ILLUMINATING SETS OF CONSTANT WIDTH

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Abstract. The problem of illuminating the boundary of sets having constant width is considered and a bound for the number of directions needed is given. As a corollary, an estimate for Borsuk's partition problem is inferred. Also, the illumination number of sufficiently symmetric strictly convex bodies is determined.

§1. Introduction. Let $x$ be a point on the boundary $\partial K$ of a convex body* $K$ in Euclidean space, $\mathbb{R}^n$. A direction $u \in S^{n-1}$ is said to illuminate $K$ at $x$ if the line $\{x + tu | t \in \mathbb{R}\}$ "enters" $K$ in $x$. More precisely, $u$ illuminates $K$ at $x$ if $x + tu$ is an interior point of $K$, for some positive $t$. Instead of saying "$u$ illuminates $K$ at $x$" we will just say "$u$ illuminates $x$". This will cause no confusion, because the convex body $K$ should be clear from the context.

Directions $u_1, u_2, \ldots, u_m \in S^{n-1}$ are said to illuminate $K$ if every point on the boundary of $K$ is illuminated by at least one of these directions. We denote by $I(K)$ the minimal number of directions sufficient to illuminate $K$ and call it the illumination number of $K$. This concept of illumination was introduced by V. G. Boltjansky in [2]. There he proved that for convex bodies $K$, $I(K)$ is equal to $H(K)$, the number of smaller, positively homothetic copies of $K$ required to cover $K$ (see [3]).

The maximum values of $I(K)$ for convex bodies $K$ in $\mathbb{R}^n$, are unknown when $n > 2$. By the above result of Boltjansky, Hadwiger's conjecture, about the covering of a set by homothetic copies of it, is equivalent to

$$I(\text{convex body in } \mathbb{R}^n) \leq I(n\text{-dimensional parallelogram}) = 2^n.$$ 

See [3] for a discussion of this conjecture.

A set of constant width $d$ is a convex body such that the distance between any two distinct parallel supporting hyperplanes of it is $d$ (see [5, pp. 122–131], [4]). In this note we prove

**Theorem 1.** If $W$ is a set of constant width in $\mathbb{R}^n$ then

$$I(W) < 5n\sqrt{n}(4 + \log n)\left(\frac{3}{2}\right)^{n/2}.$$ 

The research exposed in this note was done while I was at the Hebrew University in Jerusalem, as a student of Professor Gil Kalai. I am grateful to Kalai for his interest, encouragement and advice, and for valuable improvements of the original manuscript.

* A convex body is a compact convex set that has interior points.

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Using this theorem, and the equivalence $I(W) = H(W)$, we get

**Corollary 2.** Every set of constant width $W \subseteq \mathbb{R}^n$ can be covered by less than $5n\sqrt{n(4 + \log n)}(3/2)^{n/2}$ homothetic copies of itself, having some homothety coefficient $\alpha$, $0 < \alpha < 1$.

Since every bounded set of positive diameter is contained in a set of constant width having the same diameter (see [5, p. 126]), we have the following estimate for Borsuk’s partition problem.

**Corollary 3.** Every set of diameter $d$ ($0 < d < \infty$) in $\mathbb{R}^n$ can be covered by less than $5n\sqrt{n(4 + \log n)}(3/2)^{n/2}$ sets having smaller diameters.

To the best of our knowledge, this is, asymptotically, the best bound known for Borsuk’s partition problem (see [7] for references and results concerning this problem). The approach to Borsuk’s problem through the the illumination problem is suggested in [3] and [8, p. 420].

Theorem 1 is proved using a probabilistic argument: The probability that a “small” region in $\partial W$ will be illuminated by a random, uniformly distributed, direction is estimated from below. A straightforward computation then shows that if “enough” directions are chosen randomly, uniformly, and independently, the probability that they will completely illuminate $W$ is nonzero. An essential part of the proof relies on finding lower bounds for volumes of spherical sets of a certain type. In [11] analogous results for sets in $\mathbb{R}^n$ give lower bounds for the volumes of sets having constant width.

The only known result in the direction opposite to Theorem 1 is that $I(K) \geq n + 1$ for every convex body $K \subseteq \mathbb{R}^n$ ([3]). It is not known if there is a set of constant width $W \subseteq \mathbb{R}^n$, that cannot be illuminated by $n + 1$ directions.

Rogers [10] has shown that every $K \subseteq \mathbb{R}^n$, having diameter $0 < d < \infty$ and invariant under the group of congruences that leave invariant an $n$-dimensional regular simplex, can be partitioned into $n + 1$ subsets, each with diameter $< d$. Inspired by this, we prove in Section 4 that if $K$ is a strictly convex body† invariant under a group of orthogonal transformations that is generated by reflections through hyperplanes and acts irreducibly‡ on $\mathbb{R}^n$, then $I(K) = n + 1$. This can give an alternate proof of Rogers’ result.

§2. First we will introduce some notation and give a condition for the illumination of a single boundary point. Throughout this note, $K$ will denote an arbitrary convex body, and $W$ will denote a set of constant width in $\mathbb{R}^n$. For a set $A \subseteq S^{n-1}$ we define

$$A^+ = \{u \in S^{n-1} | u \cdot v > 0 \text{ for all } v \in A\}.$$

†A strictly convex body is a convex body whose boundary contains no line segment.
‡“$G$ acts irreducibly on $\mathbb{R}^n$” means that there is no subspace of $\mathbb{R}^n$, other than 0 and $\mathbb{R}^n$, which is invariant under all elements of $G$. 
When \( x \) is a boundary point of a convex body \( K \), we use \( N_K(x) \) to denote the set of inward normal unit vectors of \( K \) at \( x \):

\[
N_K(x) = \{ u \in S^{n-1} | u \cdot p \geq u \cdot x \text{ for all } p \in K \}.
\]

Notice that \( N_K(x) \) is nonempty for \( x \in \partial K \).

**Lemma 4.** Let \( K \) be a convex body in \( \mathbb{R}^n \), let \( x \) be a boundary point of \( K \), and let \( u \in S^{n-1} \). Then \( x \) is illuminated by the direction \( u \), if, and only if, \( u \in N_K(x)^+ \).

**Proof.** Suppose \( u \in N_K(x)^+ \). If the line \( L = \{ x + tu | t \in \mathbb{R} \} \) contains no interior points of \( K \), then it is a supporting line of \( K \). Therefore there is a hyperplane, \( H = \{ p \in \mathbb{R}^n | p \cdot w = x \cdot w \} \), supporting \( K \) and containing \( L \). Thus one of the vectors \( \pm w/\|w\| \) is in \( N_K(x) \). That is a contradiction to \( u \in N_K(x)^+ \), because \( H \cap L \) implies that \( w \cdot u = 0 \). Therefore \( L \) contains interior points of \( K \). Pick any \( v \in N_K(x) \). \( K \) is contained in the half space \( \{ p \in \mathbb{R}^n | p \cdot v \geq x \cdot v \} \). Since \( u \cdot v > 0 \), this means that the points \( x + tu \) with \( t < 0 \) are not in \( K \). Thus the interior points of \( L \) correspond to positive values of \( t \), and \( u \) illuminates \( x \).

Now suppose that \( u \) illuminates \( x \). Let \( t \) be a positive number such that \( p = x + tu \) is an interior point of \( K \). Because \( p \) is an interior point, for every \( v \in N_K(x) \) we have \( p \cdot v < x \cdot v \), which implies \( u \cdot v > 0 \), so \( u \in N_K(x)^+ \).

From now on, we work with an arbitrary, but fixed, set of constant width \( W \subset \mathbb{R}^n \). The lemma above shows that if \( E \) is a subset of the boundary of \( W \), then one direction can illuminate \( E \), if, and only if,

\[
\bigcap_{x \in E} N_W(x)^+ = \left( \bigcup_{x \in E} N_W(x) \right)^+ \nonumber
\]

is nonempty. The following proposition will help us find subsets of \( \partial W \) that are “easily” illuminated. For a subset \( A \subset S^{n-1} \) define \( U_W(A) \) to be the union of the sets \( N_W(x), x \in \partial W \), that intersect \( A \):

\[
U_W(A) = \bigcup_{N_W(x) \cap A \neq \emptyset} N_W(x). \nonumber
\]

A direction in \( U_W(A)^+ \) illuminates every point \( x \in W \) that satisfies

\[
N_W(x) \cap A \neq \emptyset. \nonumber
\]

In order to show that when \( A \) is chosen properly these points are “easily” illuminated, we want to prove that \( U_W(A)^+ \) is “large”. Our means of doing so is by estimating the diameter of \( U_W(A) \). (We view \( S^{n-1} \) with the metric induced by the Euclidean metric in \( \mathbb{R}^n \). The diameters of subsets of \( S^{n-1} \) refer to this metric.)

**Proposition 5.** Let \( A \) be a nonempty subset of \( S^{n-1} \). Then

\[
\text{diameter } U_W(A) \leq 1 + \text{diameter } A. \nonumber
\]

**Proof.** Since \( U_W(A) \) does not change when we replace \( W \) with a positively homothetic copy of itself, we may, and will, assume that \( W \) has constant width.
1, and therefore also diameter 1. Let \( v_1, v_2 \) be unit vectors in \( U_w(A) \). By the definition of \( U_w(A) \), there are points \( x_1, x_2 \in \partial W \) such that \( v_i \in N_w(x_i) \) and 
\( N_w(x_i) \cap A \neq \emptyset \) for \( i = 1, 2 \). Suppose \( u_i \) is in \( N_w(x_i) \cap A \), \( i = 1, 2 \). Since \( u_i \) is an inward normal of \( W \) at \( x_i \), and since \( W \) has constant width 1, the hyperplane 
\[ \{ p \in \mathbb{R}^n \mid p \cdot u_i = x_i \cdot u_i + 1 \} \]
 is a supporting hyperplane of \( W \). The only point on this hyperplane whose distance from \( x_i \) is not greater than 1 is \( x_i + u_i \). Since diameter \( W = 1 \), we conclude that \( x_i + u_i \in \partial W \) for \( i = 1, 2 \). Therefore
\[
1 = (\text{diameter } W)^2 \geq \| (x_1 + u_1) - x_2 \|_2^2 = \| x_1 - x_2 \|_2^2 + 2u_1 \cdot (x_1 - x_2) + 1
\]
and
\[
1 = (\text{diameter } W)^2 \geq \| (x_2 + u_2) - x_1 \|_2^2 = \| x_2 - x_1 \|_2^2 + 2u_2 \cdot (x_2 - x_1) + 1.
\]
Summing these inequalities and rearranging we get
\[
(u_1 - u_2) \cdot (x_2 - x_1) \geq \| x_2 - x_1 \|_2^2.
\]
Because \((u_1 - u_2) \cdot (x_2 - x_1) \leq \| u_1 - u_2 \| \| x_2 - x_1 \|_2\), this implies \( \| u_1 - u_2 \| \geq \| x_2 - x_1 \|_2 \).
So
\[
\| x_2 - x_1 \|_2 \leq \text{diameter } A. \quad (2.1)
\]
As with \( x_i + u_i \), the points \( x_i + v_i \) lie in \( \partial W \), and therefore
\[
1 \geq \| (x_1 + v_1) - (x_2 + v_2) \| \geq \| v_1 - v_2 \| - \| x_2 - x_1 \|.
\]
Using (2.1), this implies
\[
1 + \text{diameter } A \geq \| v_1 - v_2 \|.
\]
We use \( \mu \) to denote the standard probability measure on \( S^{n-1} \). Define
\[
g(n, d) = \inf \{ \mu(A^+) \mid A \subseteq S^{n-1}, \text{diameter } A \leq d \}\).
\]
Let \( N(n, \varepsilon) \) be the number of sets having diameter \( \varepsilon \) that is required to cover \( S^{n-1} \). The core of this note is:

**Proposition 6.** For \( 0 < \varepsilon < \sqrt{2} - 1 \) we have
\[
I(W) \leq 1 + \frac{\log N(n, \varepsilon)}{-\log (1 - g(n, 1 + \varepsilon))}.
\]

**Proof.** It is easily verified that \( 0 < g(n, 1 + \varepsilon) < 1 \) (if \( \emptyset \neq A \subseteq S^{n-1} \) then \( A^+ \) is contained in a hemisphere, so that \( \mu(A^+) \leq \frac{1}{2} \). If also diameter \( A = d < \sqrt{2} \) then \( A^+ \) contains a spherical cap of radius \( \sqrt{2} - d \) around any point of \( A \)), so that the right-hand side is well defined. Let \( M \) be a natural number satisfying
\[
M > \frac{\log N(n, \varepsilon)}{-\log (1 - g(n, 1 + \varepsilon))}.
\]
It is sufficient to show that \( M \) directions can illuminate \( W \). Set \( N = N(n, \varepsilon) \), and let \( A_1, \ldots, A_N \) be a covering of \( S^{n-1} \) with sets of diameter \( \varepsilon \). By Proposition 5, we have diameter \( U_w(A_i) \leq 1 + \varepsilon \), and therefore
\[
\mu(U_w(A_i)^+) \geq g(n, 1 + \varepsilon), \quad i = 1, 2, \ldots, N.
\]
Pick $M$ directions $u_1, \ldots, u_M$ at random, uniformly and independently distributed on $S^{n-1}$. Take any $i, j, 1 \leq i \leq N$, $1 \leq j \leq M$. The probability that $u_j$ will be in $U_w(A_i)^+$ is $\mu(U_w(A_i)^+)$, which is at least $g(n, 1+\varepsilon)$. Therefore the probability that $U_w(A_i)^+$ will contain none of the points $u_1, \ldots, u_M$ is at most $(1 - g(n, 1+\varepsilon))^M$. Thus the probability $p$ that at least one $U_w(A_i)^+$, $1 \leq l \leq N$ will contain no points of $u_1, \ldots, u_M$ satisfies

$$p \leq \sum_{l=1}^{N} (1 - g(n, 1+\varepsilon))^M < N(1 - g(n, 1+\varepsilon))^\log N / -\log (1 - g(n, 1+\varepsilon)) = 1.$$ 

This shows that one can choose $M$ directions, so that each set $U_w(A_i)^+$, $l = 1, \ldots, N$, contains at least one of them. Let $v_1, \ldots, v_M$ be such directions, and let $x$ be a point of $\partial W$. We claim that one of these directions illuminates $x$. Since $N_w(x)$ is nonempty, and the sets $A_1, \ldots, A_N$ cover $S^{n-1}$, one of them, say $A_i$, intersects $N_w(x)$. By the definition of $U_w(A_i)$, we have $N_w(x) \subset U_w(A_i)$. So that

$$N_w(x)^+ \supseteq U_w(A_i)^+.$$ 

$U_w(A_i)^+$ contains at least one of $v_1, \ldots, v_M$, say $v_k$. We have

$$v_k \in U_w(A_i)^+ \subset N_w(x)^+$$

and therefore, by Lemma 4, $v_k$ illuminates $x$. This shows that the directions $v_1, \ldots, v_M$ illuminate $W$.

In order to deduce Theorem 1 from Proposition 6, we only have to estimate $g(n, 1+\varepsilon)$ from below, and $N(n, \varepsilon)$ from above. The former will be done in the next section, and the latter is dealt with by the following well known fact.

**Lemma 7.** $N(n, \varepsilon) \leq (1 + 4/\varepsilon)^n$.

**Proof.** Let $E$ be a maximal subset of $S^{n-1}$ having the property that $\|u - v\| > \frac{1}{2}\varepsilon$ for $u \neq v$ in $E$. The maximality of $E$ shows that the balls with radius $\frac{1}{2}\varepsilon$ and centers in $E$ cover $S^{n-1}$, therefore

$$|E| \geq N(n, \varepsilon).$$

All the balls $B(u, \varepsilon/4)$, $u \in E$, are disjoint and are contained in the ball $B(0, 1+\varepsilon/4)$. Comparing volumes gives

$$|E|(\varepsilon/4)^n \leq (1 + \varepsilon/4)^n,$$

or

$$|E| \leq \left(\frac{4}{\varepsilon} + 1\right)^n.$$

**Remark.** Better estimates for $N(n, \varepsilon)$ are known (see [9]), but do not seem to contribute any significant improvement to Theorem 1.

§3. In this section we give a lower bound for $g(n, d)$, and prove Theorem 1.
Proposition 8. Let \( d > 0 \) and let \( A \) be a nonempty subset of \( S^{n-1} \) having diameter \( \leq d \). Suppose \( u \in S^{n-1} \), \( a > 0 \) and \( A \) is contained in the half-space \( \{ p \in R^n | p \cdot u \geq a \} \), then

\[ A^+ \cup TA^+ \supset D_0(u, \arctan(2a/d)), \]

where \( T : R^n \rightarrow R^n \) is the reflection through the line determined by \( u, -u \):

\[ Tp = 2(p \cdot u)u - p, \]

and \( D_0(u, \psi) \) is the open spherical cap consisting of all unit vectors having an angle with \( u \), which is smaller than \( \psi \).

Proof. Suppose \( x \) is a point in \( S^{n-1} \) but not in \( A^+ \cup TA^+ \), and let \( \theta \) be the angular distance between \( x \) and \( u \), \( 0 \leq \theta \leq \pi \). Write

\[ x = (\cos \theta)u + (\sin \theta)v, \quad (3.1) \]

where \( v \) is a unit vector orthogonal to \( u \) (we ignore the trivial case \( n = 1 \)). Since \( x \notin A^+ \), there is a point \( y \in A \) with

\[ 0 \geq y \cdot x = y \cdot u \cos \theta + y \cdot v \sin \theta. \quad (3.2) \]

Since \( T^{-1} = T \) and \( x \notin TA^+ \) we have \( Tx \notin A^+ \). Thus there is a point \( z \in A \) with

\[ 0 \geq z \cdot Tx = z \cdot u \cos \theta - z \cdot v \sin \theta. \quad (3.3) \]

Summing (3.2) and (3.3), and using \( \|y - z\| \leq d \), \( \sin \theta \geq 0 \), we have

\[ 0 \geq (y \cdot u + z \cdot u) \cos \theta + (y - z) \cdot v \sin \theta \geq (y \cdot u + z \cdot u) \cos \theta - d \sin \theta. \quad (3.4) \]

Temporarily suppose that \( \theta < \frac{1}{2} \pi \). Then \( \cos \theta > 0 \), so by (3.4)

\[ \tan \theta \geq \frac{y \cdot u + z \cdot u}{d} \geq \frac{2a}{d}. \]

The last inequality is justified by the hypothesis that \( A \subset \{ p \in R^n | p \cdot u \geq a \} \).

Whether or not \( \theta \leq \frac{1}{2} \pi \), we have \( \theta \geq \arctan(2a/d) \). This shows that \( A^+ \cup TA^+ \supset D_0(u, \arctan(2a/d)). \)

Proposition 9.

\[ g(n, d) = \frac{1}{\sqrt{8\pi n}} \left( \frac{3 + (2n + 1) d^2 - (2n + 2)}{4 n + 4 - 2 d^2 n} \right)^{-\frac{(n-1)/2}{2}} \]

for \( 0 < d \leq \sqrt{2} \).

Proof. Let \( 0 < d \leq \sqrt{2} \) and let \( A \) be a nonempty subset of \( S^{n-1} \) having diameter \( \leq d \). By Jung's theorem [5, p. 111], there is, in \( R^n \), a ball with radius \( d \sqrt{n/(2n+2)} \) containing \( A \). Let \( q \) be the center of this ball. Write \( q = tu \) with \( t \geq 0 \) and \( u \in S^{n-1} \). For every \( x \in A \) we have

\[ d^2 \frac{n}{2n + 2} \geq \|x - q\|^2 = \|x - tu\|^2 = 1 - 2tx \cdot u + t^2. \quad (3.5) \]

Since \( 1 - 2tx \cdot u + t^2 = 1 - (x \cdot u)^2 + (x \cdot u - t)^2 \geq 1 - (x \cdot u)^2 \), inequality (3.5)
implies
\[ d^2 \frac{n}{2n+2} \geq 1 - (\mathbf{x} \cdot \mathbf{u})^2. \] (3.6)

From (3.5), \( t \geq 0 \) and \( d \leq \sqrt{2} \) it can be seen that \( \mathbf{x} \cdot \mathbf{u} \geq 0 \), so, using (3.6), we obtain
\[ \mathbf{x} \cdot \mathbf{u} \geq \sqrt{1 - d^2 \frac{n}{2n+2}}. \]

Set
\[ a = \sqrt{1 - d^2 \frac{n}{2n+2}}. \] (3.7)

The above argument shows that \( A \) is contained in the half-space
\[ \{ \mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} \cdot \mathbf{u} \geq a \}. \]

Proposition 8 can be applied, yielding
\[ A^+ \cup TA^+ \supset D_0(\mathbf{u}, \arctan(2a/d)). \]

Now, since \( T \) is an orthogonal transformation, we have \( \mu(A^+) = \mu(TA^+) \), and
\[ \mu(A^+) = \frac{1}{2} \mu(A^+) + \mu(TA^+) \geq \frac{1}{2} \mu(A^+ \cup TA^+) \]
\[ \geq \frac{1}{2} \mu \left( D_0(\mathbf{u}, \arctan \frac{2a}{d}) \right) = \frac{1}{2} \frac{\text{Vol}_{n-1} D_0(\mathbf{u}, \arctan 2a/d)}{\text{Vol}_{n-1} S^{n-1}} \]
\[ = \frac{\text{Vol}_{n-1} D_0(\mathbf{u}, \arctan 2a/d)}{2n \Omega_n}. \] (3.8)

Here \( \Omega_n \) denotes the volume of the \( n \)-dimensional unit ball, and \( \text{Vol}_{n-1} \) is the \( (n-1) \)-dimensional volume.

Let \( D' \) be the orthogonal projection of \( D_0(\mathbf{u}, \arctan 2a/d) \) to the hyperplane \( \{ \mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} \cdot \mathbf{u} = 0 \} \). Obviously we have
\[ \text{Vol}_{n-1} D_0(\mathbf{u}, \arctan \frac{2a}{d}) \geq \text{Vol}_{n-1} D'. \] (3.9)

\( D' \) is an \( (n-1) \)-dimensional ball having radius
\[ \sin \left( \arctan \frac{2a}{d} \right) = \left( 1 + \frac{d^2}{4a^2} \right)^{-1/2}, \]
so
\[ \text{Vol}_{n-1} D' = \Omega_{n-1} \left( 1 + \frac{d^2}{4a^2} \right)^{-(n-1)/2}. \] (3.10)

Using (3.8), (3.9), (3.10), we get
\[ \mu(A^+) \geq \frac{\Omega_{n-1}}{2n \Omega_n} \left( 1 + \frac{d^2}{4a^2} \right)^{-(n-1)/2}. \] (3.11)
Now

\[
\frac{\Omega_{n-1}}{\Omega_n} = \frac{\pi^{(n-1)/2}/\Gamma((1+n)/2)}{\pi^{n/2}/\Gamma(1+n/2)} = \frac{\Gamma(1+n/2)}{\sqrt{\pi} \Gamma((1+n)/2)}
\]

(3.12)

where \( \Gamma \) is the Gamma function. Since \( \log \Gamma \) is convex ([1, p. 12]), we have

\[
\Gamma(1+n/2) \Gamma(n/2) \geq \Gamma((1+n)/2)^2,
\]

and therefore

\[
\frac{\Gamma(1+n/2)}{\Gamma((1+n)/2)} \geq \frac{\Gamma(1+n/2)}{\sqrt{\Gamma(1+n/2) \Gamma(n/2)}} \sqrt{\frac{\Gamma((1+n)/2)^2}{\Gamma(1+n/2) \Gamma(n/2)}}
\]

\[
= \sqrt{\frac{\Gamma(1+n/2)}{\Gamma(n/2)}} = \sqrt{\frac{n}{2}}.
\]

(3.13)

Using (3.11), (3.12), (3.13), we get

\[
\mu(A^+) \geq \frac{1}{2n} \sqrt{\pi} \sqrt{\frac{n}{2}} \left(1 + \frac{d^2}{4a^2}\right)^{-(n-1)/2}.
\]

And after substituting the value of \( a \),

\[
\mu(A^+) \geq \frac{1}{\sqrt{8\pi n}} \left(1 + \frac{d^2}{4-2d^2n/(n+1)}\right)^{-(n-1)/2}
\]

\[
= \frac{1}{\sqrt{8\pi n}} \left(\frac{3}{2} + \frac{(2n+1)d^2 - 2n - 2}{4n + 4 - 2d^2n}\right)^{-(n-1)/2}.
\]

Now proving Theorem 1 is just a matter of putting the pieces together.

**Proof of Theorem 1.** Since \( t < -\log (1-t) \) for \( 0 < t < 1 \), Proposition 6 implies

\[
I(W) < 1 + \frac{\log N(n, \varepsilon)}{g(n, 1+\varepsilon)}, \quad 0 < \varepsilon < \sqrt{2} - 1.
\]

Choose

\[
\varepsilon = \sqrt{1 + \frac{1}{2n+1}} - 1.
\]

From Lemma 7 and Proposition 9 we get

\[
I(W) < 1 + \frac{\log N(n, \varepsilon)}{g(n, 1+\varepsilon)} \leq 1 + \frac{\log \left(1 + \frac{4}{\varepsilon}\right)}{g\left(n, \sqrt{\frac{2n+2}{2n+1}}\right)}
\]

\[
\leq 1 + \sqrt{8\pi n} \left(\frac{3}{2}\right)^{(n-1)/2} n \log \left(1 + \frac{4}{\varepsilon}\right)
\]

\[
= 1 + 4n\sqrt{\pi n/3} \left(\frac{3}{2}\right)^{n/2} \log \left(1 + \frac{4}{\varepsilon}\right).
\]
Since one easily sees that \( \varepsilon > 1/(4n+3) \), we have

\[
I(W) < 1 + 4n\sqrt{\pi n/3} \log (13 + 16n) \left( \frac{3}{2} \right)^{n/2} \leq 5n\sqrt{n(4 + \log n)} \left( \frac{3}{2} \right)^{n/2}
\]

Remarks. 1. In [11] we give a lower bound for the volumes of sets of constant width in \( \mathbb{R}^n \), using results analogous to Propositions 8 and 9.
2. The factor \( 5n\sqrt{n(4 + \log n)} \) in Theorem 1 should not be taken seriously. It can be improved with some more careful estimates. However, any improvement of the exponential factor, \( (3/2)^{n/2} \), would be interesting. A possible way to do this may be to try to get a better lower bound for \( g(n, d) \). An advance in this direction may lead to better estimates for the minimal volume of a set having constant width 1 in \( \mathbb{R}^n \).

§4.

Theorem 10. Let \( K \subset \mathbb{R}^n \) be a strictly convex body, invariant under a group of orthogonal transformations \( G \) that is generated by reflections through hyperplanes and acts irreducibly on \( \mathbb{R}^n \). Then \( I(K) = n + 1 \).

We preface the proof with a few definitions and a lemma. A unit vector \( r \) is called a root of \( G \) if the orthogonal reflection through the subspace orthogonal to \( r \) is an element of \( G \). We denote this reflection by \( S_r \):

\[
S_r x = x - 2(\langle x, r \rangle) r, \quad x \in \mathbb{R}^n.
\]

If \( v, r \) are roots of \( G \) then \( S_r v \) is also a root of \( G \), because \( S_r S_v = S_r S_r S_r \). In particular \( -r = S_r r \) is a root.

Lemma 11. Let \( G \) be as in Theorem 10. If \( n > 1 \) then there are \( n + 1 \) roots of \( G, r_0, \ldots, r_n, \) such that every nonzero \( x \in \mathbb{R}^n \) has a negative inner product with at least one of them.

At least when \( G \) is finite this follows from known results (see [6]).

Proof of the Lemma. We will say that a set of vectors \( \{v_0, \ldots, v_m\} \) is almost independent if every proper subset of it is linearly independent but \( \{v_0, \ldots, v_m\} \) is linearly dependent. It is easily checked that \( \{v_0, \ldots, v_m\} \) is almost independent, if, and only if, \( v_1, \ldots, v_m \) are linearly independent and \( v_0 \) is a linear combination of \( v_1, \ldots, v_m \) with nonzero coefficients.

Because of the hypotheses on \( G \) and because \( n \geq 2 \), \( G \) has at least one root, \( r \). \{\( r, -r \)\} is an almost independent set. Suppose \( \{r_0, \ldots, r_m\} \) is the largest almost independent set of roots of \( G \). Let \( U \) be the subspace generated by \( r_0, \ldots, r_m \). We claim that \( U = \mathbb{R}^n \) and therefore \( n = m \). Let \( r \) be a root of \( G \) and suppose that \( U \) is not invariant under \( S_r \). This implies \( r \notin U \) and \( S_r r_i \neq r_i \) for some \( i = 0, 1, \ldots, m \). Assume, without loss of generality, that \( S_r r_0 \neq r_0 \). \( r_0 \) is a linear combination of \( r_1, \ldots, r_m \) with nonzero coefficients. Since \( S_r r_0 - r_0 \) is a nonzero multiple of \( r \), \( S_r r_0 \) is a linear combination of \( r, r_1, \ldots, r_m \) with nonzero coefficients. Because \( r, r_1, \ldots, r_m \) are linearly independent, this means
that the set \( \{S_2, r_0, r, r_1, \ldots, r_m\} \) is almost independent, contradicting the choice of \( \{r_0, \ldots, r_m\} \).

This forces us to conclude that \( U \) is invariant under the reflections generating \( G \), and therefore under every transformation in \( G \). \( G \) acts irreducibly on \( \mathbb{R}^n \) and \( U \neq 0 \). This implies \( U = \mathbb{R}^n \) and \( n = m \).

Because \( \{r_0, \ldots, r_n\} \) is almost independent, it is possible to write

\[
0 = \sum_{i=0}^n a_i r_i
\]

with all coefficients nonzero. We replace some of the roots \( r_i \) by their negatives, to have all the coefficients \( a_i \) positive. If \( x \) is any nonzero vector, we must have \( x \cdot r_i \neq 0 \) for some \( i = 0, 1, \ldots, n \), because the \( r_i \) span \( \mathbb{R}^n \). Since \( 0 = x \cdot 0 = \sum_{i=0}^n a_i (x \cdot r_i) \) and \( a_i > 0 \), we must have \( x \cdot r_i < 0 \) for some \( i \).

**Proof of Theorem 10.** As mentioned in Section 1, \( I(K) \geq n + 1 \) holds for every convex body in \( \mathbb{R}^n \), thus we only need to show that \( I(K) \leq n + 1 \). The Theorem is obviously true when \( n = 1 \), so we assume \( n > 1 \).

Let \( r_0, \ldots, r_n \) be the roots of \( G \) guaranteed by the lemma and let \( x \in \partial K \). First observe that the origin is necessarily an interior point of \( K \) so \( x \neq 0 \). For some \( r_i \), \( x \cdot r_i < 0 \). We claim that this \( r_i \) illuminates \( x \). \( x - 2(x \cdot r_i)r_i = S_{r_i}x \in K \), since \( S_{r_i} \in G \). For some positive \( t \), \( x + tr_i \) is an interior point of \( K \), because \( x \), \( x - 2(x \cdot r_i)r_i \in K \), \( K \) is strictly convex and \( -2(x \cdot r_i) > 0 \). This verifies the claim and we see that \( r_0, \ldots, r_n \) illuminate \( K \).

**Remark.** A set of constant width is strictly convex; therefore Theorem 10 applies to (sufficiently symmetric) sets of constant width.

**References**


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