Chapter 2

Projection Matrices

2.1 Definition

Definition 2.1  Let \( x \in E^n = V \oplus W \). Then \( x \) can be uniquely decomposed into
\[
x = x_1 + x_2 \quad \text{(where } x_1 \in V \text{ and } x_2 \in W).\]
The transformation that maps \( x \) into \( x_1 \) is called the projection matrix (or simply projector) onto \( V \) along \( W \) and is denoted as \( \phi \). This is a linear transformation; that is,
\[
\phi(a_1y_1 + a_2y_2) = a_1\phi(y_1) + a_2\phi(y_2)
\]
for any \( y_1, y_2 \in E^n \). This implies that it can be represented by a matrix. This matrix is called a projection matrix and is denoted by \( P_{V,W} \). The vector transformed by \( P_{V,W} \) (that is, \( x_1 = P_{V,W}x \)) is called the projection (or the projection vector) of \( x \) onto \( V \) along \( W \).

Theorem 2.1  The necessary and sufficient condition for a square matrix \( P \) of order \( n \) to be the projection matrix onto \( V = \text{Sp}(P) \) along \( W = \text{Ker}(P) \) is given by
\[
P^2 = P.
\]

We need the following lemma to prove the theorem above.

Lemma 2.1  Let \( P \) be a square matrix of order \( n \), and assume that (2.2) holds. Then
\[
E^n = \text{Sp}(P) \oplus \text{Ker}(P)
\]
and
\[ \text{Ker}(P) = \text{Sp}(I_n - P). \] (2.4)

**Proof of Lemma 2.1.** (2.3): Let \( x \in \text{Sp}(P) \) and \( y \in \text{Ker}(P) \). From \( x = Pa \), we have \( Px = P^2a = Pa = x \) and \( Py = 0 \). Hence, from \( x + y = 0 \Rightarrow Px + Py = 0 \), we obtain \( Px = x = 0 \Rightarrow y = 0 \). Thus, \( \text{Sp}(P) \cap \text{Ker}(P) = \{0\} \). On the other hand, from \( \dim(\text{Sp}(P)) + \dim(\text{Ker}(P)) = \) \( \text{rank}(P) + (n - \text{rank}(P)) = n \), we have \( E^n = \text{Sp}(P) \oplus \text{Ker}(P) \).

(2.4): We have \( Px = 0 \Rightarrow x = (I_n - P)x \Rightarrow \text{Ker}(P) \subset \text{Sp}(I_n - P) \) on the one hand and \( P(I_n - P) \Rightarrow \text{Sp}(I_n - P) \subset \text{Ker}(P) \) on the other. Thus, \( \text{Ker}(P) = \text{Sp}(I_n - P) \).

Q.E.D.

**Note** When (2.4) holds, \( P(I_n - P) = O \Rightarrow P^2 = P \). Thus, (2.2) is the necessary and sufficient condition for (2.4).

**Proof of Theorem 2.1.** (Necessity) For \( \forall x \in E^n, y = Px \in V \). Noting that \( y = y + 0 \), we obtain
\[ P(Px) = Py = y = Px \implies P^2x = Px \implies P^2 = P. \]

(Sufficiency) Let \( V = \{y | y = Px, x \in E^n\} \) and \( W = \{y | y = (I_n - P)x, x \in E^n\} \). From Lemma 2.1, \( V \) and \( W \) are disjoint. Then, an arbitrary \( x \in E^n \) can be uniquely decomposed into \( x = Px + (I_n - P)x = x_1 + x_2 \) (where \( x_1 \in V \) and \( x_2 \in W \)). From Definition 2.1, \( P \) is the projection matrix onto \( V = \text{Sp}(P) \) along \( W = \text{Ker}(P) \).

Q.E.D.

Let \( E^n = V \oplus W \), and let \( x = x_1 + x_2 \), where \( x_1 \in V \) and \( x_2 \in W \). Let \( P_{W \rightarrow V} \) denote the projector that transforms \( x_1 \) into \( x_2 \). Then,
\[ P_{V \rightarrow W}x + P_{W \rightarrow V}x = (P_{V \rightarrow W} + P_{W \rightarrow V})x. \] (2.5)
Because the equation above has to hold for any \( x \in E^n \), it must hold that
\[ I_n = P_{V \rightarrow W} + P_{W \rightarrow V}. \]

Let a square matrix \( P \) be the projection matrix onto \( V \) along \( W \). Then, \( Q = I_n - P \) satisfies \( Q^2 = (I_n - P)^2 = I_n - 2P + P^2 = I_n - P = Q \), indicating that \( Q \) is the projection matrix onto \( W \) along \( V \). We also have
\[ PQ = P(I_n - P) = P - P^2 = O, \] (2.6)
implying that $\text{Sp}(Q)$ constitutes the null space of $P$ (i.e., $\text{Sp}(Q) = \text{Ker}(P)$). Similarly, $QP = O$, implying that $\text{Sp}(P)$ constitutes the null space of $Q$ (i.e., $\text{Sp}(P) = \text{Ker}(Q)$).

**Theorem 2.2** Let $E^n = V \oplus W$. The necessary and sufficient conditions for a square matrix $P$ of order $n$ to be the projection matrix onto $V$ along $W$ are:

(i) $Px = x$ for $\forall x \in V$,  
(ii) $Px = 0$ for $\forall x \in W$. \hspace{1cm} (2.7)

**Proof.** (Sufficiency) Let $P_{V,W}$ and $P_{W,V}$ denote the projection matrices onto $V$ along $W$ and onto $W$ along $V$, respectively. Premultiplying (2.5) by $P$, we obtain $P(P_{V,W}x) = P_{V,W}x$, where $PP_{W,V}x = 0$ because of (i) and (ii) above, and $P_{V,W}x \in V$ and $P_{W,V}x \in W$. Since $Px = P_{V,W}x$ holds for any $x$, it must hold that $P = P_{V,W}$.

(Necessity) For any $x \in V$, we have $x = x + 0$. Thus, $Px = x$. Similarly, for any $y \in W$, we have $y = 0 + y$, so that $Py = 0$.

Q.E.D.

**Example 2.1** In Figure 2.1, $\overrightarrow{OA}$ indicates the projection of $z$ onto $\text{Sp}(x)$ along $\text{Sp}(y)$ (that is, $\overrightarrow{OA} = P_{\text{Sp}(x) \cdot \text{Sp}(y)}z$), where $P_{\text{Sp}(x) \cdot \text{Sp}(y)}$ indicates the projection matrix onto $\text{Sp}(x)$ along $\text{Sp}(y)$. Clearly, $\overrightarrow{OB} = (I_2 - P_{\text{Sp}(y) \cdot \text{Sp}(x)}) \times z$.

![Figure 2.1: Projection onto $\text{Sp}(x) = \{x\}$ along $\text{Sp}(y) = \{y\}$](image)

**Example 2.2** In Figure 2.2, $\overrightarrow{OA}$ indicates the projection of $z$ onto $V = \{x| x = \alpha_1x_1 + \alpha_2x_2\}$ along $\text{Sp}(y)$ (that is, $\overrightarrow{OA} = P_{V \cdot \text{Sp}(y)}z$), where $P_{V \cdot \text{Sp}(y)}$ indicates the projection matrix onto $V$ along $\text{Sp}(y)$. 


Figure 2.2: Projection onto a two-dimensional space $V$ along $\text{Sp}(y) = \{y\}$.

**Theorem 2.3** The necessary and sufficient condition for a square matrix $P$ of order $n$ to be a projector onto $V$ of dimensionality $r$ ($\dim(V) = r$) is given by

$$P = T \Delta_r T^{-1},$$

where $T$ is a square nonsingular matrix of order $n$ and

$$\Delta_r = \begin{bmatrix}
1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0
\end{bmatrix}.$$

(There are $r$ unities on the leading diagonals, $1 \leq r \leq n$.)

**Proof.** (Necessity) Let $E^n = V \oplus W$, and let $A = [a_1, a_2, \cdots, a_r]$ and $B = [b_1, b_2, \cdots b_{n-r}]$ be matrices of linearly independent basis vectors spanning $V$ and $W$, respectively. Let $T = [A, B]$. Then $T$ is nonsingular, since $\text{rank}(A) + \text{rank}(B) = \text{rank}(T)$. Hence, $\forall x \in V$ and $\forall y \in W$ can be expressed as

$$x = A\alpha = [A, B] \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = T \begin{pmatrix} \alpha \\ 0 \end{pmatrix},$$

$$y = A\alpha = [A, B] \begin{pmatrix} 0 \\ \beta \end{pmatrix} = T \begin{pmatrix} 0 \\ \beta \end{pmatrix}.$$
Thus, we obtain
\[ Px = x \implies PT \left( \begin{array}{c} \alpha \\ 0 \end{array} \right) = T \left( \begin{array}{c} \alpha \\ 0 \end{array} \right) = T \Delta r \left( \begin{array}{c} \alpha \\ 0 \end{array} \right), \]
\[ Py = 0 \implies PT \left( \begin{array}{c} 0 \\ \beta \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) = T \Delta r \left( \begin{array}{c} 0 \\ \beta \end{array} \right). \]
Adding the two equations above, we obtain
\[ PT \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) = T \Delta r \left( \begin{array}{c} \alpha \\ \beta \end{array} \right). \]
Since \( \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \) is an arbitrary vector in the \( n \)-dimensional space \( E^n \), it follows that
\[ PT = T \Delta r \implies P = T \Delta r T^{-1}. \]
Furthermore, \( T \) can be an arbitrary nonsingular matrix since \( V = \text{Sp}(A) \) and \( W = \text{Sp}(B) \) such that \( E^n = V \oplus W \) can be chosen arbitrarily.
(Sufficiency) \( P \) is a projection matrix, since \( P^2 = P \), and \( \text{rank}(P) = r \) from Theorem 2.1. (Theorem 2.2 can also be used to prove the theorem above.)
Q.E.D.

**Lemma 2.2** Let \( P \) be a projection matrix. Then,
\[ \text{rank}(P) = \text{tr}(P). \] (2.9)
**Proof.** \( \text{rank}(P) = \text{rank}(T \Delta r T^{-1}) = \text{rank}(\Delta r) = \text{tr}(T \Delta T^{-1}) = \text{tr}(P). \)
Q.E.D.

The following theorem holds.

**Theorem 2.4** Let \( P \) be a square matrix of order \( n \). Then the following three statements are equivalent.
\[ P^2 = P, \] (2.10)
\[ \text{rank}(P) + \text{rank}(I_n - P) = n, \] (2.11)
\[ E^n = \text{Sp}(P) \oplus \text{Sp}(I_n - P). \] (2.12)
**Proof.** (2.10) \( \rightarrow \) (2.11): It is clear from \( \text{rank}(P) = \text{tr}(P). \)
(2.11) → (2.12): Let $V = \text{Sp}(P)$ and $W = \text{Sp}(I_n - P)$. Then, $\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$. Since $x = Px + (I_n - P)x$ for an arbitrary $n$-component vector $x$, we have $E^n = V + W$. Hence, $\dim(V \cap W) = 0 \implies V \cap W = \{0\}$, establishing (2.12).

(2.12) → (2.10): Postmultiplying $I_n = P + (I_n - P)$ by $P$, we obtain $P = P^2 + (I_n - P)P$, which implies $P(I_n - P) = (I_n - P)P$. On the other hand, we have $P(I_n - P) = 0$ and $(I_n - P)P = 0$ because $\text{Sp}(P(I_n - P)) \subset \text{Sp}(P)$ and $\text{Sp}((I_n - P)P) \subset \text{Sp}(I_n - P)$. Q.E.D.

**Corollary**

$$P^2 = P \iff \text{Ker}(P) = \text{Sp}(I_n - P). \quad (2.13)$$

**Proof.** ($\Rightarrow$): It is clear from Lemma 2.1.

($\Leftarrow$): $\text{Ker}(P) = \text{Sp}(I_n - P) \Leftrightarrow P(I_n - P) = 0 \Rightarrow P^2 = P$. Q.E.D.

### 2.2 Orthogonal Projection Matrices

Suppose we specify a subspace $V$ in $E^n$. There are in general infinitely many ways to choose its complement subspace $V^c = W$. We will discuss some of them in Chapter 4. In this section, we consider the case in which $V$ and $W$ are orthogonal, that is, $W = V^\perp$.

Let $x, y \in E^n$, and let $x$ and $y$ be decomposed as $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in V$ and $x_2, y_2 \in W$. Let $P$ denote the projection matrix onto $V$ along $V^\perp$. Then, $x_1 = Px$ and $y_1 = Py$. Since $(x_2, Py) = (y_2, Px) = 0$, it must hold that

$$(x, Py) = (Px + x_2, Py) = (Px, Py)$$

$$= (Px, Py + y_2) = (Px, y) = (x, P'y)$$

for any $x$ and $y$, implying

$$P' = P. \quad (2.14)$$

**Theorem 2.5** *The necessary and sufficient condition for a square matrix $P$ of order $n$ to be an orthogonal projection matrix (an orthogonal projector) is given by*

(i) $P^2 = P$ and (ii) $P' = P$.

**Proof.** (Necessity) That $P^2 = P$ is clear from the definition of a projection matrix. That $P' = P$ is as shown above.

(Sufficiency) Let $x = P\alpha \in \text{Sp}(P)$. Then, $Px = P^2\alpha = P\alpha = x$. Let $y \in \text{Sp}(P)^\perp$. Then, $Py = 0$ since $(Px, y) = x'P'y = x'Py = 0$ must
hold for an arbitrary \( \mathbf{x} \). From Theorem 2.2, \( \mathbf{P} \) is the projection matrix onto \( \text{Sp}(\mathbf{P}) \) along \( \text{Sp}(\mathbf{P})^\perp \); that is, the orthogonal projection matrix onto \( \text{Sp}(\mathbf{P}) \). Q.E.D.

**Definition 2.2** A projection matrix \( \mathbf{P} \) such that \( \mathbf{P}^2 = \mathbf{P} \) and \( \mathbf{P}' = \mathbf{P} \) is called an orthogonal projection matrix (projector). Furthermore, the vector \( \mathbf{P}\mathbf{x} \) is called the orthogonal projection of \( \mathbf{x} \). The orthogonal projector \( \mathbf{P} \) is in fact the projection matrix onto \( \text{Sp}(\mathbf{P}) \) along \( \text{Sp}(\mathbf{P})^\perp \), but it is usually referred to as the orthogonal projector onto \( \text{Sp}(\mathbf{P}) \). See Figure 2.3.

![Figure 2.3: Orthogonal projection.](image)

**Note** A projection matrix that does not satisfy \( \mathbf{P}' = \mathbf{P} \) is called an oblique projector as opposed to an orthogonal projector.

**Theorem 2.6** Let \( \mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m] \), where \( \mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m \) are linearly independent. Then the orthogonal projector onto \( V = \text{Sp}(\mathbf{A}) \) spanned by \( \mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m \) is given by

\[
\mathbf{P} = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'.
\]  

**Proof.** Let \( \mathbf{x}_1 \in \text{Sp}(\mathbf{A}) \). From \( \mathbf{x}_1 = \mathbf{A}\alpha \), we obtain \( \mathbf{P}\mathbf{x}_1 = \mathbf{x}_1 = \mathbf{A}\alpha = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{x}_1 \). On the other hand, let \( \mathbf{x}_2 \in \text{Sp}(\mathbf{A})^\perp \). Then, \( \mathbf{A}'\mathbf{x}_2 = 0 \implies \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{x}_2 = 0 \). Let \( \mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 \). From \( \mathbf{P}\mathbf{x}_2 = 0 \), we obtain \( \mathbf{P}\mathbf{x} = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{x} \), and (2.15) follows because \( \mathbf{x} \) is arbitrary.
Let \( Q = I_n - P \). Then \( Q \) is the orthogonal projector onto \( \text{Sp}(A)^\perp \), the ortho-complement subspace of \( \text{Sp}(A) \).

**Example 2.3** Let \( 1_n = (1, 1, \cdots, 1)' \) (the vector with \( n \) ones). Let \( P_M \) denote the orthogonal projector onto \( V_M = \text{Sp}(1_n) \). Then,

\[
P_M = 1_n (1_n' 1_n)^{-1} 1_n' = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}.
\]

(2.16)

The orthogonal projector onto \( V_M^\perp = \text{Sp}(1_n)^\perp \), the ortho-complement subspace of \( \text{Sp}(1_n) \), is given by

\[
I_n - P_M = \begin{bmatrix} 1 - \frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1 - \frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & 1 - \frac{1}{n} \end{bmatrix}.
\]

(2.17)

Let

\[
Q_M = I_n - P_M.
\]

(2.18)

Clearly, \( P_M \) and \( Q_M \) are both symmetric, and the following relation holds:

\[
P_M^2 = P_M, \quad Q_M^2 = Q_M, \quad \text{and} \quad P_M Q_M = Q_M P_M = O.
\]

(2.19)

**Note** The matrix \( Q_M \) in (2.18) is sometimes written as \( P_M^\perp \).

**Example 2.4** Let

\[
x_R = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x = \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix}, \quad \text{where} \quad \bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j.
\]

Then,

\[
x = Q_M x_R,
\]

(2.20)

and so

\[
\sum_{j=1}^{n} (x_j - \bar{x})^2 = ||x||^2 = x' x = x_R' Q_M x_R.
\]

The proof is omitted.
2.3 Subspaces and Projection Matrices

In this section, we consider the relationships between subspaces and projectors when the $n$-dimensional space $\mathbb{E}^n$ is decomposed into the sum of several subspaces.

2.3.1 Decomposition into a direct-sum of disjoint subspaces

**Lemma 2.3** When there exist two distinct ways of decomposing $\mathbb{E}^n$,

\[ E^n = V_1 \oplus W_1 = V_2 \oplus W_2, \]  

(2.21)

and if $V_1 \subset W_2$ or $V_2 \subset W_1$, the following relation holds:

\[ E^n = (V_1 \oplus V_2) \oplus (W_1 \cap W_2). \]  

(2.22)

**Proof.** When $V_1 \subset W_2$, Theorem 1.5 leads to the following relation:

\[ V_1 + (W_1 \cap W_2) = (V_1 + W_1) \cap W_2 = E^n \cap W_2 = W_2. \]

Also from $V_1 \cap (W_1 \cap W_2) = (V_1 \cap W_1) \cap W_2 = \{0\}$, we have $W_2 = V_1 \oplus (W_1 \cap W_2)$. Hence the following relation holds:

\[ E^n = V_2 \oplus W_2 = V_2 \oplus V_1 \oplus (W_1 \cap W_2) = (V_1 \oplus V_2) \oplus (W_1 \cap W_2). \]

When $V_2 \subset W_2$, the same result follows by using $W_1 = V_2 \oplus (W_1 \cap W_2)$. Q.E.D.

**Corollary** When $V_1 \subset V_2$ or $W_2 \subset W_1$,

\[ E^n = (V_1 \oplus W_2) \oplus (V_2 \cap W_1). \]  

(2.23)

**Proof.** In the proof of Lemma 2.3, exchange the roles of $W_2$ and $V_2$. Q.E.D.

**Theorem 2.7** Let $P_1$ and $P_2$ denote the projection matrices onto $V_1$ along $W_1$ and onto $V_2$ along $W_2$, respectively. Then the following three statements are equivalent:

(i) $P_1 + P_2$ is the projector onto $V_1 \oplus V_2$ along $W_1 \cap W_2$.

(ii) $P_1 P_2 = P_2 P_1 = O$.

(iii) $V_1 \subset W_2$ and $V_2 \subset W_1$. (In this case, $V_1$ and $V_2$ are disjoint spaces.)

**Proof.** (i) $\rightarrow$ (ii): From $(P_1 + P_2)^2 = P_1 + P_2$, $P_1^2 = P_1$, and $P_2^2 = P_2$, we
have $P_1P_2 = -P_2P_1$. Pre- and postmultiplying both sides by $P_1$, we obtain
$P_1P_2 = -P_1P_2P_1$ and $P_1P_2P_1 = -P_2P_1$, respectively, which imply
$P_1P_2 = P_2P_1$. This and $P_1P_2 = -P_2P_1$ lead to $P_1P_2 = P_2P_1 = O$.

(ii) $\rightarrow$ (iii): For an arbitrary vector $x \in V_1$, $P_1x = x$ because $P_1x \in V_1$.
Hence, $P_2P_1x = P_2x = 0$, which implies $x \in W_2$, and so $V_1 \subset W_2$. On
the other hand, when $x \in V_2$, it follows that $P_2x \in V_2$, and so $P_1P_2x =
P_1x = 0$, implying $x \in W_2$. We thus have $V_2 \subset W_2$.

(iii) $\rightarrow$ (ii): For $x \in E^n$, $P_1x \in V_1$, which implies $(I - P_2)P_1x = P_1x$,
which holds for any $x$. Thus, $(I - P_2)P_1 = P_1$, implying $P_1P_2 = O$.
We also have $x \in E^n \Rightarrow P_2x \in V_2 \Rightarrow (I - P_1)P_2x = P_2x$, which again
holds for any $x$, which implies $(I - P_1)P_2 = P_2 \Rightarrow P_1P_2 = O$. Similarly,
$P_2P_1 = O$.

(ii) $\rightarrow$ (i): An arbitrary vector $x \in (V_1 \oplus V_2)$ can be decomposed into
$x = x_1 + x_2$, where $x_1 \in V_1$ and $x_2 \in V_2$. From $P_1x_2 = P_1P_2x = 0$
and $P_2x_1 = P_2P_1x = 0$, we have $(P_1 + P_2)x = (P_1 + P_2)(x_1 + x_2) =
P_1x_1 + P_2x_2 = x_1 + x_2 = x$. On the other hand, by noting that $P_1 =
P_1(I - P_2)$ and $P_2 = P_2(I - P_1)$ for any $x \in (W_1 \cap W_2)$, we have
$(P_1 + P_2)x = P_1(I - P_2)x + P_2(I - P_1)x = 0$. Since $V_1 \subset W_2$ and
$V_2 \subset W_1$, the decomposition on the right-hand side of (2.22) holds. Hence,
we know $P_1 + P_2$ is the projector onto $V_1 \oplus V_2$ along $W_1 \cap W_2$ by Theorem
2.2.

Q.E.D.

Note In the theorem above, $P_1P_2 = O$ in (ii) does not imply $P_2P_1 = O$.
$P_1P_2 = O$ corresponds with $V_2 \subset W_1$, and $P_2P_1 = O$ with $V_1 \subset W_2$ in (iii). It
should be clear that $V_1 \subset W_2 \iff V_2 \subset W_1$ does not hold.

Theorem 2.8 Given the decompositions of $E^n$ in (2.21), the following three
statements are equivalent:

(i) $P_2 - P_1$ is the projector onto $V_2 \cap W_1$ along $V_1 \oplus W_2$.
(ii) $P_1P_2 = P_2P_1 = P_1$.
(iii) $V_1 \subset V_2$ and $W_2 \subset W_1$.

Proof. (i) $\rightarrow$ (ii): $(P_2 - P_1)^2 = P_2 - P_1$ implies $2P_1 = P_1P_2 + P_2P_1$.
Pre- and postmultiplying both sides by $P_2$, we obtain $P_2P_1 = P_2P_1P_2$
and $P_1P_2 = P_2P_1P_2$, respectively, which imply $P_1P_2 = P_2P_1 = P_1$.

(ii) $\rightarrow$ (iii): For $\forall x \in E^n$, $P_1x \in V_1$, which implies $P_1x = P_2P_1x \in V_2$,
which in turn implies $V_1 \subset V_2$. Let $Q_j = I_n - P_j$ $(j = 1, 2)$. Then,
$P_1P_2 = P_1$ implies $Q_1Q_2 = Q_2$, and so $Q_2x \in W_2$, which implies $Q_2x =
Q_1Q_2x \in W_1$, which in turn implies $W_2 \subset W_1$. 


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(iii) → (i): From $V_1 \subset V_2$, for $\forall x \in E^n$, $P_1 x \in V_1 \subset V_2 \Rightarrow P_2(P_1 x) = P_1 x \Rightarrow P_2 P_1 = P_1$. On the other hand, from $W_2 \subset W_1$, $Q_2 x \in W_2 \subset W_1$ for $\forall x \in E^n \Rightarrow Q_1 Q_2 x = Q_2 x \Rightarrow Q_1 Q_2 Q_2 \Rightarrow (I_n - P_1)(I_n - P_2) = (I_n - P_2) \Rightarrow P_1 P_2 = P_1$.

(ii) → (i): For $x \in (V_2 \cap W_1)$, it holds that $(P_2 - P_1)x = Q_1 P_2 x = Q_1 x = x$. On the other hand, let $x = y + z$, where $y \in V_1$ and $z \in W_2$. Then, $(P_2 - P_1)x = (P_2 - P_1)y + (P_2 - P_1)z = P_2 Q_1 y + Q_1 P_2 z = 0$. Hence, $P_2 - P_1$ is the projector onto $V_2 \cap W_1$ along $V_1 + W_2$. Q.E.D.

Note As in Theorem 2.7, $P_1 P_2 = P_1$ does not necessarily imply $P_2 P_1 = P_1$. Note that $P_1 P_2 = P_1 \iff W_2 \subset W_1$, and $P_2 P_1 = P_1 \iff V_1 \subset V_2$.

**Theorem 2.9** When the decompositions in (2.21) and (2.22) hold, and if

$$P_1 P_2 = P_2 P_1,$$

then $P_1 P_2$ (or $P_2 P_1$) is the projector onto $V_1 \cap V_2$ along $W_1 + W_2$.

**Proof.** $P_1 P_2 = P_2 P_1$ implies $(P_1 P_2)^2 = P_1 P_2 P_1 P_2 = P_2^2 P_2 = P_2 P_2$, indicating that $P_1 P_2$ is a projection matrix. On the other hand, let $x \in V_1 \cap V_2$. Then, $P_1(P_2 x) = P_1 x = x$. Furthermore, let $x \in W_1 + W_2$ and $x = x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$. Then, $P_1 P_2 x = P_1 P_2 x_1 + P_1 P_2 x_2 = P_2 P_1 x_1 + 0 = 0$. Since $E^n = (V_1 \cap V_2) \oplus (W_1 \oplus W_2)$ by the corollary to Lemma 2.3, we know that $P_1 P_2$ is the projector onto $V_1 \cap V_2$ along $W_1 \oplus W_2$. Q.E.D.

Note Using the theorem above, (ii) → (i) in Theorem 2.7 can also be proved as follows: From $P_1 P_2 = O$

$$Q_1 Q_2 = (I_n - P_1)(I_n - P_2) = I_n - P_1 - P_2 = Q_2 Q_1.$$ 

Hence, $Q_1 Q_2$ is the projector onto $W_1 \cap W_2$ along $V_1 \oplus V_2$, and $P_1 + P_2 = I_n - Q_1 Q_2$ is the projector onto $V_1 \oplus V_2$ along $W_1 \cap W_2$.

If we take $W_1 = V_1^\perp$ and $W_2 = V_2^\perp$ in the theorem above, $P_1$ and $P_2$ become orthogonal projectors.
Theorem 2.10  Let $P_1$ and $P_2$ be the orthogonal projectors onto $V_1$ and $V_2$, respectively. Then the following three statements are equivalent:

(i) $P_1 + P_2$ is the orthogonal projector onto $V_1 \oplus V_2$.
(ii) $P_1 P_2 = P_2 P_1 = O$.
(iii) $V_1$ and $V_2$ are orthogonal.

Theorem 2.11  The following three statements are equivalent:

(i) $P_2 - P_1$ is the orthogonal projector onto $V_2 \cap V_1^\perp$.
(ii) $P_1 P_2 = P_2 P_1 = P_1$.
(iii) $V_1 \subset V_2$.

The two theorems above can be proved by setting $W_1 = V_1^\perp$ and $W_2 = V_2^\perp$ in Theorems 2.7 and 2.8.

Theorem 2.12  The necessary and sufficient condition for $P_1 P_2$ to be the orthogonal projector onto $V_1 \cap V_2$ is (2.24).

Proof. Sufficiency is clear from Theorem 2.9. Necessity follows from $P_1 P_2 = (P_1 P_2)'$, which implies $P_1 P_2 = P_2 P_1$ since $P_1 P_2$ is an orthogonal projector. Q.E.D.

We next present a theorem concerning projection matrices when $E^n$ is expressed as a direct-sum of $m$ subspaces, namely

$$E^n = V_1 \oplus V_2 \oplus \cdots \oplus V_m.$$  \hspace{1cm} (2.25)

Theorem 2.13  Let $P_i$ ($i = 1, \cdots, m$) be square matrices that satisfy

$$P_1 + P_2 + \cdots + P_m = I_n.$$  \hspace{1cm} (2.26)

Then the following three statements are equivalent:

$$P_i P_j = O \quad (i \neq j).$$  \hspace{1cm} (2.27)
$$P_i^2 = P_i \quad (i = 1, \cdots m).$$  \hspace{1cm} (2.28)
$$\text{rank}(P_1) + \text{rank}(P_2) + \cdots + \text{rank}(P_m) = n.$$  \hspace{1cm} (2.29)
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**Proof.** (i) → (ii): Multiply (2.26) by $P_i$.

(ii) → (iii): Use $\text{rank}(P_i) = \text{tr}(P_i)$ when $P_i^2 = P_i$. Then,

$$\sum_{i=1}^{m} \text{rank}(P_i) = \sum_{i=1}^{m} \text{tr}(P_i) = \text{tr} \left( \sum_{i=1}^{m} P_i \right) = \text{tr}(I_n) = n.$$ 

(iii) → (i), (ii): Let $V_i = \text{Sp}(P_i)$. From $\text{rank}(P_i) = \text{dim}(V_i)$, we obtain $\text{dim}(V_1) + \text{dim}(V_2) + \cdots + \text{dim}(V_m) = n$; that is, $E^n$ is decomposed into the sum of $m$ disjoint subspaces as in (2.26). By postmultiplying (2.26) by $P_i$, we obtain

$$P_1 P_i + P_2 P_i + \cdots + P_i(P_i - I_n) + \cdots + P_m P_i = 0.$$ 

Since $\text{Sp}(P_1), \text{Sp}(P_2), \cdots, \text{Sp}(P_m)$ are disjoint, (2.27) and (2.28) hold from Theorem 1.4. Q.E.D.

**Note** $P_i$ in Theorem 2.13 is a projection matrix. Let $E^n = V_1 \oplus \cdots \oplus V_r$, and let

$$V_{(i)} = V_1 \oplus \cdots \oplus V_{i-1} \oplus V_{i+1} \oplus \cdots \oplus V_r.$$ 

Then, $E^n = V_i \oplus V_{(i)}$. Let $P_{i \cdot (i)}$ denote the projector onto $V_i$ along $V_{(i)}$. This matrix coincides with the $P_i$ that satisfies the four equations given in (2.26) through (2.29).

The following relations hold.

**Corollary 1**

$$P_{1 \cdot (1)} + P_{2 \cdot (2)} + \cdots + P_{m \cdot (m)} = I_n,$$  

$$P_{i \cdot (i)}^2 = P_{i \cdot (i)} \ (i = 1, \cdots, m),$$  

$$P_{i \cdot (i)} P_{j \cdot (j)} = 0 \ (i \neq j).$$

**Corollary 2** Let $P_{(i) \cdot i}$ denote the projector onto $V_{(i)}$ along $V_i$. Then the following relation holds:

$$P_{(i) \cdot i} = P_{1 \cdot (1)} + \cdots + P_{i-1 \cdot (i-1)} + P_{i+1 \cdot (i+1)} + \cdots + P_{m \cdot (m)}.$$ 

**Proof.** The proof is straightforward by noting $P_{i \cdot (i)} + P_{(i) \cdot i} = I_n$. Q.E.D.
Note The projection matrix $P_{i(i)}$ onto $V_i$ along $V(i)$ is uniquely determined. Assume that there are two possible representations, $P_{i(i)}$ and $P_{i(i)}^*$. Then,

$$P_{1(1)} + P_{2(2)} + \cdots + P_{m(m)} = P_{1(1)}^* + P_{2(2)}^* + \cdots + P_{m(m)}^*,$$

from which

$$(P_{1(1)} - P_{1(1)}^*) + (P_{2(2)} - P_{2(2)}^*) + \cdots + (P_{m(m)} - P_{m(m)}^*) = O.$$

Each term in the equation above belongs to one of the respective subspaces $V_1, V_2, \cdots, V_m$, which are mutually disjoint. Hence, from Theorem 1.4, we obtain $P_{i(i)} = P_{i(i)}^*$. This indicates that when a direct-sum of $E^n$ is given, an identity matrix $I_n$ of order $n$ is decomposed accordingly, and the projection matrices that constitute the decomposition are uniquely determined.

The following theorem due to Khatri (1968) generalizes Theorem 2.13.

**Theorem 2.14** Let $P_i$ denote a square matrix of order $n$ such that

$$P = P_1 + P_2 + \cdots + P_m. \tag{2.35}$$

Consider the following four propositions:

(i) $P_i^2 = P_i$ \quad ($i = 1, \cdots m$),

(ii) $P_iP_j = O$ \quad ($i \neq j$), and $\text{rank}(P_i^2) = \text{rank}(P_i)$,

(iii) $P^2 = P$,

(iv) $\text{rank}(P) = \text{rank}(P_1) + \cdots + \text{rank}(P_m)$.

All other propositions can be derived from any two of (i), (ii), and (iii), and (i) and (ii) can be derived from (iii) and (iv).

**Proof.** That (i) and (ii) imply (iii) is obvious. To show that (ii) and (iii) imply (iv), we may use

$$P^2 = P_1^2 + P_2^2 + \cdots + P_m^2 \quad \text{and} \quad P^2 = P,$$

which follow from (2.35).

(ii), (iii) → (i): Postmultiplying (2.35) by $P_i$, we obtain $PP_i = P_i^2$, from which it follows that $P_i^3 = P_i^2$. On the other hand, $\text{rank}(P_i^2) = \text{rank}(P_i)$ implies that there exists $W$ such that $P_i^2W_i = P_i$. Hence, $P_i^3 = P_i^2 \Rightarrow P_i^3W_i = P_i^2W_i \Rightarrow P_i(P_i^2W_i) = P_i^2W \Rightarrow P_i^2 = P_i$.
(iii), (iv) → (i), (ii): We have \( \text{Sp}(P) \oplus \text{Sp}(I_n - P) = E^n \) from \( P^2 = P \).
Hence, by postmultiplying the identity

\[
P_1 + P_2 + \cdots + P_m + (I_n - P) = I_n
\]

by \( P \), we obtain \( P_i^2 = P_i \), and \( P_i P_j = O \) \( (i \neq j) \). Q.E.D.

Next we consider the case in which subspaces have inclusion relationships like the following.

**Theorem 2.15** Let

\[
E^n = V_k \supset V_{k-1} \supset \cdots \supset V_2 \supset V_1 = \{0\},
\]

and let \( W_i \) denote a complement subspace of \( V_i \). Let \( P_i \) be the orthogonal projector onto \( V_i \) along \( W_i \), and let \( P_i^* = P_i - P_{i-1} \), where \( P_0 = O \) and \( P_k = I_n \). Then the following relations hold:

(i) \( I_n = P_1^* + P_2^* + \cdots + P_k^* \).

(ii) \( (P_i^*)^2 = P_i^* \).

(iii) \( P_i^* P_j^* = P_j^* P_i^* = O \) \( (i \neq j) \).

(iv) \( P_i \) is the projector onto \( V_i \cap W_{i-1} \) along \( V_{i-1} \oplus W_i \).

**Proof.** (i): Obvious. (ii): Use \( P_i P_{i-1} = P_{i-1} P_i = P_{i-1} \). (iii): It follows from \( (P_i^*)^2 = P_i^* \) that \( \text{rank}(P_i^*) = \text{tr}(P_i^*) = \text{tr}(P_i - P_{i-1}) = \text{tr}(P_i) - \text{tr}(P_{i-1}) \). Hence, \( \sum_{i=1}^k \text{rank}(P_i^*) = \text{tr}(P_k) - \text{tr}(P_0) = n \), from which \( P_i^* P_j^* = O \) follows by Theorem 2.13. (iv): Clear from Theorem 2.8(i). Q.E.D.

**Note** The theorem above does not presuppose that \( P_i \) is an orthogonal projector. However, if \( W_i = V_i^\perp \), \( P_i \) and \( P_i^* \) are orthogonal projectors. The latter, in particular, is the orthogonal projector onto \( V_i \cap V_{i-1}^\perp \).

### 2.3.2 Decomposition into nondisjoint subspaces

In this section, we present several theorems indicating how projectors are decomposed when the corresponding subspaces are not necessarily disjoint. We elucidate their meaning in connection with the commutativity of projectors.
We first consider the case in which there are two direct-sum decompositions of $E^n$, namely

$$E^n = V_1 \oplus W_1 = V_2 \oplus W_2,$$

as given in (2.21). Let $V_{12} = V_1 \cap V_2$ denote the product space between $V_1$ and $V_2$, and let $V_3$ denote a complement subspace to $V_1 + V_2$ in $E^n$. Furthermore, let $P_{1+2}$ denote the projection matrix onto $V_{1+2} = V_1 + V_2$ along $V_3$, and let $P_j$ ($j = 1, 2$) represent the projection matrix onto $V_j$ ($j = 1, 2$) along $W_j$ ($j = 1, 2$). Then the following theorem holds.

**Theorem 2.16** (i) The necessary and sufficient condition for $P_{1+2} = P_1 + P_2 - P_1P_2$ is

$$(V_{1+2} \cap W_2) \subset (V_1 \oplus V_3).$$  \hspace{1cm} (2.36)

(ii) The necessary and sufficient condition for $P_{1+2} = P_1 + P_2 - P_2P_1$ is

$$(V_{1+2} \cap W_1) \subset (V_2 \oplus V_3).$$  \hspace{1cm} (2.37)

**Proof.** (i): Since $V_{1+2} \supset V_1$ and $V_{1+2} \supset V_2$, $P_{1+2} - P_1$ is the projector onto $V_{1+2} \cap W_1$ along $V_1 \oplus V_3$ by Theorem 2.8. Hence, $P_{1+2}P_1 = P_1$ and $P_{1+2}P_2 = P_2$. Similarly, $P_{1+2} - P_2$ is the projector onto $V_{1+2} \cap W_2$ along $V_2 \oplus V_3$. Hence, by Theorem 2.8,

$$P_{1+2} - P_1 - P_2 + P_1P_2 = O \iff (P_{1+2} - P_1)(P_{1+2} - P_2) = O.$$

Furthermore,

$$(P_{1+2} - P_1)(P_{1+2} - P_2) = O \iff (V_{1+2} \cap W_2) \subset (V_1 \oplus V_3).$$

(ii): Similarly, $P_{1+2} - P_1 - P_2 + P_2P_1 = O \iff (P_{1+2} - P_2)(P_{1+2} - P_1) = O \iff (V_{1+2} \cap W_1) \subset (V_2 \oplus V_3).$  \hspace{1cm} Q.E.D.

**Corollary** Assume that the decomposition (2.21) holds. The necessary and sufficient condition for $P_1P_2 = P_2P_1$ is that both (2.36) and (2.37) hold.

The following theorem can readily be derived from the theorem above.

**Theorem 2.17** Let $E^n = (V_1 + V_2) \oplus V_3$, $V_1 = V_{11} \oplus V_{12}$, and $V_2 = V_{22} \oplus V_{12}$, where $V_{12} = V_1 \cap V_2$. Let $P_{1+2}^*$ denote the projection matrix onto $V_1 + V_2$ along $V_3$, and let $P_1^*$ and $P_2^*$ denote the projectors onto $V_1$ along $V_3 \oplus V_{22}$ and onto $V_2$ along $V_3 \oplus V_{11}$, respectively. Then,

$$P_1^*P_2^* = P_2^*P_1^*$$  \hspace{1cm} (2.38)
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\[ P_{1+2}^* = P_1^* + P_2^* - P_1^*P_2^*. \]  

**Proof.** Since \( V_{11} \subset V_1 \) and \( V_{22} \subset V_2 \), we obtain

\[ V_{1+2} \cap W_2 = V_{11} \subset (V_1 \oplus V_3) \text{ and } V_{1+2} \cap W_1 = V_{22} \subset (V_2 \oplus V_3) \]

by setting \( W_1 = V_{22} \oplus V_3 \) and \( W_2 = V_{11} \oplus V_3 \) in Theorem 2.16.

**Another proof.** Let \( y = y_1 + y_2 + y_{12} + y_3 \in E^n \), where \( y_1 \in V_{11}, y_2 \in V_{22}, y_{12} \in V_{12}, \) and \( y_3 \in V_3 \). Then it suffices to show that \( (P_1^*P_2^*)y = (P_2^*P_1^*)y \). Q.E.D.

Let \( P_j (j = 1, 2) \) denote the projection matrix onto \( V_j \) along \( W_j \). Assume that \( E^n = V_1 \oplus W_1 \oplus V_3 = V_2 \oplus W_2 \oplus V_3 \) and \( V_1 + V_2 = V_{11} \oplus V_{22} \oplus V_{12} \) hold. However, \( W_1 = V_{22} \) may not hold, even if \( V_1 = V_{11} \oplus V_{12} \). That is, (2.38) and (2.39) hold only when we set \( W_1 = V_{22} \) and \( W_2 = V_{11} \).

**Theorem 2.18** Let \( P_1 \) and \( P_2 \) be the orthogonal projectors onto \( V_1 \) and \( V_2 \), respectively, and let \( P_{1+2} \) denote the orthogonal projector onto \( V_{1+2} \). Let \( V_{12} = V_1 \cap V_2 \). Then the following three statements are equivalent:

(i) \( P_1P_2 = P_2P_1 \).  
(ii) \( P_{1+2} = P_1 + P_2 - P_1P_2 \).  
(iii) \( V_{11} = V_1 \cap V_{12}^\perp \) and \( V_{22} = V_2 \cap V_{12}^\perp \) are orthogonal.

**Proof.** (i) \(\rightarrow\) (ii): Obvious from Theorem 2.16.

(ii) \(\rightarrow\) (iii): \( P_{1+2} = P_1 + P_2 - P_1P_2 \Rightarrow (P_{1+2} - P_1)(P_{1+2} - P_2) = (P_{1+2} - P_2)(P_{1+2} - P_1) = O \Rightarrow V_{11} \) and \( V_{22} \) are orthogonal.

(iii) \(\rightarrow\) (i): Set \( V_3 = (V_1 + V_2)^\perp \) in Theorem 2.17. Since \( V_{11} \) and \( V_{22} \), and \( V_1 \) and \( V_{22} \), are orthogonal, the result follows. Q.E.D.

When \( P_1, P_2, \) and \( P_{1+2} \) are orthogonal projectors, the following corollary holds.

**Corollary** \( P_{1+2} = P_1 + P_2 - P_1P_2 \iff P_1P_2 = P_2P_1 \).

### 2.3.3 Commutative projectors

In this section, we focus on orthogonal projectors and discuss the meaning of Theorem 2.18 and its corollary. We also generalize the results to the case in which there are three or more subspaces.
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Theorem 2.19 Let $P_j$ denote the orthogonal projector onto $V_j$. If $P_1P_2 = P_2P_1$, $P_1P_3 = P_3P_1$, and $P_2P_3 = P_3P_2$, the following relations hold:

\[ V_1 + (V_2 \cap V_3) = (V_1 + V_2) \cap (V_1 + V_3), \quad (2.40) \]

\[ V_2 + (V_1 \cap V_3) = (V_1 + V_2) \cap (V_2 + V_3), \quad (2.41) \]

\[ V_3 + (V_1 \cap V_2) = (V_1 + V_3) \cap (V_2 + V_3). \quad (2.42) \]

Proof. Let $P_{1+(2 \cap 3)}$ denote the orthogonal projector onto $V_1 + (V_2 \cap V_3)$. Then the orthogonal projector onto $V_2 \cap V_3$ is given by $P_2P_3$ (or by $P_3P_2$). Since $P_1P_2 = P_2P_1 \Rightarrow P_1P_2P_3 = P_2P_3P_1$, we obtain

\[ P_{1+(2 \cap 3)} = P_1 + P_2P_3 - P_1P_2P_3 \]

by Theorem 2.18. On the other hand, from $P_1P_2 = P_2P_1$ and $P_1P_3 = P_3P_1$, the orthogonal projectors onto $V_1 + V_2$ and $V_1 + V_3$ are given by

\[ P_{1+2} = P_1 + P_2 - P_1P_2 \quad \text{and} \quad P_{1+3} = P_1 + P_3 - P_1P_3, \]

respectively, and so $P_{1+2}P_{1+3} = P_{1+3}P_{1+2}$ holds. Hence, the orthogonal projector onto $(V_1 + V_2) \cap (V_1 + V_3)$ is given by

\[ (P_1 + P_2 - P_1P_2)(P_1 + P_3 - P_1P_3) = P_1 + P_2P_3 - P_1P_2P_3, \]

which implies $P_{1+(2 \cap 3)} = P_{1+2}P_{1+3}$, Since there is a one-to-one correspondence between projectors and subspaces, (2.40) holds.

Relations (2.41) and (2.42) can be similarly proven by noting that $(P_1 + P_2 - P_1P_2)(P_2 + P_3 - P_2P_3) = P_2 + P_1P_3 - P_1P_2P_3$ and $(P_1 + P_3 - P_1P_3)(P_2 + P_3 - P_2P_3) = P_3 + P_1P_2 - P_1P_2P_3$, respectively.

Q.E.D.

The three identities from (2.40) to (2.42) indicate the distributive law of subspaces, which holds only if the commutativity of orthogonal projectors holds.

We now present a theorem on the decomposition of the orthogonal projectors defined on the sum space $V_1 + V_2 + V_3$ of $V_1$, $V_2$, and $V_3$.

Theorem 2.20 Let $P_{1+2+3}$ denote the orthogonal projector onto $V_1 + V_2 + V_3$, and let $P_1$, $P_2$, and $P_3$ denote the orthogonal projectors onto $V_1$, $V_2$, and $V_3$, respectively. Then a sufficient condition for the decomposition

\[ P_{1+2+3} = P_1 + P_2 + P_3 - P_1P_2 - P_2P_3 - P_3P_1 + P_1P_2P_3 \quad (2.43) \]
to hold is

\[ P_1P_2 = P_2P_1, \ P_2P_3 = P_3P_2, \text{ and } P_1P_3 = P_3P_1. \] (2.44)

**Proof.** \( P_1P_2 = P_2P_1 \Rightarrow P_{1+2} = P_1 + P_2 - P_1P_2 \) and \( P_2P_3 = P_3P_2 \Rightarrow P_{2+3} = P_2 + P_3 - P_2P_3 \). We therefore have \( P_{1+2}P_{2+3} = P_{2+3}P_{1+2} \). We also have \( P_{1+2+3} = P_{(1+2)+(1+3)} \), from which it follows that

\[
P_{1+2+3} = (P_{1+2} + P_{1+3} - P_{1+2}P_{1+3}) = (P_1 + P_2 - P_1P_2) + (P_1 + P_3 - P_1P_3) - (P_2P_3 + P_1 - P_1P_2P_3) = P_1 + P_2 + P_3 - P_1P_2 - P_2P_3 - P_1P_3 + P_1P_2P_3.
\]

**An alternative proof.** From \( P_1P_{2+3} = P_{2+3}P_1 \), we have \( P_{1+2+3} = P_1 + P_{2+3} - P_1P_{2+3} \). If we substitute \( P_{2+3} = P_2 + P_3 - P_2P_3 \) into this equation, we obtain (2.43). Q.E.D.

Assume that (2.44) holds, and let

\[
P_1 = P_1 - P_1P_2 - P_1P_3 + P_1P_2P_3,
P_2 = P_2 - P_2P_3 - P_1P_2 + P_1P_2P_3,
P_3 = P_3 - P_1P_3 - P_2P_3 + P_1P_2P_3,
P_{12(3)} = P_1P_2 - P_1P_2P_3,
P_{13(2)} = P_1P_3 - P_1P_2P_3,
P_{23(1)} = P_2P_3 - P_1P_2P_3,
\]

and

\[ P_{123} = P_1P_2P_3. \]

Then,

\[ P_{1+2+3} = P_1 + P_2 + P_3 + P_{12(3)} + P_{13(2)} + P_{23(1)} + P_{123}. \] (2.45)

Additionally, all matrices on the right-hand side of (2.45) are orthogonal projectors, which are also all mutually orthogonal.

**Note** Since \( P_1 = P_1(I_n - P_{2+3}) \), \( P_2 = P_2(I_n - P_{1+3}) \), \( P_3 = P_3(I - P_{1+2}) \), \( P_{12(3)} = P_1P_2(I_n - P_3) \), \( P_{13(2)} = P_1P_3(I_n - P_2) \), and \( P_{23(1)} = P_2P_3(I_n - P_1) \),
the decomposition of the projector $P_{123}$ corresponds with the decomposition of
the subspace $V_1 + V_2 + V_3$

$$V_1 + V_2 + V_3 = V_1 \oplus V_2 \oplus V_3 \oplus V_{12(3)} \oplus V_{13(2)} \oplus V_{23(1)} \oplus V_{123}, \quad (2.46)$$

where $V_1 = V_1 \cap (V_2 + V_3)^{\perp}$, $V_2 = V_2 \cap (V_1 + V_3)^{\perp}$, $V_3 = V_3 \cap (V_1 + V_2)^{\perp}$,
$V_{12(3)} = V_1 \cap V_2 \cap V_3^{\perp}$, $V_{13(2)} = V_1 \cap V_2^{\perp} \cap V_3$, $V_{23(1)} = V_1^{\perp} \cap V_2 \cap V_3$, and
$V_{123} = V_1 \cap V_2 \cap V_3$.

Theorem 2.20 can be generalized as follows.

**Corollary** Let $V = V_1 + V_2 + \cdots + V_s (s \geq 2)$. Let $P_V$ denote the orthogonal
projector onto $V$, and let $P_j$ denote the orthogonal projector onto $V_j$. A
sufficient condition for

$$P_V = \sum_{j=1}^{s} P_j - \sum_{i<j} P_i P_j + \sum_{i<j<k} P_i P_j P_k + \cdots + (-1)^{s-1} P_1 P_2 P_3 \cdots P_s$$

(2.47)

to hold is

$$P_i P_j = P_j P_i \ (i \neq j). \quad (2.48)$$

**2.3.4 Noncommutative projectors**

We now consider the case in which two subspaces $V_1$ and $V_2$ and the cor-
responding projectors $P_1$ and $P_2$ are given but $P_1 P_2 = P_2 P_1$ does not
necessarily hold. Let $Q_j = I_n - P_j$ ($j = 1, 2$). Then the following lemma holds.

**Lemma 2.4**

$$V_1 + V_2 = \text{Sp}(P_1) \oplus \text{Sp}(Q_1 P_2)$$

(2.49)

$$= \text{Sp}(Q_2 P_1) \oplus \text{Sp}(P_2). \quad (2.50)$$

**Proof.** $[P_1, Q_1 P_2]$ and $[Q_2 P_1, P_2]$ can be expressed as

$$[P_1, Q_1 P_2] = [P_1, P_2] \begin{bmatrix} I_n & -P_2 \\ O & I_n \end{bmatrix} = [P_1, P_2]S$$

and

$$[Q_2 P_1, P_2] = [P_1, P_2] \begin{bmatrix} I_n & O \\ -P_1 & I_n \end{bmatrix} = [P_1, P_2]T.$$
Since $S$ and $T$ are nonsingular, we have
\[ \text{rank}(P_1, P_2) = \text{rank}(P_1, Q_1 P_2) = \text{rank}(Q_2 P_1, P_1), \]
which implies
\[ V_1 + V_2 = \text{Sp}(P_1, Q_1 P_2) = \text{Sp}(Q_2 P_1, P_2). \]

Furthermore, let $P_1 x + Q_1 P_2 y = 0$. Premultiplying both sides by $P_1$, we obtain $P_1 x = 0$ (since $P_1 Q_1 = O$), which implies $Q_1 P_2 y = 0$. Hence, $\text{Sp}(P_1)$ and $\text{Sp}(Q_1 P_2)$ give a direct-sum decomposition of $V_1 + V_2$, and so do $\text{Sp}(Q_2 P_1)$ and $\text{Sp}(P_2)$.

Q.E.D.

The following theorem follows from Lemma 2.4.

**Theorem 2.21** Let $E^n = (V_1 + V_2) \oplus W$. Furthermore, let
\[ V_2[1] = \{ x | x = Q_1 y, y \in V_2 \} \quad (2.51) \]
and
\[ V_1[2] = \{ x | x = Q_2 y, y \in V_1 \}. \quad (2.52) \]

Let $Q_j = I_n - P_j$ ($j = 1, 2$), where $P_j$ is the orthogonal projector onto $V_j$, and let $P^*, P^*_1, P^*_2, P^*_{1[2]}$, and $P^*_{2[1]}$ denote the projectors onto $V_1 + V_2$ along $W$, onto $V_1$ along $V_2[1] \oplus W$, onto $V_2$ along $V_1[2] \oplus W$, onto $V_1[2]$ along $V_2 \oplus W$, and onto $V_2[1]$ along $V_1 \oplus W$, respectively. Then,
\[ P^* = P^*_1 + P^*_2 \quad (2.53) \]
or
\[ P^* = P^*_{1[2]} + P^*_2 \quad (2.54) \]
holds.

**Note** When $W = (V_1 + V_2)^\perp$, $P^*_j$ is the orthogonal projector onto $V_j$, while $P^*_{j[i]}$ is the orthogonal projector onto $V_j[i]$.

**Corollary** Let $P$ denote the orthogonal projector onto $V = V_1 \oplus V_2$, and let $P_j$ ($j = 1, 2$) be the orthogonal projectors onto $V_j$. If $V_i$ and $V_j$ are orthogonal, the following equation holds:
\[ P = P_1 + P_2. \quad (2.55) \]
2.4 Norm of Projection Vectors

We now present theorems concerning the norm of the projection vector $Px$ ($x \in E^n$) obtained by projecting $x$ onto $\text{Sp}(P)$ along $\text{Ker}(P)$ by $P$.

Lemma 2.5 $P' = P$ and $P^2 = P \iff P'P = P$.

(The proof is trivial and hence omitted.)

Theorem 2.22 Let $P$ denote a projection matrix (i.e., $P^2 = P$). The necessary and sufficient condition to have

$$||Px|| \leq ||x||$$

for an arbitrary vector $x$ is

$$P' = P.$$  \hfill (2.57)

Proof. (Sufficiency) Let $x$ be decomposed as $x = Px + (I_n - P)x$. We have $(Px)'(I_n - P)x = x'(P' - P'P)x = 0$ because $P' = P \Rightarrow P'P = P'$ from Lemma 2.5. Hence,

$$||x||^2 = ||Px||^2 + ||(I_n - P)x||^2 \geq ||Px||^2.$$  \hfill (2.56)

(Necessity) By assumption, we have $x'(I_n - P'P)x \geq 0$, which implies $I_n - P'P$ is nnd with all nonnegative eigenvalues. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ denote the eigenvalues of $P'P$. Then, $1 - \lambda_j \geq 0$ or $0 \geq \lambda_j \geq 1$ ($j = 1, \cdots, n$). Hence, $\sum_{j=1}^{n} \lambda_j^2 \leq \sum_{j=1}^{n} \lambda_j$, which implies $\text{tr}(P'P)^2 \leq \text{tr}(P'P)$. On the other hand, we have

$$(\text{tr}(P'P))^2 = (\text{tr}(PP'P))^2 \leq \text{tr}(P'P)\text{tr}(P'P)^2$$

from the generalized Schwarz inequality (set $A' = P$ and $B = P'P$ in (1.19)) and $P^2 = P$. Hence, $\text{tr}(P'P) \leq \text{tr}(P'P)^2 \Rightarrow \text{tr}(P'P) = \text{tr}(P'P)^2$, from which it follows that $\text{tr}\{(P - P'P)'(P - P'P)\} = \text{tr}\{P'P - P'P - P'P + (P'P)^2\} = \text{tr}\{P'P - (P'P)^2\} = 0$. Thus, $P = P'P \Rightarrow P' = P$. \hfill Q.E.D.

Corollary Let $M$ be a symmetric pd matrix, and define the (squared) norm of $x$ by

$$||x||_M^2 = x'Mx.$$  \hfill (2.58)
The necessary and sufficient condition for a projection matrix \( P \) (satisfying \( P^2 = P \)) to satisfy

\[
\|Px\|_M^2 \leq \|x\|_M^2
\]  

for an arbitrary \( n \)-component vector \( x \) is given by

\[
(MP)' = MP.
\]  

**Proof.** Let \( M = U\Delta^2U' \) be the spectral decomposition of \( M \), and let \( M^{1/2} = \Delta U' \). Then, \( M^{-1/2} = U\Delta^{-1} \). Define \( y = M^{1/2}x \), and let \( \tilde{P} = M^{1/2}PM^{-1/2} \). Then, \( \tilde{P}^2 = \tilde{P} \), and (2.58) can be rewritten as \( \|\tilde{P}y\|^2 \leq \|y\|^2 \). By Theorem 2.22, the necessary and sufficient condition for (2.59) to hold is given by

\[
\tilde{P}^2 = \tilde{P} \implies (M^{1/2}PM^{-1/2})' = M^{1/2}PM^{-1/2},
\]  

leading to (2.60).

Q.E.D.

**Note** The theorem above implies that with an oblique projector \( P \) (\( P^2 = P \), but \( P' \neq P \)) it is possible to have \( \|Px\| \geq \|x\| \). For example, let

\[
P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

Then, \( \|Px\| = 2 \) and \( \|x\| = \sqrt{2} \).

**Theorem 2.23** Let \( P_1 \) and \( P_2 \) denote the orthogonal projectors onto \( V_1 \) and \( V_2 \), respectively. Then, for an arbitrary \( x \in E^n \), the following relations hold:

\[
\|P_2P_1x\| \leq \|P_1x\| \leq \|x\|
\]  

and, if \( V_2 \subset V_1 \),

\[
\|P_2x\| \leq \|P_1x\|.
\]

**Proof.** (2.62): Replace \( x \) by \( P_1x \) in Theorem 2.22.

(2.63): By Theorem 2.11, we have \( P_1P_2 = P_2 \), from which (2.63) follows immediately.

Let \( x_1, x_2, \ldots, x_p \) represent \( p \) \( n \)-component vectors in \( E^n \), and define \( X = [x_1, x_2, \cdots, x_p] \). From (1.15) and \( P = P'P \), the following identity holds:

\[
\|Px_1\|^2 + \|Px_2\|^2 + \cdots + \|Px_p\|^2 = \text{tr}(X'PX).
\]  

(2.64)
The above identity and Theorem 2.23 lead to the following corollary.

**Corollary**

(i) If $V_2 \subset V_1$, \( \text{tr}(X'P_2X) \leq \text{tr}(X'P_1X) \leq \text{tr}(X'X) \).

(ii) Let $P$ denote an orthogonal projector onto an arbitrary subspace in $E^n$. If $V_1 \supset V_2$,
\[
\text{tr}(P_1P) \geq \text{tr}(P_2P).
\]

**Proof.** (i): Obvious from Theorem 2.23. (ii): We have \( \text{tr}(PP_jP) = \text{tr}(PP_jP) \) \((j = 1, 2)\), and \( (P_1 - P_2)^2 = P_1 - P_2 \), so that
\[
\text{tr}(PP_1P) - \text{tr}(PP_2P) = \text{tr}(SS') \geq 0,
\]
where \( S = (P_1 - P_2)P \). It follows that \( \text{tr}(P_1P) \geq \text{tr}(P_2P) \). Q.E.D.

We next present a theorem on the trace of two orthogonal projectors.

**Theorem 2.24** Let $P_1$ and $P_2$ be orthogonal projectors of order $n$. Then the following relations hold:
\[
\text{tr}(P_1P_2) = \text{tr}(P_2P_1) \leq \min(\text{tr}(P_1), \text{tr}(P_2)). \tag{2.65}
\]

**Proof.** We have \( \text{tr}(P_1) - \text{tr}(P_1P_2) = \text{tr}(P_1(I_n - P_2)) = \text{tr}(P_1Q_2) = \text{tr}(P_1Q_2P_1) = \text{tr}(S'S) \geq 0 \), where \( S = Q_2P_1 \), establishing \( \text{tr}(P_1) \geq \text{tr}(P_1P_2) \). Similarly, (2.65) follows from \( \text{tr}(P_2) \geq \text{tr}(P_1P_2) = \text{tr}(P_2P_1) \). Q.E.D.

**Note** From (1.19), we obtain
\[
\text{tr}(P_1P_2) \leq \sqrt{\text{tr}(P_1)\text{tr}(P_2)}. \tag{2.66}
\]
However, (2.65) is more general than (2.66) because \( \sqrt{\text{tr}(P_1)\text{tr}(P_2)} \geq \min(\text{tr}(P_1), \text{tr}(P_2)) \).
2.5 Matrix Norm and Projection Matrices

Let \( A = [a_{ij}] \) be an \( n \) by \( p \) matrix. We define its Euclidean norm (also called the Frobenius norm) by

\[
\|A\| = \{\text{tr}(A'A)\}^{1/2} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{p} a_{ij}^2}.
\] (2.67)

Then the following four relations hold.

**Lemma 2.6**

\[
\|A\| \geq 0. \quad (2.68)
\]

\[
\|CA\| \leq \|C\| \cdot \|A\|, \quad (2.69)
\]

Let both \( A \) and \( B \) be \( n \) by \( p \) matrices. Then,

\[
\|A + B\| \leq \|A\| + \|B\|. \quad (2.70)
\]

Let \( U \) and \( V \) be orthogonal matrices of orders \( n \) and \( p \), respectively. Then

\[
\|UAV\| = \|A\|. \quad (2.71)
\]

**Proof.** Relations (2.68) and (2.69) are trivial. Relation (2.70) follows immediately from (1.20). Relation (2.71) is obvious from

\[
\text{tr}(V'A'U'UAV) = \text{tr}(A'AVV') = \text{tr}(A'A).
\]

Q.E.D.

**Note** Let \( M \) be a symmetric \( n 
 by \( n \) matrix of order \( n \). Then the norm defined in (2.67) can be generalized as

\[
\|A\|_M = \{\text{tr}(A'MA)\}^{1/2}.
\] (2.72)

This is called the norm of \( A \) with respect to \( M \) (sometimes called a metric matrix). Properties analogous to those given in Lemma 2.6 hold for this generalized norm.

There are other possible definitions of the norm of \( A \). For example,

(i) \( \|A\|_1 = \max_{j} \sum_{i=1}^{n} |a_{ij}| \),

(ii) \( \|A\|_2 = \mu_1(A) \), where \( \mu_1(A) \) is the largest singular value of \( A \) (see Chapter 5), and

(iii) \( \|A\|_3 = \max_{i} \sum_{j=1}^{p} |a_{ij}| \).

All of these norms satisfy (2.68), (2.69), and (2.70). (However, only \( \|A\|_2 \) satisfies (2.71).)
Lemma 2.7  Let $P$ and $\tilde{P}$ denote orthogonal projectors of orders $n$ and $p$, respectively. Then,

$$||PA|| \leq ||A|| \quad (2.73)$$

(the equality holds if and only if $PA = A$) and

$$||A\tilde{P}|| \leq ||A|| \quad (2.74)$$

(the equality holds if and only if $A\tilde{P} = A$).

Proof. (2.73): Square both sides and subtract the right-hand side from the left. Then,

\[
\begin{align*}
\text{tr}(A'A) - \text{tr}(A'PA) &= \text{tr}(A'(I_n - P)A) \\
&= \text{tr}(A'QA) = \text{tr}(QA')(QA) \geq 0 \quad \text{(where $Q = I_n - P$)}.
\end{align*}
\]

The equality holds when $QA = O \iff PA = A$.

(2.74): This can be proven similarly by noting that $||A\tilde{P}||^2 = \text{tr}(\tilde{P}A'\tilde{P}) = \text{tr}(A\tilde{P}A') = ||\tilde{P}A'||^2$. The equality holds when $QA' = O \iff \tilde{P}A' = A' \iff A\tilde{P} = A$, where $\tilde{Q} = I_n - \tilde{P}$. Q.E.D.

The two lemmas above lead to the following theorem.

Theorem 2.25  Let $A$ be an $n$ by $p$ matrix, $B$ and $Y$ $n$ by $r$ matrices, and $C$ and $X$ $r$ by $p$ matrices. Then,

$$||A - BX|| \geq ||(I_n - P_B)A||, \quad (2.75)$$

where $P_B$ is the orthogonal projector onto $\text{Sp}(B)$. The equality holds if and only if $BX = P_BA$. We also have

$$||A - YC|| \geq ||A(I_p - P_{C'})||, \quad (2.76)$$

where $P_{C'}$ is the orthogonal projector onto $\text{Sp}(C')$. The equality holds if and only if $YC = AP_{C'}$. We also have

$$||A - BX - YC|| \geq ||(I_n - P_B)A(I_p - P_{C'})||. \quad (2.77)$$

The equality holds if and only if

$$P_B(A - YC) = BX \text{ and } (I_n - P_B)AP_{C'} = (I_n - P_B)YC \quad (2.78)$$
or

\[(A - BX)P_{C'} = YC \text{ and } P_B A (I_p - P_{C'}) = BX (I_n - P_{C'}). \tag{2.79}\]

**Proof.** (2.75): We have \((I_n - P_B)(A - BX) = A - BX - P_B A + BX = (I_n - P_B) A\). Since \(I_n - P_B\) is an orthogonal projector, we have \(\|A - BX\| \geq \|(I_n - P_B)(A - BX)\| = \|(I_n - P_B)A\|\) by (2.73) in Lemma 2.7. The equality holds when \((I_n - P_B)(A - BX) = A - BX\), namely \(P_B A = BX\).

(2.76): It suffices to use \((A -YC)(I_p - P_{C'}) = A(I_p - P_{C'})\) and (2.74) in Lemma 2.7. The equality holds when \((A -YC)(I_p - P_{C'}) = A - YC\) holds, which implies \(YC = AP_{C'}\).

(2.77): \(\|A - BX -YC\| \geq \|(I_n - P_B)(A -YC)\| \geq \|(I_n - P_B)A(I_p - P_{C'})\|\) or \(\|A - BX -YC\| \geq \|(A - BX)(I_p - P_{C'})\| \geq \|(I_n - P_B)A(I_p - P_{C'})\|\). The first equality condition (2.78) follows from the first relation above, and the second equality condition (2.79) follows from the second relation above.

Q.E.D.

**Note** Relations (2.75), (2.76), and (2.77) can also be shown by the least squares method. Here we show this only for (2.77). We have

\[\|A - BX -YC\|^2 = \text{tr}\{(A - BX -YC)'(A - BX -YC)\} = \text{tr}(A - YC)'(A - YC) - 2\text{tr}(BX)'(A - YC) + \text{tr}(BX)'(BX)\]

to be minimized. Differentiating the criterion above by \(X\) and setting the result to zero, we obtain \(B'(A - YC) = B'BX\). Premultiplying this equation by \(B(B'B)^{-1}\), we obtain \(P_B(A - YC) = BX\). Furthermore, we may expand the criterion above as

\[\text{tr}(A - BX)'(A - BX) - 2\text{tr}(YC(A - BX)' + \text{tr}(YC)(YC)'.\]

Differentiating this criterion with respect to \(Y\) and setting the result equal to zero, we obtain \(C(A - BX) = CC'Y'\) or \((A - BX)C' = YCC'.\) Postmultiplying the latter by \((CC')^{-1}C'\), we obtain \((A - BX)P_{C'} = YC\). Substituting this into \(P_B(A - YC) = BX\), we obtain \(P_B A (I_p - P_{C'}) = BX (I_n - P_{C'})\) after some simplification. If, on the other hand, \(BX = P_B(A - YC)\) is substituted into \((A - BX)P_{C'} = YC\), we obtain \((I_n - P_B)A P_{C'} = (I_n - P_B)YC\). (In the derivation above, the regular inverses can be replaced by the respective generalized inverses. See the next chapter.)
2.6 General Form of Projection Matrices

The projectors we have been discussing so far are based on Definition 2.1, namely square matrices that satisfy \( P^2 = P \) (idempotency). In this section, we introduce a generalized form of projection matrices that do not necessarily satisfy \( P^2 = P \), based on Rao (1974) and Rao and Yanai (1979).

**Definition 2.3** Let \( V \subset E^n \) (but \( V \neq E^n \)) be decomposed as a direct-sum of \( m \) subspaces, namely \( V = V_1 \oplus V_2 \oplus \cdots \oplus V_m \). A square matrix \( P_j^* \) of order \( n \) that maps an arbitrary vector \( y \) in \( V \) into \( V_j \) is called the projection matrix onto \( V_j \) along \( V_{(j)} = V_1 \oplus \cdots \oplus V_{j-1} \oplus V_{j+1} \oplus \cdots \oplus V_m \) if and only if

\[
P_j^*x = x \quad \forall x \in V_j \quad (j = 1, \ldots, m)
\]  

(2.80)

and

\[
P_j^*x = 0 \quad \forall x \in V_{(j)} \quad (j = 1, \ldots, m).
\]  

(2.81)

Let \( x_j \in V_j \). Then any \( x \in V \) can be expressed as

\[
x = x_1 + x_2 + \cdots + x_m = (P_1^* + P_2^* + \cdots P_m^*)x.
\]

Premultiplying the equation above by \( P_j^* \), we obtain

\[
P_i^*P_j^*x = 0 \quad (i \neq j) \quad \text{and} \quad (P_j^*)^2x = P_j^*x \quad (i = 1, \ldots, m)
\]

(2.82)

since \( \text{Sp}(P_1), \text{Sp}(P_2), \ldots, \text{Sp}(P_m) \) are mutually disjoint. However, \( V \) does not cover the entire \( E^n \) (\( x \in V \neq E^n \)), so (2.82) does not imply \( (P_j^*)^2 = P_j^* \) or \( P_i^*P_j^* = O \) (\( i \neq j \)).

Let \( V_1 \) and \( V_2 \in E^3 \) denote the subspaces spanned by \( e_1 = (0,0,1)' \) and \( e_2 = (0,1,0)' \), respectively. Suppose

\[
P^* = \begin{bmatrix}
a & 0 & 0 \\
b & 0 & 0 \\
c & 0 & 1
\end{bmatrix}.
\]

Then, \( P^*e_1 = e_1 \) and \( P^*e_2 = 0 \), so that \( P^* \) is the projector onto \( V_1 \) along \( V_2 \) according to Definition 2.3. However, \( (P^*)^2 \neq P^* \) except when \( a = b = 0 \), or \( a = 1 \) and \( c = 0 \). That is, when \( V \) does not cover the entire space \( E^n \), the projector \( P_j^* \) in the sense of Definition 2.3 is not idempotent. However, by specifying a complement subspace of \( V \), we can construct an idempotent matrix from \( P_j^* \) as follows.
2.7. EXERCISES FOR CHAPTER 2

**Theorem 2.26** Let $P^*_j$ $(j = 1, \cdots, m)$ denote the projector in the sense of Definition 2.3, and let $P$ denote the projector onto $V$ along $V_{m+1}$, where $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ is a subspace in $E^n$ and where $V_{m+1}$ is a complement subspace to $V$. Then,

$$ P_j = P^*_j P \ (j = 1, \cdots, m) \quad \text{and} \quad P_{m+1} = I_n - P \quad (2.83) $$

are projectors (in the sense of Definition 2.1) onto $V_j \ (j = 1, \cdots, m + 1)$ along $V^*_j = V_1 \oplus \cdots \oplus V_{j-1} \oplus V_{j+1} \oplus \cdots \oplus V_m \oplus V_{m+1}$.

**Proof.** Let $x \in V$. If $x \in V_j \ (j = 1, \cdots, m)$, we have $P_j^*P_jx = P^*_jx = x$. On the other hand, if $x \in V_i \ (i \neq j, \ i = 1, \cdots, m)$, we have $P_j^*P_jx = P^*_jx = 0$. Furthermore, if $x \in V_{m+1}$, we have $P_j^*P_jx = 0 \ (j = 1, \cdots, m)$. On the other hand, if $x \in V$, we have $P_{m+1}x = (I_n - P)x = x - x = 0$, and if $x \in V_{m+1}$, $P_{m+1}x = (I_n - P)x = x - 0 = x$. Hence, by Theorem 2.2, $P_j \ (j = 1, \cdots, m + 1)$ is the projector onto $V_j$ along $V_j$.

Q.E.D.

2.7 Exercises for Chapter 2

1. Let $\tilde{A} = \begin{bmatrix} A_1 & 0 \\ O & A_2 \end{bmatrix}$ and $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$. Show that $P_A P_A = P_A$.

2. Let $P_A$ and $P_B$ denote the orthogonal projectors onto Sp($A$) and Sp($B$), respectively. Show that the necessary and sufficient condition for Sp($A$) = $\{\text{Sp}(A) \cap \text{Sp}(B)\} \oplus \{\text{Sp}(A) \cap \text{Sp}(B) \perp\}$ is $P_A P_B = P_B P_A$.

3. Let $P$ be a square matrix of order $n$ such that $P^2 = P$, and suppose

$$ \|Px\| = \|x\| $$

for any $n$-component vector $x$. Show the following:

(i) When $x \in (\text{Ker}(P))^\perp$, $Px = x$.

(ii) $P' = P$.

4. Let Sp($A$) = Sp($A_1$) $\oplus \cdots \oplus$ Sp($A_m$), and let $P_j \ (j = 1, \cdots, m)$ denote the projector onto Sp($A_j$). For $\forall x \in E^n$:

(i) Show that

$$ \|x\|^2 \geq \|P_1x\|^2 + \|P_2x\|^2 + \cdots + \|P_mx\|^2 \quad (2.84) $$

(Also, show that the equality holds if and only if Sp($A$) = $E^n$.)

(ii) Show that Sp($A_i$) and Sp($A_j$) ($i \neq j$) are orthogonal if Sp($A$) = Sp($A_1$) $\oplus$ Sp($A_2$) $\oplus \cdots \oplus$ Sp($A_m$) and the inequality in (i) above holds.

(iii) Let $P_{[j]} = P_1 + P_2 + \cdots + P_j$. Show that

$$ \|P_{[m]}x\| \geq \|P_{[m-1]}x\| \geq \cdots \geq \|P_{[2]}x\| \geq \|P_{[1]}x\|.$$

5. Let $E^n = V_1 \oplus W_1 = V_2 \oplus W_2 = V_3 \oplus W_3$, and let $P_j$ denote the projector onto $V_j$ ($j = 1, 2, 3$) along $W_j$. Show the following:

(i) Let $P_i P_j = O$ for $i \neq j$. Then, $P_1 + P_2 + P_3$ is the projector onto $V_1 + V_2 + V_3$ along $W_1 \cap W_2 \cap W_3$.

(ii) Let $P_1 P_2 = P_2 P_1$, $P_1 P_3 = P_3 P_1$, and $P_2 P_3 = P_3 P_2$. Then $P_1 P_2 P_3$ is the projector onto $V_1 \cap V_2 \cap V_3$ along $W_1 + W_2 + W_3$.

(iii) Suppose that the three identities in (ii) hold, and let $P_{1+2+3}$ denote the projection matrix onto $V_1 + V_2 + V_3$ along $W_1 \cap W_2 \cap W_3$. Show that

$$P_{1+2+3} = P_1 + P_2 + P_3 - P_1 P_2 - P_2 P_3 - P_1 P_3 + P_1 P_2 P_3.$$  

6. Show that

$$Q_{[A,B]} = Q_A Q_{AB},$$

where $Q_{[A,B]}$, $Q_A$, and $Q_{AB}$ are the orthogonal projectors onto the null space of $[A, B]$, onto the null space of $A$, and onto the null space of $Q_A B$, respectively.

7. (a) Show that

$$P_X = P_{X_A} + P_{X(X'X)^{-1}B},$$

where $P_X$, $P_{X_A}$, and $P_{X(X'X)^{-1}B}$ are the orthogonal projectors onto $\text{Sp}(X)$, $\text{Sp}(X A)$, and $\text{Sp}(X (X'X)^{-1}B)$, respectively, and $A$ and $B$ are such that $\text{Ker}(A^\perp) = \text{Sp}(B)$.

(b) Use the decomposition above to show that

$$P_{[X_1, X_2]} = P_{X_1} + P_{Q_{X_1} X_2},$$

where $X = [X_1, X_2]$, $P_{Q_{X_1} X_2}$ is the orthogonal projector onto $\text{Sp}(Q_{X_1} X_2)$, and $Q_{X_1} = I - X_1 (X_1'X_1)^{-1}X_1'$.

8. Let $E^n = V_1 \oplus W_1 = V_2 \oplus W_2$, and let $P_1 = P_{V_1, W_1}$ and $P_2 = P_{V_2, W_2}$ be two projectors (not necessarily orthogonal) of the same size. Show the following:

(a) The necessary and sufficient condition for $P_1 P_2$ to be a projector is $V_{12} \subset V_2 \oplus (W_1 \cap W_2)$, where $V_{12} = \text{Sp}(P_1 P_2)$ (Brown and Page, 1970).

(b) The condition in (a) is equivalent to $V_2 \subset V_1 \oplus (W_1 \cap V_2) \oplus (W_1 \cap W_2)$ (Werner, 1992).

9. Let $A$ and $B$ be $n$ by $a$ ($n \geq a$) and $n$ by $b$ ($n \geq b$) matrices, respectively. Let $P_A$ and $P_B$ be the orthogonal projectors defined by $A$ and $B$, and let $Q_A$ and $Q_B$ be their orthogonal complements. Show that the following six statements are equivalent: (1) $P_A P_B = P_B P_A$, (2) $A'B = A' P_B P_A B$, (3) $(P_A P_B)^2 = P_A P_B$, (4) $P_{[A,B]} = P_A + P_B - P_A P_B$, (5) $A' Q_B Q_A B = O$, and (6) $\text{rank}(Q_A B) = \text{rank}(B) - \text{rank}(A'B)$.
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