CHAPTER 1

STORAGE AND FLOW OF FLUIDS AND ELECTRICITY

In this chapter, hydraulic and electric phenomena will be introduced and described with concepts known from the physics of dynamical systems and processes. We start with fluids in systems of tanks, pipes and pumps, and extend the description to electrical processes by making use of analogical reasoning. Dynamical models will be constructed that share the same underlying structure even though the phenomena are not at all alike—at least not superficially. The form of the conceptualization, and the tools used to express it, are the same as those used to create models of mechanical systems (Chapter 3). They will be used to build a theory of the dynamics of heat in Part II.

1.1 PHENOMENA AND MODELS IN FLUIDS AND ELECTRICITY

We all are familiar with the flow of water in simple settings, such as the filling or discharging of tanks through pipes. By looking at some special examples, we will be able to identify the elements of a physical theory which allow us to calculate such things as the current of water through a pipe, the pressure at various points in the fluid, the storage of water and associated values of pressure, and the time required to discharge a storage element. Beyond the immediate application, the analysis will tell us that the description of systems and processes is similar to what we know from electricity. By comparing hydraulic and electrical systems, we shall learn about the importance of analogies between different fields of physics.

Fluids and electricity demonstrate most clearly how humans conceptualize processes. Fluidlike quantities that are stored and can flow, and differences of their intensities which I interpret as a kind of tension or driving force, are the basic concepts. In this section, processes of the creation and the equilibration of such driving forces demonstrate how we perceive similarities in otherwise dissimilar phenomena.

1.1.1 Differences, Driving Forces, and Flows

Communicating tanks. Consider two cylindrical tanks connected by a hose at their bottoms, and filled with some oil. We let the oil flow from one tank into the other and measure the fluid levels in the tanks as functions of time (Fig. 1.1).

It turns out that the level of oil in one of the tanks decreases while the other level increases. This is so since the oil flows from where the level is higher to where it is lower. The shape of the measured curves tells us that the process runs fast at the beginning,
slows down and comes to a standstill when the levels have become equal (Fig. 1.1, right). We say the system has reached *equilibrium*. There is *dynamics* as long as we have a difference of levels in the two tanks—the level difference is conceptualized as the *driving force* of the flow of fluid.

Note that the final common level is not the average of the initial levels if the tanks have different cross sections (Fig. 1.1). This demonstrates that levels become equal, and not quantities of liquid.

A closer look at the flow of fluids from one storage element to another shows yet another important aspect of these phenomena. If you fill one of the tanks with oil having a somewhat higher or lower density, the final levels in the tanks will not be the same—even though the flow process has stopped. If we maintain that this is due to the fact that the driving force of the process has become zero, the driving force cannot be measured by the difference of levels of liquids but rather by pressure differences.

**Communicating balloons.** A third experiment clarifies the situation. Instead of tanks, let us use toy balloons filled with air at different pressures. The balloons are allowed to communicate. If we measure the air pressure in the balloons as functions of time, we see that pressures equilibrate (Fig. 1.2) whereas quantities of air or levels do not (there are no levels of air to speak of in this experiment). In summary, the three cases described above tell us that we can interpret pressure differences as *driving forces of flows of fluids*.

**Two capacitors in an electric circuit.** In a circuit having two electric capacitors with a resistor between them (Fig. 1.3), we can observe a process which demonstrates similar behavior. We charge the capacitors to different voltages in the open circuit, close the circuit and measure the electric potential differences (voltages) as functions of time. We get curves analogous to the ones in the diagrams of Fig. 1.1 and Fig. 1.2—see Fig. 1.3. The interpretation of this phenomenon is analogous to that of fluid processes, and it is well known from electricity. We imagine a quantity responsible for electric processes—compared to quantity of fluid—which we call *electric charge*. 
This quantity is said to flow from a place having a higher electrical intensity—called electric potential—to one having a lower value of intensity. The process stops when the potential differences across the capacitors have become equal. Note that a potential difference has its own name; it is called voltage (the German word for this is tension). Observations like the one described here lead us to interpret electric potential differences (voltages) as driving forces of the flow of electric charge.

If differences, i.e., driving forces, decay spontaneously as we have seen in these experiments, they have to be reestablished if they are supposed to again make something happen. In nature, on our planet, this works mostly through the action of solar radiation. Chemical and thermal differences are created which in turn create pressure and gravitational differences that drive the oceans and the atmosphere. As long as the Sun shines on us, these vital tensions can be maintained. In engineering we build devices that either set up differences or make use of them.

Driving charge apart. A simple electric experiment shows how this can work in technical settings. Add a battery to the circuit having two capacitors and a resistor (see Fig. 1.4, left). Initially, the devices are uncharged, the electric potential differences across the capacitors are zero; we have electrical equilibrium. If we now connect the battery, i.e., close the circuit, the potentials of the capacitors are driven apart—a potential difference is established (Fig. 1.4, right).

1. In old mythic cultures like Egyptian and Babylonian, people seem to have captured this understanding in their cosmologies. The world begins with a separation of the sky from the earth. In Egypt, it was the god Shu (air) that supports Nut (heavens) from falling to Geb (earth). In Babylonian mythology, it was the wind that separated heaven and earth. Dynamics is rooted in the tension between the poles of the polarities that govern nature and society.
Differences beget differences—there has to be a potential difference in the battery to make that device establish an electric potential difference. This initial tension is chemical in nature. There are chemicals in the battery that react as a result of their innate chemical difference. Later, in Chapter 6, I will take up the issue of chemical processes and formalize the idea of chemical differences by introducing the notion of chemical potentials. At this point, we probably realize how we can understand thermal dynamical processes: they are the result of the difference between hot and cold.

1.1.2 System Dynamics Models

The descriptions of the phenomena discussed above contain the seeds of formal explanations of physical processes. Take the case of communicating tanks in Fig. 1.1. A word model of the system and its processes might go like this. There is a pressure difference of the oil across the connecting pipe. As a result, oil will flow through the pipe from the point of higher to the point of lower pressure. By resisting the flow, the pipe regulates the current. This makes the volume of oil decrease in the first container, and increase in the second. Since the quantity of oil in a container sets up the pressure due to the action of gravity, the pressure difference between the tanks will go down, which will make the current of oil through the pipe decrease, and this will make the levels of oil in the tanks change more slowly, and so on.

Graphical modeling tools. There are system dynamics modeling tools that support us in translating these words into diagrams and equations. Liquid flowing from tank 1 to tank 2 diminishes the volume of liquid in tank 1 and increases it in tank 2. This basic idea is expressed graphically with storage and flow symbols (Fig. 1.5) whose combination represents laws of balance. The stored liquid sets up a pressure which depends upon the quantity of fluid and the size of the tank which is symbolized by a capacitance: The greater the volume the higher the pressure, with the capacitance as the factor relating one to the other. The flow, finally, depends upon the pressure difference and a material factor that tells us how hard it is for the fluid to flow (this factor is called flow resistance).

The structure of the model has now been represented graphically. We still need to cast our ideas into formal expressions. The software used to draw the diagram lets us enter the relevant relations in mathematical form. Here is a concrete example:

\[
\begin{align*}
\frac{d \text{ (Volume}_1\text{)}}{dt} & = - \text{Flow} , \quad \text{Volume}_1(0) = 10 \\
\frac{d \text{ (Volume}_2\text{)}}{dt} & = \text{Flow} , \quad \text{Volume}_2(0) = 2 \\
\text{Pressure difference} & = \text{Pressure}_2 - \text{Pressure}_1
\end{align*}
\]
Flow = \frac{-\text{Pressure difference}}{\text{Resistance}}
Pressure_1 = \frac{\text{Volume}_1}{\text{Capacitance}_1}
Pressure_2 = \frac{\text{Volume}_2}{\text{Capacitance}_2}
Resistance = 100
Capacitance_1 = 2
Capacitance_2 = 1

In this example, values and units are arbitrary. Numerical software solves this set of equations leading to pressures (and volumes and flow) as functions of time as shown in the graph of Fig. 1.5. Quite obviously, the numerical result shows the same behavior as experimental values obtained in the example of Fig. 1.1. In practice, we use data to determine parameters of the model such as the flow resistance: parameter values are changed and the simulations are fitted to experimental values. In the example discussed here, the ideas built into the model give close to perfect agreement between model and reality. Therefore, we accept the assumptions that (a) the pressure of oil in a tank is proportional to its volume, and (b) the flow of oil through the pipe is proportional to the driving force (pressure difference).

**Extending and changing models.** Extending and changing the model produced above becomes quite intuitive and simple when we use the graphical modeling tools available to us today. Imagine that one of the tanks in Fig. 1.1 had an additional outlet, or we used water instead of oil. In the first instance, we would simply add a flow element to one of the tanks in the model diagram of Fig. 1.5 and express the flow in terms of the appropriate pressure difference and a resistance. In the second case, we would have to change the form of the relation between pressure differences and flows (it would no longer be linear).

**Analogical reasoning.** The model constructed for this first example can be transferred to electrical systems (Fig. 1.6). Take the example of driving electric charge from one capacitor to another with the help of a battery (or power supply), as in Fig. 1.4. You can use the system dynamics model constructed for communicating tanks (Fig. 1.5, left), change the names of the variables (replace volume by charge, pressure by potential or voltage), and add the effect of the power supply by specifying a fixed value of a driving potential difference which is added to the potential difference between the capacitors, and the model is completed. Equations for potentials and for the flow do not have to be changed. The correspondence between fluid and electric dynamical phenomena will become even more evident in an example from physiology discussed in some detail below.

**Structure of dynamical models.** The model—represented by the equations listed above—has a structure worth noting. There are two laws of balance of volume of oil including two initial values (the volumes at the beginning of the process). We observe that the flow leaving the first tank must be equal to the flow entering the second storage element. This is a special case of the balance of volume which we might call the interaction rule. A relation between pressure differences in a circuit tells us how to relate the pressure difference across the pipe to those across the fluid columns in the tanks. Then there are three constitutive relations for pressures of stored fluids and for the flow, and three parameters. The laws of balance and the relation between pressure differences have a generic structure that is the same in all systems. The constitutive laws, on the other hand, depend upon circumstances. To be specific, if we had let water flow through the pipe, the flow law would be different from what we used for oil.
1.2 A COMPLEX CASE: THE BLOOD CIRCULATORY SYSTEM

The concepts and methods used in Section 1.1 for creating an understanding of dynamical systems are basic, yet they suffice for some fairly sophisticated models of real complex cases. This will be demonstrated by constructing models of the blood circulatory system. At the same time we will see how analogical reasoning is employed in an active field of research and development. Researchers in physiology use the language of electric circuits to describe their models of the blood circulatory system.

1.2.1 Description of the System

Historically, it was not self evident that one could understand the blood circulatory system in terms of hydraulics. Today, we are accustomed to seeing the heart as a pump and the vessels as pipes for the flow of blood. Briefly said, the circulatory system consists of a single circuit having two pumps made up by the two main chambers (ventricles) of the heart (Fig. 1.7).

The blood circulatory system. The right ventricle of the heart pumps blood through the lungs (through the pulmonary circuit) where it is replenished with oxygen. From there it flows to the left side of the heart into the atrium and then into the main chamber called the left ventricle. From there the blood is pumped through vessels through the body. The “pipe” leading away from the heart is called the aorta which branches off into arteries, which branch into finer vessels and capillaries. There are several branches of this body circuit (called the systemic circuit) going through the torso with its organs, through the legs and arms, and through the head. After the blood has brought oxygen, nutrients, and water to body parts and has taken up waste products, it flows back through the veins toward the right ventricle of the heart. If we want to understand blood pressure properly, we need to understand how the aorta functions.

Blood pressure. We have probably all had our blood pressure measured at one time or another. An air filled cuff is put around our upper arm. The pressure of the air is increased until the cuff fits tightly. Then the pressure is slowly decreased and the doctor listens for changes of sound and then reports something like “130 over 80.” The upper value is called the systolic pressure, the lower one is the diastolic pressure (Fig. 1.7).

First, we have to remember that the values reported are typically given in units of mmHg (millimeters of mercury column, i.e., the pressure that supports a column mea-
suring so many mm of mercury; 1 bar correspond to about 760 mmHg). Second, a value of zero means “ambient” pressure, i.e., the average pressure in the body which is close to ambient air pressure.

Why are there two values reported in blood pressure measurements? Why isn’t the lower value simply zero or close to zero, considering that the pressure in the left ventricle reaches a value close to zero at some point during the cardiac cycle? The aorta and its properties are responsible for this. The aorta is a flexible vessel, in contrast to arteries and capillaries. This means that the aorta functions not only as a pipe but also as a storage vessel—very much like a balloon or a membrane accumulator. It stores the blood that comes from the left ventricle for some time and releases it relatively steadily into the rest of the circuit, i.e., into the arteries and capillaries. These basically work as simple pipes that let the blood pass. Their hydraulic function is described by stating that they set up a resistance to the flow of the relatively viscous blood. The blood pressure measured at the doctor’s office is close to the pressure measured in the aorta. This quantity varies rhythmically between a high value (somewhat lower than the maximum pressure in the heart) and a lower one. Since there always is blood in the aorta, this lower (systolic) value is never close to zero (Fig. 1.7).

1.2.2 An Electric Circuit Modeling the Action of the Aorta

In summary, the left ventricle is the intermittent pump for the systemic circuit, the aorta is like a windkessel, and the rest of the blood vessels serve as a (branching) pipe (see Fig. 1.8, top left). Between the pump and the windkessel there is the aortic valve which must make sure that blood does not flow back into the left ventricle. Such a system can be modeled successfully by a physical model using electric elements—power supply, capacitors, resistors, and a diode (Fig. 1.8, bottom left). Operating the circuit with a variable voltage of the power supply that mimics the pressure of blood in the left ventricle leads to a voltage across the capacitor analogous to the blood pressure in the aorta (see the graph on the right in Fig. 1.8).

2. Around 1660, windkessel pumps were invented to smooth water flows through fire hoses. Without these, the flows directed at a fire would be as intermittent as the pressure differences set up by hand operated pumps used in old fire fighting systems. (Windkessel is German for air or wind chamber.)

![Figure 1.8: The windkessel model of the systemic circuit (top left). It consists of a pump that takes liquid from the environment (left), a short pipe with a valve, a container, and a (long) pipe leading back to the environment. Analogous electric circuit (bottom left) having a power supply, diode, resistors, and a capacitor modeling the aorta. Operating the electric circuit with a variable voltage of the power supply leads to data shown on the right.](image-url)
To produce the equivalent electric circuit for the hydraulic windkessel, it is important to identify voltages with pressure differences and to make sure that the relation between potential differences in a circuit (i.e., the loop rule) is adhered to. Here, we have two loops, one leading from the ambient air through the pump and the valve to the tank and back to the environment, the second leading from the environment through the tank and the long pipe back to the ambient air (Fig. 1.9). The intermittently driven pump is represented by a power supply that sets up a voltage as a function of time resembling the pressure difference of the pump.

System dynamics model. Creating a system dynamics model of the electric circuit—and by analogy for the heart-aorta-vessels part of the systemic blood flow system—is rather simple (Fig. 1.10, left). The fit of simulations and data can be made close to perfect for the electric circuit (Fig. 1.10, right) whereas the model will always be too simple to represent the physiological system very well. Still, it gives us a clear understanding of important processes in the blood flow system.

There is a single law of balance of electric charge of the capacitor (analogous to volume; see the reservoir symbol in the system dynamics diagram of Fig. 1.10). Charge can flow through two resistive elements (pipes), one associated with the circuit through the power supply (pump), the other going through the second resistive element (long pipe). If we use the ideas applied before, the flows through the resistive elements are expressed in terms of the voltages (pressure differences) across them, and the respective resistances. The voltage across the capacitor (pressure of the fluid in the tank) is calculated with the help of the charge and the capacitance. The valve is imitated by a diode. Finally, the resistive voltages are calculated from the other voltages using the loop rule.

The model equations consist of a law of balance, two loop rules relating voltages, and five constitutive relations for the resistors, the diode, the capacitor, and the power supply:

\[
\frac{d \text{ (Charge)}}{dt} = \text{IQ}_1 - \text{IQ}_2, \quad \text{Charge}(0) = 0
\]

\[
\text{UR}_1 = \text{US} - \text{UD} - \text{UC}
\]

\[
\text{UR}_2 = \text{UC}
\]

\[
\text{IQ}_1 = \text{IF} (\text{UR}_1 > 0) \text{ THEN } \text{UR}_1/R1 \text{ ELSE } 0
\]

\[
\text{IQ}_2 = \text{UR}_2/R2
\]
1.2.3 Limitations of the Model

The model for the electric circuit representing the windkessel system is quite good. The only real shortcoming is the strongly simplified model of the diode. Here, we assume the diode to be ideal: it does not let any charge pass for voltages across the element that are smaller than 0.82 V; above that, it is an ideal conductor. This is a crude approximation of the real behavior, but apparently it suffices.

If we transfer the model to the physiological case (by simply changing names of variables and parameters, and by applying proper values for the hydraulic parameters), limitations become more apparent. Experience with such models shows that the expressions for the capacitive pressure of the blood in the aorta, and for the flow through the aortic valve, could be improved upon. The aorta is a pressure vessel having an elastic wall which typically leads to a nonlinear relation between pressure and volume stored. The flow from the left ventricle into the aorta is quite complex (narrowing of the conduit, a valve, relatively high flow speed) so that the flow relation turns out to be nonlinear as well. Using a relation for turbulent flow instead of laminar flow comes closer to reality.

Another shortcoming is demonstrated by the shape of the simulated aortic pressure. With a pressure of the blood in the left ventricle taken from measurements (see the diagram in Fig. 1.7), the simulated pressure curve looks more like that in Fig. 1.10. There is no hump near the maximum values, with a slower decrease after the closing of the aortic valve. We do get this behavior, however, if we apply a more realistic representation of the aorta in our model. Rather than taking the aorta as a single piece (with the blood having a single value of the pressure), we divide the vessel into several elements. The dynamics of the fluid is represented by storage in each element and flow from element to element (as in Fig. 1.1 and Fig. 1.5). This pseudo-finite-element model yields results that are better already.

Finally, there is one more important limitation in all of the models produced so far. If we limit our concepts to those of storage and flow through resistive elements, we will never understand why blood flows backward for a brief period during the cardiac cycle (see the flow data in the diagram of Fig. 1.7). Note that this does not mean that blood flows back through the valve into the heart of the animal. Rather, the liquid flows back and forth in the aorta, mostly forward, but backward as well. We will discuss how to understand this later in this chapter (Section 1.6).

The limitations discussed have to do with the constitutive laws for storage and flow applied in our models. The generic laws of balance of amounts of liquid or electric charge, and the concept of pressure differences and voltages in circuits are unaffected, and so is the basic assumption that fluid and electric phenomena can be described by similar basic structures in dynamical models. We will see in this book how we can apply this type of analogical reasoning to construct a theory of the dynamics of heat.
1.3 Water, Charge, and Driving Forces

We have been fairly successful in producing practical models without much regard for formalisms. In the present and the following sections, I will present a more formal discussion of what we can learn from dynamical fluid and electric systems and the modeling exercises just described. I will introduce the basic quantities needed to formulate ideas, and discuss the most important relations between these quantities that make up our models of dynamical systems.

1.3.1 Basic Explanatory Schemas and Physical Quantities

Let us review the structure of explanations used in our models of fluid and electric dynamical systems discussed so far. A process is visualized as the result of the storage and flow of fluidlike quantities such as amount of fluid or electric charge. A flow going through a potential difference—pressure difference or electric potential difference—from high to low is said to run by itself. Potentials are visualized as levels, and a potential difference is conceptualized as a driving force. Flows can be forced through a potential difference going in the opposite direction—up instead of down—by a pump. Finally, flows through conduits are resisted, and if a fluidlike quantity is stored, it literally sets up a tension (a potential difference) in the system.3

The interaction of fluidlike quantities and differences of levels is the source of dynamics in the world. If differences decay, we end up in equilibrium where processes stop; if the processes build up new differences, dynamics continues in waves through chains of systems. The schemas listed here (substances, intensities, driving forces, resistances, tensions) are aspects of a gestalt of natural processes constructed by human perception. They form the basis of our understanding of nature.

Waterfalls and process diagrams. The relationship between fluidlike quantities and their associated driving forces (potential differences) can be cast in graphical form. In a voluntary process, a fluidlike quantity flows from a higher to a lower level of the potential—like water in a waterfall (Fig. 1.11, left). The reverse is a fluidlike quantity being forced through a potential difference from a low to a high level, like water being pumped uphill. This I call an involuntary process (Fig. 1.11, center).

Figure 1.11: Both water and electrical charge (X) may be imagined to flow by themselves from regions of higher potentials (ϕ) to regions of lower ones. Potentials are conceptualized as levels. IX is the flow of quantity X. Coupling of processes is represented in process diagrams (right; this is a diagram for an ideal water pump). IQ and IV represent flows of charge and of volume, respectively. P stands for pressure.

3. What is still missing from this description is an explicit reference to the force or power of a process which can be taken as the source of our concept of energy (see Chapter 2).
Physical systems often couple processes where a voluntary process drives an involuntary one (Fig. 1.11, right). An example of this is an electrically driven water pump. Here, electric charge flowing downhill drives the flow of water from low to high pressure. The diagram used to represent the coupling of phenomena is called a process diagram.

Fundamental or primitive quantities. Our first question must be which physical quantities we can use as the basis for a quantitative description of the flow and storage of fluids and electric charge. From the foregoing discussion of explanatory schemas it should be clear that we need to formalize the concepts of amounts of fluidlike quantities, their flows, and associated levels of fluids or electricity.

There is an important difference between fluidlike quantities and intensities which helps us in keeping them apart. The former scale with the size of a system whereas the latter do not. Take a body carrying electric charge and divide it into two equal parts. Half of the body carries half of the charge. The electric potential, however, is the same for both parts of the body (if it were not, charge would flow and rearrange upon bringing the two parts of the body together again). The difference between the two types of quantities is reflected in the terms extensive or additive for amounts of fluid or charge, and intensive for pressure and electric potentials.

Measures of amounts. To be concrete, let us consider water in a container. There are several possible choices for measures of its amount, namely, volume of a certain amount of the liquid, its mass or amount of substance (Chapter 6). These measures are related:

\[ n = \frac{1}{M_0} m \]

\[ V = \rho m \]

\[ n = \frac{1}{M_0} m \]

\[ V, m, \text{ and } n \text{ stand for volume, mass, and amount of substance, respectively. The conversion factors are the density } \rho \text{ and the molar mass } M_0. \text{ Standard SI units of volume, mass, and amount of substance are } \text{m}^3, \text{kg, and mole. We shall express most of what follows in terms of the volume of a body of water. Using the volume of an amount of fluid is the basis of descriptions of hydraulic processes. Amount of substance will become important when we turn to a chemically oriented discussion of transport processes. Mass is sometimes preferred in mechanical engineering models.} \]

The physical quantity which measures an amount of electricity is well known: it is the electric charge. A capacitor stores a certain amount of charge, just as a container stores a certain volume of water. In contrast to quantities of fluids, quantities of charge can take positive and negative values.

Measures of flows. If we want to set up a theory of the flow and balance of water and charge, we need a primitive quantity which describes their transport. For this purpose we conceive of the rate of flow of water or charge into or out of a system, measured in terms of a new quantity which we call the flux or the current of water or charge.
The rate at which water flows can be expressed in terms of the *volume flux* or *current of volume*, i.e., the volume of water flowing past a measuring device per time. Its unit is m$^3$/s. Alternatively, we may employ the flux of mass or the flux of amount of substance. Again, for practical purposes, we shall choose the first of these measures for simple fluid processes. In electricity, the quantity analogous to volume flux is the *current of charge* whose unit is Ampere (A).

We shall use the symbol $I$ for fluxes or currents. Since there will be many fluxes for different physical phenomena, indices will be used to distinguish between them. Here, the index $V$ stands for volume, $Q$ for charge. So $I_V$ and $I_Q$ stand for currents of volume and charge, respectively.

**Measures of levels.** Using volume and volume current, we are able to say something about an amount of water, namely the amount of it stored in a system, and the rate at which it is flowing. These quantities do not suffice for a complete theory of the phenomena associated with containers of water and currents flowing in and out. They do not tell us anything about why water should be flowing at all. In electrical circuits as well, we need a quantity which is responsible for setting up currents of charge in the first place. As we have already discussed, we introduce *potentials* or, figuratively speaking, *levels* to make sense of this aspect of our conceptualization of reality. In fluids, *pressure* takes this role; in electricity, it is the *electric potential*. Pressure is measured in Pascal (Pa), electric potential in Volt (V).

**Derived quantities.** Only hindsight can tell us if we have chosen the right quantities as the fundamental ones for a given range of phenomena. This means that we have to accept a certain choice, define new quantities on its basis, build a theory, and work out its consequences. If we are satisfied with the results compared to what nature demonstrates, we call the theory a successful one.

For the following, let us define derived quantities related to volume and volume current. Again, the case of electricity is analogous. An important derived quantity for the description of dynamical phenomena is the *rate of change* of stored quantities of fluid (volume) or charge; it measures how fast quantities change. The rate of change of volume is visualized as the slope of the graphical representation of the function $V(t)$ or, more precisely, the slope of a tangent to the function at a chosen point of time (Fig. 1.12). Symbols for rates of change of volume are $dV/dt$ or the letter $V$ with a dot on top (read as $V$-dot).

\[
\dot{V} = \frac{dV}{dt}
\]  

(1.2)

Often we are interested in the overall change of the volume of water in a container as the result of a process lasting for a period of time. For this purpose we define the *change of volume* which is simply given by

\[
\Delta V(t_1 \rightarrow t_2) = V(t_2) - V(t_1)
\]  

(1.3)

The change of volume is related to the rate of change. It is the integral over time of the rate of change of volume:

\[
\Delta V = \int_{t_1}^{t_2} \dot{V} dt
\]  

(1.4)
Finally, we need a measure of how much water has flowed across a surface with a current in a given period. We shall call this quantity the *volume exchanged* or *volume transported*. It is defined as the integral over time of the flux of volume:

\[
V_e = \int_{t_1}^{t_2} I_v \, dt \quad \text{(1.5)}
\]

As mentioned, analogous expressions hold for quantities relating to electric charge.

### 1.3.2 Accounting for Volume and Charge

Fluidlike quantities such as volume, amount of substance, or electric charge accumulate in systems, and there is a simple law of accounting for such quantities called a *law of balance*. Consider water. If the volume of water is a conserved quantity (as should be the case for incompressible fluids not subject to chemical conversions) the volume stored in a given system can change only due to the transport of water across the surface of the system (Fig. 1.13). There must be currents or fluxes of volume with respect to the system, and they alone are responsible for the change of the contents of the system. They determine how fast the volume changes:

*The rate of change of the volume of water in the system must be equal to the sum of all fluxes associated with the currents of water crossing the surface:*

\[
\frac{dV}{dt} = I_{V,\text{net}} \quad \text{(1.6)}
\]

This is taken as one of the most fundamental relations of a theory of dynamics. Here, \( I_{V,\text{net}} \) is the *sum of all currents* with respect to the system chosen (Fig. 1.13). Note that the quantity we call *flux* has the dimensions of the quantity which is flowing divided by time. If we write the law of balance in this form, we have implicitly assumed that the flux of a quantity flowing into a system should be given a positive sign.

Equ.(1.6) is *not* a definition of the currents or fluxes of volume. The quantities occurring in this equation are fundamentally different, related only by an interesting property of fluidlike quantities. If we could no longer assume the volume of water to be a conserved quantity, we would have to change the law of balance. We would be forced to account for other means of changing the volume, by introducing other terms in Equ.(1.6). For now, let us assume that this is not necessary.

The definitions of change and of transported quantities (Equations (1.4) and (1.5)) allow us to express the law of balance of volume in the following form:

\[
\Delta V = V_{e,\text{net}} \quad \text{(1.7)}
\]

I call Equ.(1.6) the dynamical or *instantaneous form* of the law of balance of volume, whereas Equ.(1.7) is the *integrated form*. Identifying processes with flows, and expressing laws of balance, is the first important step in systems analysis.

In the case of electricity, the laws and definitions are identical. Since charge is a strictly conserved quantity, the rate of change of the charge of a body must be equal to the sum of all currents of charge with respect to this body. In other words, the law of balance of electric charge looks exactly like Equ.(1.6), with volume replaced by charge:

**Figure 1.13:** A system is a region of space occupied by a physical object. It is separated from the surroundings by its surface. A fluidlike quantity which we imagine to be stored in the surface can change as a consequence of transports across its boundary. Currents leaving a system are given negative fluxes. In some cases, transport across the surface is the only means of changing the contents.
The change of charge and the amount of charge exchanged in a process are defined analogously to Equations (1.4) and (1.5). Again, the law of balance in integrated form looks just like the expression in Equ.(1.7).

There is a special expression of the instantaneous law of balance of charge that is often used in modeling of electric circuits. Consider a junction of three or more wires in a composite circuit. Charge flowing toward the junction from one or more wires then leaves directly through the other wires. This is so because the junction does not store charge. The charge of the junction is and remains equal to zero. Therefore, the rate of change of charge of the junction must be zero as well, which means that for a junction,

$$0 = I_{Q_1} + I_{Q_2} + \ldots$$

This is called the junction rule or Kirchhoff’s First Law. Clearly, the same relation must hold for the balance of volume of a liquid applied to a junction made of pipes.

1.3.3 Pressure Differences and Voltages

Pressure differences and voltages are considered causes for processes. Alternatively, we may look at processes leading to potential differences. Identifying such differences and related processes is the second integral part of systems analysis.

**Pressure differences in circuits.** The pressure of a fluid changes from point to point in a closed hydraulic circuit. To make use of this observation, choose a few important points in the system (usually at the inlets and outlets of elements such as pipes, pumps, and tanks). Label *pressure differences*\(^5\) from point to point (Fig. 1.14) by introducing arrows and symbols $\Delta P_{AB}$, etc.:

$$\Delta P_{AB} = P_B - P_A$$

---

\(^5\) Note that there are two different kinds of (mathematical) differences in nature. One refers to a change in time as in Equ.(1.3). Then there are also differences of a quantity associated with their change along a path as in Equ.(1.10). I typically call the former *change* and the latter *difference.*
We can draw a diagram looking like a “landscape” (Fig. 1.14). When we are back at the origin, the pressure is the same. Therefore, the sum of all pressure differences in a closed circuit must be equal to zero (loop rule, Kirchhoff’s Second Law):

\[ \Delta P_{AB} + \Delta P_{BC} + \Delta P_{CD} + \ldots = 0 \]  

Electric potential and voltages. The electric potential measures the intensity of the electric state of a system at a point (these systems can be material bodies such as wire, but also electric fields). Its unit is V (Volt), the symbol used is \( \phi \) or \( \phi_{el} \). Electric potential does not have an absolute zero point (in contrast to pressure). This means that only the differences of electric potentials are important.

Electric potential differences are called voltages (see below). By itself, positive electric charge will flow from points of high to low electric potential, whereas electrons flow from lower toward higher electric potentials. The potential difference is the difference of potentials at two different points A and B of a system (Fig. 1.15), independent of the physical reasons for the difference:

\[ \Delta \phi_{AB} = \phi_B - \phi_A \]  

The negative potential difference is called the voltage between points A and B:

\[ U_{AB} = -\Delta \phi_{AB} \]  

This means that the voltage is positive across a resistor in the direction of the flow of (positive) charge, whereas it is negative across a battery in the direction of flow.

Potential differences (voltages) in closed electric circuits. The potential changes from point to point in a closed electric circuit. To make use of this observation, choose a few important points in the system such as the circuit of Fig. 1.15. Label potential differences (voltages) from point to point by arrows and by symbols \( U_{AB} \), etc.

\[ U_{AB} + U_{BC} + U_{CD} + \ldots = 0 \]  

Figure 1.15: Electric potential as a function of position in a circuit containing two capacitors, a battery, and a resistor. The values will change in the course of time, but the form of the level diagram will basically remain the same.
1.4 SOME CONSTITUTIVE RELATIONS IN FLUIDS AND ELECTRICITY

The relations discussed in Section 1.3 are basic or generic: they take the same general form in different systems and processes. Laws of balance and relations between level differences along a path in a system are independent of special circumstances. This is not so for the relations between potential differences, stored quantities, and currents. The latter depend upon special circumstances—upon how a system is built, the materials used, etc. For this reason we call the relations needed to express the terms in laws of balance *special, material, or constitutive laws*. 

1.4.1 Resistive Transports of Fluids and Charge

When a fluid flows through a pipe, its pressure drops in the direction of flow because of fluid friction. This pressure drop is called a *resistive pressure difference* $\Delta P_R$, and it is characteristic of the flow which, in turn, depends upon fluid properties and pipe dimensions. Charge flows through a resistor (or a conductor) from higher to lower electric potentials, so there is a resistive voltage $U_R$ associated with this process.

**Process diagram of resistive transports.** Since a fluid goes from high to low pressure (different from what it does in a pump), or charge flows from high to low electric potentials, we say that the level differences are driving the process. We know that in resistive transports, the process caused by the flow of fluid or charge consists of the production of heat (Fig. 1.16).

**Flow characteristic for fluids.** The relation between the resistive pressure drop $\Delta P_R$ and the associated volume current is called the *flow characteristic* (Fig. 1.17). (Note that here the current $I_v$ measures the flow *through* the system.) It allows us to calculate flows if we know the associated pressure difference, or vice-versa. There are two types of flow (laminar and turbulent) leading to two different characteristic curves.
Analogous characteristic relations can be constructed for electric conductors. The simplest conductors are made of ohmic materials where the relation between resistive voltage and electric currents is linear as in the case of laminar flow (Fig. 1.17, left). When we let electricity pass through a wire that can change its temperature drastically—as in the filament of an incandescent light bulb—characteristic curves look more like those on the right in Fig. 1.17 (even though the material is still ohmic; the deviation from linearity is not due to a basic change in transport mechanism, but to increasing temperature). A completely different type of transport of charge is observed in diodes which are made of combinations of semiconductors. Here, the electric current grows exponentially with increasing voltage across the diode.

**Laminar Flow.** For laminar flow, the characteristic relation is linear. In this case, we can write the flow law with the help of a hydraulic conductance $G_V$ (units $m^3/(s \cdot Pa)$) or its inverse, the hydraulic resistance $R_V$ (units $Pa \cdot s/m^3$):

$$I_V = -G_V \Delta P_R \quad \text{or} \quad I_V = -\frac{1}{R_V} \Delta P_R \quad (1.15)$$

There is an expression for the hydraulic conductance or resistance for laminar flow in pipes with circular cross section which is called the law of Hagen and Poiseuille:

$$R_V = \frac{8 \mu l}{\pi r^4} \quad (1.16)$$

$r$ and $l$ are the radius and length of the pipe, $\mu$ is the viscosity of the fluid. The viscosity of a fluid tells us how “thick” it is. Viscosity is discussed in some more detail in Chapter 3.

**Turbulent flow.** In turbulent flow, the flow increases less rapidly with an increase of the associated pressure difference (diagram on the right of Fig. 1.17). The turbulent characteristic function is close to the square root function for many practical cases:

$$I_V = k \sqrt{\Delta P_R} \quad (1.17)$$

This simple relation suffices as a first approximation. $k$ is called the turbulent flow factor. This factor is similar to a conductance, however, the terms resistance and conductance are only used for laminar flow.

**Ohmic transport of charge.** The transport of charge in metallic conductors satisfies a simple relation. For small enough voltages or electric currents, the current is strictly proportional to the potential difference across the conductor. Therefore, the characteristic relation is linear. In this case, we can write the flow law with the help of a conductance $G$ (units $A/V = 1/Ohm$) or its inverse, the resistance $R$ ($V/A = Ohm = \Omega$):

$$I_Q = GU_R \quad \text{or} \quad I_Q = \frac{1}{R} U_R \quad (1.18)$$
There is an expression for the conductance or resistance for ohmic transport in conductors having constant cross section:

\[ R = \frac{\rho_{el}}{A} \quad (1.19) \]

\( l \) and \( A \) are the length and cross section of the conductor, respectively, and \( \rho_{el} \) is the resistivity of the material. The resistivity basically measures how hard it is for charge to flow through the conductor. The inverse of resistivity is called electrical conductivity: \( \sigma = \frac{1}{\rho_{el}} \) (unit: S/m, S: siemens).

### 1.4.2 Storage of Fluids and Charge

Constitutive laws specifying currents have to do with transport phenomena. We also need a means of saying something about the process of storing water (or electrical charge). It is customary to introduce a quantity which expresses the relationship between a change of the amount of fluid contained in a system and the change of the associated potential, i.e., the change of pressure. It allows us to relate the change of system content to the possibly more easily measured potential. In electricity we are interested in the relationship between the charge contained in a system and the voltage.

**Capacitive characteristic in fluid systems.** If fluids are stored in tanks or pressure vessels, the pressure difference (normally) increases with an increasing amount of stored fluid. In other words, there is a relation between the volume stored and the associated pressure difference (which we call a capacitive pressure difference \( \Delta P_C \)). The relation is called a capacitive characteristic (Fig. 1.19).

**Elastance and hydraulic capacitance.** The characteristic can be expressed mathematically if we introduce the elastance \( \alpha_V \), i.e., the factor which tells us how easy it is to increase the pressure with a given amount of fluid:

\[ \dot{P}_C = \alpha_V \dot{V} \]
\[ \Delta P_C = \alpha_V \Delta V \quad \text{if} \quad \alpha_V = \text{const.} \quad (1.20) \]

\( \alpha_V \) is equal to the slope of a tangent to the characteristic curve (Fig. 1.19). This means that the elastance measures the stiffness of container walls (in the case of pressure vessels) or the inverse of the cross section of a tank. The unit of elastance is Pa/m³. Alternatively, we can introduce the hydraulic capacitance \( C_V \) (units m³/Pa) which is defined as the inverse of the elastance \( (C_V = 1/\alpha_V) \):

\[ \dot{V} = C_V \dot{P}_C \]
\[ \Delta V = C_V \Delta P_C \quad \text{if} \quad C_V = \text{const.} \quad (1.21) \]

For a liquid of density \( \rho \) in an open container the capacitance is

\[ C_V = \frac{A(h)}{(\rho g)} \quad (1.22) \]

Equ.(1.21) suggests a way to determine volume changes from pressure changes if the capacitance is known (as a function of pressure). For constant capacitance, we simply multiply the pressure difference by the capacitance. Geometrically, this corresponds...
to the calculation of the area of a rectangle. This tells us that, in general, the change of
volume associated with a change of pressure is equal to the area between the capaci-
tance – pressure function and the pressure axis (Fig. 1.20).

![Electric capacitive characteristic.](image)

Electric capacitive characteristic. If charge is stored in a capacitor, the voltage in-
creases with increasing amount of stored charge. In other words, there is a relation be-
tween the charge stored and the associated voltage (which we call a capacitive voltage
$U_C$). The relation is called a capacitive characteristic (Fig. 1.21, left). Another way of
representing the relation is by drawing a fluid image (Fig. 1.21, right), an imaginary
tank with charge inside where the level represents the voltage $U_C$. In general, the char-
eteristic is nonlinear. A linear characteristic is related to a constant capacitance. The
cross section of the imaginary tank represents the capacitance of the capacitor.

![Elastance and capacitance.](image)

Elastance and capacitance. The characteristic relation can be expressed mathema-
tically if we introduce the elastance $\alpha_Q$, i.e., the factor which tells us how easy it is
to increase the voltage with a given amount of charge:

$$
\dot{U}_C = \alpha_Q \dot{Q} \\
U_C = \alpha_Q Q \quad \text{if} \quad \alpha_Q = \text{const.} 
$$

(1.23)

$\alpha_Q$ is equal to the slope of a tangent to the characteristic curve (Fig. 1.21). This means
that the elastance measures the “stiffness” of the storage system. The unit of elastance
is V/C.

Alternatively, we can introduce the electric capacitance $C_Q$ or simply $C$ (units $C/V =
F$ (Farad)) which is defined as the inverse of the elastance ($C = 1/\alpha_Q$):

$$
\dot{Q} = C_Q \dot{U}_C \\
Q = C_Q U_C \quad \text{if} \quad C_Q = \text{const.} 
$$

(1.24)
1.4.3 Pumps and Batteries

Pumps, batteries, generators, and solar cells drive fluid or electric processes, i.e., they set up pressure differences and voltages. The driving process can be electric or mechanical in the case of pumps. Batteries are driven chemically, generators mechanically, and solar cells get their input from solar radiation.

Process diagram and characteristic of pumps. Pumps come in many different types and forms, ranging from the heart to microengineered or large industrial pumps. Here we are only interested in their overall performance. Pumps make fluids flow, and they increase their pressure. This simple fact is best represented in a process diagram of the type shown in Fig. 1.22. The process diagram used to describe the operation of a pump can be used to introduce the notion of the energy delivered to the fluid by the pump (Chapter 2).

Figure 1.22: As a fluid is forced through a pump, its pressure is made to go up. Figuratively speaking, the fluid current is forced uphill.

We define the operation of a pump by describing the relation between the pressure difference $\Delta P_{\text{p}}$ set up and the flow through the device. An ideal pump might be described by assuming a constant pressure difference. Real pumps commonly have a more complicated type of characteristic (Fig. 1.23). If we model a real water pump as consisting of an ideal part that sets up a pressure difference, followed by a resistive element due to turbulent flow, we should get a parabolic pressure-flow relation (this corresponds well to what we see in the diagram on the left of Fig. 1.23).

Figure 1.23: Measured characteristic curves of a water pump (left) and a typical 4.5 V battery. Whereas the water pump pumps water, a battery pumps electric charge.

Characteristics of batteries. Characteristic diagrams of batteries are simple linear curves: The voltage across the terminals decreases with increasing electric current (see Fig. 1.23, right). The formal description of the characteristic is

$$U_B = U_0 - R_i I_Q$$  \hspace{1cm} (1.25)
1.4 SOME CONSTITUTIVE RELATIONS IN FLUIDS AND ELECTRICITY

$U_B$ is the voltage measured across the terminals, $U_0$ is the maximum possible voltage (the open circuit voltage), $I_Q$ is the current through the battery, and $R_i$ is the internal resistance of the battery. A derivation of Equ.(1.25) uses ohm’s relation for resistive transports of charge.

1.4.4 Gravity and Height Differences

In the case of fluids, gravity—the Earth’s gravitational field—plays an important role if we do not restrict our view to purely horizontal processes. The action of gravity leads to particular constitutive relations for fluids in tanks and (vertical) pipes.

**Hydrostatic pressure.** For fluids which are “stacked” in a gravitational field, i.e., systems where the weight of the fluid is responsible for a pressure difference, there is a simple relation between pressure difference and height difference (Fig. 1.24). It can be derived from the observations which are summarized in Fig. 1.25:

$$\Delta P_{\text{grav}} = -\rho g (h_B - h_A) \quad (1.26)$$

This relation can be used to calculate pressure differences (and capacitive characteristics) for fluid tanks (Fig. 1.20 and Equ.(1.22)). It is correct for constant density only.

**Pressure gradients.** According to the example of hydrostatic pressure in an incompressible liquid (Equ.(1.26)), the pressure gradient in the upward direction is

$$\frac{dP}{dh} = -\rho g \quad (1.27)$$

As observed in Fig. 1.25, the pressure gradient is proportional to the density of the liquid. Furthermore, it must depend upon the strength of gravity ($g$). The negative sign tells us that the pressure decreases if we go upward.

**Pressure in the Earth’s atmosphere.** The pressure-height relation for the atmosphere is not linear: the pressure of the air drops exponentially with height above ground. This is so since the fluid is a gas whose density changes with pressure (and temperature). Nevertheless, the expression for the vertical pressure gradient is the same as that for incompressible fluids, i.e., Equ.(1.27) still holds.

If we know how the density of the air depends upon pressure and temperature, the re-
lation for hydrostatic equilibrium (Equ.(1.26)) can be solved. Even though this is not realistic, one often considers the case of an isothermal atmosphere (an atmosphere where the temperature does not change in the vertical direction). If this is the case, the density is proportional to the pressure (see Problem 7 in Chapter 5) leading to an exponential pressure-height relation:

\[ P(h) = P(0)e^{-h/k} \]  

(1.28)

For the Earth’s atmosphere, the factor \( k \) is about 7000 m. This means that the pressure decreases by a factor of \( e \) for every 7000 m, or by a factor of 2 for every 5000 m. Even though our atmosphere is not isothermal, the result is useful for quick estimates.

**The gravitational potential.** Pressure differences are interpreted as hydraulic driving forces, pressures are hydraulic “levels.” For fluids stacked in the gravitational field, vertical pressure differences are the result of gravity. They are calculated according to Equ.(1.26) or Equ.(1.27). If we multiply the pressure difference by the volume of a certain quantity of liquid which we imagine to be transported from a height \( h_1 \) to a height \( h_2 \) (Fig. 1.26), we have

\[ \Delta PV = \rho g \Delta h V = g \Delta h \rho V = g \Delta h m \]

(1.29)

This result is interpreted as follows (Fig. 1.26). If we look at gravitational processes as the transfer of the mass of a substance from a level 1 to a level 2, the right hand side of Equ.(1.29) represents mass \( m \) going from a gravitational level \( gh_1 \) to a level \( gh_2 \). Therefore, \( g \Delta h \) is interpreted as the gravitational driving force, and \( gh \) is the so-called gravitational potential.

### 1.5 Behavior of RC Models

The models we can construct by using laws of balance, loop rules, and resistive and capacitive constitutive relations are called RC models. \( R \) and \( C \) stand for resistance and capacitance, respectively. Adding pumps and batteries does not change anything about the nature of the models we have been building. As we will see, the behavior of such models is relatively simple. In particular, they do not admit oscillatory solutions. We will see in the following section how to extend models to create an understanding of oscillatory behavior and wave motion.

#### 1.5.1 Charging and Discharging Single Storage Elements

Systems made up of containers and pipes (or capacitors and resistors) show relatively simple behavior. Complex behavior is often the result of the interaction of several simple elements. For the simplest systems—those having constant values of capacitance and resistance—analytic solutions of the model equations can be obtained. Solutions are combinations of exponential functions of time. In the case of draining straight-walled tanks through horizontal pipes with laminar flow (Fig. 1.27, top) we get

\[ \Delta P(t) = \Delta P_0 e^{-t/RC_V} \]

(1.30)
If an empty tank is charged (Fig. 1.27, bottom), the solution of the model is

$$\Delta P(t) = \Delta P_{\text{Max}} \left( 1 - \exp \left( -\frac{t}{R_V C_V} \right) \right)$$  \hspace{1cm} (1.31)

Equ.(1.30) and Equ.(1.31) also hold for \( h(t) \) and \( V(t) \), and they work for the equivalent electric circuits. We only have to substitute electric for hydraulic variables.

The functions reported above are the solutions of the differential equations resulting from the combination of all the relevant model equations. Here is an example of the draining of the oil tank. We have a single storage element with a single flow. Therefore, the law of balance of volume is

$$\frac{dV}{dt} = I_V$$

The volume is related to the capacitive pressure difference, the flow results from the resistive pressure difference along the pipe, and the pressure differences are equal:

$$V = C_V \Delta P_C$$
$$I_V = -\frac{1}{R_V} \Delta P_R$$
$$\Delta P_C = \Delta P_R$$

If we introduce these equations into the law of balance of volume, we obtain a single differential equation:

$$C_V \frac{d \Delta P_C}{dt} = -\frac{1}{R_V} \Delta P_C$$

An initial condition has to be added to this differential equation: \( \Delta P_C(0) = \rho g h_0 \). Solving the initial value problem leads to a function of the type seen in Equ.(1.30).
1.5.2 Time Constants

The behavior (fluid level as a function of time) for the simple cases of draining and filling of a tank is shown in the accompanying graphs (Fig. 1.27). The solutions of the model are exponential functions. A measure of how fast (or slow) the process is, is the time it would take for the tank to drain or to fill were the level to continue to change at the initial rate. This time span is called the capacitive time constant $\tau_C$ of the system. In a period equal to one time constant, the level of fluid in the system shown on the left in Fig. 1.27 drops to $1/e = 0.37$ times the initial level. The analytic solutions in Equ.(1.30) and Equ.(1.31) demonstrate that

$$\tau_C = RC$$  \hspace{1cm} (1.32)

1.5.3 Conductive Transports Through Chains of $RC$ Elements

If we combine storage elements and resistors (or conductors) in long chains (see Fig. 1.28), we obtain models of a phenomenon that is very common in natural and technical settings: diffusion. Diffusion is usually associated with the transport of substances through matter. An everyday example is the spreading of a drop of ink on blotting paper, or salt dissolved in a layer of water slowly migrating into other parts. Diffusion is particularly important in biology where many processes depend on the transport of chemicals through bodies (see Chapter 6 for more detail).

Diffusion is very much a physical process. As we have seen, fluidlike quantities such as charge or amount of substance can be transported (later we will add heat and momentum to the list). One form of transport is the flow through conducting materials. A fluidlike quantity is present in matter (it is stored), and it is conducted through it (Fig. 1.28). The combination of effects leads to diffusion.
Consider several tanks in a row connected by pipes as in Fig. 1.28. This is the simplest and most vivid of systems whose behavior mimics diffusion. The tanks are for temporary storage of a fluid, and the pipes let the fluid pass according to the law of conduction (or resistance): The flow depends upon the difference of fluid levels in the storage elements. Alternatively, we can build an equivalent electric circuit that demonstrates analogous behavior (Fig. 1.28, middle). This is a model of a thin electric conductor such as a wire. It can be considered to consist of storage elements for charge placed one after another along a line. The wire also acts as the conductor. If we place some charge near the middle of such a conductor, electricity will spread toward the ends as shown in the diagram of Fig. 1.28 (bottom, right).

1.6 O SCILLATORY PROCESSES AND WAVES

We still do not understand an important element of the behavior of dynamical systems. I have pointed this out when we discussed the blood flow system (Section 1.2.2). Blood flows backward in the aorta for brief periods during each cardiac cycle—the flow oscillates back and forth. The windkessel model in the form presented in Fig. 1.10, however, does not lead to oscillations.

This shortcoming is no problem for the example of how oil flows from one tank into another through a pipe. However, if we change the system to a U-pipe containing a less viscous liquid (Fig. 1.29), oscillations occur. So we need to understand what we have neglected in our discussions so far.

1.6.1 Starting and Stopping Currents

Contrary to what we would consider realistic, we have assumed that currents follow pressure differences directly and in accordance with a simple flow relation such as the ones in Equ.(1.15) and Equ.(1.17) (and Equ.(1.18) for electricity). To be concrete, the model of the equilibration of levels in two communicating tanks (Fig. 1.5) predicts an immediate rise of the current from zero to its maximum value upon opening the valve of the connecting pipe (Fig. 1.30).

This is unrealistic. We know from experience that a current of water starting in a pipe takes a noticeable amount of time to reach its maximum. The same is true of electric currents, even though the delay there may be so much shorter that we think we can neglect the effect in all our models. This is not so, however. The phenomenon of induction, as it is called in electricity, is present all the time, and it can be made very noticeable by introducing electromagnets in the form of solenoids in our circuits.

Both in fluids and in electricity, currents have to be driven to start up, or put more generally, to change. Driving means we need a pressure difference or a voltage for the effect to occur. The beginning phase of draining of a tank through a pipe (Fig. 1.31) demonstrates how we can understand this phenomenon. As we open the pipe (take the finger off the end of the pipe) to let the fluid flow, a pressure difference is established along the pipe. (Even though this process also takes some time, we shall assume, and reasonably so, that this happens quickly compared to the rise of the current we are interested in.) This pressure difference results from the difference of pressures at the inlet of the pipe at the tank and the pressure of the air at the outlet. It will slowly decrease in time due to the draining of the tank. (This is the phenomenon we already understand on the basis of our $RC$ models.)
What we need to understand is this: there is a real pressure difference $\Delta P_{AB}$ along the fluid in the pipe. At the very beginning, when the flow is still zero, the resistive pressure difference which we have been considering so far in this chapter must also be zero. So, there is an actual pressure difference that has nothing to do with the actual flow. Rather, the pressure difference leads to an acceleration of the fluid: the fluid in the tank presses more strongly upon the fluid in the pipe from behind than does the air at the outlet from the front. Now, as the current of fluid increases, the resistive pressure difference associated with it increases as well. As a result, the part of the actual pressure difference $\Delta P_{AB}$ remaining, i.e., the quantity $\Delta P_{AB} - \Delta P_R$, also decreases. This is the part that continues to accelerate the fluid. The flow will increase more slowly as time goes on, just as indicated in the lower diagram of Fig. 1.31. Finally, the inductive effect—the changing of the current due to the driving force $\Delta P_{AB} - \Delta P_R$—must have come to a halt.

The pressure difference $\Delta P_{AB} - \Delta P_R$ is called the inductive pressure difference $\Delta P_L$:

$$\Delta P_{AB} = \Delta P_R + \Delta P_L \quad (1.33)$$

Hydraulic induction results from the inertia of the fluid flowing through a pipe (electric induction is caused by the magnetic field of a solenoid). Therefore, it should be possible to derive the relation between rates of change of currents and inductive pressure differences on the basis of the laws of mechanics. Newton’s law (i.e., the balance of momentum; see Chapter 3) lets us calculate the accelerating effect of the inductive pressure difference:

$$-A_p \Delta P_L = A_p \rho \frac{dV}{dt}$$

where $A_p$ is the cross section of the pipe, and $l$ is its length. We assume the flow speed to be uniform over the entire cross section of the pipe. Therefore, the pressure difference may be expressed in terms of the rate of change of the volume flux (or volume current):

---

7. While energy is stored in the magnetic field associated with a current of charge, energy is stored in the flowing water (kinetic energy). The form of the relationship between energy, current, and inductance is the same in hydraulics and electricity (see Chapter 2).
The factor $l p/\rho A_p$ is called the hydraulic inductance of the fluid in the pipe (in analogy to the electromagnetic inductance of a solenoid in electric circuits):

$$L_v = \frac{l p}{\rho A_p}$$  \hspace{1cm} (1.34)

This factor measures how hard it is to change a current. Now we can write the law of induction, i.e., the relation that shows in what way an inductive pressure difference leads to a rate of change of a current:

$$\Delta P_L = -L_v \frac{dI_v}{dt}$$  \hspace{1cm} (1.35)

In electricity, we have an analogous relation:

$$\Delta \Phi_{el,L} = -L \frac{dQ}{dt}$$  \hspace{1cm} (1.36)

Note that these equations are constitutive laws, not some sort of law of balance of a current. Hydraulic and electric phenomena are clearly comparable. As in the case of electric currents, decreasing a current of water induces a positive potential difference which tends to oppose the change of volume flux.

If we want to know how a laminar fluid current through a pipe (or, in analogy, an electric current through a solenoid having ohmic resistive properties) changes in time, we simply combine Equ.(1.33) and Equ.(1.35) to obtain

$$\Delta P_{AB} = -R_v I_v - L_v \frac{dI_v}{dt}$$  \hspace{1cm} (1.37)

I have written the relation for the hydraulic case, but you may translate it to fit electric phenomena by simply substituting charge for volume. The ratio of inductance and resistance has the dimension of time. Therefore, $L/R$ is the characteristic time scale on which currents change in an inductive circuit (the inductive time constant). If we assume a constant value for the total pressure difference, the solution of Equ.(1.37) for the starting of a current is an exponential of the type shown in the lower diagram of Fig. 1.27. The formal solution proves that $L/R$ is the inductive time constant.

### 1.6.2 Oscillations

The explanation of the phenomena associated with starting and stopping of currents is the missing link for an understanding of oscillatory systems. If we combine electric or hydraulic capacitors and inductors, oscillations are possible. In an electromagnetic LCR-circuit, charge oscillates between storage and flow with a frequency which depends mainly on $L$ and $C$. Water may be made to oscillate in a U-tube in just the same manner (Fig. 1.29). By calculating the capacitance and the inductance of the container
we can find the frequency of oscillation. There is another hydraulic setup which has an electronic equivalent. If the flow of water from an artificial lake to the turbines of an electric power plant has to be stopped abruptly for any reason the pressure may rise to such a level that the pipes rupture. For this reason a hydraulic capacitor is built in parallel to the system, namely a tower (surge tank) which is filled rapidly with the water rushing down the pipes.

Let me derive the equations of a simple oscillator by creating a system dynamics model for a fluid in a U-pipe (Fig. 1.29). The U-pipe is structurally similar to two communicating tanks described in Fig. 1.1. So let us start with the system dynamics model presented in Fig. 1.5. The relations it represents are still valid. Only the flow relation has to be turned around: since the current will be calculated by integrating its rate of change (which we obtain from the law of induction), we can use the flow to compute the resistive pressure difference (Fig. 1.32). The difference of the capacitive pressures yields the total pressure difference along the pipe ($\Delta P_p$). The latter minus the resistive pressure difference yields the inductive pressure difference $\Delta P_L$. This is used to calculate the rate of change of the current. This completes the system dynamics model shown in Fig. 1.32. Here, $C$ is the capacitance of one of the sides of the pipe, $L$ is the inductance of the fluid in the pipe, and $R$ is its resistance. Properties are assumed to be constant.

The equations that make up the model are three differential equations (one for each of the stocks), the loop rule, and constitutive relations for the rate of change of the current, the capacitive pressures, and the resistive relation. These equations can be rearranged to yield two initial value problems (first order ordinary differential equations with initial conditions). If we express these equations with two of the three state variables, namely $P_{C1}$ and $I_V$, we obtain

$$\frac{dP_{C1}}{dt} = -\frac{1}{C}I_V, \quad P_{C1}(0) = P_{C1,0}$$

$$\frac{dI_V}{dt} = -\frac{1}{L}(-2P_{C1} + RI_V), \quad I_V(0) = I_{V,0} \quad \text{(1.38)}$$

It is customary to combine such equations still further so we have a single second order differential equation for one of the variables. If we choose $P_{C1}$, we arrive at

\[\text{Figure 1.32: A system dynamics model diagram for a fluid in a U-tube. Note the integrator for the current (the combination of flow symbol for the rate of change of the current and the stock for the current itself). The inductive pressure difference is calculated from the capacitive and resistive pressure differences.}\]
1.6 Oscillatory Processes and Waves

The oscillatory behavior of certain physical systems can be described by the following differential equation:

\[
\frac{d^2 P_{C_1}}{dt^2} + \frac{R}{L} \frac{dP_{C_1}}{dt} + \frac{1}{LC_{\text{tot}}} P_{C_1} = 0
\]  

(1.39)

with proper initial values for \(P_{C_1}\) and \(dP_{C_1}/dt\). This is the well-known linear equation for damped oscillations. If \(R = 0\), the oscillation is undamped, and its period is

\[
T = 2\pi \sqrt{\frac{C_{\text{tot}}}{L}}
\]  

(1.40)

Note that the total capacitance, i.e., the capacitance of both sides of the U-pipe taken together, is equal to half the capacitance of one side, so \(C_{\text{tot}} = C/2\).

1.6.3 Inductive Model for Blood Flow

Notice how induction leads to an explanation of the back-flow of blood in the aorta of a mammal (Fig. 1.7). Assume the aorta to be a storage element (it has flexible walls), a resistor or conductor (it lets viscous blood pass), and an inductor (the fluid demonstrates inertia). A simple model of the dynamic behavior of the aorta is one that divides the vessel into two sections (Fig. 1.33). The storage elements and the flow between them can now be modeled like a system made up of a fluid in a U-tube (Fig. 1.32). The difference is that we have an input to and an output from the U-tube (intermittent input from the heart, smoothed output to the body).

The model contains quite a number of parameters which have to be determined if we want reasonable simulation results. It is important to have data to estimate at least some of these values (see the graph in Fig. 1.7). The aortic pressure curve and the flow allow us to estimate the capacitance of the aorta and the systemic resistance (we obtain

Figure 1.33: If we divide the aorta in the blood circulatory system into two elements, the windkessel model looks like the illustration at the top. The flow between the two tanks can be made to exhibit both resistive and inductive properties. Bottom: A system dynamics model diagram of this system.
values of 2·10⁻⁹ m³/Pa for the former, and 2·10⁸ Pa·s/m³ for the latter). The difference between the pressures in the left ventricle and the aorta yields an estimate of the resistance of valve and entrance to the aorta (or, if we model the flow there as turbulent, the flow factor as in Eqn.(1.17); we obtain roughly 4·10⁻⁶ m³/(s·Pa⁰.⁵)).

We still need starting values for the inductance and resistance of the aorta. If we assume a hose of 50 cm length and 1 cm diameter for the aorta of a sheep, the inductance should be about 6·10⁶ Pa·s²/m³ (Eqn.(1.34)). For laminar flow, the resistance would be roughly 7·10⁶ Pa·s/m³ (Eqn.(1.16)). If we use the inductance and the capacitance of the aorta to estimate the period of oscillation of blood in this vessel as in Eqn.(1.40), we obtain 0.35 s. This can be compared to the oscillations visible in the flow data. Armed with such values, we can simulate the model in Fig. 1.33, and adjust parameters a little more. Despite this overly simplified model, we obtain results that do mimic many of the important real features (Fig. 1.34).

**Figure 1.34**: Data for blood pressure (left) and blood flow (right) in the aorta of a sheep (circles in the diagrams; see also Fig. 1.7). Simulation results for the model in Fig. 1.33 have been superimposed. The results are far from perfect, but they show behavior not visible in RC models discussed earlier. There are wave-like features both in the pressure and in the flow.

### 1.6.4 Wave Propagation in Chains of LCR Elements

The inductive two-tank windkessel model is a step toward an explanation of another important phenomenon—wave propagation. If we divide a conductor into many elements and model each element as a storage device with a resistor and an inductor between each of them (Fig. 1.35), electric charge or a fluid will be transported in a wavelike manner through the chain. Whereas in RC chains (Fig. 1.28), a substance diffuses and basically does not flow backwards, here the fluidlike quantity can be reflected at an end of the chain and come back, and the amplitude can become higher in an element further down the chain, something that does not happen in diffusion. Most importantly, one can identify a definite finite speed of wave propagation. This is noteworthy since the equations predict infinite speed for diffusion—something that is quite unrealistic and unphysical (see Chapter 13).

**Figure 1.35**: A chain of RCL elements can serve as a model for a conductor that admits wavelike transports. The black rectangles symbolize inductors.

A system dynamics representation of the chain of RCL elements in Fig. 1.35 is fairly simple to achieve. We start with the laws of balance of charge for two neighboring ca-
pacitors with a flow between them. This flow is calculated on the basis of the law of induction which necessitates knowledge of the inductive voltage $U_L$ between the capacitors. This inductive voltage is a part of the total voltage between capacitors, i.e., $U_{C(i+1)} - U_{C,i}$ if there is a resistive element. The part taken by the resistive element is calculated from $U_R = RI_Q$. Once we have a model for this part of the complete chain, elements can be copied and joined which leads to a model diagram such as the one shown in Fig. 1.36.

Qualitative reasoning can give us an idea of which quantities the wave speed should depend upon (for a derivation of the wave equation and the speed of propagation see Section 3.6). Resistive properties of the chain lead to attenuation of the wave; they can be made smaller (even equal to zero, in theory) which lessens the damping but should not affect the wave otherwise. So we do not expect the resistance to determine the speed of propagation of a wave, at least not significantly. Inductance and capacitance, on the other hand, seem to be the factors that are directly responsible for wavelike behavior. We know that they determine the frequency of oscillation between two capacitors (Equ.(1.40)).

Take an electric coaxial cable and represent it in a simplified manner by an RCL chain as in Fig. 1.35. Assume that we have chosen 10 elements to model the cable. What keeps us from using 20 or 40 or still more element? Models with successively more elements should be better representations of the cable, not fundamentally different ones. Within limits set by how well a number of elements models the cable, each mod-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.36.png}
\caption{Top: Diagram of dynamical model of a chain of LCR elements as in Fig. 1.35. Bottom: Simulation of electric current at the center of a 1 m cable, represented by 40 RCL elements. Total capacitance: 1.0 F, total inductance: 1.0 H, resistance equal to zero. Input: A short pulse at $t = 0.025$ s. Note that it takes about 0.5 s for the input pulse to arrive at $x = 0.5$ m, and 1.5 s for the wave to reflect at the end of the cable and come back to the same point. This corresponds to a wave speed of 1.0 m/s.}
\end{figure}
el should yield the same value for the propagation speed of the wave. Now, if we take 20 instead of 10 elements for the same physical object, we have to reduce the capacitances and inductances by a factor of 2. If we were to use a pair of values to calculate the wave speed, different models would lead to different results which contradicts our assumption. Therefore, it seems reasonable to assume that capacitance per length and inductance per length are the determining factors for wave speed:

\[ c \sim \sqrt{\frac{l}{C L}} \]  

(1.41)

c denotes the speed of propagation, and \( l \) stands for the length of the cable. The form is reasonable on grounds of physical units since the product of \( C \) and \( L \) has units of time squared. The product of capacitance and inductance per length has units of \( s^2/m^2 \), so Equ.(1.41) has correct units. Playing around with a dynamical model of a wave guide can confirm this hunch and tells us that the factor in Equ.(1.41) must be equal to one (see the simulation results in Fig. 1.36).

Both the models for diffusion in Fig. 1.28 and for wave propagation in Fig. 1.35 tell an important story for the later parts of this book. When we divide a body transporting a fluid-like quantity into ever smaller elements, we arrive at a model of spatially continuous processes. Waves will be taken up again briefly in Chapter 3, and continuous systems will be investigated in quite some detail in Parts III and IV of this book.

**EXERCISES AND PROBLEMS**

1. Two currents of water are flowing into a fountain. The first changes linearly from 2.0 liters/s to 1.0 liters/s within the first 10 s. The second has a constant magnitude of 0.50 liters/s. In the time span from the beginning of the 4th second to the end of the 6th second, the volume of the water in the fountain decreases by 0.030 m\(^3\). (a) Calculate the volume flux of the current leaving the fountain. (b) How much water will be in the fountain after 10 s, if the initial volume is equal to 200 liters?

2. Consider a tank having two outlets as in Fig. P.2. There is a constant inflow. Take vegetable oil as the liquid. Assume the tank to be half filled initially. How will the level of oil in the tank change in the course of time? Sketch a diagram showing different possibilities and explain how the different cases depend upon the magnitude of the inflow.

3. A battery is used to charge a capacitor through a resistor. The voltage across the capacitor has been measured as a function of time (Fig. P.3). The parameters of the system are to be determined with the help of a dynamical model. Resistance of the resistor, capacitance, internal resistance of the battery and open circuit (oc) voltage are unknown. (a) Sketch the diagram of a system dynamics model. (b) Formulate all equations of the model. (c) The open circuit voltage can be determined directly from the data. Why? How? Determine the \( \text{oc} \) voltage. (d) Determine the time constant of the circuit. (e) More experiments will be needed for a determination of all parameters. Describe possible measurements for a sufficient number of additional parameters. Show how the parameters can be determined. Assume that the elements of the circuit plus other similar elements are available for experimentation. Volt meters and ammeters and power supplies are available.

4. Derive the expression for the hydraulic capacitance of a U-pipe (as in Fig. 1.29). If you look at each of the sides of the U-pipe as a capacitor, are these capacitors connected in parallel or in series?

5. Two tanks (see Fig. P.5) contain oil with a density of 800 kg/m\(^3\) and a viscosity of 0.20 Pa\cdot s. Initially, in the container having a cross section of 0.010 m\(^2\), the fluid stands at a level
of 10 cm; in the second container (cross section 0.0025 m²) the level is 60 cm. The hose connecting the tanks has a length of 1.0 m and a diameter of 1.0 cm. (a) What is the volume current right after the hose has been opened? (b) Calculate the pressure at A, B, C, and D at this point in time. The pressure of the air is equal to 1.0 bar, and C is in the middle of the hose. (c) Sketch the levels in the containers as a function of time. (d) Sketch an electric circuit which is equivalent to the system of containers and pipe. (e) Sketch a pressure profile (pressure as a function of position) for a path leading from A to D; include a point C* at the other end of the pipe from point B.

6. A hydraulic windkessel system (Fig. P.6 left) consisting of an ideal pump, a valve and two identical containers is to be transformed into an electric system to be modeled formally (use and ideal diode for the valve). There is a short pipe between the containers. The diagram, Fig. P.6 right, shows the voltages across the (ideal) power supply (square wave signal) and across the two capacitors. Values of resistances and capacitances are assumed to be constant. The resistance of the resistor corresponding to the pipe leading to the environment is 1000 Ω.

(a) Sketch, label, and explain an electric circuit diagram that corresponds to the windkessel model. (b) Sketch the diagram of a system dynamics model for the electric windkessel circuit. Use electric symbols for quantities. (c) Formulate all relevant equations of the model. (d) Shortly after switching on the power supply, the capacitor voltages become constant (roughly between 75 s and 80 s). Why? Give a formal explanation. Use this to determine the missing resistances. (e) Use the behavior of the system to show that the capacitance of a single capacitor must be roughly equal to 5 mF. (f) What is the electric current between the capacitors shortly before 80 s?

7. The figure (Fig. P.7.1) shows an electric circuit. Resistances and capacitances are constant. The capacitance of the first capacitor is \( C_1 = 1.0 \cdot 10^{-4} \) F. Create a hydraulic model for the circuit; perform the calculations for the electric quantities. Initially, the first capacitor is charged, the second one is uncharged. At \( t = 0 \) s, the first switch (S1) is closed, the second one stays open. The diagram, Fig. P.7.2, shows the voltage across the first capacitor.

(a) Sketch a hydraulic system that corresponds to the electric circuit (b) Sketch the diagram of a system dynamics model for the hydraulic system. (c) Formulate all the equations of
the hydraulic model. (d) Sketch the voltage across the second capacitor for $0 \leq t \leq 40$ and explain the result. (e) Determine the capacitance of the second capacitor. (f) Determine the resistance of the resistor between the capacitors. (g) At $t = 40$ s, the second switch is closed as well. What will the currents through the circuit be and what are the rates of change of the voltages of the two capacitors right after closing of $S2$? Assume the resistance of the second resistor to be equal to that of the first. (h) Sketch the voltages across the capacitors all the way to 80 s and explain your result.

8. For the circuit diagram shown in Fig. P.8, (a) sketch a hydraulic system made up of containers, pipes, and pumps which would be equivalent to the electrical system shown here; (b) write down the equations governing the processes taking place in the circuit. Solve the differential equation for the current through the second resistor and demonstrate that the time constant of the system is equal to $\tau = CR_1R_2/(R_1 + R_2)$.

9. An inventory contains a quantity $M$ of a certain product. It is filled by a production rate $P$ and drained by a sales rate $S$ (the latter is assumed to be constant). There is a desired quantity ($E$) of the product in the inventory. When the stored quantity $M$ differs from $E$, the production rate is changed. $E$ is constant. Make the following assumptions regarding the change of $P$. The rate of change of $P$ is proportional to the difference between desired storage $E$ and the actual stored quantity $M$. We introduce a constant “inertia” or delay factor $L$ to calculate the relation between inventory difference and rate of change of $P$ (larger inertia leads to a smaller rate of change of $P$). (a) Sketch the diagram of a system dynamics model that represents the word model formulated above. (b) Formulate all equations of the model. Which equation is a law of balance? Which equation resembles an inductive relation? Which quantity is analogous to a pressure difference or a voltage? What is the unit of $L$? (c) What are the two differential equations of the model? What are the associated initial conditions? (d) Show that the two differential equations in (c) can be transformed into a single second order differential equation:

$$\frac{d^2M}{dt^2} + \frac{1}{L}M = \frac{1}{L}E$$

(e) What is the solution to the differential equation in (d) for $E = 0$? Is this the solution to a damped or an undamped oscillation? What is the period of oscillation? (f) What is the solution to the equation in (d)?

10. The ideal power supply of the circuit shown in Fig. P.10.1 is turned on at $t = 0$. The voltage of the power supply is set to 10 V. The curves in the graph of Fig. P.10.2 show the electric currents through the branches containing the (ideal) inductive element, the resistor, and the capacitor.

(a) Show which curve in the graph belongs to which element in the circuit. (b) Use the data in the graph to determine the inductance, resistance, and capacitance. (c) Formulate all equations of the model of the circuit (including initial conditions). (d) Convert the model to a single second order differential equation for the voltage across the capacitor, including initial conditions.
The Dynamics of Heat
A Unified Approach to Thermodynamics and Heat Transfer
Fuchs, H.U.
2010, XII, 734 p. 229 illus., Hardcover
ISBN: 978-1-4419-7603-1