Preface – Volume 1

Partial differential equations appear in both physics and geometry. Within mathematics they unite the areas of complex analysis, differential geometry and calculus of variations. The investigation of partial differential equations has contributed substantially to the development of functional analysis. Although a relatively uniform treatment of ordinary differential equations is possible, multiple and quite diverse methods are available for partial differential equations. With this two-volume textbook we intend to present the entire domain PARTIAL DIFFERENTIAL EQUATIONS – so rich in theories and applications – to students on the intermediate level.

We presuppose a basic knowledge of the analysis, as it is conveyed in the beautiful lectures [Hi1] and [Hi2] by S. Hildebrandt or in our lecture notes [S1] and [S2] or in W. Rudin's influential textbook [Ru]. For the convenience of the reader we further develop foundations from the analysis in a form adequate to the theory of partial differential equations. Therefore, this textbook can be used for a course extending over several semesters. We have intended to present the theory in the same form we know from books on complex analysis or differential geometry. In our opinion, gaining a deep understanding of the subject replaces the need for exercises, which are implicitly present in our text and may be supplemented from other books. By excluding exercises, we instead focus on presenting a complete and self-contained theory.

A survey of all the topics is provided by the table of contents, which naturally reflects the interests of the author. For advanced readers, each chapter may be studied independently from the others. In selecting the topics of our lectures and consequently for our textbooks, I tried to follow the advice of one of the first great scientists at the University of Göttingen, namely G.C. Lichtenberg: *Teach the students h o w to think and not always w h a t to think!* When I was a student at Göttingen, I admired the commemorative plaques throughout the city in honor of many great physicists and mathematicians. In this spirit, I attribute the results and theorems in our compendium to the persons who – to the best of my knowledge – created them.

The original version of this textbook, *Friedrich Sauvigny: Partielle Differentialgleichungen der Geometrie und der Physik – Grundlagen und Integraldarstellungen – Unter Berücksichtigung der Vorlesungen von E. Heinz*, was first published in 2004 by *Springer-Verlag*. A translated and expanded version of this monograph followed in 2006 as *Springer-Universitext*, namely *Friedrich Sauvigny: Partial Differential Equations 1*, and we are now presenting a second edition of this textbook.

In Chapter 1 we treat the *Differentiation and Integration on Manifolds*, where we use the improper Riemannian integral. After the Weierstraß approximation theorem in Section 1, we introduce differential forms in Section 2 as functionals on surfaces – parallel to [Ru]. The calculus rules for differential forms are immediately derived from the determinant laws and the transformation formula for multiple integrals. With the partition of unity and an adequate approximation we prove the Stokes integral theorem for manifolds in Section 4, which may possess singular boundaries of capacity zero besides their regular boundaries. In Section 5 we especially obtain the Gaussian integral theorem for singular domains as in [H1], which is indispensable for the theory of partial differential equations. After the discussion of contour integrals in Section 6, we shall follow [GL] in Section 7 and represent A. Weil's proof of the Poincaré lemma. In Section 8 we shall explicitly construct the $*$-operator for certain differential forms in order to define the Beltrami operators. Finally, we represent the Laplace operator in n-dimensional spherical coordinates.

In Chapter 2 we shall constructively supply the *Foundations of Functional Analysis*. Having presented Daniell's integral in Section 1, we shall continue the Riemannian integral to the Lebesgue integral in Section 2. The latter is distinguished by convergence theorems for pointwise convergent sequences of functions. We deduce the theories of Lebesgue measurable sets and functions in a natural way; see the Sections 3 and 4. In Section 5 we compare Lebesgue's with Riemann's integral. Then we consider Banach and Hilbert spaces in Section 6, and in Section 7 we present the Lebesgue spaces $L^p(X)$ as classical Banach spaces. Especially important are the selection theorems with respect to almost everywhere convergence due to H. Lebesgue and with respect to weak convergence due to D. Hilbert. Following ideas of J. v. Neumann we investigate bounded linear functionals on $L^p(X)$ in Section 8. For this Chapter 1 have profited from a seminar on functional analysis, offered to us as students by Professor Dr. E. Heinz in Göttingen.

In Chapter 3 we shall study topological properties of mappings in $\mathbb{R}^n$ and solve nonlinear systems of equations. In this context we utilize Brouwer's degree of mapping, for which E. Heinz has given an ingenious integral representation (compare [H8]). Besides the fundamental properties of the degree of mapping, we obtain the classical results of topology. For instance, the theorems of Poincaré on spherical vector-fields and of Jordan-Brouwer on topological spheres in $\mathbb{R}^n$ appear. The case $n = 2$ reduces to the theory of the winding number. In this chapter we essentially follow the first part of the lecture on fixed point theorems [H4] by E. Heinz.

In Chapter 4 we develop the theory of holomorphic functions in one and several complex variables. Since we utilize the Stokes integral theorem, we easily attain the well-known theorems from the classical theory of functions in the Sections 2 and 3. In the subsequent paragraphs we also study solutions of the inhomogeneous Cauchy-Riemann differential equation, which has been completely investigated by L. Bers and I. N. Vekua (see [V]). In Section 6 we assemble statements on pseudoholomorphic functions, which are similar to holomorphic functions as far as the behavior at their zeroes is concerned. In Section 7 we prove the Riemannian mapping theorem with an extremal method due to Koebe and investigate in Section 8 the boundary behavior of conformal mappings. Furthermore, we consider the discontinuous behavior of Cauchy's integral across the boundary in Section 9 and solve a Dirichlet problem for plane harmonic mappings. In this chapter we have profited from the beautiful lecture [Gr] on complex analysis by H. Grauert.

Chapter 5 is devoted to the *Potential Theory in* $\mathbb{R}^n$. With the aid of the Gaussian integral theorem we investigate Poisson's differential equation in Section 1 and Section 2, and we establish an analyticity theorem. With Perron's method we solve the Dirichlet problem for Laplace's equation in Section 3. Starting with Poisson's integral representation, we develop the theory of spherical harmonic functions in $\mathbb{R}^n$; see Section 4 and Section 5. This theory was founded by Legendre, and we owe this elegant representation to G. Herglotz. In this chapter as well, I was able to profit decisively from the lecture [H2] on partial differential equations by my academic teacher, Professor Dr. E. Heinz in Göttingen.

In Chapter 6 we consider *Linear Partial Differential Equations in* $\mathbb{R}^n$. We prove the maximum principle for elliptic differential equations in Section 1 and apply this central tool on quasilinear, elliptic differential equations in Section 2 (compare the lecture [H6]). In Section 3 we turn to the heat equation and present the parabolic maximum-minimum principle. Then in Section 4, we study characteristic surfaces and establish an energy estimate for the wave equation. In Section 5 we solve the Cauchy initial value problem of the wave equation in $\mathbb{R}^n$ for the dimensions $n = 1, 3, 2$. With the aid of Abel's integral equation we solve this problem for all $n \geq 2$ in Section 6 (compare the lecture [H5]). Then we consider the inhomogeneous wave equation and an initial-boundary-value problem in Section 7. For parabolic and hyperbolic equations we recommend the textbooks [GuLe] and [J]. Finally, we classify the linear partial differential equations of second order in Section 8. We discover the Lorentz transformations as invariant transformations for the wave equation (compare [G]).

With Chapters 5 and 6, we intend to give a geometrically oriented introduction into the theory of partial differential equations, without assuming prior functional analytic knowledge.

In this second edition of our monograph *Partial Differential Equations 1*, we have carefully revised Volume 1 and added Section 9 on the Boundary Behavior of Cauchy's Integral in Chapter 4. We shall present a revised version of our book *Partial Differential Equations 2* as well, where we shall add a new chapter on *Boundary Value Problems from Differential Geometry.* The topics of the new Chapter 13 are listed in the table of contents for our enlarged second edition of Volume 2.

We follow the Total Order Code of the Universitext series, however, adapt this to the present extensive contents. Since we see our books as an entity, we count the chapters throughout our two volumes from 1-13. In each section individually, we count the equations and refer to them by a single number; when we refer to an equation in another section of the same chapter, say the m-th section, we have to add *Section m*; when we refer to an equation in the m-th section of another chapter, say the l-th chapter, we have to add *Section m in Chapter l.*

We assemble *definitions, theorems, propositions, examples* to the expression *environment*, which is borrowed from the underlying TEX-file. Individually in each section, we count these environments consecutively by the number n, and denote the n-th environment within the m-th section by *Environment m.n.* Thus we have attributed a pair of integers $m.n$ to all definitions, theorems, propositions, and examples, which is unique within each chapter and easy to find. Referring to these environments throughout both books, we proceed as described above for the equations.

We add Figure 1.1 – Figure 1.9 to our Volume 1 and Figure 2.1 – Figure 2.11 to our Volume 2, which mostly represent portraits of mathematicians. This small photo collection of some scientists, who have contributed to the theory of Partial Differential Equations, already shows that our area is situated in the center of modern mathematics and possesses profound interrelations with geometry and physics.

This textbook PARTIAL DIFFERENTIAL EQUATIONS has been developed from lectures that I have been giving in the Brandenburgische Technische Universität at Cottbus from the winter semester 1992/93 to the present semester. The monograph, in part, builds upon the lectures (see [H1] – [H6]) of Professor Dr. Dr.h.c. E. Heinz, whom I was fortunate to know as his student in Göttingen from 1971 to 1978 and as postdoctoral researcher in his *Oberseminar* from 1983 to 1989. As an assistant in Aachen from 1978 to 1983, I very much appreciated the elegant lecture cycles of Professor Dr. G. Hellwig (see [He1] – [He3]). Since my research fellowship at the University of Bonn in 1989/90, an intensive scientific collaboration with Professor Dr. Dr.h.c.mult. S. Hildebrandt has developed, which continues to this day (see [DHS] and [DHT2] with the list of references therein). All three of these excellent representatives of mathematics will forever have my sincere gratitude and my deep respect!

Here I gratefully acknowledge the valuable and profound advice of Priv.-Doz. Dr. Frank Müller (Universität Duisburg-Essen) for the original edition *Partielle Differentialgleichungen* as well as the indispensable and excellent assistance of Dipl.-Math. Michael Hilschenz (BTU Cottbus) for the present edition *Partial Differential Equations 1.* Furthermore, my sincere thanks are devoted to Mrs. C. Prescott (Berlin) improving the English style of this second edition. Moreover, I would like to thank cordially Herr Clemens Heine (Heidelberg) and Mr. Jörg Sixt (London) as well as Mrs. Lauren Stoney (London) and all the other members of Springer for their helpful collaboration and great confidence.

Cottbus, February 2012 *Friedrich Sauvigny*

Springer

http://www.springer.com/978-1-4471-2980-6

Partial Differential Equations 1
Foundations and Integral Representations
Sauvigny, F.
2012, XV, 447 p. 16 illus., Softcover
ISBN: 978-1-4471-2980-6