Chapter 2
Brief Introduction

The techniques and results contained in this monograph arose as, and from, the solution of a long-standing problem on the interface between complex analysis and partial differential equations, namely the (local) analytic hypoellipticity of the "$\overline{\partial}$-Neumann problem" at the boundary of a strictly pseudoconvex domain in $\mathbb{C}^n$. This problem was introduced in its modern form by J.J. Kohn in [K1], where he proved local C^∞ hypoellipticity by way of what amounted, in modern language, to a "subelliptic" estimate for test functions satisfying the so-called $\overline{\partial}$-Neumann boundary conditions ($v \in \mathcal{D}(\overline{\partial}^*)$). This led to the canonical solution of $\overline{\partial}$ on such domains and hence to the solution of the Cousin problem concerning domains of holomorphy. Another approach to the solution of the Cousin problem had been used by Hörmander [Hö4] but is of quite a different character, not involving boundary regularity.

The question of local real analytic regularity of the $\overline{\partial}$-Neumann problem on strictly pseudoconvex domains or slightly more general domains was achieved independently and simultaneously by the present author [T4], [T5] and F. Treves [Tr4]. I shall not comment on Treves' solution until Chap. 14, both because it is extremely different from ours, and, we believe, because it lacks the flexibility and elementary nature of our proof. Indeed, in the succeeding 32 years, a whole host of other problems has proved amenable to our constructions with relatively minor modifications, and I prefer to present those here.

It is this flexibility and enduring success of the present method that have encouraged me to present the story to a wider audience. I will thus explain the origins of the problem as we go along but not dwell on the complex analysis or even on the symplectic geometry involved but rather focus on what has proved to be the most difficult aspect of the work for the reader, namely learning not to become lost in the details.

And details there are. But we believe strongly that once one understands the motivation and meaning of the proof, the details are "mere details", and could be filled in at one's own pace.

It is my fervent hope that the present approach will fulfill this aim.

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Thus the monograph will have an unconventional flow, or at least its flow will differ from that of most "mathematics books". There will of course be theorems and proofs, but the initial portion will be devoted to a detailed, and extremely intuitive, analysis of the simplest imaginable model that our approach is designed to attack, and even there the result was unknown prior to our work in 1978.

There will be rigorous constructions, but even more space devoted in the first few chapters to the intuitive understanding of *why* these constructions work and, in fact, are the only ones that can work. For the miracle that gradually makes itself visible is that the same basic construction can be easily adapted to numerous more general, and more degenerate, situations.

Only after we feel certain that the serious reader has come to deeply appreciate the value and subtleties of the simplest case will the presentation become more "mathematical" and conventional, with tougher calculations but ones that should not feel more difficult once the first parts have been mastered.

In places, once the new and most difficult material has been introduced and worked through, there will come moments when the proof proceeds precisely as the model case did, where certain concepts and constructions have changed in detail, but not in essential flavor, and in particular have not changed in ways that would affect the argument of the model case. In such situations, we will not repeat all the details of the model case, but will merely substituting the entities that have changed only in ways that do not affect the remaining argument.

We hope and trust that such arguments will not be misunderstood as "hand-waving"; it is also our conviction that at certain points to include every detail would only obfuscate the proof, not elucidate it.

And one more important point: research in this field is ongoing. There are important situations in which analytic hypoellipticity (AHE), even global analytic hypoellipticity (GAHE), fails although many had hoped that it would hold, for example in the case of weakly pseudoconvex domains in complex analysis (cf. [Chr1], [Chr3], [Chr6]), and there are several important open conjectures (the embedding question for strongly pseudoconvex "CR" manifolds of dimension 5, which would follow from a suitable nonlinear AHE result, and the so-called conjecture of Treves, which concerns a certain "Poisson–Treves" stratification of the characteristic manifold, which I shall discuss in due course).

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