al. [17] show that a "generic" system of polynomials that includes every monomial of a specified system of polynomials will have as many or more solutions as the specified polynomial system. The *linear product decomposition* of a specified system of polynomials is a way of constructing this generic polynomial system that includes all of the monomials of the specified system, so that it allows convenient computation of the number of roots. Each polynomial in the linear product decomposition consists of polynomials formed by the products of linear combinations of the variables and all of the monomials of the corresponding original polynomial.

Let $\langle x, y, 1\rangle$ represent the set of linear combinations of parameters x, y and 1, which means a typical term is $\alpha x + \beta y + \gamma \in \langle x, y, 1\rangle$, where α, β and γ are arbitrary constants. Using this notation, we define the product of $\langle x, y, 1\rangle\langle u, v, 1\rangle$ as the set of linear combinations of the product of the elements of the two sets, that is

$$\langle x, y, 1\rangle\langle u, v, 1\rangle = \langle xu, xv, yu, yv, x, y, u, v, 1\rangle. \tag{2.116}$$

This product commutes, which means $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$, and it distributes over unions, such that $\langle x\rangle\langle y\rangle \cup \langle x\rangle\langle z\rangle = \langle x\rangle(\langle y\rangle \cup \langle z\rangle) = \langle x\rangle\langle y, z\rangle$. Furthermore, we represent repeated factors using exponents, so $\langle x, y, 1\rangle\langle x, y, 1\rangle = \langle x, y, 1\rangle^2$.

In order to illustrate the construction of the linear product decomposition consider the synthesis equations of the TS chain presented in the previous chapter, given by

$$\left(\mathbf{P}^i - \mathbf{B}\right) \cdot \left(\mathbf{P}^i - \mathbf{B}\right) = R^2, \quad i = 1, \ldots, 7, \tag{2.117}$$

where the dot denotes the vector dot product. Now subtract the first equation from the rest in order to eliminate R^2. This reduces the problem to six equations in the unknowns $\mathbf{z} = (x, y, z, u, v, w)$, given by

$$\mathscr{S}_j(\mathbf{z}) = \left(\mathbf{P}^{j+1} \cdot \mathbf{P}^{j+1} - \mathbf{P}^1 \cdot \mathbf{P}^1\right) - 2\mathbf{B} \cdot \left(\mathbf{P}^{j+1} - \mathbf{P}^1\right) = 0, \quad j = 1, \ldots, 6. \tag{2.118}$$

We now focus attention on the monomials formed by the unknowns.

Recall that $\mathbf{P}^i = [A_i]\mathbf{p} + \mathbf{d}_i$ where $[A_i]$ and $\mathbf{d}_i$ are known, so it is easy to see that

$$2\mathbf{B} \cdot \left(\mathbf{P}^{j+1} - \mathbf{P}^1\right) \in \langle u, v, w\rangle\langle x, y, z, 1\rangle. \tag{2.119}$$

It is also possible to compute

$$\begin{aligned} \mathbf{P}^{j+1} \cdot \mathbf{P}^{j+1} - \mathbf{P}^1 \cdot \mathbf{P}^1 &= 2\mathbf{d}_{j+1} \cdot [A_{j+1}]\mathbf{p} - 2\mathbf{d}_1 \cdot [A_1]\mathbf{p} + \mathbf{d}_{j+1}^2 - \mathbf{d}_1^2 \\ &\in \langle x, y, z, 1\rangle. \end{aligned} \tag{2.120}$$

Thus, we find that each of the equations in (2.118) has the monomial structure given by

$$\langle x, y, z, 1\rangle \cup \langle u, v, w\rangle\langle x, y, z, 1\rangle \subset \langle x, y, z, 1\rangle\langle u, v, w, 1\rangle. \tag{2.121}$$

This allows us to construct a generic set of polynomials as a product of linear factors that contains our synthesis equations as a special case, that is

$$Q(\mathbf{z}) = \begin{Bmatrix} (a_1x + b_1y + c_1z + d_1)(e_1u + f_1v + g_1w + h_1) \\ \vdots \\ (a_6x + b_6y + c_6z + d_6)(e_6u + f_6v + g_6w + h_6) \end{Bmatrix} = 0, \tag{2.122}$$

where the coefficients are known constants. This is the *linear product decomposition* of the synthesis equations for the TS chain.

This linear product decomposition provides a convenient way to determine a bound on the number of solutions for the original polynomial system. This is done by assembling all combinations of the linear factors, one from each equation, that can be set to zero and solved for the unknown parameters. The number of combinations that yield solutions is the LPD bound for the original polynomial system.

In the example above, select three factors $a_ix + b_iy + c_iz + d_i = 0$ from the six equations, and combine this with the three factors $e_iu + f_iv + g_iw + h_i = 0$ in the remaining equations. A solution of this set of six linear equations is a root of (2.122). Thus, we find that this system has $\binom{6}{3} = 20$ solutions, which matches the known result for (2.118).

In the following sections, we determine the synthesis equations for each of the seven spatial serial chains with a reachable surface. We evaluate its total degree,

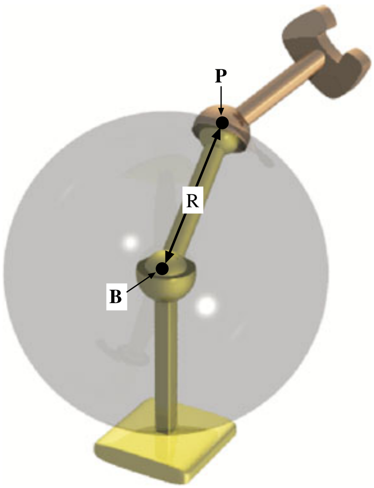


Fig. 2.5 A sphere traced by a point at the wrist center of a TS serial chain

compute its linear product decomposition bound, and then numerically solve a generic problem to find the number of articulated chains that reach a specified set of displacements.

2.11.2 The Sphere

We now return to our opening example in which a point $\mathbf{P} = (X, Y, Z)$ constrained to lie on a sphere of radius R around the point $\mathbf{B} = (u, v, w)$, Fig. 2.5. This means its coordinates satisfy the equation

$$(X-u)^2 + (Y-v)^2 + (Z-w)^2 - R^2 = (\mathbf{P}-\mathbf{B})^2 - R^2 = 0. \qquad (2.123)$$

We now consider $\mathbf{P}^i$ to be the image of a point $\mathbf{p} = (x, y, z)$ in a moving frame M that takes positions in space defined by the displacements $\hat{\mathbf{Q}}_i$, $i = 1, \ldots, n$.

This problem has seven parameters, therefore we can evaluate (2.123) on $n = 7$ displacements. We reduced these equations to the set of six quadratic polynomials,

$$\mathscr{S}_j : \left(\mathbf{P}^{j+1^2} - \mathbf{P}^{1^2}\right) - 2\mathbf{B} \cdot \left(\mathbf{P}^{j+1} - \mathbf{P}^1\right) = 0, \quad j = 1, \ldots, 6. \qquad (2.124)$$

This system has total degree of $2^6 = 64$.

We have already seen that the system (2.124) has the linear product decomposition

$$\mathscr{S}_j \in \langle x, y, z, 1\rangle\langle u, v, w, 1\rangle|_j = 0, \quad j = 1, \ldots, 6. \qquad (2.125)$$

From this we can compute the LPD bound $\binom{6}{3} = 20$. Parameter elimination yields a univariate polynomial of degree 20, so we see that this bound is exact. Innocenti

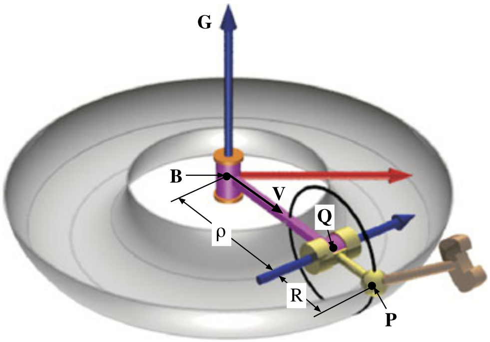


Fig. 2.6 The circular torus traced by the wrist center of a right RRS serial chain

[18] presents an example that results in 20 real roots. Also see Liao and McCarthy [19] and Raghavan [20].

The conclusion is that given seven arbitrary spatial positions there can be as many as 20 points in the moving body that have positions lying on a sphere. For each real point, it is possible to determine an associated TS chain.

2.11.3 The Circular Torus

A circular torus is generated by sweeping a circle around an axis so its center traces a second circle. Let the axis be $L(t) = \mathbf{B} + t\mathbf{G}$, with Plucker coordinates $\mathsf{G} = (\mathbf{G}, \mathbf{B} \times \mathbf{G})$. See Fig. 2.6. Introduce a unit vector $\mathbf{v}$ perpendicular to this axis so the center of the generating circle is given by $\mathbf{Q} - \mathbf{B} = \rho\mathbf{v}$. Now define $\mathbf{u}$ to be the unit vector in the direction $\mathbf{G}$, then a point $\mathbf{P}$ on the torus is defined by the vector equation,

$$\mathbf{P} - \mathbf{B} = \rho\mathbf{v} + R(\cos\phi\mathbf{v} + \sin\phi\mathbf{u}), \tag{2.126}$$

where ϕ is the angle measured from $\mathbf{v}$ to the radius vector of the generating circle.

An algebraic equation of the torus is obtained from (2.126) by first computing the magnitude

$$(\mathbf{P} - \mathbf{B})^2 = \rho^2 + R^2 + 2\rho R\cos\phi. \tag{2.127}$$

Next compute the dot product with $\mathbf{u}$, to obtain

$$(\mathbf{P} - \mathbf{B}) \cdot \mathbf{u} = R\sin\phi. \tag{2.128}$$

Finally, eliminate $\cos\phi$ and $\sin\phi$ from these equations, and the result is

$$\mathbf{G}^2\big((\mathbf{P} - \mathbf{B})^2 - \rho^2 - R^2\big)^2 + 4\rho^2\big((\mathbf{P} - \mathbf{B}) \cdot \mathbf{G}\big)^2 = 4\rho^2\mathbf{G}^2R^2. \tag{2.129}$$

This is the equation of a circular torus. It has 11 parameters, the scalars ρ and R, and the three vectors $\mathbf{G}$, $\mathbf{P}$ and $\mathbf{B}$.

In contrast to what we have done previously, here we set the magnitude of **G** to a constant, in order to simplify the polynomial (2.129),

$$\mathscr{G}: \mathbf{G}\cdot\mathbf{G} = 1. \tag{2.130}$$

Unfortunately, this doubles the number of solutions since $-\mathbf{G}$ and $\mathbf{G}$ define the same torus, however, it reduces this polynomial from degree sixth to degree four.

Let $[T_i] = [A_i, \mathbf{d}_i]$ be a specified set of displacements, so we have the 10 positions $\mathbf{P}^i = [T_i]\mathbf{p}$ of a point $\mathbf{p} = (x, y, z)$ that is fixed in the moving frame M. Evaluating (2.129) on these points, we obtain the polynomial system

$$\begin{aligned}&\mathscr{T}_i\colon \big((\mathbf{P}^i-\mathbf{B})^2-\rho^2-R^2\big)^2+4\rho^2\big((\mathbf{P}^i-\mathbf{B})\cdot\mathbf{G}\big)^2-4\rho^2R^2=0, \quad i=1,\dots,10,\\&\mathscr{G}\colon \mathbf{G}\cdot\mathbf{G}-1=0.\end{aligned} \tag{2.131}$$

The total degree of this system is $2(4^{10}) = 2{,}097{,}152$.

In order to simplify the polynomials $\mathscr{T}_i$ we introduce the parameters

$$\mathbf{H} = 2\rho\mathbf{G} \quad \text{and} \quad k_1 = \mathbf{B}^2 - \rho^2 - R^2, \tag{2.132}$$

which yields the identity

$$4\rho^2R^2 = \mathbf{H}^2\left(\mathbf{B}^2 - \frac{\mathbf{H}^2}{4} - k_1\right). \tag{2.133}$$

Substitute these relations into $\mathscr{T}_i$ to obtain

$$\begin{aligned}&\mathscr{T}_i'\colon \big((\mathbf{P}^i)^2-2\mathbf{P}^i\cdot\mathbf{B}+k_1\big)^2+\big((\mathbf{P}^i-\mathbf{B})\cdot\mathbf{H}\big)^2-\mathbf{H}^2\left(\mathbf{B}^2-\frac{\mathbf{H}^2}{4}-k_1\right)=0,\\&i=1,\dots,10.\end{aligned} \tag{2.134}$$

It is difficult to find a simplified formulation for these equations, even if we subtract the first equation from the remaining in order to cancel terms.

Expanding the polynomial $\mathscr{T}_i'$ and examining each of the terms, we can identify the linear product decomposition

$$\mathscr{T}_i' \in \langle x, y, z, h_1, h_2, h_3, 1\rangle^2 \langle x, y, z, h_1, h_2, h_3, u, v, w, k_1, 1\rangle^2. \tag{2.135}$$

This allows us to compute the LPD bound on the number of roots as

$$\mathrm{LPD} = 2^{10}\sum_{j=0}^{6}\binom{10}{j} = 868{,}352. \tag{2.136}$$

Our POLSYS_GLP algorithm obtained 94,622 real and complex solutions for a random set of specified displacements. However, this problem needs further study to provide an efficient way to evaluate and sort the large number of right RRS chains.

2.12 Summary

The exponential form of the kinematics equations of the chain are reformulated using Clifford algebra exponentials to obtain an efficient and systematic set of design equations. These design equations can be obtained as special cases of those for 2C, 3C, 4C and 5C serial chains. The solution process is demonstrated by determining the structural parameters of a CCS serial chain so that it reaches an arbitrarily specified 12 position task trajectory. While individual solutions can be obtained numerically, these synthesis equations have not yet been formulated for complete solution by numerical homotopy. The complexity of this problem is illustrated by 5R chain synthesis to reach 21 task positions, which requires the solution of 130 equations in 130 unknowns.

We have also formulated the synthesis equations the special cases of serial chains that can position a spherical wrist center on an algebraic surface. A linear product decomposition provides a bound for the number of solutions to these equations. We focus on the SS and RRS which show the challenges of solving the synthesis equations. While algebraic techniques yield 20 solutions for the SS chain, numerical homotopy was needed to compute the over 90,000 solutions for the RRS chain.

This chapter shows that we can derive the synthesis equations for spatial serial chains, however, they are complex and difficult to solve.

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